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THESIS

SOME BOOLEAN REPRESENTATIONS

OF THE PROPOSITIONAL CALCULUS

by

Leonard A. Snider

1st Lt.

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Submitted in partial fulfillment of
the requirements for the degree of

MASTER OF SCIENCE
with major in
Mathematics

United States Naval Postgraduate School
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ABSTRACT

Boolean algebra has long played a well known role in the development of mathematical logic, but even in the propositional calculus there are many problems still to be investigated. Among these is the question of the feasibility of identifying the statement calculus with a Boolean algebra other than the (0, 1) algebra. The theory of Boolean algebra as required for a study of the algebraic aspects of logic is formulated. Characteristics of the equality relation are discussed and the propositional calculus is outlined with early emphasis on the equivalence classes, $[0]$ and $[1]$. The concepts of truth values, truth functions and truth sets are developed. Through these concepts, the statement calculus is identified with a Boolean algebra consisting of more than two elements.

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1. Boolean Algebra

As necessary introductory material in the development of the present thesis, this section defines and characterizes a Boolean algebra in terms of its set-theoretical aspects. A summarization of the properties of Boolean algebra relative to the discussion is included.

A Boolean algebra is defined as a non-empty set with two binary operations, $\cup$ (union) and $\cap$ (intersection), and one unary operation, ' (complementation). (These operations are called union, intersection and complementation because they will have the same properties as the corresponding operations in the special Boolean algebra of sets.)

The following set of axioms [16] are assumed as characterizing the operations of union, intersection and complementation. (The number of axioms in this set is not a minimum number required to establish a Boolean algebra but is satisfactory for the development of the material in this section.)

i. $A \cup B = B \cup A$; $A \cap B = B \cap A$

ii. $A \cup (B \cup C) = A \cup B \cup C$; $A \cap (B \cap C) = (A \cap B) \cap C$

iii. $(A \cap B) \cup B = B$; $A \cap (A \cup B) = A$

iv. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$; $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

v. $(A \cap A') \cup B = B$; $(A \cup A') \cap B = B$

To relate the Boolean algebra to set theory, we define a field of sets as any non-empty class $\mathcal{B}$ of subsets of a fixed non-empty space X, where $\mathcal{B}$ is closed with respect to the set-theoretic union, intersection and complementation. Then every field of sets is a Boolean algebra with the Boolean operations $\cup$, $\cap$, ' .

As a particular example [5], let $P(X)$ be the power set of X, i.e.

the class of all subsets of X. P(X) is a field of sets as defined above and, therefore, a Boolean algebra.

As a consequence of the above axioms, various results can be established and are summarized below.

The idempotent laws $A = AUA$ and $A = A \cap A$ follow directly from the axioms.

From the absorption laws, it follows that if one of $A \cap B = A$ or $AUB = B$ holds, then so does the other relation. In case

$$A \cap B = A \quad \text{and} \quad AUB = B ,$$

we say that A is contained in B and write $A \subseteq B$ where the relation $\subseteq$ is called the (Boolean) inclusion.

If the Boolean algebra $\mathcal{B}$ is a field of sets, then the Boolean inclusion $\subseteq$ coincides with the set-theoretic inclusion and has the following properties (those which define a partial ordering [17],

i. $A \subseteq A$ (reflexive)

ii. if $A \subseteq B$ and $B \subseteq A$, then $A = B$ (antisymmetric)

iii. if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$ (transitive)

where A, B, C are arbitrary elements of the Boolean algebra $\mathcal{B}$.

The element $A \cap A'$ is called the zero element (or zero) and is denoted by 0. It can be shown [16] from the axioms and the inclusion relation that $A \cap A'$ does not depend on the choice of A in $\mathcal{B}$ and that the zero is unique.

The element $A \cup A'$ (which also may be shown to be independent of the choice of A and unique) is called the unit element and will be denoted by 1.

A familiar principle in Boolean algebra is the duality principle,

which follows primarily from the symmetrical nature of the operations $\cup$ and $\cap$. For example the set axioms remain unchanged if $\cup$ is replaced by $\cap$ and $\cap$ is replaced by $\cup$. The substitutions of $\cup$ by $\cap$ and $\cap$ by $\cup$ transforms the unit element into the zero element and the zero element into the unit. Therefore, given a statement about $\cup$, $\cap$, 0, 1, the dual statement is obtained by substituting $\cap$ for $\cup$, $\cup$ for $\cap$, 1 for 0, and 0 for 1. As a consequence of the principle of duality, we can restate the axioms using only $\cap$ and $'$. And, as duality appears in every development of the Boolean theory, a choice between two possible approaches should be made; the dual then follows as a natural consequence. In the development of logic, emphasis is placed on truth and probability (with "1" for "true"). The dual approach would emphasize falsity and refutability (with "0" for "false").

Closely related to the concept of duality, the De Morgan formulas provide a convenient method for transforming relations involving $\cup$ and $'$ into relations involving $\cap$ and $'$ in the following manner:

$$(A \cup B)' = A' \cap B' \; ; \; (A \cap B)' = A' \cup B'$$

The element $A \cap B'$ is denoted by $A-B$ and called the <u>difference</u> of A and B. If the Boolean algebra is a field of sets, then $A - B$ coincides with the set-theoretic difference of sets A and B. Some useful properties of difference are:

$$A \subseteq B \text{ if and only if } A - B = 0$$

and

$$A \subseteq B \text{ if and only if } A' \cup B = 1$$

Elements A, B in B are <u>disjoint</u> if $A \cap B = 0$. It then follows that

$$A \cap (B-A) = 0$$

since $A \cap (B - A) = A \cap (B \cap A') = B \cap (A \cap A') = B \cap 0 = 0$

2. Equality.

Throughout the first section, the equality symbol " = " was tacitly used in the sense of logical identity. Through everyday usage, the equality relation is considered as a synonym for either identity or a qualified degree of likeness. But as Stoll [17] has pointed out, if the relation is restricted to elements having identical form, then, in general, it is not possible to generate a Boolean algebra. Set relations, however, can be developed under " = " as identity by use of the Axiom of Extent: If A and B are sets and if, for all x, $x \in A$ if and only if $x \in B$, then $A = B$. For example, through the Axiom of Extent, it can be shown that the commutative relation, $A \cup B = B \cup A$, is a consequence.

