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APPLICATION OF Z-TRANSFORM METHODS TO MARKOV

CHAIN PROBLEMS, WITH A MAINTENANCE EXAMPLE

by

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U. S. NAVAL CIVIL ENGINEERING LABORATORY
Port Hueneme, California

APPLICATION OF Z-TRANSFORM METHODS TO MARKOV

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ABSTRACT

This Technical Note develops an application of the theory of the
Z-transform to the problem of finding the nth power of a transition matrix,
motivated by the need of such a solution in the theory of Markov Chains.
To illustrate the theory, an example involving vehicle maintenance is
presented in detail. The results of the example may be compared with those
of Phillp M. Morse, "Markov Processes", in Notes On Operations Research,
1959, The Technology Press, Massachusetts Institute of Technology.

INTRODUCTION

Often in Markov chain problems it is desired to find the nth power, P^n, of a transition matrix P. P^n can be found by using Z-transforms as follows:

Let Π_n be the row vector,

$$\Pi_n = \left[\Pi_{1n} , \Pi_{2n} , \cdots , \Pi_{mn} \right]$$

where Π_{jn} is the probability of being in state j after n transitions. (In general, brackets [ ] will enclose a matrix or vector.)

Let $P = [P_{ij}]$ be the matrix of transition probabilities of going from state i to state j in one transition.

Then the Markov equation is

$$\Pi_{n+1} = \Pi_n P .$$

In particular,

$$\Pi_1 = \Pi_0 P ,$$

$$\Pi_2 = \Pi_1 P = \Pi_0 P^2 .$$

In general,

$$\Pi_n = \Pi_0 P^n .$$

Let the Z-transform be defined as

$$Z\left\{ y_n \right\} = \sum_{n=0}^{\infty} y_n z^n , \text{ assuming that the series converges, where}$$

$\left\{ y_n \right\}$ is a sequence of numbers $y_0, y_1, y_2 , \ldots$. Since Z is a linear operator,

$$Z (P) = \left[Z (p_{ij}) \right] ;$$

so $Z\left(\Pi_{n+1} \right) = \sum_{n=0}^{\infty} \Pi_{n+1} z^n$

$$= \frac{1}{z} \sum_{n=0}^{\infty} \Pi_{n+1} z^{n+1}$$

$$= \frac{1}{z} \left(\sum_{n=1}^{\infty} \Pi_n z^n \right)$$

$$= \frac{1}{z} \left[\left(\sum_{n=0}^{\infty} \Pi_n z^n \right) - \Pi_0 \right]$$

$$= \frac{1}{z} Z \left(\Pi_n \right) - \frac{1}{z} \Pi_0$$

and, since

$$\Pi_{n+1} - \Pi_n P = 0$$

and
$$Z \left(\Pi_n P \right) = Z \left(\Pi_n \right) P ,$$

$$\frac{1}{z} Z \left(\Pi_n \right) - \frac{1}{z} \Pi_0 - Z \left(\Pi_n \right) P = 0.$$

Multiply by z and transposing Π_0 :

$$Z \left(\Pi_n \right) - z Z \left(\Pi_n \right) P = \Pi_0$$

so
$$Z \left(\Pi_n \right) \cdot [I - z P] = \Pi_0$$

and
$$Z \left(\Pi_n \right) = \Pi_0 [I - z P]^{-1} \quad \text{if the inverse exists.}$$

Taking inverse transforms:

$$\Pi_n = \Pi_0 Z^{-1} \left([I - zP]^{-1} \right)$$

so
$$P^n = Z^{-1} \left([I - zP]^{-1} \right) .$$

Hence, we have an algorithm for computing $\left\{ P^n \right\}$, where P^n is the sequence of matrices

$$\left[\left\{ p_{ij}{}^k \right\} \right]$$

where $p_{ij}{}^k$ is the element in the ith row and jth column of $[P]^k$, k = 1, 2, ... Hereafter, the generic term P^n will be used.

