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ON THE USE OF A LYAPUNOV FUNCTION TO DETERMINE LOGIC FOR A SATURATING TYPE CONTROL OF A NON-LINEAR SYSTEM

WATSON W. GOLDSMITH

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ON THE USE OF A LYAPUNOV FUNCTION TO DETERMINE LOGIC FOR A SATURATING TYPE CONTROL OF A NCN-LINEAR SYSTEM

Watson W. Goldsmith

ON THE USE OF A LYAPUNOV FUNCTION

TO DETERMINE LOGIC

FOR A SATURATING TYPE CONTROL

OF A NON-LINEAR SYSTEM

by

Watson W. Goldsmith // Lieutenant, United States Navy

Submitted in partial fulfillment of the requirements for the degree of

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MASTER OF SCIENCE IN ELECTRICAL ENGINEERING

United States Naval Postgraduate School Monterey, California

LIBRARY U.S. NAVAL POSTGRADUATE SCHOOL MONTEREY, CALIFORNIA

ON THE USE OF A LYAPUNOV FUNCTION

TO DETERMINE LOGIC

FOR A SATURATING TYPE CONTROL

OF A NON-LINEAR SYSTEM

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Watson W. Goldsmith

This work is accepted as fulfilling

the thesis requirements for the degree of

MASTER OF SCIENCE

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United States Naval Postgraduate School

ABSTRACT

A method is developed, using the quadratic form of canonical uncoupled state variables as a Lyapunov function to determine the switching logic for a quasi-optimum control of a non-linear dynamic system.[#] Analytic arguments are presented to support the method, which is capable of being extended to any order system and any number of controls, subject to certain practical limitations noted in the analysis.

A computational scheme for determining the canonical state variables and control functions is presented and a topological interpretation is made of the choice of variables used for the control logic calculation.

The controller based on the method described is applied to a nonlinear second-order system. Phase trajectories for both the uncontrolled and controlled systems are obtained by means of a digital computer simulation. Various aspects of the theoretical limitations of the method presented are investigated and the experimental results are analyzed with respect to the theoretical predictions.

#The system is postulated to be described by a system of differential equations of known form.

TABLE OF CONTENTS

Section	Title				
1.	Introduction	1			
2.	Theory	5			
3. 1	Application of Theory to Design of a Single Controller	15			
4	Simulation of Control of a Non-Linear Second Order System	20			
5. (Conclusions	36			
1 1	Appendix I - Flow Charts and FORTRAN Language Program for Digital Computer Simulation	37			
1 :	Appendix II - Graphs of Simulated System Trajectories	44			
1	Bibliography	68			

.

LIST OF ILLUSTRATIONS

Figure		Page
(1-1)	Geometric Interpretation of the Various Types of Stability	1
(2-1)	Control Function Ui As an Ideal Relay	9
(2-2)	Control Function U _i as a Relay With Dead Zone	9
(2-3)	Plot of $y_i(t)$ vs. Time For $u_i = -a_i \operatorname{sgn} (\underline{E}^T \underline{y})_i$, Stable Eigenvalue	10
(2-4)	Plot of $y_{i}(t)$ vs. Time For $u_{i} = -a_{i} \operatorname{sgn} (\underline{E}^{T}\underline{y})_{i}$, Unstable Eigenvalue, $y_{i}(0) > \left \frac{a_{i}}{\lambda_{i}} \right $	11
(2-5)	Plot of $y_i(t)$ vs. Time For $u_i = -a_i \operatorname{sgn} (\underline{E}^T \underline{y})_i$, Unstable Eigenvalue, $y_i(0) < \left \frac{a_i}{\lambda_i} \right $	11
(2-6)	Control Function u _i With the Characteristics of a Saturating Amplifier	12
(2-7)	Geometric Interpretation of Successive Linear Approximation Process	14
(4-1)	Phase Portrait For an Uncontrolled VanderPol Equation	20
(4-2)	Root Locus Plot for Equation (4-2), $-\infty < k < \infty$	21
(4-3)	Root Locus of the Uncontrolled VanderPol Equation in Stable Limit Cycle	22
(4-4)	System Optimum Switching Line	33
(4-5)	Root Locus for Uncontrolled System, Initial Conditions x = .5, x = .5	35
(4-6)	Root Locus for Controlled System, Initial Conditions $x = .5, \dot{x} = .5$	35 •

1. Introduction.

The stability of dynamic systems whose equations of motion are such that their solutions are difficult or impossible to obtain in closed form may successfully be analyzed by means of Lyapunov's Second Method.

We have a given dynamic system described by the set of differential equations as

$$\dot{\mathbf{x}} = \underline{g}(\mathbf{x}); \ \underline{g}(0) = 0$$
. (1-1)

Then, referring to Fig. (1-1), the origin, according to LaSalle and Lefschetz⁵ is:

stable whenever for each $R \checkmark A$ there is an $r \And R$ such that if a path (a motion) g* initiates at a point x° of the spherical region S(r) then it remains in the spherical region S(R) ever after; that is, a path starting in S(r) never reaches the boundary sphere H(R) of S(R);

asymptotically stable whenever it is stable and in addition every path g⁺ starting inside some $S(R_{o})$, $R_{o} \gg o$, tends to the origin as time increases indefinitely;

unstable whenever for some R and any r, no matter how small, there is always in the spherical region S(r) a point x such that the path g⁺ through x reaches the boundary sphere H(R).



Fig. $(1-1)^5$

Geometric Interpretation of the Various Types of Stability.

Now suppose we have a given dynamic system described by the set of differential equations written in vector form as

$$\frac{\dot{\mathbf{x}}}{\mathbf{x}} = \underline{g}(\mathbf{x}, \mathbf{u}) \tag{1-2}$$

or

$$\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{D}\mathbf{u} + \mathbf{g}^{*}(\mathbf{x},\mathbf{u}) \tag{1-3}$$

where F is a constant square matrix, D is the control distribution matrix, <u>u</u> is the control function vector, and $\underline{g}^{*}(x,u)$ contains terms of a second and higher power in x and u. The linearized form of the equation is

$$\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{D}\mathbf{u} \tag{1-4}$$

and can be said to adequately describe the motion of the system for small values of \underline{x}_{\bullet}

According to Kalman and Bertram⁴, a Lyapunov function may be defined as some scalar function of the state variables $V(\underline{x})$ with the following properties: (a) $V(\underline{x}) > 0$, $\tilde{V}(\underline{x}) < 0$ when $\underline{x} \neq \underline{x}_{\Theta}$, where \underline{x}_{Θ} is the equilibrium state, and (b) $V(\underline{x}) = \tilde{V}(\underline{x}) = 0$ when $\underline{x} = \underline{x}_{\Theta}$.

We wish to apply Lyapunov's Second Method to the design of a controller so as to return the state vector (\underline{x}) to a position of equilibrium (\underline{x}_{e}) in some optimum manner. No loss of generality will result if we choose (\underline{x}_{e}) to be the origin of the state space, and henceforth the equilibrium position will be considered as such.

Lyapunov stated that if a Lyapunov function V(x) exists (being positive definite) in some neighborhood of the origin, and if $-\dot{V}(x)$ is also positive definite in some neighborhood of the origin, the system is asymptotically stable⁵.

In general the method employed will be to make $\tilde{V}(x)$ as negative as possible in order to drive the system to the origin as quickly as possible.

The Second Method of Lyapunov leaves open to the individual investigator the choice of a particular Lyapunov function to be used in the analysis of a given system. For a given system then, there are possibly an unlimited number of such functions. The quadratic form of the canonical state variables as the particular Lyapunov function employed in the design of the controller was governed by the simplification obtained in the centrol logic.

Analysis and design for a controller of a linear, stationary system using the quadratic form procedes in a straightforward manner. In the case of a non-linear system however, the limitation that the linear approximation is sufficiently accurate only within some region surrounding the equilibrium position poses a rather severe restriction on the extent of the state space for which a given control can be useful.

In order to overcome this limitation to some degree the controller was designed in such a manner that the control logic was obtained by successive linear approximations to the actual system at points on the system trajectory.

For best performance of the system the time involved in the calculation of the control logic should be infinitesimal so that there would be no appreciable time delay in the application of the proper control function. That is, the system state vector would not have moved from the point (\underline{x}_1) , where the state variables were sampled, to a new position on the trajectory, (\underline{x}_2) where the control function was actually applied.

In actual practice a finite calculation time is of course necessary so that the state vector will always have moved from the point of sampling before the new value of control function can be applied.

For plants with long time constants, such as might exist in chemical or metallurgical processes or ship stabilization systems, presently

available digital computers have computation speeds adequate to successfully provide on line, real time control.

Application of this method to plants with short time constants will depend upon increased speeds of the computers and the development of more refined methods of calculation.