A further analysis of the equality relation [18] leads to the conclusion that equality satisfies the axioms leading to the definition of an <u>equivalence relation</u> as a relation r in a set A such that:

i. x r x for all x in A (reflexive)

ii. If x r y, then y r x (symmetric)

iii. If x r y and y r z, then x r z (transitive)

The main feature of equivalence relations is that it divides all members of a set A into disjoint subsets called equivalence classes, denoted by $[x]$, and defined by $[x] = \{y \in A \mid x \; r \; y\}$ with the basic properties:

i. $x \in [x]$

ii. if x r y, then $[x] = [y]$ and conversely.

As a partition of a set A is a disjoint collection $\mathcal{A}$ of nonempty and distinct subsets of A such that each member of A is a

member of exactly one member of $\mathcal{A}$, it follows that the collection of distinct equivalence classes is a partition of A.

The fact that an equivalence relation defines a partition (and conversely) demonstrates that the reflexive, symmetric and transitive axioms are adequate to assure the desired separation into classes, and hence that they characterize equality. Consequently, any equivalence relation may be called an equality relation.

Throughout this investigation of the properties of Boolean algebra and the propositional calculus to follow, the problem will be to identify elements which, although not of identical form, exhibit a certain likeness denoted by the basic relation, = . This equals relation must lead to a partition of the basic set, and is therefore an equivalence relation.

It is then necessary to be more explicit about the meaning of equality and so we re-define a Boolean algebra $\mathcal{B}$ to include = as a primitive term as follows:

$$\mathcal{B} = (B, \cup, \cap, ', =)$$

such that the following axioms are satisfied:

i. = is an equivalence relation in A

ii. If $A = B$, then $A \cap C = B \cap C$ for all C

iii. If $A = B$, then $A' = B'$

iv. Axioms of Section 1 are satisfied.

Then it can be shown that if $A = B$, it follows that $A \cup C = B \cup C$ for all C as ii. and iii. above postulate a substitution principle.

Thus to use a Boolean algebra as a model it is necessary to interpret equality as well as the other elements in the definiens.

3. Propositional Calculus.

Historically, George Boole developed the theory of Boolean algebra (Algebra of Logic) as an aid in investigating the "laws of thought". It is not surprising then that there are many important applications of Boolean algebra to the theory of Mathematical Logic.

This section is intended to develop the necessary concepts of propositional calculus and to show its connection with Boolean algebra.

The usual way to see the connection between Boolean algebra and logic is to begin by examining the manner in which sentences are combined by means of sentential connectives [17]. Let S_0 be an arbitrary non-empty set containing at least two members (these members of S_0 are to be called <u>prime statements</u>). If $p \in S_0$, then associated with p we must assign a truth value T or F. Then let $\wedge$, $\vee$, $'$, be distinct objects not contained in S_0 (where, intuitively, $\vee$, $\wedge$, $'$, are to be thought of as the logical connectives "or", "and", "not", respectively).

The set S_0 can then be extended to a set S (the members of which are to be called <u>composite statements</u>) by adjoining all statements which can be formed by using the sentential connectives in all possible (but finite) ways. Then if p, q are in S_0, p', $p \wedge q$, $p \vee q$ are in S so that S is closed under the operations $'$, $\wedge$, $\vee$.

The propositional calculus is concerned with the truth values of composite statements in terms of one of the truth-value assignments (T for "true" or F for "false") to the prime statements and in terms of the interrelations of the truth values of composite statements having some prime components in common.

If p and q are in S_0, then $p \wedge q$ and $q \wedge p$ are distinct

elements in S but truth value seems to require that if p and q are sentences, then "p and q" and "q and p" should have a certain degree of likeness (or the same truth value).

Two propositions of the propositional calculus are said to be <u>equivalent</u> (eq) if they have the same truth value, so that in the intuitive example above

$$p \wedge q \quad eq \quad q \wedge p$$

which corresponds with the law of Boolean algebra

$$A \cap B = B \cap A$$

Here, then the elements $\cap$ and $=$ of the Boolean algebra correspond with the elements $\wedge$ and eq of the propositional calculus. Similarly, the elements $\cup$ and $'$ correspond with $\vee$ and $'$ respectively, and the laws of Boolean algebra also hold for the logic of propositions. In addition, there exists an interpretation of Boolean algebra axioms as follows:

B	$\cup$	$\cap$	$'$	1	0	$=$
S	$\vee$	$\wedge$	$'$	$p \vee p'$	$p \wedge p'$	eq

Thus the propositional calculus is a Boolean algebra and henceforth Boolean symbols and logic symbols will be freely interchanged.

The elements of S which correspond with the 0 and 1 of the Boolean algebra are $p \wedge p'$ and $p \vee p'$, respectively. Considering the elements of S as sentences, a statement of the form $p \wedge p'$ ("both p and not p") is intuitively "false". A statement of the form $p \vee p'$ ("p or not p"), on the other hand would seem to be "true" - and is so

treated in classical Aristotlean logic.

The truth value of composite statements is defined in accordance with "truth tables". Truth table definitions for the connectives "not" (negation), "and" (conjunction), and "or" (disjunction) are given in the following table. [4]

		Negation	Conjunction	Disjunction
p	q	p'	$p \wedge q$	$p \vee q$
T	T	F	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	F

Table 1
Negation, Conjunction and Disjunction

These definitions are intuitively clear. If a statement is true, its negation is false (and vice versa). For the conjunction of two statements to be true, it is necessary that both prime statements be true. The disjunction is used in the inclusive or meaning: "p is true or q is true or both are true".

All the 16 truth functions of two variables can be expressed in terms of $\vee$, $\wedge$, and '. For example, "if...then..." (material implication; conditional), and "if and only if" (material equivalence; biconditional) which are consequences of the definitions of Table 1, are listed in the following table:

		Conditional	Biconditional
p	q	$p' \vee q$	$(p \wedge q) \vee (p' \wedge q')$
T	T	T	T
T	F	F	F
F	T	T	F
F	F	T	T

Table 2
Conditional and Biconditional

The conditional $p' \vee q$ is commonly denoted $p \rightarrow q$, so that $p \rightarrow q$ is an alternate way of indicating material implication. $p \rightarrow q$ represents the assertion that for each possible pair of corresponding values p_o, q_o of the statements p and q, "either p_o is false or, if p_o is true, then q_o is true also".

The biconditional is commonly denoted by the symbol $p \leftrightarrow q$. Then $p \leftrightarrow q$ asserts "either p and q are true or p and q are false." In comparing the biconditional with identity, it is seen $p = q$ means that p and q are the same statements while $p \leftrightarrow q$ means that p and q have the same truth value.

