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A GRADIENT INEQUALITY FOR
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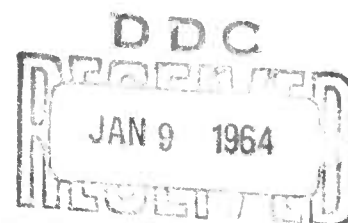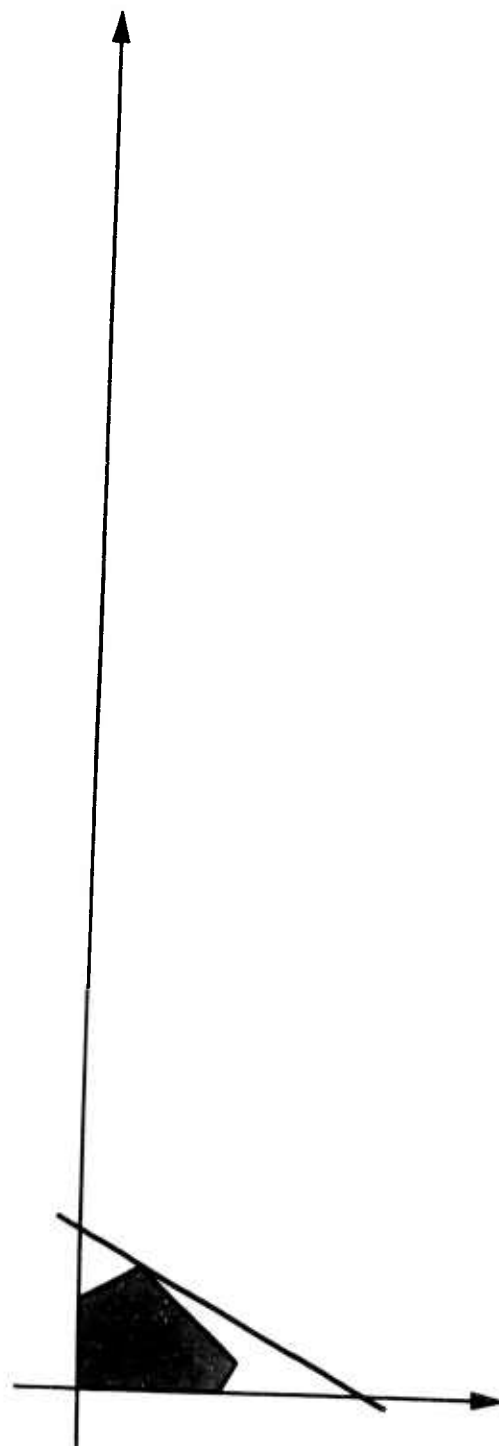
by

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A GRADIENT INEQUALITY FOR NON-DIFFERENTIABLE FUNCTIONS

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A GRADIENT INEQUALITY FOR NON-DIFFERENTIABLE FUNCTIONS

I. Introduction

In mathematical programming the following result is very useful:

THEOREM 1

Assumption: K is a nonempty open subset of R^n,[1] $f : K \to R$, f is differentiable on K,[2] X is a convex subset of K.

Conclusion: f is convex on X if, and only if, $f(x) - f(y) \geq (x - y)^T f'(y)$ for all $x, y \in X$,

where $f'(y)$ denotes the gradient of f at y.

The above theorem is stated, in a slightly weaker form, in [8, Vol. 1, p. 405]; for the sake of completeness we present a proof of Theorem 2 in Appendix A. A related result, extremely useful in programming theory, and discussed in [3] is:

THEOREM 2

Assumption: K is an open subset of R^n, $f : K \to R$, f is differentiable on K, X is a convex subset of K, f is convex on X, $x_0 \in X$.

Conclusion: $f(x_0) \leq f(x)$ for all $x \in X$ if, and only if, $(x - x_0)^T f'(x_0) \geq 0$ for all $x \in X$.

---

[1] R^n denotes the set of all column n-tuples with real number components, we write R in place of R^1.

[2] At each point of K, all first partial derivatives exist.