2. Theory.

If we have given a Lyapunov function $V(x) \approx x^{T} P x$ we may assert that⁴ the equilibrium state $\mathbf{x}_{\Theta} = 0$ of a linear dynamic system

$$\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} \tag{2-1}$$

is asymptotically stable if and only if given any symmetric positivedefinite matrix Q, there exists a symmetric, positive definite matrix P which is the unique solution of the set of linear equations described by

$$\mathbf{F}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{F} = -\mathbf{Q} . \tag{2-2}$$

A simple procedure for calculating the control logic attributed to R.W. Bass¹ procedes as follows: Choose Q > 0 arbitrarily. P > 0 may then be calculated from (2-2). $\|\mathbf{x}\|^2 P$ will then be a Lyapunov function for the system (2-1). We now choose u(t) so as to make $\dot{V}(x)$ as negative as possible. Let

$$V(\mathbf{x}) = \underline{\mathbf{x}}^{\mathrm{T}} \mathbf{P} \underline{\mathbf{x}} = ||\mathbf{x}||^{2} \mathbf{P}, \text{ then}$$

$$\dot{\mathbf{V}}(\mathbf{x}) = \underline{\dot{\mathbf{x}}}^{\mathrm{T}} \mathbf{P} \underline{\mathbf{x}} + \underline{\mathbf{x}}^{\mathrm{T}} \mathbf{P} \underline{\dot{\mathbf{x}}} \cdot$$
(2-3)

We wish to consider the effect of control on V. Taking the transpose of (1-4) we obtain

$$\underline{\dot{\mathbf{x}}}^{\mathrm{T}} = \underline{\mathbf{x}}^{\mathrm{T}} \mathbf{F}^{\mathrm{T}} + \underline{\mathbf{u}}^{\mathrm{T}} \mathbf{D}^{\mathrm{T}}$$
(2-4)

and

$$\mathbf{\tilde{V}}(\mathbf{x}) = \mathbf{x}^{\mathrm{T}} \mathbf{F}^{\mathrm{T}} \mathbf{P} \mathbf{x} + \mathbf{x}^{\mathrm{T}} \mathbf{P} \mathbf{F} \mathbf{x} + \mathbf{u}^{\mathrm{T}} \mathbf{D}^{\mathrm{T}} \mathbf{P} \mathbf{x} + \mathbf{x}^{\mathrm{T}} \mathbf{P} \mathbf{D} \mathbf{u}$$

or

$$\mathbf{\dot{v}}(\mathbf{x}) = \underline{\mathbf{x}}^{\mathrm{T}}(\mathbf{F}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{F})\underline{\mathbf{x}} + 2\underline{\mathbf{u}}^{\mathrm{T}}\mathbf{D}^{\mathrm{T}}\mathbf{P}\underline{\mathbf{x}}$$

o. To To.

Since P is symmetric, $D^{T}P = PD$. Simplifying this gives

$$\dot{\mathbf{V}}(\mathbf{x}) = -\underline{\mathbf{x}}^{\mathrm{T}}\mathbf{Q}\underline{\mathbf{x}} + 2\underline{\mathbf{u}}^{\mathrm{T}}\mathbf{D}^{\mathrm{T}}\mathbf{P}\underline{\mathbf{x}}$$

or
$$\dot{\mathbf{V}}(\mathbf{x}) = -\|\mathbf{x}\|^{2}\mathbf{Q} + 2\underline{\mathbf{u}}^{\mathrm{T}}\mathbf{D}^{\mathrm{T}}\mathbf{P}\underline{\mathbf{x}} \quad (2-5)$$

To

To make $\dot{V}(x)$ as negative as possible we want the term $2u^{T}D^{T}Px$ to be as negative as possible. Therefore the control logic is given as

$$u_{i}(t) = -a_{i} \operatorname{sgn} (D^{T} P_{x})_{i} \qquad (2-6)$$

where the a_i are constants denoting the maximum allowable magnitudes of each individual control effort u_i . It is obvious from (2-6) that in order to make the term $2\underline{u}^T\underline{D}^T\underline{P}\underline{x}$ as negative as possible the individual controller $u_i(t)$ must always exert a <u>maximum effort</u>, and in such a direction that the term

$$-a_i \operatorname{sgn}(D^T P \underline{x})_i$$

is, in fact, negative. Thus the control logic becomes merely logic for detecting the appropriate switching points of the "bang-bang" controller $u_i(t)$.

In the case where there is only one controller the vector \underline{u} may be of the form such that $\underline{u}^{T} = (0,0,0...u)$. The system equation is then of the form

where the b_i are the coefficients of the characteristic equation of the unforced system, taken in reverse order.

If we make the equivalence transformation from the physical state space to a canonical state space denoted by the matrix equation

$$\underline{y} = G\underline{x} \tag{2-8}$$

then by differentiating we get

$$\dot{\mathbf{y}} = \mathbf{G}\mathbf{\mathbf{x}} \tag{2-9}$$

and equation (1-4) becomes

$$\underline{\mathbf{y}} = \mathbf{G}^{-1}\mathbf{F}\mathbf{G}\mathbf{y} + \mathbf{G}^{-1}\mathbf{D}\mathbf{u} \, .$$

We choose G such that

$$G^{-1}FG = \Lambda$$
 (2-10)

where Λ is a diagonal matrix with the same eigenvalues as $\mathbf{F}^{\#}$.

For the system with one controller such as described by (2-7) we may further choose the particular transformation matrix G such that

$$G^{-1}Du = E$$

where $\underline{E}^{T} = (a, a, a, ...a)$. This has the effect of weighting the value of all the state variables equally in determining the sign of u(t).

If the P of (2-2) is chosen to be the identity matrix I and the \bigwedge matrix substituted for F, equation (2-2) becomes

$$\Lambda^{\mathrm{T}_{\mathrm{I}}} + \mathrm{I} \Lambda = -\mathrm{Q}. \tag{2-11}$$

If all the real parts of the eigenvalues of F are negative, so then are all those of Λ , and Q is positive definite. Also

$$\hat{\mathbf{V}}(\mathbf{x}) = \underline{\mathbf{y}}^{\mathrm{T}}(\mathbf{\Lambda}^{\mathrm{T}} + \mathbf{\Lambda})\underline{\mathbf{y}} + 2\underline{\mathbf{E}}^{\mathrm{T}}\underline{\mathbf{y}} \cdot (2-12)$$

The first term on the right side of (2-12) is again negative and the control logic is given by the second term. For a single controller, we have

$$u(t) = -a \operatorname{sgn} \left(\sum_{i}^{n} y_{i} \right) .$$
 (2-13)

In other words the switching function is such that the control must change sign whenever the sum of the n canonical state variables changes sign.

Consider the given form

 $y = \Lambda y + G^{-1}Du$.

#It is always possible to perform the equivalence transformation (2-10) provided F has no repeated eigenvalues. A system described by an nth order differential equation having repeated roots may be transformed into a partially uncoupled set of 1st order differential equations. It will be assumed that the systems considered in this paper have no multiple roots. If we take the LaPlace transform we get

$$s\underline{Y}(s) - \underline{y}(0) = \Lambda \underline{Y}(s) \Rightarrow G^{-1}\underline{Du}(s)$$

or

$$\underline{Y}(s)(sI - \Lambda) = \underline{y}(0) + G^{-1}\underline{Du}(s) . \qquad (2-14)$$

Since from (2-6) the control variable should assume only values of $+ \underline{a}$ or $-\underline{a}$, we stipulate that it is a constant during a given control period (between switchings). We then have the LaPlace identity

$$(G^{-1}D\underline{u})(s) = \frac{\underline{a}}{s}$$

Solving (2-14) for a given component y_i we get

$$Y_i(s)(s - \lambda_i) = y_i(0) + \frac{a_i}{s}$$

or

$$Y_{i}(s) = \frac{y_{i}(0)}{(s - \lambda_{i})} + \frac{a_{i}}{s(s - \lambda_{i})}$$
 (2-15)

Then the time domain solution of (2-15) is

$$y_i(t) = y_i(0)e^{\lambda_i t} - \frac{a_i}{\lambda_i} (1 - e^{\lambda_i t})$$

or

$$y_i(t) = \left\{ y_i(0) + \frac{a_i}{\lambda_i} \right\} e^{\lambda_i t} - \frac{a_i}{\lambda_i} . (2-16)$$

An interesting result becomes apparent from this solution as t increases without bound. For the case where λ_i is negative (a consequence of Q being positive definite) we see that

$$y_i(t) \rightarrow \frac{a_i}{\lambda_i}; t \rightarrow \infty$$

This result indicates that although this form of controller is useful in making $\dot{V}(\mathbf{x})$ more negative in an asymptotically stable system, it will not result in $V(\mathbf{x}) \rightarrow 0$ as $\mathbf{t} \rightarrow \boldsymbol{\infty}$ if u is a function of the type seen in Fig. (2-1).