The following table combines the truth tables of Tables 1 and 2 with the remainder of the 16 different truth functions of two prime statements. [9]

1	2	3	4	5
0	0 0 0 0	F	Inconsistent	False; never
1	0 0 0 1	$p \wedge q$	Disjunction	And; both
2	0 0 1 0	$p \wedge q'$		Only p
3	0 0 1 1	p		p
4	0 1 0 0	$p' \wedge q$		Only q
5	0 1 0 1	q		q
6	0 1 1 0	$(p \wedge q') \vee (p' \wedge q)$	Symmetric Difference	Or else
7	0 1 1 1	$p \vee q$	Conjunction	either
8	1 0 0 0	$p' \wedge q'$	Double stroke	neither
9	1 0 0 1	$(p \wedge q) \vee (p' \wedge q')$	Biconditional	alike; if and only if
10	0 1 0	q'		not q
11	1 0 1 1	$p \vee q'$	Conditional	if q, then p
12	1 1 0 0	p'		not p
13	1 1 0 1	$p' \vee q$	Conditional	if p, then q
14	1 1 1 0	$p' \vee q'$	Scheffer stroke	nor (neither...,nor)
15	1 1 1 1	T	Tautology	True; always

Table 3
The 16 Truth Functions of Two Statements

The truth table definitions are arranged in a convenient row form with 0 for F and 1 for T. Column 1 is the decimal equivalent of the binary number in column 2. Column 2 is the 2^{2^n} possible truth functions of $n = 2$ propositions. Column 3 are the possible combinations of the sentential connections $\vee$, $\wedge$, $'$. Column 4 lists the usual names

of the expressions listed in Column 3. Column 5 lists the common verbal expressions applied to the propositions of column 3 [15].

For example, 13 is the decimal equivalent of 1101 as $1.2^3 + 1.2^2 + 0.2 + 1 = 13$. This binary number represents $p' \vee q$ as the only zero in 1101 occurs in the column where $p = 1$ and $q = 0$, thus corresponding with the truth table for the conditional ("if p, then q") as derived in Table 2.

Some of the logical functions defined in Table 3 are of equal importance with the basic ones defined in Table 1. #6 is known as the exclusive or and asserts "p or q but not both". The exclusive or is normally symbolized by $p \triangle q$ and called the symmetric difference. #8, called the double stroke, uses the symbol, $p \uparrow q$. #14 is called the Scheffer stroke of p and q, asserts "either p' or q' or both" and is denoted by the symbol, $p \downarrow q$.

A proposition is a tautology (or is valid) if its truth value is T under all assignments of truth values to its prime statements. (#15 of Table 3 is then a tautology as it is true for all assignments of T or F to p or q.)

The traditional approach to establishing a set of tautologies (a subset of S) is to establish an initial set of tautologies and then provide a rule for new tautologies from the old [5]. As part of this procedure some abbreviations of admissible sequences (members of S) are formed:

i. if p and q are members of S, then write $p \wedge q$ for $((p') \vee (q'))'$

ii. write $p \rightarrow q$ for $p' \vee q$

iii. write $p \leftrightarrow q$ for $p \rightarrow q \wedge q \rightarrow p$

Then an initial set of tautologies consists of the sequences of one of the following forms:

i. $p \vee p \rightarrow p$

ii. $p \rightarrow p \vee q$

iii. $p \vee q \rightarrow q \vee p$

iv. $(p \rightarrow q) \rightarrow ((r \rightarrow p) \rightarrow (r \rightarrow q))$

Then new tautologies can be formed by the following rule of inference (modus ponens): If p is a tautology and if $p \rightarrow q$ is a tautology, then q is a tautology.

Case #1 of Table 3 is an illustration of the class of propositions called inconsistent (propositions which are false for all assignments of T or F to the prime statements). All the other propositions (including #15) are consistent as they are true for some assignment of T or F to the prime statements.

One proposition implies another if there is no assignment of truth values which makes the first proposition true and the second false. (This logical implication is illustrated by #11, $p \vee q'$ for $q \rightarrow p$, and by #13, $p' \vee q$ for $p \rightarrow q$.)

Two propositions are equivalent if they imply each other. Combining this definition with the discussion on tautologies leads to a mechanical procedure for deciding if a proposition is a tautology by an examination of its truth table: Two propositions p and q are called logically equivalent if $p \leftrightarrow q$ is a tautology.

It can then be shown that the logical equivalence of propositions is an equivalence relation for:

i. $(p \leftrightarrow p) \rightarrow (p \leftrightarrow p) = (p' \vee p) \wedge (p' \vee p) \rightarrow (p' \vee p) \wedge (p' \vee p)$

$= (p \vee p)' \vee (p' \vee p)$

$= 0 \vee 1 = 1$

ii. $(p \leftrightarrow q) \rightarrow (q \leftrightarrow p) = \left[(p' \vee q) \wedge (q' \vee p)\right]' \vee \left[(q' \vee p) \wedge (p' \vee q)\right]$

$= \left[(p \vee q) \wedge (p' \vee q')\right] \vee \left[(p \vee q') \wedge (p' \vee q)\right]$

$= (p \vee 1) \wedge (q' \vee 1) \wedge (q \vee 1) \wedge (p' \vee 1) = 1$

iii. $(p \leftrightarrow q) \wedge (q \leftrightarrow r) \rightarrow (p \leftrightarrow r)$

$= \left[(p' \wedge q') \vee (p \wedge q)\right]' \vee \left[(q' \wedge r') \vee (q \wedge r)\right]'$

$\vee \left[(p' \wedge r') \vee (p \wedge r)\right]$

$= \left[(p \vee q) \wedge (p' \vee q')\right] \vee \left[(q \vee r) \wedge (q' \vee r')\right] \vee \left[1\right] = 1$

The propositional calculus is, then, characterized by:

i. the logical equivalence of propositions as an equivalence relation,

ii. the subset of tautologies, and

iii. the set S of all formulas of the theory based on the two-valued algebra of propositions.

4. Congruent Algebras

The set of axioms established for the Boolean algebra included the equality relation and the substitution principle. When the propositional calculus was identified as a Boolean algebra with equality meaning logical equivalence such a relation is a natural congruence relation. In this study of Boolean algebra as related to propositional calculus, there are reasons for introducing an equivalence relation, θ, other than the natural one. Then attention can be centered on $B/_\theta$ as the basic set whose elements are the θ-equivalence classes.

The relation θ in B is called a congruence relation [17] if:

i. θ is an equivalence relation in B

ii. If $a\theta b$, then $a\wedge c\ \theta\ b\wedge c$ for all c

iii. If $a\theta b$, then $a'\theta b'$

A congruence relation is called proper if θ is not the universal relation in B. (If θ is the universal relation, then $a\theta b$ if and only if $a,b\in B$ implies $[a] = B$ for all a in B. Then $B/_\theta = \{B\}$ which cannot constitute a Boolean algebra.)