The proof of Theorem 2 is straightforward; if we know that $(x - x_0)^T f'(x_0) \geq 0$ whenever $x \in X$ then a direct application of Theorem 1, with $y = x_0$, yields: $f(x_0) \leq f(x)$ for all $x \in X$. Conversely, if x_0 minimizes f on X then for each $x \in X$ and $\lambda \in (0, 1)$, since $\lambda x + (1 - \lambda)x_0 \in X$, we must have: $f(x_0) \leq f(\lambda x + (1 - \lambda)x_0) = f(x_0 + \lambda(x - x_0))$. As a consequence we have:

$$\frac{f(x_0 + \lambda(x - x_0)) - f(x_0)}{\lambda} \geq 0 \qquad \text{whenever } \lambda \in (0, 1) \quad ,$$

letting λ approach zero we obtain $(x - x_0)^T f'(x_0) \geq 0$. \hfill Q.E.D.

We shall be concerned here with generalizing Theorem 2 by relaxing the hypothesis that f is differentiable everywhere on K (and thus we shall actually be able to omit mentioning K), though keeping the requirement that f be convex on X. In case we know only that f is convex, we still wish to obtain nontrivial characterizations of the fact that f has a minimum on X and, if possible, obtain some information about the minimizing point (i.e., x_0). We will be able to obtain such characterizations when X and f are of a special form, though f may fail to be differentiable.

In what follows we shall frequently apply the so-called Minkowski-Farkas lemma and we list it here for reference:

Lemma 1: Let A be any real $m \times n$ matrix and let $b \in R^n$. The following statements are equivalent:

(i) $b^T \pi \leq 0$ whenever $A^T \pi \leq 0$, $\pi \in R^m$, [†]

(ii) there exists an $x \in R^n$ such that $x \geq 0$, $Ax = b$

[†] A vector inequality means that the inequality indicated holds for each component.

II. Discussion of Special Problem

Suppose we are given an n-tuple $a = (a_1, \ldots, a_n)$, an $n \times n$ real symmetric positive semi-definite matrix C and a real $m \times n$ matrix A. We define:

(1)
$$X = R^n \cap \{x \mid Ax \leq 0\}$$
$$f(x) = ax + (x^T C x)^{\frac{1}{2}} \quad , \qquad \text{for } x \in R^n$$

It is immediate that X is a convex cone (it is in fact a "finite cone"), also X is obviously nonempty because $0 \in X$. In addition, it follows readily from a well known inequality [4 . Theorem 1 — (ii)] that f is a convex function on R^n.

With f and x as defined in (1) and $x_0 \in X$ we should like to obtain a statement similar to the conclusion in Theorem 2. Some direct considerations lead us to believe that the situation is rather complicated! We note that $0 \in X$ and $f(0) = 0$, furthermore, since X is a cone, we observe that if for any $x \in X$ we had $f(x) < 0$ then indeed f is unbounded below on X. Thus the only situation in which a minimum exists is when a minimizing x is $x_0 = 0$, and of course f does not have a well defined gradient at the origin except in case $C = 0$.

Pursuing the above line of reasoning let us assume that $f(x) = ax + (x^T C x)^{\frac{1}{2}} \geq 0$ whenever $x \in X$ and that, in addition, the minimum of f (i.e., zero) is taken on at some point x_0 such that $x_0^T C x_0 > 0$, and thus the gradient of f is defined at x_0. Summarizing the preceding, x_0 satisfies:

(2)
$$x_0 \in R^n \quad , \quad Ax_0 \leq 0 \quad , \quad ax_0 + (x_0^T C x_0) = 0 \quad , \quad x_0^T C x_0 > 0 \quad .$$

Using (2) the condition that $(x - x_0)^T f'(x_0) \geq 0$ whenever $Ax \leq 0$ may be written as:

$$(3) \qquad \left[a + \frac{x_0^T C}{(x_0^T C x_0)^{\frac{1}{2}}} \right] x \geq 0 \qquad \text{whenever} \quad Ax \leq 0 \quad .$$