Fig. (2-1) Control Function U_i As An Ideal Relay.

In fact it will result in a condition of chatter-motion around the origin as the controller becomes dominant at points in the state space where the $y_i < ||e||$: where ||e|| assumes some particular small value.

This objectionable feature may be overcome in the case of asymptotically stable systems by choosing the control function of the form shown in Fig. (2-2) below so that the controller is effectively disengaged within some region arbitrarily close to the origin, allowing the system to reach the equilibrium condition in an unforced manner.



Fig. (2-2) Control Function U_i As A Relay With Dead Zone.

If the uncontrolled system were such that one (or more) roots contained positive real parts, (in this case -Q can no longer be positive definite) then for some particular points in state space, it is still

possible that

$$\dot{v}(x) = -||x^2||Q$$
.

However $\dot{V}(x) > 0$ and remains so.

Consider a plant governed by the equations $\frac{dx/dt = F_{X} + Du(t)}{|u_{i}(t)| \leq a_{i} < \infty \quad (i = 1, \dots, m) \quad (2-17)}$ where the control variables are subject to the constraints, $|u_{i}(t)| \leq a_{i} < \infty \quad (i = 1, \dots, m) \quad (2-17)$ This is essentially the relay or saturating servo problem. The problem is to return every initial state to the origin. It is well known that if F has eigenvalues with positive real parts then there are some states which cannot be returned to the origin by any control subject to the constraints $(2-17)^{4}$.

If a controller of the form described is applied to the unstable system, it is apparent from (2-5) that control may be maintained so long as the inequality (2-18) below exists.² That is

$$-\|\mathbf{x}\|^{2}Q + 2\underline{u}^{T}D^{T}P\underline{x} < 0. \qquad (2-18)$$

From the solutions (2-16) the behavior of each $y_i(t)$ for 0 < t < k, where k denotes the switching period is as in Fig. (2-3).









Fig. (2-5) Plot of $y_i(t)$ vs. Time For $u_i = -a_i \operatorname{sgn} (\underline{E}^T \underline{y})_i$, Unstable Eigenvalue, $y_i(0) < \left| \frac{a_i}{\lambda_i} \right|$

In the event that the system distribution matrix D were such that the controls were uncoupled, i.e. for each y_i there were a corresponding u_i , then it would be possible to design a controller whose individual elements changed sign at the zero crossings of the individual state variables. From Fig. (2-5) it can be seen that the element would switch at t = T. It is possible to calculate $T_i = T_i(y_i(0))$ from (2-16) for the case where $\lambda_i > 0$, $y_i(0) > 0$, $u_i < 0$.

$$y_{i}(t) = \left\{ y_{i}(0) - \frac{a_{i}}{\lambda_{i}} \right\} e^{\lambda_{i}T} + \frac{a_{i}}{\lambda_{i}}, let y_{i}(t) = 0$$

then

$$\left\{y_{i}(0) - \frac{a_{i}}{\lambda_{i}}\right\} e^{\lambda_{i}T} = -\frac{a_{i}}{\lambda_{i}}$$

giving

$$\mathbf{e}^{\boldsymbol{\lambda}_{i}T} = - \frac{\mathbf{a}_{i}}{\boldsymbol{\lambda}_{i}} \left\{ \frac{\boldsymbol{\lambda}_{i}}{\boldsymbol{\lambda}_{i}\mathbf{y}_{i}(0)-\mathbf{a}_{i}} \right\}$$

and

$$\mathbf{r} = \frac{1}{\lambda_{i}} \ln \left\{ \frac{-\mathbf{a}_{i}}{\lambda_{i} \mathbf{y}_{i}(0) - \mathbf{a}_{i}} \right\}. \qquad (2-19)$$

24

From (2-19) it is clear that for $y_i(0) = 0$, $T = \frac{1}{\lambda_i} \ln(1) = 0$, indicating that this system also will result in chatter-motion around the origin.

Selection of a control function u_i of the form of Fig. (2-2) would be unsatisfactory in this case because the uncontrolled system is unstable at the origin. A better form of function would be that of the form of a saturating emplifier with a high gain. (See Fig. (2-6)).



Fig. (2-6) Control Function u_i With the Characteristics of a Saturating Amplifier.

Employing LaSalle and Lefschetz's definitions⁵ of stability described in Section 1., a system whose eigenvalues all have negative real parts is asymptotically stable with no control, merely stable with a "bang-bang" controller (Fig. (2-1)), and again asymptotically stable if the controller contains a dead zone as in Fig. (2-2). Given an arbitrary H(R), a system with eigenvalues lying in the right half plane is defined as unstable, and the same system with a controller of the form described may be defined as stable if X° is restricted to lie within some S(r) whose radius r is of some necessarily small value. The restrictions on the allowable values of r are necessarily related to the constraints on the magnitudes of the control functions (2-17).

Given a non-linear system described by the equation $\underline{x} = F(x)\underline{x}_{,p}$ the linear approximation of the non-linear system referred to by Kalman and Bertram⁴ may be obtained by a Taylor series expansion of the coefficient matrix F(x) at the origin. The linearized, constant, matrix F has the elements $\frac{\partial F_i(x)}{\partial x_i}$. (The matrix F is the Jacobian of F(x)with respect to x at x = x(0) = 0).

Suppose a system described by the equation

$$\dot{\mathbf{x}} = F(\mathbf{x})\mathbf{x} + \mathbf{D}\mathbf{u}$$

is to be controlled in an environment where the disturbances are of such a magnitude as to carry the state variable beyond the region of validity of the linear approximation given by (1-4). Referring to Fig. (2-7), this condition corresponds to an x° lying outside of S(r). By expanding F(x) about the point x° and obtaining the Jacobian of F(x) with respect to x at $\underline{x} = x^{\circ}$ we may obtain a linear approximation at x°

$$\dot{\mathbf{x}} = \mathbf{F} \mathbf{x} + \mathbf{D} \mathbf{u} . \qquad (2-20)$$



Fig. (2-7) Geometric Interpretation of Successive Linear Approximation Process.

Control logic based on this approximation will be valid while the trajectory remains within the spherical region S(p). If $S(p^n)$ is the region of validity of the linear approximation due to the n^{th} sampling, then by an iterative process such that the trajectory from X^n to X^{n+1} always remains inside $S(p^n)$ we may obtain an optimum control policy based on the particular Lyapunov function employed for the complete trajectory for x^0 to $x_e = 0$.

3. Application of Theory to Design of a Single Controller.

It was shown in Section 2. that the control logic resulting from the choice of the quadratic form of the uncoupled state variables as the system Lyapunov function is of the form

$$u(t) = -a \operatorname{sgn} \left\{ \sum_{i=1}^{n} y_{i} \right\}$$

when the system has only one controller. Thus, the control problem is reduced to determining the sign of the sum of canonical state variables and insuring that the control, u(t), assumes the opposite sign. The steps involved in this process consist of the following:

- 1. Measuring the physical state variables (x).
- 2. Calculating the canonical state variables (\underline{y}) from the measured (\underline{x}) .
- 3. Forming $\sum y_i$.

4. Applying the proper control $u(t) \approx -a \operatorname{sgn} y_i$.

As was shown in Section 2., the variables (\underline{y}) may be calculated by the equation

$$\underline{\mathbf{y}} = \mathbf{G}^{-1}\underline{\mathbf{x}} \,. \tag{3-1}$$

The problem then reduces to evaluating G^{-1} .

It may be shown that a matrix G such that

 $G^{-1}FG = \Lambda \tag{3-2}$

where \bigwedge is a diagonal matrix, may be formed by calculating the eigenvectors of the matrix F and constructing an array in which each eigenvector is a particular column of the array. Once a G has been obtained it is merely necessary to calculate the inverse, G^{-1} , in order to be able to calculate the state variables (y).

The elements of the diagonal matrix Λ are the eigenvalues of F. If the system is oscillatory the elements of Λ will necessarily be complex. Since the elements of the coefficient matrix F of the original differential equation are all real, consideration of (3-2) reveals that for an oscillatory system there must be complex elements in G and G^{-1} .

When G is complex, care must be taken in evaluating G^{-1} . If we let G = ($\mathbb{R}G$ + $j\mathbb{R}G$, and attempt to form $G^{-1} = (\mathbb{R}G)^{-1}$ + $j(\mathbb{R}G)^{-1}$ difficulty arises immediately. Since complex roots arise in conjugate pairs, there will be two eigenvectors (columns) of ($\mathbb{R}G$) due to the identical real parts of the conjugate pairs. These vectors will be identical, and therefore linearly related so that ($\mathbb{R}G$) will be singular. Similar reasoning shows that ($\mathbb{R}G$) will also be singular, so that it is impossible to obtain ($\mathbb{R}G$)⁻¹ and ($\mathbb{R}G$)⁻¹.

