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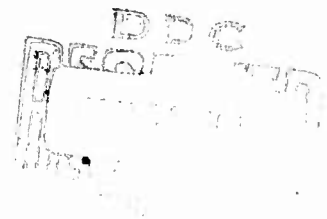
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A "String Algorithm" for Shortest Paths
in Directed Networks



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A "STRING ALGORITHM" FOR SHORTEST PATHS
IN DIRECTED NETWORKS

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The literature contains several algorithms for finding the shortest path between two nodes P and Q of a network, where the distances or arc-lengths are assumed to be positive. (For references, consult the review article by Pollack and Weibenson [3] and the book by Ford and Fulkerson [1].) Some of the algorithms, and in particular some of the analogue devices, are applicable only when the distance matrix is symmetric. As was remarked in [1] and [3], this is true of the simplest of the analogue procedures ----- the "string algorithm" reported by Minty [2]. It consists of making an inelastic string model of the network, with knots corresponding to nodes and string-lengths proportional to the corresponding distances, and then stretching the knots P and Q as far apart as is possible without breaking the string; this produces at least one straight path from P to Q , and each such straight path corresponds to a shortest path in the network.

Networks with asymmetric distance matrices are most conveniently represented by means of directed networks, in which every arc is regarded as a one-way street of the appropriate length. In the present note we describe a simple cutting procedure (related to one suggested by Thomas Seidman) which can be combined with any algorithm for undirected networks (symmetric distance matrix) so as to form a shortest-path algorithm for directed networks (asymmetric distance matrix). In particular, the cutting and stretching can be alternated to form a "string algorithm" for directed networks.

* * * * *

For each directed network N , let N^u denote the corresponding undirected network.

THEOREM Suppose that P and Q are nodes of a finite directed network N_1 which has v nodes and e arcs, and that there is a path from P to Q in N_1 . Suppose that A is an algorithm for finding shortest paths in undirected networks, and let the sequential procedure $S_1, C_1, S_2, C_2, \dots$ be as follows:

S_i) Apply A to the undirected network N_i^u to find a shortest path π_i from P to Q in N_i^u . Suppose π_i is given by

$$v_0^i \alpha_1^i v_1^i \alpha_2^i v_2^i \dots v_{h(i)-1}^i \alpha_{h(i)}^i v_{h(i)}^i,$$

where the arcs α_j^i and the nodes v_j^i are listed as they appear in traversing π_i from $P = v_0^i$ to $Q = v_{h(i)}^i$.

C_i) If π_i is also a path in N_i , terminate the procedure. If π_i is not a path in N_i , there exists a smallest index $r(i)$ and a largest index $s(i)$ (possibly the same) such that the directions of $\alpha_{r(i)}^i$ and $\alpha_{s(i)}^i$ in π_i are opposite to their directions in N_i . Let N_{i+1} be the directed network that is obtained from N_i by deleting every arc of N_i that (like $\alpha_{r(i)}^i$) ends in N_i at $v_{r(i)-1}^i$ but is not $\alpha_{r(i)-1}^i$, and deleting every arc of N_i that (like $\alpha_{s(i)}^i$) starts in N_i at $v_{s(i)}^i$ but is not $\alpha_{s(i)+1}^i$. There is a path Σ_i from P to Q in N_{i+1} such that Σ_i is actually a shortest path from P to Q in N_i .

The procedure terminates at some stage C_t for which

$$t \leq \min(v, (e+2)/2),$$

and the path π_t is a shortest path from P to Q in the directed network N_i .

(The same conclusion holds if C_1 requires only the first of the two deletions specified above, or if it requires only the second.)

Proof. Of course the algorithm itself does not involve the actual construction of Σ_i , but we require the existence of Σ_i (when C_i does not specify termination) to show that the sequential procedure $S_1, C_1, S_2, C_2, \dots$ can actually be followed and that each of the paths $\Sigma_1, \Sigma_2, \dots$ is a shortest path from P to Q in N_1 . Since N_1 is finite, the procedure must terminate at some stage C_t and then π_t is a shortest path from P to Q in N_1 .

Suppose C_i does not specify termination and let Σ_{i-1} be a shortest path from P to Q in N_i , given by

$$w_0^i \beta_1^i w_1^i \beta_2^i w_2^i \dots w_{l(i)-1}^i \beta_{l(i)}^i w_{l(i)}^i,$$

where of course $w_0^i = P$ and $w_{l(i)}^i = Q$. In constructing Σ_i , we consider the following three possibilities:

- (i) no w_j^i is equal to either $v_{r(i)-1}^i$ or $v_{s(i)}^i$;
- (ii) there exists j such that $w_j^i = v_{r(i)-1}^i$ and $j < k \Rightarrow w_k^i \neq v_{s(i)}^i$;
- (iii) there exist j and k such that $j < k$, $w_j^i = v_{r(i)-1}^i$ and $w_k^i = v_{s(i)}^i$.