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A MAINTENANCE EXAMPLE*

A bus makes N runs per week. Suppose there are two stages in the depreciation of the bus; in stage A it can still be operated, but it is apparent that it will soon need to be repaired; in stage B, it cannot do its job and must be repaired. If the bus is repaired when it is discovered to be in stage A of depreciation, the repair is not difficult, but if it is allowed to run until it breaks down (stage B) the repair will take longer. The decision to be made is when to repair the bus.

Suppose the deterioration is random, such that if it is in good condition at the beginning of a run then the probability it will still be in stage A at the end of the run will be a. Similarly, if the bus is in stage A at the beginning of a run, the chance that it will reach stage B by the end of the run is b. For simplicity, let the length of time to repair the bus in stage A be the same length of time it takes to complete one run. If stage B is reached, the bus will be out of commission until the beginning of the following week.

Two alternative maintenance doctrines suggest themselves: Doctrine 1 - repair if the bus reaches stage A; doctrine 2 - wait until stage B is reached to make repairs. The Markov matrices are:

Doctrine 1   Doctrine 2

$$
\begin{array}{c}
\\
A \\
A
\end{array}
\begin{bmatrix}
1 - a & a \\
1 & 0
\end{bmatrix}
\qquad
\begin{array}{c}
\\
A \\
A \\
B
\end{array}
\begin{bmatrix}
1 - a & a & 0 \\
0 & 1 - b & b \\
0 & 0 & 1
\end{bmatrix}
$$

To compare the two doctrines, two questions may be posed.

(1) What is the expected number of runs, n_s, the bus will make if it always starts in good condition?

This is

$$n_z = \sum_{n=0}^{n-1} \sum_{k=1}^{M} P_{1k}{}^{n}$$ over all states k in which the bus may

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*Adapted from Morse, Philip M., "Markov Processes", in Notes on Operations Research, 1959, The Technology Press. Morse uses eigenvectors to obtain his results.

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still operate, and M is the total number of states.

(2) After many runs, what is the probability of the bus not breaking down, i.e., what is the utility of the bus? This will be obtained in the solution.

SOLUTION

Doctrine 1.

$$P = \begin{bmatrix} 1-a & a \\ 1 & 0 \end{bmatrix}$$

$$[I - zP] = \begin{bmatrix} 1-z\,(1-a) & -za \\ -z & 1 \end{bmatrix}$$

$$[I - zP]^{-1} = \begin{bmatrix} \dfrac{1}{(1+za)\,(1-z)} & , & \dfrac{za}{(1+za)\,(1-z)} \\[3mm] \dfrac{z}{(1+za)\,(1-z)} & , & \dfrac{1-z\,(1-a)}{(1+za)\,(1-z)} \end{bmatrix}$$

Using partial fractions:

$$[I - zP]^{-1} = \begin{bmatrix} \dfrac{a/\,(1+a)}{1+za} + \dfrac{1/\,(1+a)}{1-z} & , & \dfrac{-a/\,(1+a)}{1+za} + \dfrac{a/\,(1+a)}{1-z} \\[4mm] \dfrac{-1/\,(1+a)}{1+za} + \dfrac{1/\,(1+a)}{1-z} & , & \dfrac{1/\,(1+a)}{1+za} + \dfrac{a/\,(1+a)}{1-z} \end{bmatrix}$$

$$z^{-1}\left([I - zP]^{-1}\right) = \frac{1}{1+a} \begin{bmatrix} 1-(-a)^{n+1} & , & a+(-a)^{n+1} \\[2mm] 1-(-a)^{n} & , & a+(-a)^{n} \end{bmatrix}$$

This points out an error in the solution of this example in Morse's article. The top line of page 100 should read:

$$(1)\ m^{n} = \frac{1}{1+a} \begin{bmatrix} 1 & a \\ 1 & a \end{bmatrix} + \frac{(-a)^{n}}{1+a} \begin{bmatrix} a & -a \\ -1 & 1 \end{bmatrix} = \cdots$$

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Then the results of both authors agree:

Now n_s can be computed:

$$n_s = \sum_{n=0}^{n-1} P^n_{11} = \frac{N}{1+a} + a\frac{1-(-a)^n}{(1+a)^2}$$

P^n can be represented as the sum of two matrices as in equation (1). The effect of the second matrix diminishes as n gets $>>0$, so after many runs P^n becomes independent of n.

$$\frac{1}{1+a}\begin{bmatrix} 1 & a \\ 1 & a \end{bmatrix}$$ is called the steady state transition matrix.