Let us form a new matrix H in the following manner. If the order of G is n, the order of H will be 2n so that for every complex element of G there will be four elements of H arranged in a square pattern. The two elements on the principal diagonal of the square have a value equal to the real part of the corresponding complex element of G. The absolute values of the remaining two elements are equal to the imaginary part of the corresponding complex element of G, the element in the lower left hand corner, taking the positive sign, while that in the upper right hand corner, the negative sign. Where the corresponding element in G is real, zeros appear in the locations in H corresponding to the missing imaginary parts.

If now we form H^{-1} , then we may obtain the elements of G^{-1} from the proper locations of H^{-1} .

However since we are interested in obtaining the (\underline{y}) corresponding to the measured (\underline{x}) , the simplest procedure is to form the 2 x n supplementary vector $(\underline{x})^*$ which has alternately the real and imaginary parts of the physical state variables. Since these state variables may

be measured and have physical significance, they may only be real, so that (\underline{x}) consists of the values of the state variables alternating with zeros. Matrix multiplication of H⁻¹ on the right by (\underline{x}) results in a 2 x n column vector whose elements are the real and imaginary parts of the canonical space variables (y).

EXAMPLE. The product of a complex $2 \ge 2$ square matrix and a column vector with real elements

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} R_{11} + jI_{11} & R_{12} + jI_{12} \\ R_{21} + jI_{21} & R_{22} + jI_{22} \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Then

$$\begin{bmatrix} \mathbf{y}_{1} \\ \mathbf{y}_{2} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{11}\mathbf{x}_{1} + \mathbf{j}\mathbf{I}_{11}\mathbf{x}_{1} + \mathbf{R}_{12}\mathbf{x}_{2} + \mathbf{j}\mathbf{I}_{12}\mathbf{x}_{2} \\ \mathbf{R}_{21}\mathbf{x}_{1} + \mathbf{j}\mathbf{I}_{21}\mathbf{x}_{1} + \mathbf{R}_{22}\mathbf{x}_{2} + \mathbf{j}\mathbf{I}_{22}\mathbf{x}_{2} \end{bmatrix}$$

Rearranging gives

$$y_{1} = (R_{11}x_{1} + R_{12}x_{2}) + j(I_{11}x_{1} + I_{12}x_{2}) \quad (3-3)$$

$$y_{2} = (R_{21}x_{1} + R_{22}x_{2}) + j(I_{21}x_{1} + I_{22}x_{2}) \quad (3-4)$$

Suppose instead we form a 4×4 matrix of the form H⁻¹. Then

$$\begin{bmatrix} \mathbf{Q} \mathbf{y}_{1} \\ \mathbf{A} \mathbf{y}_{1} \\ \mathbf{Q} \mathbf{y}_{2} \\ \mathbf{Q} \mathbf{y}_{2} \\ \mathbf{A} \mathbf{y}_{2} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{11} & \div \mathbf{j}\mathbf{I}_{11} & \mathbf{R}_{12} & \lnot \mathbf{j}\mathbf{I}_{12} \\ \div \mathbf{j}\mathbf{I}_{11} & \mathbf{R}_{11} & \div \mathbf{j}\mathbf{I}_{12} & \mathbf{R}_{12} \\ \mathbf{R}_{21} & -\mathbf{j}\mathbf{I}_{21} & \mathbf{R}_{22} & -\mathbf{j}\mathbf{I}_{22} \\ \mathbf{j}\mathbf{I}_{21} & \mathbf{R}_{21} & \mathbf{j}\mathbf{I}_{22} & \mathbf{R}_{22} \end{bmatrix} \times \begin{bmatrix} \mathbf{x}_{1} \\ \mathbf{0} \\ \mathbf{x}_{2} \\ \mathbf{0} \end{bmatrix}$$

and

&y ₁	98	^R 11 ^x 1	+	R12x2
l yl	=	jI ₁₁ x ₁	+	jI ₁₂ %2
Q . y ₂	90	R ₂₁ x ₁	+	R ₂₂ x ₂
Jy2	-	jI ₂₁ x1	+	jI ₂₂ %₂ ∘

Rearranging gives

$$y_1 = (R_{11}x_1 + R_{12}x_2) + j(I_{11}x_1 + I_{12}x_2)$$

 $y_2 = (R_{21}x_1 + R_{22}x_2) + j(I_{21}x_1 + I_{22}x_2)$ which is identical to (3-3) and (3-4).

It is evident from the above discussion that in the case of an oscillatory system the transformation

$$\underline{\mathbf{y}} = \mathbf{G}^{-1} \underline{\mathbf{x}} \tag{3-5}$$

may map a given physical state variable x_i from a point on the real axis to some point in the complex plane. Thus for a given coordinate axis in the physical n-space there is a complex plane in the canonical space.

Since, however, there is a one for one correspondence between points in the two spaces, a control which succeeds in reducing all state variables to zero in the canonical space will have also reduced the real state variables to zero.

The computation method employed to determine the transformation matrix G was based on a computer program MATSUB, programmed by Louis W. Ehrlich of the University of Texas, which calculates eigenvalues and eigenvectors of a matrix up to order 50. The general method employed is to make an initial guess for each eigenvector of the matrix F and to converge on the correct value by an iterative scheme due to E.E. Osborne⁶. The original program employed a guess of 1.0 for all components of the eigenvectors. Since it was assumed that the system matrix F was not changing at more than a moderate rate, a modification was made to the program so that the most recently calculated value for each eigenvector was selected as the initial guess for a new calculation thereby speeding the computation. The output of the MATSUB program was also modified so that the eigenvectors were arranged into two n x n matrices corresponding to the real and imaginary parts of the transformation matrix G.

From the two n x n matrices the 2n x 2n matrix H was formed and the inverse taken. The canonical state variables were then determined and the proper control calculated and applied.

The path of the system trajectory in the physical space (\underline{x}) was then calculated for a selected time interval between sampling instants by the Runge-Kutta-Gill method. Instantaneous values of the elements of the coefficient matrix F = F(x) were calculated for the new sampling point and the procedure repeated until the trajectory in the canonical space was within a pre-set epsilon region of the origin or a given time had elapsed.

Initially the ideal case was simulated by holding the solution of the system trajectory at the sampling point until the new calculation of the control function was made and the control applied. Simulation of the non-ideal case, a finite calculation time, was affected by applying the control calculated at the (n-1)th sampling, at the time of the nth sampling. This is equivalent to a controller continually applying a new control as soon as it can perform the sampling and control calculations.

Simplified flow charts of both the ideal and non-ideal case simulations are presented in Appendix I. The computer program in FORTRAN language (for the ideal case) is also included in Appendix I.

4. Simulation of Control of a Non-Linear Second Order System.

A. Background.

The non-linear VanderPol equation

$$\mathbf{x} - (1 - \mathbf{x}^2) \, \mathbf{\dot{x}} + \mathbf{x} = 0 \tag{4-1}$$

was selected as the system on which to evaluate the controller. Although it is only a second order equation it was felt that since it is quite non-linear it would be a fair test of the controller and the two dimension phase space has the advantage of making graphical analysis more simple.

The equation exhibits a stable limit cycle of the general shape shown in Fig. (4-1).



Fig. (4-1) Phase Portrait For an Uncontrolled VanderPol Equation.

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Referring to (4-1), when x = 0, the system is most unstable. For this point the roots are +.5 \pm j.866. It was determined from trajectories of the uncontrolled system obtained by digital computer analysis that when the system was in the limit cycle the maximum excursion of displacement was \approx 2.1925. This value was substantiated by an analog computer simulation. From (4-1) the eigenvalues, or roots, of the system when the displacement reaches its maximum value in the limit cycle were calculated to be -3.5235 and -.2835 so that the locus of roots of equation (4-1) during a complete circuit of its stable limit cycle is shown in Fig. (4-3).



Fig. (4-3) Root Locus of the Uncontrolled VanderPol Equation in Stable Limit Cycle.

Examination of Fig. (4-1) or equation (4-1) reveals that the system becomes unstable whenever $|\mathbf{x}| < 1.0$ and becomes stable again whenever $|\mathbf{x}| > 1.0$. Thus the equation presents the controller with the problem of controlling a system which contains at various times, stable real roots, stable complex roots, and unstable complex roots.

It is informative to observe that a stability analysis of the VanderPol equation by Lyapunov's Second Method yields precisely the same results as the more familiar methods of analysis such as the root locus.