When (i) holds, we define $\Sigma_i = \Sigma_{i-1}$. When (ii) holds, we obtain Σ_i by following π_i from P to $v_{m(i)-1}^i$ and then following Σ_{i-1} from $v_{m(i)-1}^i$ to Q . When (iii) holds, we obtain Σ_i by following π_i from P to $v_{m(i)-1}^i$, next following Σ_{i-1} from $v_{m(i)-1}^i$ to $v_{n(i)}^i$, and then following

π_i from $V_{n(i)}^i$ to Q . In each case, it is easily verified that Σ_1 has the stated properties. Thus the existence of t is established and it remains only to show that $t \leq \min(v, (e+2)/2)$.

Let us review the special properties of certain nodes and arcs of N_i relative to N_i itself and relative to N_j for $j > i$.

- (a) $V_{r(i)-1}^i \neq Q$. If $V_{r(i)-1}^i = P$, then at least one arc of N_i ends at $V_{r(i)-1}^i$ but no arc of N_j ends there. If $V_{r(i)-1}^i \neq P$, then at least two arcs of N_i end at $V_{r(i)-1}^i$ but at most one arc of N_j ends there.
- (b) $\alpha_{r(i)}^i$ ends at P or is coterminal with another arc of N_i ; $\alpha_{r(i)}^i$ does not end at Q and does not appear in N_j .
- (c) $\alpha_{r(i)-1}^i$ does not end at P or Q , and is nonexistent if $\alpha_{r(i)}^i$ ends at P . If $\alpha_{r(i)}^i$ does not end at P , then $\alpha_{r(i)-1}^i$ is coterminal with another arc of N_i but not with another arc of N_j .

We see from (a) that the t nodes $Q, V_{r(1)-1}^1, V_{r(2)-1}^2, \dots, V_{r(t-1)-1}^{t-1}$ are pairwise distinct, and consequently $t \geq v$. From (b) and (c) it follows that the arcs $\alpha_{r(1)-1}^1, \alpha_{r(1)}^1, \alpha_{r(2)-1}^2, \alpha_{r(2)}^2, \dots, \alpha_{r(t-1)-1}^{t-1}, \alpha_{r(t-1)}^{t-1}$ are pairwise distinct. If $\alpha_{r(i)}^i$ ends at P , then $\alpha_{r(i)-1}^i$ does not appear, but this happens for at most one value of i , and since at least one arc of N_1 ends at Q we conclude that $e \geq 2t - 2$.

The above reasoning completes the proof when C_i is as originally described, and also when C_i is replaced by C_i' which requires only the first of the specified deletions. Similar reasoning applied to C_i'' , which

Figure 1



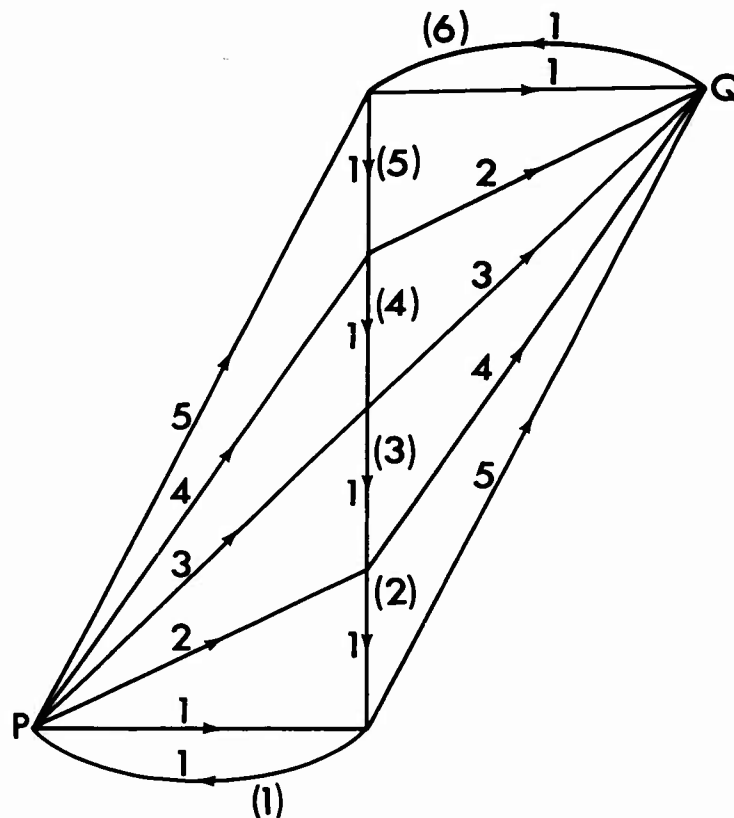
v nodes.