Various properties of proper congruence relations follow from the definition; among them are;

1) If $a\theta c$ and $b\theta d$, then $a\wedge b\theta c\wedge d$ (which is a theorem analogous with the substitutivity property discussed earlier).

If B/θ is the set of θ - equivalence classes, $\{[a]\}$, this property becomes:

2) If $[a] = [c]$ and $[b] = [d]$, then $[a\wedge b] = [c\wedge d]$

In addition, it follows from the third requirement of congruence relations that $[a]' = [a']$. Define binary operations $\oplus$, $\odot$ in $B/_\theta$

such that

i. $[a] \oplus [b] = [a + b]$

ii. $[a] \odot [b] = [a \cdot b]$

and observe that

$[a] = [b] \leftrightarrow a\theta b$ defines equality in $B/_\theta$ as set equality.

Then, by verification of the axioms of the Boolean algebra, it can be shown that B/θ is a Boolean algebra (called the Boolean algebra induced by θ).

Therefore, from a Boolean algebra, B, and a proper congruence relation θ on B, another Boolean algebra, B/θ, may be derived. The elements of B/θ are the θ-equivalence classes and the operations of B/θ are defined in terms of those of the original algebra using representatives of equivalence classes. If θ is different from the equality relation in B, then B/θ may be essentially different from B.

Further relationships of the Boolean algebra, B/θ, to the Boolean algebra, B, under a proper congruence relation will be pointed out following a discussion of the properties of homomorphisms and isomorphisms

Let B and C be Boolean algebras. A mapping g of B onto C is said to be a homomorphism [16] provided the mapping preserves intersections and complements, that is

$$g(a \cap b) = g(a) \cap g(b)$$

$$g(a') = (g(a))'$$

As a consequence of the definition,

3) $g(a-b) = g(a \cap b') = g(a) \cap g(b')$

$= g(a) \cap (g(b))' = g(a) - g(b)$

The homomorphism transforms the zero and unit of B onto the zero and

unit of C, which is ambiguously denoted also by 0 and 1:

4) $g(0) = 0$; $g(1) = 1$

as $g(0) = g(a \wedge a') = g(a) \wedge (g(a))' = 0$

and dually for $g(1) = 1$

The homomorphism g preserves the inclusion;

(if $a \subseteq b \leftrightarrow a \wedge b' = 0$, then $g(a) \subseteq g(b)$).

For if $b = a \vee b$; $g(b) = g(a \vee b) = g(a) \vee g(b)$

A one-to-one homomorphism is called an <u>isomorphism</u> and the algebra B and C are <u>isomorphic</u> if there exists an isomorphism g of B onto C. Then g^{-1} is an isomorphism of C onto B.

In order that a one-to-one mapping B onto C be an isomorphism, it is necessary and sufficient that both g and g^{-1} preserve the inclusion. (This theorem then implies (4)), [16].

A homomorphism g of B into C is an isomorphism if and only if g^{-1} (0) contains only the zero of B, that is

5) $g(a) = 0$ implies that $a = 0$

Returning to the derived Boolean algebra B/Θ, additional theorems, definitions and concepts of a proper congruence relation are presented [17].

Let Θ be a proper congruence in a Boolean algebra B and define the functions $p : B \rightarrow B/\Theta$ by $p(a) = [a]$. Then p is a homomorphism. (p is called the natural mapping of B onto B/Θ.)

The algebra B/Θ of Θ - equivalence classes is a homomorphic image of B under the natural mapping on B onto B/Θ. If the algebra C is a homomorphic image of B, then C is isomorphic to some B/Θ. Moreover, if $f : B \rightarrow C$ is the homomorphism at hand, then $f = g \circ p$ where p is the natural mapping of B onto B/Θ and g is an isomorphism of B/Θ

onto C, where $\varphi \circ \psi$ is the composition of the mappings φ and ψ.

As a consequence of these theorems it follows that the homomorphisms of a Boolean algebra are in one-to-one correspondence with the proper congruence relations on the algebra.

If θ is the equality relation, then the elements of B will map onto one of the equivalence classes $[0]$ or $[1]$. If a proposition in B is refutable, then its image in B/θ is equal to 0. If a proposition in B is provable, then its image in B/θ is equal to 1. Then, with θ the equality relation, $[1]$ is the class of all tautologies and, as stated in the last section, a necessary and sufficient condition for $p \equiv q$ is that the biconditional of p and q be a tautology.

Then as was noted in the interpretation of Boolean algebra axioms in the previous section, S becomes a Boolean algebra after identification of equivalent formulas. This Boolean algebra is called the Lindenbaum algebra of the propositional calculus [16].

The fundamental completeness theorem of the propositional calculus [12] states that the formulas obtained from the set of axioms by means of the rules of inference coincides with the class of all tautologies. The completeness theorem can be obtained from the fundamental representation theorem stating that every Boolean algebra is isomorphic to a field of sets, and conversely, the fundamental representation theorem for Boolean algebra can be directly deduced from the completeness theorem [7].

5. Alternate Approaches to Truth-Values.

In defining the propositional calculus, a primary objective was to ensure that equivalent propositions were assigned the same truth-value. At the conclusion of the previous section, it was demonstrated that any two-valued homorphism gave a truth-valuation of the elements of S upon assignment of T or F to propositions according as the equivalence class is assigned to T or F. This procedure resulted in the truth-table solution and the collapse of the Boolean algebra to the (0, 1) algebra.

Various other aspects of the truth-value problem are discussed in this section. One of these will lead to the development of a general theory of truth-values in terms of truth-functions mapping the elements of the Boolean algebra onto a "larger" algebra than the (0, 1).

A first consideration involved in solving logical problems by the propositional calculus is embodied in the method of elimination which states that the making of a statement is an assertion of the non-existence of some of the classes [13]. Thus, logical reasoning involves elimination of situations which conflict with clearly definable rules expressed by a statement.

Two statements divide the universe into four classes characterized by possession or non-possession of a specified status. These four classes are $p \wedge q$, $p \wedge q'$, $p' \wedge q$, and $p' \wedge q'$. (These four disjoint classes are the "atoms" of the Boolean algebra. Their union is a tautology.) [1]

The other truth-functions of two variables can be expressed as exact equivalents in terms of these atoms.