Rewrite (4-1) in vector form. Then

$$\frac{\mathbf{\dot{x}}}{\mathbf{\dot{x}}} = \begin{bmatrix} 0 & 1\\ -1 & (1-\mathbf{x}_1^2) \end{bmatrix} \mathbf{\dot{x}}$$
(4-2)

or

 $\mathbf{\dot{x}}_1 = \mathbf{x}_2 \tag{4-3}$

and

$$\dot{\mathbf{x}}_2 = -\mathbf{x}_1 + (1 - \mathbf{x}_1^2)\mathbf{x}_2$$
 (4-4)

Choose

 $V(x) = ||x|| = x_1^2 + x_2^2$

then

$$\dot{N}(x) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2$$
 (4-5)

and from (4-3) and (4-4),

$$\dot{\mathbf{v}}(\mathbf{x}) = 2\mathbf{x}_1\mathbf{x}_2 + 2\mathbf{x}_2(-\mathbf{x}_1 + (1-\mathbf{x}_1^2)\mathbf{x}_2)$$

or

$$\dot{v}(x) = 2x_2^2(1-x_1^2)$$
 (4-6)

so that V(x) is positive definite but $-\dot{V}(x)$ is not for $|x_1| < 1$. Therefore the region near the origin (-1 < x < 1) is unstable. This is precisely the result obtained by observing that the root locus passes into the right half plane whenever $|x_1| < 1$.

B. Procedure.

It was decided to initially investigate the action of the control with no time delay between calculation of the control and its application in order to verify that the method of control and the simulation procedure (described in Section 3.) were feasible. First a family of trajectories of the system with no control effort was obtained from various initial conditions chosen inside and outside the limit cycle

(Graphs A-1 through A-8). Trajectories were then obtained using the same initial conditions with the control system activated (Graphs B-1 through B-8). The sampling rate was set at .3 seconds and u was set at 1.0. An arbitrary epsilon region surrounding the origin was established so that the trajectory was considered to have reached the origin when the sum of the canonical state variables squared was less than .001. Whenever the trajectory entered the epsilon region of the origin the solution was terminated.

Effects on the trajectories due to varying sampling rates (Graphs C-1 through C-5) were then investigated, and finally two trajectories were obtained for the system simulating the effect of finite calculation time (Graphs D-1 and D-2). These trajectories all were started from an initial condition close to the uncontrolled limit cycle trajectory.

The simulation program included a sub-routine for the graphical presentation of the system trajectories. The graphs noted above are presented in Appendix II.

C. Tabulation of Simulation Results.

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II-a. CONTROLLED CASE: SAMPLING RATE = .30 SECONDS, u =

1.0.
1.0. T\$ 2 .27 SECONDS. 85 II-b. CONTROLLED CASE: SAMPLING RATE

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GRA FH NULIDER	C-1	N 5	C=0	G⊷∆.	с С

This solution is erroncoup. See Section 4.D.

	REMARKS		Enters a modified limit cycle.
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37 2 1.5, 11 • 14 1.5 ы К JV. TIME-DELAY CASE: INITIAL CONDITIONS

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D. Discussion of Digital Computer Simulation.

Comparison of the trajectories of the controlled and uncontrolled systems starting from the same initial conditions (Graphs A-1 through A-8 and B-1 through B-8) clearly indicates that the control system with a control effort = 1.0 and a sampling rate of .30 seconds[#] is capable of driving the system into the unstable region surrounding the origin and maintaining it within some region of lesser extent than that enclosed by the system's uncontrolled limit cycle. Referring to Fig. (1-1), if the radius of S(R) were established for this system as R = .5, then the controlled system might be defined as stable while the uncontrolled system would be defined as unstable. Graph B-1, initial conditions x = .2, $\dot{x} = .2$, illustrates the system trajectory in the region close to the origin. The chatter-motion mode of operation is readily apparent.

The theoretical analysis of Section 2., predicting that by employment of this type of controller, an unstable system could be made stable, but not asymptotically stable, and that a chatter-motion mode of operation would result around the origin, is substantiated.

Examination of Graphs C-1 through C-5 reveals that an increase in the control system sampling rate causes the trajectory, having once arrived in the neighborhood of the origin, to remain thereafter within a smaller region of the origin than the same system operating with a low sampling rate.

Curves B-3 and C-3, both trajectories for the identical initial conditions and system parameters, demonstrate remarkably different trajectories, each of which eventually arrives in a neighborhood close to the origin.

[#]The time interval between successive calculations of the control function. The Runge-Kutta integration of the trajectory employed an interval of 0.03 seconds.

Examination of the calculated data revealed that in the case of Graph B-3 the initial values of the canonical state variables were

> $y_1 = .75 - j.1561$ $y_2 = .75 + j.1561$

while in the case of Graph C-3 the values were

$$y_1 = .75 - j.1561$$

 $y_2 = .75 - j.1561$.

Since the control function $u = -1.0 \operatorname{sgn} \left(\sum \mathcal{R} y_i + \sum \mathcal{A} y_i \right)$, it is evident that u assumed a different sign in each case. That this was so can be determined by examination of the diverging paths of the trajectories in the two cases.

It seemed evident that the more direct trajectory should be the correct one, however the possibility was raised that there might not be a unique solution for the control function. Six additional runs each exactly coincided with the more direct trajectory B-3, indicating that perhaps an error in calculation led to the selection of the opposite value of control function rather than there being two possible solutions for the control function.

The computation routine had been programmed so that the inverse transformation matrix was printed at each sampling point. It was discovered that the transformation matrices in the two cases were not identical.

INITIAL INVERSE TRANSFORMATION MATRIX FOR GRAPH B-3

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0E-10 .4	40032E-00	.50000E-00
	0E-10 1E-00 0E-10	0E-1040032E∞00 1E-00 .50000E-00 - 0E-10 .40032E-00

INITIAL INVERSE TRANSFORMATION MATRIX FOR GRAPH C-3

.37180E-08	.64051E-00	.50000E-00	-40032E-00
64051E-00	.37180E-08	40032E-00	.50000E-00
• 50000E-00	.40032E-00	.65670E-09	.64051E-00
40032E-00	。50000E-00	64051E-00	.65484E-09

In order to determine if two solutions were possible, the two inverse matrices were checked against the original matrix to see if both were valid.

The original matrix was calculated by hand in the same manner that the computer was programmed to do.

The procedure is illustrated below. The initial F matrix was $\begin{bmatrix} 0 & 1 \\ -1 & -1.25 \end{bmatrix}$ and the initial eigenvalues were -.625 ± j.7806. Then for the first eigenvalue

$$\begin{bmatrix} 0 & 1 \\ -1 & -1.25 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (-.625 + j_7806) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

or

$$\mathbf{x}_2 = (-.625 + j.7806)\mathbf{x}_1$$

 $\mathbf{x}_1 = (-.625 - j.7806)\mathbf{x}_2$

so that

^x 2	6¢	1.0			
xl	81	625	ma	j.7806	o

In a similar manner, for the second eigenvalue

 $x_2 = 1.0$ $x_1 = -.625 + j.7806$.

Then the 2n x 2n matrix becomes

625 .7806 _1 0	7806 625 0 1	625 7806 1	。7806 625 0 1	
Ó	1	0	1	c

When this matrix is multiplied by the inverses, B=3 and C=3, only the inverse B=3 results in the identity matrix, demonstrating that the

calculation resulting in C-3 is not an independent correct solution.

Comparison of the false trajectory with the correct one indicated the possibility of there being at least two other instances of erroneous calculations besides the one at the origin. These are indicated by the abrupt changes in direction (outward) of the trajectory at the points x = .65, y = 0.6, and x = -.5, y = .45. Calculations similar to the one carried out above verified the fact that erroneous determinations of the correct control function had been made.

The similarity of the trajectories of curves C-1, C-2 and D-1 in the fourth quadrant raises the interesting comparison between the control obtained by Lyapunov's method and the method of optimum control employing the calculation of the optimum switching lines in negative time as discussed by I. Flugge-Lotz and H.A. Titus³.

We may speculate that there is some optimum switching line for this system, emanating from the origin in a manner similar to that in Fig. (4-4).



Fig. (4-4) System Optimum Switching Line.

The optimum trajectory for this system would then proceed from the initial condition to its first intersection with the optimum switching line, change the sign of its controller, and proceed along the optimum switching line to the origin.

It appears evident that the method of switching logic discussed in this paper selects a switching time too soon in the cases of curves C-1 and C-2. In the case of curve D-2 the system appeared to be in a state of indecision. The switchings at points A and C undoubtedly would have brought the trajectory initially closer to the origin than the one finally selected at E. The switchings at B and D served only to hinder the system response. Investigation of the computer data again indicated erroneous calculations made for switchings at points B, C and D, although the resulting decision at C proved to be a correct one.