$2(v-1)$ arcs, all of the same length.

$$t = v = (e+2)/2$$

$(\pi_1$ follows upper arcs except at $\alpha_{r(i)}^i$)

Figure 2



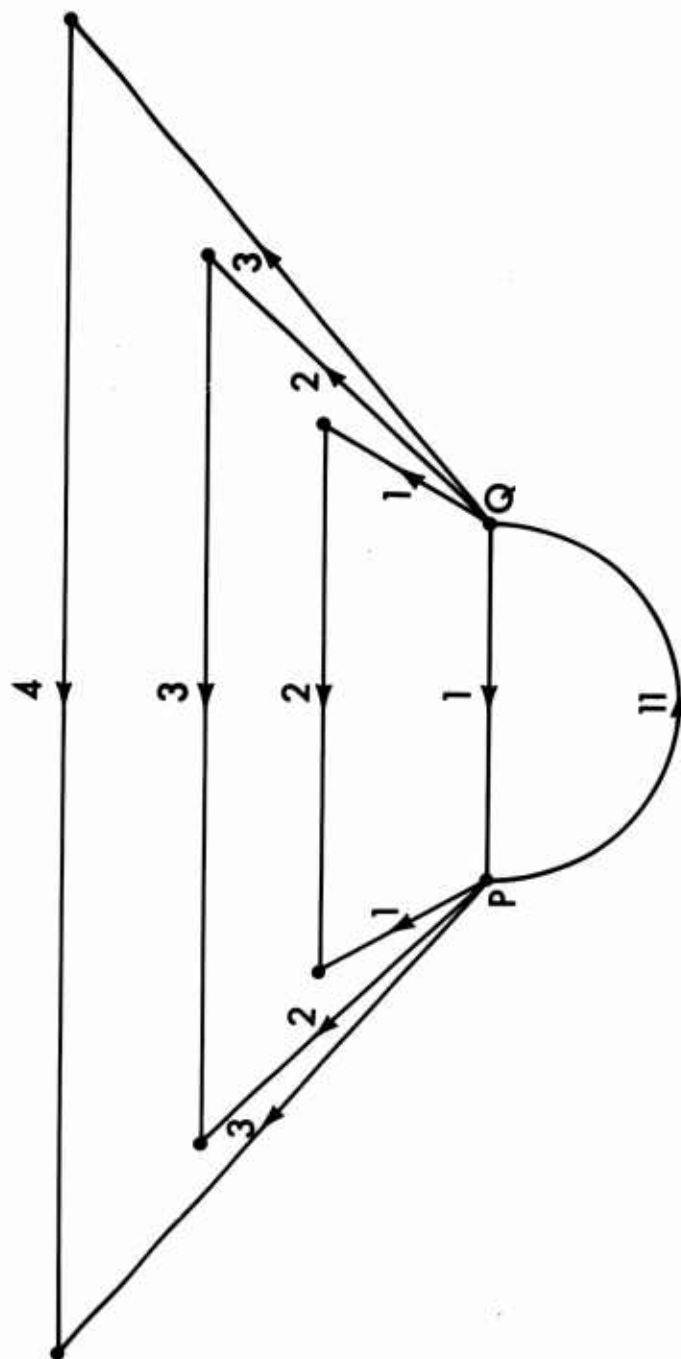
v nodes.

$3v - 5$ arcs, with lengths as indicated.

$\alpha_{r(i)}^i (= \alpha_{s(i)}^i)$ indicated by (i)

$$t = v$$

Figure 3



$2n$ nodes.

$3n - 1$ arcs.

$$t = n + 1 = \frac{v+2}{2} = \frac{e+4}{3}$$

(π_i is uniquely determined)

REFERENCES

1. L. R. Ford, Jr. and D. R. Fulkerson, Flows in Networks, Princeton University Press, 1962.
2. G. J. Minty, "A Comment on the Shortest-Route Problem," *Opns. Res.* 5, 724 (1957).
3. M. Pollack and W. Wiebenson, "Solutions of the Shortest-Route Problem-- A Review," *Opns. Res.* 8, 224-230 (1959).