Up to this point, we have assumed that x_0 satisfies (2) and that $0 = f(x_0) \leq f(x)$ whenever $x \in X$; we wrote down (3) as a statement "to be contemplated" and have not said anything about the truth or falsity of (3). However, it is quite trivial to show that when x_0 satisfies (2) and if (3) is true then, necessarily, $f(x_0) \leq f(x)$ for all $x \in X$. That is, from the fact that

$$(4) \qquad x_0^T C x \leq (x_0^T C x_0)^{\frac{1}{2}} (x^T C x)^{\frac{1}{2}}$$

which holds for any x_0 , $x \in R^n$ [5, Lemma 1] and (3) we conclude that $x \in R^n$ and $Ax \leq 0$ implies $f(x) = ax + (x^T C x)^{\frac{1}{2}} \geq 0$. The converse statement, that one can conclude (3) from (2) and the fact that $f(x) \geq 0$ whenever $x \in X$ follows from Theorem 2 and the fact that f is then differentiable and convex in some small neighborhood of x_0 . We have thus obtained a characterization of a minimizing x_0 which satisfies $x_0^T C x_0 > 0$. Before attempting to dispose of minimizing x_0's satisfying $x_0^T C x_0 = 0$ (and thus, by (4), $C x_0 = 0$) we observe that (3) is of the form: $Ax \leq 0$ implies $dx \leq 0$, thus by Lemma 1 we know that (3) is equivalent to the statement:

There exists $\pi \in R^m$ such that

$$(5) \qquad \pi \geq 0 \quad \text{and} \quad \pi^T A + a + \frac{x_0^T C}{(x_0^T C x_0)^{\frac{1}{2}}} = 0 \quad .$$

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We observe next that if there exists an x_0 satisfying (2) and (5) then by a suitable normalization we should be able to assume $x_0^T C x_0 = 1$; specifically, letting $z = (x_0^T C x_0)^{-\frac{1}{2}} x_0$ we obtain from (2) and (5) the conditions:

$$z \in R^n , \quad \pi \in R^m$$
$$(6) \qquad Az \le 0 , \quad \pi \ge 0$$
$$\pi^T A + a + z^T C = 0 , \quad z^T C z = 1 .$$

Again, by using (4), it is trivial to show that if there exist π and z satisfying (6) then $f(x) \ge 0$ whenever $x \in X$. In fact, and this is very crucial to our development, we note that the same is true if we relax (6) to read:

$$z \in R^n , \quad \pi \in R^m$$
$$(6') \qquad Az \le 0 , \quad \pi \ge 0$$
$$\pi^T A + a + z^T C = 0 , \quad z^T C z \le 1 .$$

We note that (6') is a more "realistic" statement in view of the fact that the z needed to satisfy (6) may yield $Cz = 0$. That is, we should really like to conclude (6') from the fact that $ax + (x^T C x)^{\frac{1}{2}} \ge 0$ whenever $x \in R^n$ and $Ax \le 0$; this is indeed the case and follows directly from:

THEOREM 3

Assumption: (i) $C,$ [†] is a real symmetric $n \times n$ matrix, A is an $m \times n$ real matrix, $u^T \in R^n$.

                (ii) $(ux)^2 \le x^T \Gamma x$ whenever $x \in R^n$ and $ux \ge 0$, $Ax \le 0$.

Conclusion: There exist $z \in R^n$, $\pi \in R^m$ such that $\pi \ge 0$, $Az \le 0$,
               $u = \pi^T A + z^T C$, $z^T C z \le 1$.

The proof of Theorem 3 will be found in Appendix B.

[†] Not necessarily positive semi-definite.

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As mentioned above, a direct consequence of Theorem 3 is:

Lemma 2: Let f and X be as in (1), then $f(x) \geq 0$ for all $x \in X$ if, and only if, there exist π and z satisfying (6').

Proof: The sufficiency of (6') is, as outlined above, quite clear. Assuming that for each $x \in X$ we have $f(x) \geq 0$ and letting $u = -a$, we see that the assumptions of Theorem 3 hold while the conclusion of Theorem 3 is precisely (6').