For example:

i. $T = (p \wedge q) \vee (p \wedge q') \vee (p' \wedge q) \vee (p' \wedge q')$

ii. $p \vee q = ((p \wedge q) \vee (p \wedge q') \vee (p' \wedge q)) \wedge (p' \wedge q')'$

iii. $p \leftrightarrow q = ((p \wedge q) \vee (p' \wedge q')) \wedge ((p' \wedge q) \wedge (p \wedge q'))'$

iv. $p \rightarrow q = ((p \wedge q) \vee (p' \wedge q) \vee (p' \wedge q')) \wedge (p \wedge q')'$

These relations illustrate the following theorem: [14]

If B is a Boolean algebra with n (finite) elements, then B is isomorphic to B_1, the Boolean algebra of all subclasses of the class of all atoms in B.

Thus, the 16 truth-functions of two variables are expressed in forms distinct with respect to truth-value. If one of T or F is assigned to the prime statements, then the Boolean algebra (0, 1) would result. For example, if $p \in [0]$ and $q \in [1]$, then (using the decimal equivalents of Table 3):

$[0] = (0, 1, 2, 3, 4, 8, 9, 10)$

$[1] = (5, 6, 7, 11, 12, 13, 14, 15)$

As a further example of expressing the truth-functions in terms of the others, it has been shown [12] that $p' \wedge q'$ and $p' \vee q'$ are both statements in terms of which all the other functions may be expressed. For example, the Scheffer stroke connective, $p \downarrow q = p' \vee q'$, has the following relations with the other truth-functions: [15]

0	0
1	$(p \downarrow q) \downarrow (p \downarrow q)$
2	$\left[p \downarrow (p \downarrow q)\right] \downarrow \left[p \downarrow (p \downarrow q)\right]$
3	p
4	$\left[q \downarrow (p \downarrow q)\right] \downarrow \left[q \downarrow (p \downarrow q)\right]$
5	q
6	$\left[q \downarrow (p \downarrow q)\right] \downarrow \left[p \downarrow (p \downarrow q)\right]$
7	$(p \downarrow p) \downarrow (q \downarrow q)$
8	$\left[(p \downarrow p) \downarrow (q \downarrow q)\right] \downarrow \left[(p \downarrow p) \downarrow (q \downarrow q)\right]$
9	$(p \downarrow p) \downarrow (q \downarrow q) \downarrow (p \downarrow q)$
10	$(q \downarrow q)$
11	$q \downarrow (p \downarrow q)$
12	$p \downarrow p$
13	$p \downarrow (p \downarrow q)$
14	$p \downarrow q$
15	1

Table 4

Truth-functions expressed in items of the
Scheffer stroke connective

In the testing of theorems by machines, which is a central problem of modern logic, complete truth tables are essential. But if a machine is built primarily for solving problems in which one wishes to know what truth values for individual terms are uniquely determined by a given set of statements, it is possible to establish a process that dispenses with the requirement of scanning truth tables. Truth tables can be scanned rapidly if only a few terms are involved, but as the number of terms

increases, the scanning time increases at an accelerating rate. If the scanning procedure is eliminated, the exact status of each variable being investigated by a logic machine would be shown after each new statement is entered.

The digital computer solves problems in the propositional logic by assigning binary numbers to the various truth functions. These truth numbers for the 16 truth functions of two variables are shown in column 2 of Table 3. Truth numbers for various relations can be combined by simple arithmetical rules to determine the validity, consistency, or inconsistency of compound statements. For example if the final truth number is 1111, then the compound statement is a tautology.

As an example of "mechanized reasoning" in finding an "answer" to a logical problem involving a large number of prime statements, consider a problem with the following rules: [11]

If B, then C

A if and only if D

A or else B

The logic machine is then set up in the following fashion:

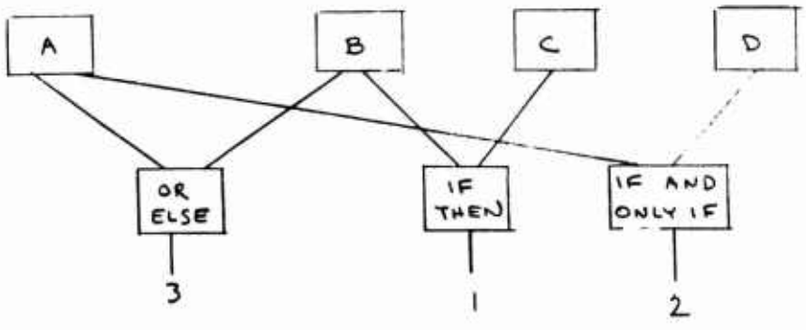

Diagram 1

where the OR ELSE, IF THEN, IF AND ONLY IF circuits perform electronically the definitions of Table 3 and where the leads 3, 1, 2 will be live (= 1) if the corresponding rules are satisfied, or will be grounded (= 0) if the rules are not satisfied.

To arrive at a solution, a trial solution is assumed and then modified by a "feedback" process to remove features inconsistent with the rules. For example, suppose the initial configuration is A'B'C'D'. The first rule (if B, then C) is satisfied so point 1 is live; the second rule (A if and only if D) is satisfied so point 2 is live; but the third rule (A or else B) is not satisfied, so point 3 is grounded, indicating that a change is required in the status of A or B. This change could be accomplished by a "feedback" signal from the ground at point 3 as shown in the following modification to diagram 1:

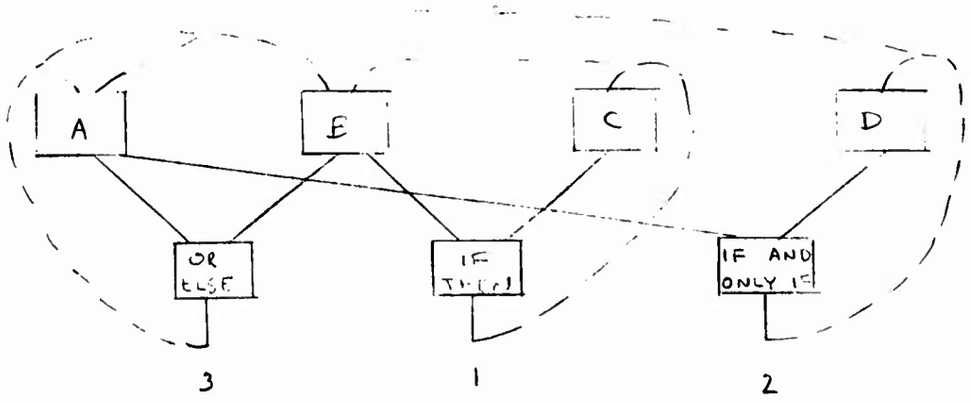

Diagram 2

If the status of A were changed, then the condition to be tested would be AB'C'D', for which rule 2 is not satisfied, requiring a change to AB'C'D. AB'C'D does satisfy all the rules and is therefore an answer to the problem. If all the answers (the other two are AB'CD and A'BCD') are desired, a new initial configuration would have to be

entered and the process repeated.