Note that each row is identical. The elements p_{ij} of any row i represent the steady state probabilities of being in state j. For doctrine 1, then, the utility of the bus is 1/ 1+a.

Doctrine 2.

$$P = \begin{bmatrix} 1-a & a & 0 \\ 0 & 1-b & b \\ 0 & 0 & 1 \end{bmatrix}$$

$$[I - z P]^{-1} =$$

$$\begin{bmatrix} \dfrac{1}{1-z(1-a)} & , & \dfrac{az}{[1-z(1-a)][1-z(1-b)]} & , & \dfrac{abz^2}{K\,z} \\ 0 & , & \dfrac{1}{1-z(1-b)} & , & \dfrac{bz}{[1-z(1-b)][1-z]} \\ 0 & , & 0 & , & \dfrac{1}{1-z} \end{bmatrix}$$

where $K = (1-z(1-a))(1-z(1-b))(1-z)$.

Using partial fractions:

$$[I - zP]^{-1} =$$

$$\begin{bmatrix} \dfrac{1}{1-z(1-a)} & , & \dfrac{a/(b-a)}{1-z(1-a)} + \dfrac{a/(a-b)}{1-z(1-b)} & , & \dfrac{a/(a-b)}{1-z(1-b)} + \dfrac{-b/(b-a)}{1-z(1-a)} + \dfrac{1}{1-z} \\\\ 0 & , & \dfrac{1}{1-z(1-b)} & , & \dfrac{-1}{1-z(1-b)} + \dfrac{1}{1-z} \\\\ 0 & , & 0 & , & \dfrac{1}{1-z} \end{bmatrix}$$

$$Z^{-1}([1-zP]^{-1}) = P^n = \begin{bmatrix} (1-a)^n, & a\,\dfrac{(1-b)^n - (1-a)^n}{a+b} & , & 1 + \dfrac{b(1-a)^n - a(1-b)^n}{a-b} \\\\ 0 & , & (1-b)^n & , & 1 - (1-b)^n \\\\ 0 & , & 0 & , & 1 \end{bmatrix}$$

so $n_s = \displaystyle\sum_{n=0}^{N-1} \left[P^n_{11} + P^n_{12} \right]$

$$= \dfrac{a^2 \left[1 - (1-b)^N \right] - b^2 \left[1 - (1-a)^N \right]}{ab(a-b)}$$

Breaking P^n into two matrices:

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$$P^n = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} (1-a)^n & , & a\dfrac{(1-b)^n - (1-a)^n}{a-b} & , & \dfrac{b(1-a)^n - a(1-b)^n}{a-b} \\ 0 & , & (1-b)^n & , & -(1-b)^n \\ 0 & , & 0 & , & 0 \end{bmatrix}$$

so, if N is large, the utility of the bus is zero.

RECOMMENDATIONS

As before stated, this Technical Note is not a development of the theory of Z-transforms, but an application of them to Markov Chain theory.

Markov Chains are useful for analyzing any probabilistic or random process which may take on a finite number of possible states. Examples of such processes are:

1. Machine maintenance. Here, the breakdown of operating machines is random. The states may be "operating" and "non-operating".

2. Inventory control. Demands for maintenance materials, construction materials, etc. are random. The states of an inventory system might be inventory levels.

3. Replacement of items that fail. Light bulbs in a building fail at random times. A Markov Chain can be used to represent this process. The states of the process are perhaps the number of light bulbs which have failed.

Since so many such random processes occur it is not difficult to find other applications within the sphere of Shore Operation maintenance.

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