III. A Nonhomogeneous Problem

Let us modify (1) so that X is defined by the inequalities $Ax \leq b$, rather than by $Ax \leq 0$, where b is a fixed vector in R^m. First of all, it need no longer be true that $x_0 = 0 \in X$, in fact X may be empty and to dispose of this difficulty we shall assume the contrary, that there exists on $x \in R^n$ such that $Ax \leq b$ or equivalently (according to Lemma 1) that:

(7)   There is no $\pi \in R^m$ such that $\pi \geq 0$, $\pi^T A = 0$, $\pi^T b < 0$.

Since X need no longer be a cone, the minimum of $f(x) = ax + (x^T C x)^{\frac{1}{2}}$ on X (when it exists) may be any real number; let us assume that M is a lower bound of f on X, i.e.,

(8) $ax + (x^T C x)^{\frac{1}{2}} \geq M$ whenever $x \in R^n$ and $Ax \leq b$.

With $b = 0$ and $M = 0$ we would be in the situation discussed in Section II with Lemma 2 applicable. We shall see now that the more general condition (8) may also be characterized in a fashion analogous to the homogeneous case. Towards this let us consider the relations:

-6-

$$z \in R^n \ , \quad \pi \in R^m \ , \quad \lambda \in R$$

$$\text{(9)} \qquad \pi \geq 0 \ , \quad Az \leq \lambda b$$

$$\pi^T A + a + z^T C = 0 \ , \quad \pi^T b + M \leq 0 \ , \quad z^T C z \leq 1 \ ,$$

and:

THEOREM 4

The statement (8) is true if, and only if, there exist z, π and λ satisfying (9).

Proof: If π, z, λ satisfy (9) and $x \in R^n$ is such that $Ax \leq b$ then:

$$0 = \pi^T A x + ax + z^T C x \leq \pi^T b + ax + z^T C x$$

$$\leq -M + ax + (z^T C z)^{\frac{1}{2}} (x^T C x)^{\frac{1}{2}}$$

$$\leq -M + ax + (x^T C x)^{\frac{1}{2}} .$$

and thus $M \leq ax + (x^T C x)^{\frac{1}{2}} .$

Conversely, let us assume that (8) is true. We assert that the statement

$$\text{(10)} \qquad \begin{array}{l} \text{If } x \in R^n \ , \quad \eta \in R \text{ are such that} \\[6pt] Ax - \eta b \leq 0 \ , \quad \eta \geq 0 \text{ then } \eta M \leq ax + (x^T C x)^{\frac{1}{2}} \ , \end{array}$$

is true. Clearly, when $\eta > 0$ (10) holds, simply consider $\eta^{-1} x$ and compare (10) with (8) which is assumed true. In case $\eta = 0$, we should like to conclude $ax + (x^T C x)^{\frac{1}{2}} \geq 0$ from $Ax \leq 0$; the last follows from the facts that (i) the statement (7) is true and (ii) $ax + (x^T C x)^{\frac{1}{2}}$ is convex, homogeneous and bounded below on the set $\{ x \mid Ax \leq b \}$. We next note that (10) is precisely analogous to the homogeneous statement in Lemma 2, applying Lemma 2 we see that from the fact that (10) holds it follows that there must exist π, z and λ satisfying (9).   Q.E.D.

IV. Concluding Remarks

The following remarks are in order:

A. Lemma 2 is precisely the principal theorem of [5] where it is demonstrated directly by using a certain "pseudo-norm" on the set of sequences of points in R^n, together with Lemma 1.   Here Lemma 2 depends ultimately on the theorem of Frank-Wolfe [7] that any quadratic function bounded below on a polygonal convex set achieves its minimum.

B. A significant interpretation of Lemma 2 is to think of it as a direct generalization of the Minkowski-Farkas theorem (our Lemma 1), because the former reduces to the latter when $C = 0$.