In the search for equivalence classes other than $[0]$ and $[1]$, the following definition is made:

Let $\mathcal{F}_0 = \{f \mid f : S_0 \rightarrow \{T,F\}\}$. The members of $\mathcal{F}_0$ are to be called truth functions [17] and consist of all mappings of prime statements into T,F where $f(p) = T$ is to be interpreted as "the prime statement p has truth value T".

The discussion and examples of this section have been intended to imply the feasibility of identifying the propositional calculus with a Boolean algebra other than the (0,1) algebra. Here, for example, no assumption as to the number of elements of B have been made and each element may have an arbitrarily assigned truth value.

So far, the discussion has been concerned with decidable propositions, those which are either true or false. Another kind of problem arises with a formula like

$$x + y = 5$$

where x and y are variable symbols representing arbitrary numbers. Formulas of this type represent propositional functions [9]. Replacement of the variable symbols x and y by arbitrary but specific numbers always results in a unique proposition to which the words "true" or "false" can be applied.

A propositional function may then be defined as a formula which contains one or more variable symbols whose allowable values are the members of some specific set. The propositional function becomes a proposition for any substitution of allowable values of the variables. Then the truth function, f (p), of the propositional function, p, becomes a

member of $\mathcal{F}$ if a determination of $f(p) = T$ or $f(p) = F$ can be made.

As an aid in developing further equivalence classes, a <u>partially ordered system</u> [2] is defined as any set P with a binary relation such that

i. $p \leq p$, for all $p \in P$ (reflexive)

ii. if $p \leq q$, and $q \leq p$, then $p = q$ (antisymmetric)

iii. if $p \leq q$ and $q \leq r$, then $p \leq r$ (transitive)

It has been shown [1] that the partial ordering relation symbol "$\leq$" is equivalent to the material implication symbol "$\rightarrow$", so that for truth functions the following properties hold:

i. $0 \leq f \leq 1$

ii. $f \leq f$ for all f

iii. If $f \leq g$ and $g \leq f$, then $f = g$

iv. If $f \leq g$ and $g \leq h$, then $f \leq h$

v. $f \leq g$ if and only if $f \wedge g = f$

vi. $f \leq g$ if and only if $f \vee g = g$

vii. $f \leq g$ if and only if $f \wedge g' = 0$

viii. $f \leq g$ if and only if $f' \vee g = 1$

These properties are clearly satisfied for the Boolean algebra $(0, 1)$. Thus $f \leq g$ is equivalent to the requirement that whenever f has the truth value 1, the value of g must also be 1. This requirement can be rephrased as "whenever f is true, g must also be true," which is equivalent to $f \rightarrow g$.

This ordering relation can then be applied to the truth functions of Table 3 using the property (v), $f \leq g$ if and only if $f \wedge g = f$,

to obtain the conditional relations:

$0 \leq 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15$

$1 \leq 3, 5, 7, 9, 11, 13, 15$

$2 \leq 3, 6, 7, 10, 11, 14, 15$

$3 \leq 7, 11, 15$

$4 \leq 5, 6, 7, 12, 13, 14, 15$

$5 \leq 7, 13, 15$

$6 \leq 7, 14, 15$

$7 \leq 15$

$8 \leq 9, 10, 11, 12, 13, 14, 15$

$9 \leq 11, 13, 15$

$10 \leq 11, 14, 15$

$11 \leq 15$

$12 \leq 13, 14, 15$

$13 \leq 15$

$14 \leq 15$

$15 \leq 15$

These relations then determine 16 classes such that given any conditional composite statement with one truth function as the antecedent and another as the consequent, then it can be immediately determined if the statement is a tautology. For example, $8 \leq 11$ (i.e. $f(p' \wedge q') \longrightarrow f(q' \vee p)$) is a tautology.

The partial order relation is not an equivalence relation, as the former has the antisymmetric property and the latter requires symmetry. The "$\leq$" was defined by abstracting the properties of order for real numbers. Every pair of two real numbers a and b are comparable.

However, for set inclusion, $\subseteq$, it is possible to have an incomparable pair of subsets. For example, p and p' (or any truth function and its complement) are not comparable as they are never simultaneous 1 or 0.

Returning to the development of equivalence classes for the members f of $\mathfrak{F}_o$, truth values for formulas can be developed in the following manner:

For any $f \in \mathfrak{F}_o$, with $p, q \in S_o$, define

1) $f(p') = \begin{cases} T & \text{if } f(p) = F \\ F & \text{if } f(p) = T \end{cases}$

2) $f(p \vee q) = \begin{cases} T \text{ if } f(p) = T \text{ or } f(q) = T \\ F \text{ otherwise} \end{cases}$

3) $f(p \wedge q) = \begin{cases} T \text{ if } f(p) = T \text{ and } f(q) = T \\ F \text{ otherwise} \end{cases}$

Then if r is any formula in $S - S_o$, f(r) can be derived from the above definitions through a finite sequence of members of S_o using the sentential connectives $\wedge$, $\vee$, $'$. For example if $r = p \vee q$ where $p = p_1' \wedge q_1$ and $q = p_1 \wedge q_1'$, then

$$f(r) = f(p) \vee f(q) = f(p_1' \wedge q_1) \vee f(p_1 \wedge q_1')$$

$$= \begin{cases} T & \text{if } f(p_1' \wedge q_1) = T \text{ or } f(p_1 \wedge q_1') = T \\ F & \text{otherwise} \end{cases}$$

$$= \begin{cases} T \begin{cases} \text{if } f(p_1') = T \text{ and } f(q_1) = T, \text{ or} \\ \text{if } f(p_1) = T \text{ and } f(q_1') = T \end{cases} \\ F \quad \text{otherwise} \end{cases}$$

$$= \begin{cases} T \begin{cases} \text{if } f(p) = F \text{ and } f(q) = T, \text{ or} \\ \text{if } f(p) = T \text{ and } f(q) = F \end{cases} \\ F \quad \text{otherwise} \end{cases}$$

which agrees with the truth values of the symmetric difference function.

Usually, as in Tables 1, 2 and 3, f is omitted and it is understood that a particular truth function f is being examined. If values of f(p) and f(q) were specified, then exactly one row (column of Table 3) would be sufficient to determine the value of f(r).