Graphs D-1 and D-2, showing the simulation of the system operating under non-ideal conditions, again substantiate the fact that higher sampling rates generally produce more satisfactory operation of the system. It is interesting to note that the controlled system operating with a time delay of .3 seconds eventually settles into a stable limit cycle of approximately one half the size of the uncontrolled system.

The effect of the controller on the excursions of the system roots as the state point travels along a trajectory for a particular initial condition is demonstrated in Figs. (4-5) and (4-6). The numbers indicate time in seconds to reach the position indicated.

The root locus for the uncontrolled system starting from initial conditions x = .5, $\dot{x} = .5$ is illustrated in Fig. (4-5). The roots attain their most stable positions after the trajectory has settled into its stable limit cycle.



Fig. (4-5) Root Locus for Uncontrolled System, Initial Conditions x = .5, $\dot{x} = .5$.

As the controlled trajectory approaches the region of chattermotion surrounding the origin, the root locations depart very little from their most unstable position ($\pm .5 \pm j.866$) because maximum instability occurs when x = 0. The most stable root positions on the controlled system root locus occur early in the solution while the trajectory is relatively far from the origin.



Fig. (4-6) Root Locus for Controlled System, Initial Conditions x = .5, $\mathring{x} = .5$.

5. Conclusions.

The operation of the controller in regulating a simulated nonlinear system described by the VanderPol equation supports the theoretical predictions developed in the design. The solution obtained by this method is seen to be less than optimum because the logic does not cause the control to switch at the optimum switching curve. It is apparent from the figures that the sampling rate for determining the eigenvalues, and hence changes in control action, is critical for this to be an effective control design procedure. With the fine sampling the system disturbances were rapidly and effectively controlled.

There exist today very few direct techniques for the control of non-linear systems. The procedures studied here provide such a technique. The method has the advantage of being applicable to nonlinear time varying systems of any order. It does however require a digital computer in the control link.

APPENDIX I

FLOW CHARTS AND FORTRAN LANGUAGE PROGRAM FOR DIGITAL COMPUTER SIMULATION



Fig. I-1 FLOW CHART FOR IDEAL CASE SIMULATION.



Fig. I-2 FLOW CHART FOR FINITE CALCULATION TIME SIMULATION

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JDB GDLDSMITT' OPVAN 1
PROGRAM DATA
DIMENSIGH AR(20,20),AI(20,20),GR(20,20),GI(20,20),
IXR(20),XDDT(20),XI(900),YI(900),YALJES(2,20),
2DUB(40,40),DUBINV(40,40),YEXP(40),XEXP(40,2),
3X2(100),Y2(100),X3(100),Y3(100),X4(100),Y4(100)
COMMON AR,AI,GR,GI,VALUES
DOD READ 51,MMAEFF,U,EPS,DT,DELTA,TF,ALRS,ALIS,(XR(J),J=1,MM)
MM = ORDER OF SYSTEM
F = CDEFFICIENT OF SECOND TERM OF VANDERPOL EQN
MM = ORDER OF SYSTEM
F = CDEFFICIENT OF SECOND TERM OF VANDERPOL EQN
MM = ORDER OF CONTROL EFFORT
EPS = RADIUS OF EPSILGN REFION OF VANDERPOL EQN
U = MAGNITUDE OF CONTROL EFFORT
EPS = RADIUS OF EPSILGN REFION OF ORIGIN
DT = TIME INTERVALS FOR RUNGE-KUTTA ROUTINE
DELTA = SAMPLING RATE
TF = TIME LIMIT FOR SOLUTION
ALRS, ALIS = COORDINATES IN IMAGINARY PLANE OF ARBITRARY OFFSET FROM
CC1GIN FOR EIGENVALUE CALCULATION IN GSUB
XR(J) = PHYSICAL SPACE STATE VARIABLES
IF(99999-MM) 70C2,7002,7001
201 FORMAT(150-2, 2E13-5)
200 FORMAT(120HITIME X(1) X(2) SIGMA(1) OMEGA(1) SIGMA(2)
AR (1.1) ARE ELEMENTS OF DIFF EON COFFE MATRIX
     .. JOB GOLDSMITH OPVAN 1
         7000
   51
1201
1200
    203
12003
12005
7002
1052
                           (I,J) ARE ELEMENTS OF DIFF EQN COEFF MATRIX
FORMAT(F6.2,2F8.4,8E11.4,E10.4)
FORMAT(1H019X29HINVERSE TRANSFORMATION MATRIX//)
FORMAT(E22.5,3E12.5)
FORMAT(1H0)
                           STOP
READ 1052;((AR(I,J),J=1,MM),I=1,MM)
FORMAT(4F10.0)
PRINT 1200
                           LINE=4
TO = 0.0
                          NUMPTS=0
            NUMPTS=0

LAG = 0

DO 200 J = 1,MM

DD 200 I = 1,MM

GR(I,J)=1.0+

GI(1,J) = 0.0

AI(I,J) = 0.0

T = TO

GSUB CALCULATE'S EIGENVALUES AND EIGEN VECOTRS OF AR AND FORMS

MATRIX G, IN TWO PORTIONS, GR AND GI

DO 30 I=1,MM

M = 2*I

K = M.-1

XEXP (K,1) = XR(I)
 200
COUR
     1000
            K = M -1
XEXP (K,1) = XR(I)
FOLLOWING ROUTINE FORMS DOUBLE SIZE TRANSFORMATION MATRIX CALLED DUB
30 XEXP (M,1) = 0.0
DD 31 I=1, MM
M = 2*I
K = M-1
DO 31 I= 1, MM
С
                      K = M-1

DO 31 J = 1, MM

L= 2*J -1

N= 2*J

DUB ((I, I)) = GR(I, J)

DUB (M_1N) = GR(I, J)

DUB (K_2N) =-GI(I, J)

DUB (M_1L) = GI(I, J)

NN = MM * 2
           NN = MM * 2

CALL GAUSS 3 (NN , 1E-06., DUB, DUBINV , KEY)

GAUSS 3 FINDS INVERSE OF DUB = DUBINV

2 PRINT 3

STOP
C
                       PRINT
                      FORMAT(19H GR MATRIX SINGULAR)
                3
          I CONTINUE
FORMS CANONICAL STATE VARIABLES YEXP (CONTAINS BOTH REAL AND IMAG. TERMS
DO 32 I=1,NN
YEXP(I)= 0.0
Ĉ
```

```
DD 32 J =1,NN

YEXP(I) = YEXP(I) + DUBINV(I,J) * XEXP(J,1)

YSQR = 0.0

DD 1003 I = 1,NN

NS SQUARE OF CANONICAL RADIUS VECTOR

YSQR = YSQR + YEXP(I) * YEXP(I)

IF(66 - LINE) 1400,1401,1401

PRINT 1200

INNE =0
             32
           FORMS
      1003
     1400
                         INE=4
                     PRINT 1202, T, XR(1), XR(2), ((VALUES(I, J), I=1,2), J=1,2), (YEXP(I), I=1,
      1401
                  PRINT 1202, 1, XR(1), XR(2), ((VALUES(I, J), I

PRINT 1203

PRINT 1204, ((DUBINV(I, J), J=1, NN), I=1, NN)

LINE = LINE + 9

IF(72 - LINE) 1402, 1403, 1403

PRINT 1200
     1402
                     LINE = 4
GO TO 1404
         GO TO 1404

403 PRINT 1205

LINE = LINE + 3

TESTS FOR CLOSENESS TO ORIGIN

404 IF(EPS - YSQR)1005,1303,1303

005 YRSUM = 0.0

CALCULATION OF CONTROL FUNCTION

DO 1006 I = 1,NN

06 YRSUM = YRSUM + YEXP(I)

UOP = - SIGNFIU,YRSUM)

802 IF(ABSF(VALUES(2,1)+VALUES(2,2))-10.E-6) 702,702,701

702 IF (VALUES(2,1)) 700,701,701

700 VALUES(2,1)= -VALUES(2,1)

VALUES(2,2)= -VALUES(2,2)

701 CONTINUE

10EL = T + DELTA
     1403
     1404
     1005
 C.
    1006
       802
702
700
       701
                   TDEL = T + DELTA

IF(TDEL - T - DT) 1300 , 1301 , 1301

IF(TDEL - TF) 1302,777,777

AR(MM,1) = F

AR(MM,2) = E * (1.0 - XR(1) * XR(1)

GC TO 1000;

CONTINUE
     1304
    1300
                                                                                                                        XR(1))
        GC TG 1000

S03 CONTINUE

PRINT 21202

202 FORMAT(26H-TRAJECTORY INSIDE EPSILON///)

PRINT 41202

202 FORMAT(25H) T RADIUS VECTOR///)

PRINT 1202 , T , YSQR

77 CONTINUE

CALL GRAPH (NUMPTS,X1,Y1,8)