C. One can think of Lemma 2 as expressing the fact that a certain convex set is closed. The set T in question consists of all negatives of a's for which there exist z and π satisfying (6'), these in turn are the tangent planes of the convex function $g(x) = (x^T C x)^{\frac{1}{2}}$ over the cone $\{x | Ax \leq 0 \}$. In general, for arbitrary convex (and even continuous) g this set of tangents need not be closed; e.g., if g exhibits "assymptotic" behavior, a limit of tangents need not be a tangent.

D. In view of the preceding and also because of the relation of Lemma 2 to Theorem 2, it would be of interest to generalize Lemma 2 to a larger class of pairs (X, f) , e.g., keeping X a finite cone and allowing f to take the form $f(x) = ax + \sum_{k=1}^{P} (x^T C_k x)^{\frac{1}{2}}$, where each C_k is positive semi-definite.

APPENDIX A
Proof of Theorem 1

All quanitities are as defined in Theorem 1. We first demonstrate the simpler half of the proof that if

$$(11) \qquad f(x) - f(y) \geq (x - y)^T f'(y) \qquad \text{whenever } x, y \in X,$$

then f is convex. Suppose u, v are elements of X and $\lambda \in [0, 1]$, let $x = u$ and $y = \lambda u + (1 - \lambda)v$ then from (11) we get:

$$(12) \qquad f(\lambda u + (1 - \lambda)v) \leq f(u) - (1 - \lambda)(u - v)^T f'(y).$$

Similarly, letting $x = v$ and $y = \lambda u + (1 - \lambda)v$ we obtain from (11):

$$(13) \qquad f(\lambda u + (1 - \lambda)v) \leq f(v) + \lambda(u - v)^T f'(y).$$

Multiplying (12) by λ, and (13) by $(1 - \lambda)$, then adding, we obtain the required inequality.

We show next that if f is convex on X then (11) must hold. First, we show the above true for $n = 1$, then reduce the general case to the one dimensional one. Let us assume that K is an open subset of R and that f is differentiable on K, then the convexity of f on a convex subset X of K is simply the statement that:

$$(14) \qquad f(\beta) \leq \left(\frac{\gamma - \beta}{\gamma - \alpha}\right) f(\alpha) + \left(\frac{\beta - \alpha}{\gamma - \alpha}\right) f(\gamma)$$

whenever $\alpha < \beta < \gamma$ and $\alpha, \gamma \in X$.

It is readily seen that (14) is equivalent to:

$$(15) \quad \frac{f(\beta) - f(\alpha)}{\beta - \alpha} \leq \frac{f(\gamma) - f(\beta)}{\gamma - \beta}$$

whenever $\alpha < \beta < \gamma$ and $\alpha, \gamma \in X$.

Letting first β approach γ and then, independently, letting β approach α we obtain from (15):

$$(16) \quad \left. \begin{array}{l} \dfrac{f(\gamma) - f(\alpha)}{\gamma - \alpha} \leq f'(\gamma) \\[3mm] f'(\alpha) \leq \dfrac{f(\gamma) - f(\alpha)}{\gamma - \alpha} \end{array} \right\} \quad \text{if } \alpha < \gamma \text{ and } \alpha, \gamma \in X.$$

which is in fact equivalent to (15). Lastly, we note that (16) is equivalent to (11). We just demonstrated that if $n = 1$ and f satisfies the assumptions of Theorem 1 then (11) must hold whenever f is convex.

Finally, suppose n is any positive integer and f is convex on X and satisfies the assumptions of Theorem 1. Since K is open, for fixed $x, y \in X \subset K$, there must exist an open interval I_{xy} containing $[0, 1]$ and such that $\lambda x + (1 - \lambda) y$ is in K whenever λ is in I_{xy} . Defining the function h_{xy} on I_{xy} by

$$h_{xy}(\lambda) = f(\lambda x + (1 - \lambda) y) \quad , \quad \text{for } \lambda \in I_{xy} \quad ,$$

one checks immediately that h_{xy} must be convex on $[0, 1]$. Consequently, since Theorem 1 holds for $n = 1$, we know that:

$$(17) \quad h_{xy}(1) - h_{xy}(0) \geq (1 - 0) h'_{xy}(0)$$

which is precisely the statement (11).　　　　　　　　　　Q. E. D.