We define an equivalence relation in S by:

if $p, q \in S$, then $p = q \leftrightarrow f(p) = f(q)$ for all $f \in \mathfrak{F}_o$. (Two propositions p and q are equivalent if and only if truth table values are identical.)

Then as $f(p \vee p') = \begin{cases} T & \text{if } f(p) = T \text{ or } f(p') = T \\ F & \text{otherwise} \end{cases}$

$= T$ if $f(p) = T$ or $f(p) = F$

$= T$ for all $f \in \mathfrak{F}_o$, let $1 = p \vee p'$ for $p \in S_o$.

Similarly, let $0 = p \wedge p'$, and it then follows that

$f(1) = T$ for all $f \in \mathfrak{F}_o$, and

$f(0) = F$ for all $f \in \mathfrak{F}_o$

Thus there exists a function, f, assigning to each formula of S of the propositional calculus a truth-value, T or F. For the formulas of Tables 1 and 2, these assignments can be expressed in the following form:

i. $f(p') = T$ if and only if $f(p) = F$

ii. $f(p \vee q) = T$ if and only if $f(p)$ or $f(q) = T$

iii. $f(p \wedge q) = F$ if and only if $f(p)$ or $f(q) = F$

iv. $f(p \rightarrow q) = F$ if and only if $f(p) = T$ and $f(q) = F$

or $f(p \rightarrow q) = T$ if and only if $f(p) \leq f(q)$

v. $f(p \leftrightarrow q) = T$ if and only if $f(p) = f(q)$

The truth table method provides an adequate means of expressing all possible truth functions of any number of prime statements. If $f \in \mathfrak{F}_o$ is selected then the statement calculus reduces to the (0,1) algebra. Otherwise, there will be more than two equivalence classes by the above definition. Thus the propositional calculus is adequate to express all truth functions and is said to be functionally complete [3]

The 16 truth functions of two arguments, $f_0(p,q),\ldots,f_{15}(p,q)$ of Table 3 are of three different kinds. Tautologous truth functions are functions whose values are true regardless of the truth or falsehood of their arguments. Contradictory truth functions are functions whose values are false regardless of the truth or falsehood of their arguments. Contingent truth functions are functions which are true for some values of their arguments and false for others.

Another aspect of truth values is developed through the concept of a truth set, which is defined as follows: [10]

Let $\mathcal{U}$ be a set of logical possibilities, and p, q r... be statements relative to $\mathcal{U}$; let P, Q, R,... be the subsets of $\mathcal{U}$ for which statements p, q, r,.... are respectively true; then we call P, Q, R,... the truth sets of statements p, q, r,....

For the example on "mechanized reasoning" in this section, $\mathcal{U}$ is the set of 16 logical possible combinations of the letters A, B, C, D and their negation; p, q, r are the statements "if B, then C", "A if and only if D", and "A or else B", respectively; and the truth sets P,Q,R are AB'C'D, AB'CD, and A'BCD'.

The concept of the equality of two functions becomes clearer if thought of in terms of truth sets. Let f and g be two functions

defined on the same domain D. The statement $f(x) = g(x)$ determines a certain truth set consisting of elements x for which the two functions happen to have the same value. The truth set may be empty, if the two functions have no common value. The truth set may be all of D, which then implies that $f = g$. Thus, the two functions are equal if $f(x) = g(x)$ has as its truth set the entire domain.

The definition of truth set above is equivalent to the following formulation of the problem: (Note: Material is from class notes taken in a course on Boolean Algebra presented by Dr. Peter W. Zehna of the Naval Postgraduate School, Monterey. A search of the literature by the author of this thesis failed to show that this approach is available elsewhere.)

Let S be the Boolean algebra of the propositional calculus. For every $p \in S$, define the truth set of p, denoted $\mathcal{T}_p$, by

$$\mathcal{T}_p = \left\{ f \in \mathcal{T}_o \mid f(p) = T \right\}$$

then $\mathcal{T}(p) \subseteq \mathcal{T}_o$ for all $p \in S$.

The truth set of p has the following properties:

1) $\mathcal{T}_{p'} = (\mathcal{T}_p)'$

Proof: $f \in \mathcal{T}_{p'} \Leftrightarrow f(p') = T \Leftrightarrow f(p) = F \Leftrightarrow f \notin \mathcal{T}_p \Leftrightarrow f \in (\mathcal{T}_p)'$

2) $\mathcal{T}_{p \vee q} = \mathcal{T}_p \cup \mathcal{T}_q$

Proof: $f \in \mathcal{T}_{p \vee q} \Leftrightarrow f \in \mathcal{T}_p$ or $f \in \mathcal{T}_q \Leftrightarrow f(p) = T$ or $f(q) = T \Leftrightarrow f(p \vee q) = T \Leftrightarrow f \in \mathcal{T}_p \cup \mathcal{T}_q$

3) $\mathfrak{F}_{p \wedge q} = \mathfrak{F}_p \cap \mathfrak{F}_q$

Proof: $f \in \mathfrak{F}_{p \wedge q} \Leftrightarrow f \in \mathfrak{F}_p$ and $f \in \mathfrak{F}_q \Leftrightarrow f(p) = T$

and $f(q) = T \Leftrightarrow f(p \wedge q) = T \Leftrightarrow f \in \mathfrak{F}_p \cap \mathfrak{F}_q$

4) $p = q \Leftrightarrow \mathfrak{F}_p = \mathfrak{F}_q$

Proof: i) Suppose $p = q$ Then $f(p) = f(q)$ for all $f \in \mathfrak{F}$. Thus if $f \in \mathfrak{F}_p$, then $f(p) = T = f(q)$ so that $f \in \mathfrak{F}_q$, and conversely. Hence $\mathfrak{F}_p = \mathfrak{F}_q$

ii) Suppose $\mathfrak{F}_p = \mathfrak{F}_q$. If $f(p) = T$, then $f \in \mathfrak{F}_p$, and so $f \in \mathfrak{F}_q$, and $f(q) = T$. If $f(p) = F$, then $f \notin \mathfrak{F}_p$, then $f \notin \mathfrak{F}_q$, so $f(q) = F$. Hence, $p = q$.