GO TO 7000

RKUTTA CALCULATES CONTROLLED SYSTEM TRAJECTORY FOR SAMPLING PERIOD

S01 CALL RKUTTA (MM , T , XR , DT , UOP)

T = T + DT
    1303
21202
41202
       777
C
    1301
                   PRINT 1200
   1405
                 PRINT 1200

LINE = 4

PRINT 1202,T,XR(1),XR(2)

LINE = LINE + 1

IF(900 - NUMPTS) 1304,750,750

NUMPTS = NUMPTS + 1

X1(NUMPTS) = XR(1)

Y1(NUMPTS) = XR(2)

CO 10 1304
   1406
      750
                   END
                  SUBROUTINE RKUTTA (M, T , XR ,DT , UOP)
DIMENSION XR(20),AK(4,20),XDOT(20), XC(20), C(4)
COMMON AR ,AI,GR,GI,VALUES
                 COMMON AR
C(1) =0.0
C(2) =0.5
C(3) = 0.5
C(4) =1.0
DD 1050 I
TC = T + C
                  1051
                                                                                                          I-1,J), UOP, M)
                                                                                                         1-1
                  CALL
                                 DERIV
                                                      TTC
                                                                 , XC , XDGI
```

```
DO 1050 J =1,M
AK (1,J) = DT * XDOT(J)
DO 1052 J =1,M
XR(J) =XR(J) +(AK(1,J) +2.0* AK(2,J) + 2.0*AK(3,J) +AK(4,J))/6.
      1050
       1052
                                 AR(3) =AR(3) #(AR(1,3) #2.0* AR(2,3) * 2.0*AR(3,3) * AR(4,3,7) * 2.0*AR(3,3) * AR(4,3,7) * 2.0*AR(3,3) * AR(4,3,7) * 2.0*AR(3,3) * CAR(4,3,7) * CAR(4,3,7) * CAR(4,3,7) * CAR(3,3) * C
                                        END
                                SUBROUTINE GSUB (M, ALRS, ALIS)
ODIMENSION AR(20,20), AI(20,20), BR(20,20), BI(20,20), CR(20,20),
ICI(20,20), XR(20), XI(20), YR(20), YI(20), VALUES(2,20), GR(20,20),
2GI(20,20)
                                       COMMON CR , CI, GR, GI, VALUES
                                      N = M
LOOP =
DD 519
D0 519
                                                                                      1
                                                                                 I=1,N
             922
                                    DO 519 J=1, N

BR(I, J) = CR(I, J)

AR(I, J) = CR(I, J)

BI(I, J) = CI(I, J)

AI(I, J) = CI(I, J)

AI(I, J) = CI(I, J)

ASSIGN 917 TO IA

ASSIGN 811 TO ID

MM=M

GO TO 535

GO TO ID

ASSIGN 914 TO ID

ASSIGN 530 TO IA

ASSIGN 50 TO IA

ASSIGN 523 TO IC

ISL=-1

GO TO 92
                                                                                 J=1,N
              519
             917
811
                                        GO TO 92
                                         ISL=0
               523
                                      ALR = ALRS
ALI = ALIS
                            9
                                        ALI
IT=1
            403 DO 504 I=1,N
NEXT TWO INSTRUCTIONS DESIGNATE INITIAL GUESSES FOR EIGENVECTOR
PRECEDING EIGENVECTOR SOLUTION
XR(I)=GR(I;LOOP)
504 XI(I)=GI(I;LOOP)
4 DO 5 I=1,N
AR(I,I)=AR(I,I)-ALR
5 AI(I,I)=AI(I,I)-ALR
Ç,
                                      AR(I,I)=AR(I,I)-ALR

AI(I,I)=AI(I,I)-ALI

IJ=1

BIG=0.

DD 13 I=1,N

YR(I)=0.

DD 1] J=1,N

YR(I)=YR(I)+AR(I,J)*XR(J)-AI(I,J)*XI(J)

YI(I)=YR(I)+AI(I,J)*XR(J)+AR(I,J)*XI(J)

AM=YR(I)*YR(I)+YI(I)*YI(I)

EXAM-BIG) 13,13,12

BIG=AM
                            5
                      10
                      11
                                         BIG=AM
                      12
                                        JJ=I
CONTINUE
IF(BIG)
ICT=1000
                      13
                                                                                              109,106,109
                106
                                         JJ = N
ISL = 1
GO TO 92
                109 RQNR=0.
                                         RQNI=0.
ROD=0.
DD 14 I=1.N
                                      RONR=RONR+XR(I)*YR(I)+XI(I)*YI(I)
RONI=RONI+XR(I)*YI(I)-XI(I)*YE(I)
ROD=ROD+XR(I)*XR(I)+XI(I)*XI(I)
AMUR=RONR/ROD
                       14
                                                                                                                                                                                                                                                                     28
                                                                                                                                                                                                                                              42
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AMUL=RQNI/RQD
                      AMM=AMUR * AMUR * AMUI * AMUI

TS=0

DC I5 I=1,N

OTS=TS+(YR(I)-AMUR*XR(I)+AFUI*XI(I))**2+

1(YI(I)-AMUR*XI(I)-AMUI*XR(I))**2

DO 16 I=1,N

XR(I)=(YR(JJ)*YR(I)+YI(JJ)*YI(I))/BIG

XI(I)=(YR(JJ)*YI(I)-YI(JJ)*YI(I))/BIG

XI(JJ)=1.0

XI(JJ)=1.0

XI(JJ)=0.0

IF (TS / RQD - 10.E-4) 20,20,18

IF(IJ - 20) 19,20,20

IF (IJ - 20) 19,20,20

IJ=IJ+1

GO 10 10

ICT=IJ

MIT2 = IJ + 20

ALR=AMUR+ALR

ALI=AMUI+ALI
                               ANM=AMUR*AMUR+AMUI*AMUI
            81
            150
            16
     11,1
13
19
          20
                      MIT2 = IJ + 20
ALR=AMUR+ALR
ALI=AMUI+ALI
MM=N
DG 310 I=1,N
AR(I,I)=AR(I,I)-AMUR
AI(I,I)=AR(I,I)-AMUT
GD TO 29
DO 100 I=1,N
AR(I,I)=AR(I,I)-ALR
AI(I,I)=AR(I,I)-ALI
IJ=IJ+1
DO 27 I=2,MM
IMT=I-T
DO 27 J=1,IMT
FM=AR(I,J)*AR(J,J)+AI(I,J)*AI(I,J)
SM=AR(J,J)*AR(J,J)+AI(J,J)*AI(J,J)
SM=AR(J,J)*AR(J,J)+AI(J,J)*AI(J,J)
IF (FM-SM) 24,24,24,22
JO 23 K=J,MM
T1=AR(J,K).
T2=AI(J,K)=AI(I,K)
AR(J,K)=AI(I,K)
AR(J,K)=AI(I,K)
AR(I,K)=T1
AI(I,K)=T1
AI(I,K)=T1
XI(I)=XR(I)
XR(I)=XI(I)
XR(I)=XI(I)
XR(I)=T1
F(FM) 25,27,25
IF (FM) 25,27,25
IF (FM) 90,27,20
    310
          99
  100
29
535
         21
        22
        23
                      SM=T1
IF (SM) 25,27,25
IF (FM) 90,27,00
RR=(AR(I,J)*AR(J,J)+AI(I,J)*AI(J,J))/SM
RI=(AR(J,J)*AI(I,J)*AI(J,J))/SM
DD 26 K=J,MM
AR(I,K)=AR(I,K)-RR*AR(J,K)*RI*AI(J,K)
AI(I,K)=AI(I,K)-RR*AI(J,K)-RI*AR(J,K)
AI(I,K)=AI(I,K)-RR*AI(J,K)-RI*AR(J,K)
AI(I,J)=0.
XR(I,J)=0.
XR(I)=XR(I)-RR*XR(J)*RI*XI(J)
XI(I)=XI(I)-RR*XI(J)-RI*XR(J)
CONTINUE
GD TD IA
SMALL=1000.
DD 28 K=1,MM
IKK=K
       24
25
       <u>90</u>
      26
      27
530
                        IKK=K

T1=AR(K,K)*AR(K,K)+AI(K,K)*AI(K,K)