5) $\mathfrak{F}_0 = \emptyset$; $\mathfrak{F}_1 = \mathfrak{F}$

Proof: i) $f \in \mathfrak{F}_0 \Rightarrow f(0) = T \Rightarrow \mathfrak{F}_0 = \emptyset$

ii) $f \in \mathfrak{F}_1 \Rightarrow f(1) = T \Rightarrow \mathfrak{F}_1 = \mathfrak{F}$

(Then: $p \in S$ is a tautology $\Leftrightarrow \mathfrak{F}_p = \mathfrak{F}$)

6) $p \rightarrow q$ is a tautology $\Leftrightarrow \mathfrak{F}_p \subseteq \mathfrak{F}_q$

Proof: i) Suppose $p \rightarrow q$ is a tautology. Then $\mathfrak{F}_{p \rightarrow q} = \mathfrak{F} = \mathfrak{F}_{p' \vee q}$; and $(\mathfrak{F}_{p' \vee q})' = \emptyset = \mathfrak{F}_{p \wedge q'} = \mathfrak{F}_p \cap \mathfrak{F}_{q'}$

$= \mathfrak{F}_p \cap (\mathfrak{F}_q)' \Rightarrow \mathfrak{F}_p \subseteq \mathfrak{F}_q$

ii) Suppose $\mathfrak{F}_p \subseteq \mathfrak{F}_q$. (Then the converse follows by reversing the above steps.)

7) If $\{p_1, p_2, \ldots, p_n\}$ are inconsistent, then

$$\bigcap_{i=1}^{n} \mathfrak{F}_{p_i} = \emptyset$$

Proof: $\{p_1, \ldots, p_n\}$ inconsistent $\Leftrightarrow$ for all $f \in \mathfrak{F}$,

$f(p_i) = F$ for some $i \Leftrightarrow$ for all $f \in \mathfrak{F}$,

$f \in \mathfrak{F}_{p'_i}$ for some $i \Leftrightarrow$ for all $f \in \mathfrak{F}$,

$$f \in \bigcup_{i=1}^{n} \left(\mathfrak{F}_{p_i}\right) = \left(\bigcap_{i=1}^{n} \mathfrak{F}_{p_i}\right)' \Leftrightarrow \mathfrak{F} \subseteq \left(\bigcap_{i=1}^{n} \mathfrak{F}_{p_i}\right)'$$

$$\Leftrightarrow \bigcap_{i=1}^{n} \mathfrak{F}_{p_i} = \emptyset$$

6. Summary and Recommendations.

This thesis has considered some of the relations between Boolean algebra and the propositional calculus. In particular, the concepts of truth value, truth functions and truth sets were defined and developed.

The Boolean algebra operations of union, intersection and complementation were shown to have the same properties as the corresponding operations in the algebra of sets. Later these operations were shown to be analogs of the logical sentential connectives, "or", "and", and "not".

The equality symbol was discussed in its various uses as identity in form or as showing a specified degree of likeness in an equivalence relation. Characteristics of equivalence relations and partitions were discussed. Section 2 ended with the conclusion that it was necessary to interpret equality in using Boolean algebra as a model.

The propositional calculus was outlined in a standard way to develop the necessary concepts and to show its interpretation as a Boolean algebra. A "truth table" was established in terms of the connectives "or", "and", and "not" for the 16 truth-functions of two variables. The properties of these functions were discussed and the table was used to illustrate various concepts throughout the thesis. The main characteristics of the propositional calculus were summarized at the conclusion of Section 3.

Properties of congruence relations, homomorphisms and isomorphisms were then discussed. From a Boolean algebra, B, and a proper congruence θ on B, an induced Boolean algebra B/θ was developed. Section 4 concluded with the observation that if θ is the equality relation, then the elements of the Boolean algebra, B, will map onto one of the

equivalence classes, $[0]$ or $[1]$.

Section 5 developed concepts designed to show that a Boolean algebra of the statement calculus may consist of more than the (0, 1) algebra. Background material and examples led to a discussion of truth functions. Truth functions were developed from fundamental concepts of mapping statements into the set $\{T, F\}$. As another aspect of the problem truth sets were introduced and their properties studied.

Most of the material of the thesis could have been developed within the framework of free Boolean algebras [6]. A subset S of an algebra A generates A if S is not included in any proper subalgebra of A. If, moreover, every function from S into an algebra B has a (necessarily unique) extension that is a homomorphism from A to B, then S is a set of free generators of A. An algebra is free if it has a set of free generators.

The following are illustrations of some concepts of free Boolean algebras related to the material of this thesis:

1) Every four-element Boolean algebra is a free Boolean algebra with one free generator [16].

Hence all of the following sub-algebras of the Boolean algebra of Table 3 are examples of free Boolean algebras:

(0, 1, 14, 15)	(0, 5, 10, 15)
(0, 2, 13, 15)	(0, 6, 9, 15)
(0, 3, 12, 15)	(0, 7, 8, 15)
(0, 4, 11, 15)	

with either of the two elements distinct from 0, 15 serving as a free generator.

2) A finite Boolean algebra is free if and only if it has 2^{2^n} elements. It then has n free generators and 2^n atoms.

Thus, for $n = 2$, the 16 elements would be generated by the two free generators, p and q, and would consist of the four atoms, $p \wedge q$, $p \wedge q'$, $p' \wedge q$, and $p' \wedge q'$.

The main concern of this thesis is with the propositional calculus and its associated algebraic aspects in the theory of Boolean algebra. More detailed studies in logic belong to the monadic functional calculus, the pure first-order functional calculus, and the functional calculus with equality whose algebraic aspects are found in monadic algebras, polyadic algebras, and cylindric algebras respectively [6].

A monadic (Boolean) algebra is a pair $(A, \exists)$, where A is a Boolean algebra and $\exists$ is a quantifier on A. The monadic algebra is generalized to the concept of a quantifier algebra $(A, I, \exists)$ where I is a set of valuables.

The theory of cylindric algebras (which could also be called quantifier algebras) aims at providing a class of algebraic structures that bear the same relations to (first-order) predicate logic as the class of Boolean algebras bears to sentential logic [8].

Quantifier algebras are found to be an inefficient logical tool as they do not allow for treatment of transformation of variables. This limitation then leads to the development of polyadic algebras $(A, I, S, \exists)$ with S a function from transformations on I to A.

The algebraization of various portions of predicate logic has its origins in the nineteenth century. Pierce and Schroder developed the logic of binary predicates. Tarski expanded the subject further in the

form of modern algebraic theory dealing with structures called relation algebras [19].

As a recommendation for further study, an investigation of the concepts and methods of free Boolean algebras should prove helpful in establishing algebraic structures for logic. For example, a next step following the material of this thesis would be the introduction and study of quantifiers. Then, the examination of the predicate calculus would coincide with the study of Boolean algebras with unions and intersections corresponding to the logical quantifiers. The algebras of predicate calculi are the free algebras of this class of Boolean algebras [16].

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