IF (T1) 750,752,750

IF (T1-SMALL) 751,28,28

SMALL=T1
750
751
                      CONTINUE
GC TO IB
      28
```

752	IZ=IKK IF (ISL) 753,30,30
50	107=2000 DD 974 I=1.MM
974 753	XR(I)=0.0 XI(I)=0.0 YR(IZ)=1.0 YI(IZ)=0.0
	BIG=I.0 JE (17-MM) 33.32.33
32	12Z=2 GD T0 95
33	IZZ=IZ+1 DO 31 I=IZZ,MM
31	YR(1)=0. YI(1)=0. IZZ=MM-IZ+2 IE (17-1) 95-19-95
40	IZZ=1
95	DO 46 I=IZZ, MM II=MM-I+1 KK=II+1 SB=0.
	SI=0. I=(1-1) 42.44.42
42	DD 43 K=KK, MM SR=SR+AR(II,K)*YR(K)-AI(II,K)*YI(K)
43	SI = SI + AR(II,K) + YI(K) + AI(II,K) + YR(K) TI = AR(II,II) + AR(II,II) + AI(II,II) + AI(II,II)
	$\begin{array}{l} & \forall R(II) = (AR(II_{2}II) * (XR(II) - SR) + AI(II_{2}II) * (XI(II) - SI)) / T \\ & \forall I(II) = (AR(II_{2}II) * (XI(II) - SI) - AI(II_{2}II) * (XR(II) - SR)) / T \end{array}$
	$AM = YR(II) = \frac{YR(II) + YI(II)}{IF (AM-BIG)} = \frac{46,46,45}{46,45}$
45	JJ=II BIG=AM
4649	CONTINUE DO 47 I=1, MM
47	XR(I) = (YR(JJ) * YR(I) + YI(JJ) * YI(I)) / BIG XI(I) = (YR(JJ) * YI(I) - YI(JJ) * YR(I)) / BIG
~ ^	XR(JJ)=1.0. XI(JJ)=0.0.
92	$\begin{array}{c} DU & GOT & I = I_{0}N \\ DO & GOT & J = I_{0}N \\ DO & GOT & J = I_{0}N \\ DO & J = I_{0}N \\ D & J = I_{$
601	AK[1,0] = BK[(1,0]) $A[(1,0]] = BI[(1,0])$ $T[(1,0]] = BI[(1,0])$
755	GO TO IC
50	
55	DD 52 I=1, N
	YI(I) = 0.
51	YR(I) = YR(I) + AR(I,K) * XR(K) - AI(I,K) * XI(K) YI(I) = YI(I) + AR(I,K) * XI(K) + AI(I,K) * XI(K)
21	ALR = ALR + XR(I) + YR(I) + XI(I) + YI(I)
52	SUM = SUM + XR(I) * XR(I) + XI(I) * XI(I)
	ALI=ALI/SUM AM=ALR*ALR*ALI*ALI
83	TS=0. DO 53 I=1,N
	TI = YR(I) - ALR * XR(I) + ALI * XI(I) T2 = YI(I) - ALR * XI(I) - ALI * XR(I)
53 93	TS=TS+T1+T1+T2+T2 IF (TS / SUM - 10.E-14) 60,60,301
301	IF (IJ-MIT2) 99,400,400 IF (IT - 3) 402,990,402
402	ALR = ALR + .1

. 434

ALI = ALI + .1 IT=IT+1 CO TO 4 IT=IT+1 CO TO 4 CO TO 4 ISL=0 VALUES(1,LOOP) = ALR VALUES(2,LOOP) = ALI IF (JJ-N) 61,65,61 T1=XR(JJ) T2=XI(JJ) XR(JJ)=XR(N) XI(JJ)=XR(N) XR(JJ)=XR(N) XR(JJ)=XR(J) XR(JJ)=X 63 65 A! {K, N} = T2 N=N-1 DD 66 I= ; N DD 66 J= ; N AR { [i , J } = AR { [i , J] - XR { [] } * AR { (N+1, J) + XI { [] } * AI { (N+1, J) } } AI { [i , J] = AI { [i , J] } - XR { [] } * AI { (N+1, J) - XI { [] } * AR { (N+1, J) } } DO 600 J= ; N DO 600 J= ; N BR { [i , J] = AR { [I , J] } AI { [i , J] = AI { [i , J] } } DO 702 J= ; M AR { [i , J] = CR { [I , J] } AI { [i , J] = CR { [I , J] } } AI { [i , J] = CR { [I , J] } AI { [i , J] = CR { [I , J] } } AI { [i , J] = CR { [I , J] } - ALR AI { [i , I] = AR { [I , J] } - ALR AI { [i , I] = AI { [i , I] } - ALI CONTINUE Mi=M ASSIGN 911 TO IA GD TO 535 ASSIGN 530 TO IA ISL=-1 DO 703 I = 1, M XR { [] = 0, 700 915 914 ISL=-1 915 DO TOJ I=1,M XR(I)=0. ASSIGN 753 ID IB ASSIGN 704 TO IC GO TO 530 916 ASSIGN 525 TO IC CO TO 92 704 DO 920 I = 1,M NEXT TWO INSTRUCTIONS FROM TRANSFORMATION MATRICES GR AND GI FROM NEXT TWO INSTRUCTIONS FROM TRANSFORMATION MATRICES GR AND GI FROM MEXT TWO INSTRUCTIONS FROM TRANSFORMATION MATRICES GR AND GI FROM MEXT TWO INSTRUCTIONS FROM TRANSFORMATION MATRICES GR AND GI FROM SEIGENVECTORS XR AND XI GR(1,LOOP) = XR(I) 20 GI(1,LOOP) = XR(I) ASSIGN 40 TO IB LOOP = LOOP + 1 525 IF(N-1) 921,67,523 67 ALR=AR(1,1) ALI=AI(1,1) VALUES(1,LOOP) = ALR VALUES(2,LOOP) = ALI N=0 CO TO 700 67 GD TO 700 GD TO 4000

XMAX = -10. ** 20 DD 1000 I = J,M IF (XMAX - VALUES(1,I)) 3,3,1000 XMAX = VALUES(1,I) MX= I CONTINUE IF (MX-J) 1,4000,1 VALUES(1,MX) = VALUES(1,J) VALUES(1,J) = XMAX SAVE = VALUES(2,MX) VALUES(2,MX) = VALUES(2,J) VALUES(2,J) = SAVE DD 3000 K=1,M SAVE = GR(K,MX) GR(K,MX)= GR(K,J) CR(K,J) = SAVE SAVE = GI(K,MX) GI(K,MX)= GI(K,J) GI(K,J) = SAVE CONTINUE GD TO 992 PRINT 991 FORMAT(15H NO CONVERGENCE) EMD 3 1000 l une 3000 4000 990 PRINT 991 FORMAT(15H NO CONVERGENCE) END SUBROUTINE GAUSS3(N, EP, A, X, KER) DIMENSION A(40, 40), X(40, 40) DO I I=1,N° DO I J=1,N° DO I J=1,N° X(1,J)=0.0 DO Z K=1,N X(K,K)=1.0 DO 34 L=1,N KP=0 991 992 1 2 10 KP = 0Z=0.0 DO 12 K=L.N IF(Z-ABSF(A(K,L)))11,12,12 Z=ABSF(A(K,L)) 110 11 KP=K CONTINUE IF(L-KP)13,20,20 DO 14 J=L,N 12 13 DU 14 J=L,M Z=A(L,J) A(L,J)=A(KP,J) A(KP,J)=Z DD 15 J=1,N Z=X(L,J) X(L,J)=X(KP,J) X(KP,J)=Z IF(ABSF(A(L,L))=EP)50,50,30 IF(L=N)31,34,34 14 1500m IF(L-N)31,34,34 LP1=L+1 DD 36 K=LP1,N IF(A(K,L))32,36,32 RATID=A(K,L)/A(L,L) DD 33 J=LP1,N A(K,J)=A(K,J)-RATIO*A(L,J) DD 35 J=1,N X(K,J)=X(K,J)-RATIO*X(L,J) CONTINUE CONTINUE DD 43 I=1,N LI=N+1-I 32 33 35640 II = N + 1 - IDD 43 J=1,N S=0.0 IF(II-N)41,43,43 IIP]=II+1 41 DO 42 K=IIP1,N S=S+A(II,K)*X(K,J) X(II,J)=(X(II,J)-S)/A(II,II) KER=1 GO TO 75 KER=2 42 50 CONTINUE 75 END .

APPENDIX II

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GRAPHS OF SIMULATED SYSTEM TRAJECTORIES

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X EXTS SI Y AXIS SI STUTE SEAR S UFLOCITY UFF CON ROL FITOR		

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	TREE SPRCE SOLUTION FOR UN ELOCITY VERSUDI DISPLACEM MIRCL EFFORT = 0.0	
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-001 -000		
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		GRAPH A-6










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000		105	810		020 0	
000		305	810	015	020 0	
		305	810	016	020 0	
000 - 005 - 005			810	016	020 0	
000			010	015	020 0	
000				015	020 0	
000			010		020	
000 -005 -005 -005					020	
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