It should be noted that in the first half of the above proof no use was made of the properties of the gradient of f, thus, in fact, the following theorem is true.

THEOREM 5

Let f, K and X satisfy the assumption of Theorem 1, then f is convex on X if, and only if, there exists a function $G : K \to R^n$ and such that

$$f(x) - f(y) \geq (x - y)^T G(y) \quad \text{for all } x, y \in X.$$

Similarly, one notes that the following is true:

THEOREM 6

Let K be a nonempty open convex subset of R^n, $f : K \to R$, then f is convex and differentiable on K if, and only if, there exists a continuous function $G : K \to R^n$ such that

$$f(x) - f(y) \geq (x - y)^T G(y) \quad \text{for all } x, y \in K.$$

APPENDIX B
Proof of Theorem 3

Consider the linear inequalities:

$$z \in R^n , \quad \pi \in R^m$$

(18)

$$\pi \geq 0 , \quad Az \leq 0 , \quad u^T = A^T \pi + Cz ,$$

and let P consist of all ordered pairs (π, z) satisfying (18). We show first that P is nonempty. Since (18) represents a system of linear inequalities, the fact that P is empty is equivalent, by Lemma 1, to the fact that there exist x and y satisfying:

$$x \in R^n , \quad y \in R^m$$

(19)

$$y \geq 0 , \quad Ax \leq 0 , \quad Cx = A^T y , \quad ux > 0 .$$

Thus, if P were empty, we would get from (19):

$$x^T Cx = \pi^T Ax \leq 0 ,$$

consequently $x^T Cx \leq 0$ which together with $ux > 0$ and $Ax \leq 0$ contradicts the assumption of Theorem 3. Thus P is nonempty.

Next, if for some $(\pi, z) \in P$ we had $z^T Cx \leq 1$ then we would have the desired conclusion of Theorem 3; assume that $z^T Cz > 1$ whenever $(\pi, z) \in P$. Applying the result in [7] that a quadratic function defined on a polyhedral convex set attains its minimum, we know that there exist $(\pi_0, z_0) \in P$ such that $1 < z_0^T Cz_0 \leq z^T Cz$ for each $(\pi, z) \in P$. Furthermore, it is clear that

(20) $$(z_0^T C)z_0 \leq (z_0^T C)z \qquad \text{whenever } (\pi, z) \in P$$

Applying Lemma 1 to (20) (essentially, making use of duality in linear programming), we see that there must then exist π and x satisfying:

$$
\begin{aligned}
&x \in R^n \ , \quad \pi \in R^m \\
&Ax \leq 0 \ , \quad \pi \geq 0 \ , \quad Az_0 \leq 0 \\
&u^T = A^T \pi_0 + C z_0 \ , \quad Cx = A^T \pi + C z_0 \ , \quad ux = z_0^T C z_0 \ .
\end{aligned}
\tag{21}
$$

The following relations then are consequences of (21):

$$
\begin{aligned}
z_0^T C z_0 &= z_0^T C x - z_0^T A^T \pi \\
&= x^T C z_0 - \pi^T A z_0 \\
&= ux - \pi_0^T Ax - \pi^T A z_0 \\
&\geq ux \\
&= z_0^T C z_0 \ .
\end{aligned}
$$

As a result, $\pi_0^T Ax = \pi^T A z_0 = 0$ and $z_0^T C z_0 = ux = x^T C z_0$. However, $Ax \leq 0$ and $ux > 1$; thus by assumption in Theorem 3 we have $0 < (ux)^2 \leq x^T Cx$ and:

$$
\begin{aligned}
(x^T Cx)^{\frac{1}{2}} &\geq ux \\
&= z_0^T C z_0 \\
&= x^T C z_0 \\
&= x^T Cx - \pi^T Ax \\
&\geq x^T Cx \ .
\end{aligned}
$$

From the last relations we get $x^T Cx \leq 1$ and thus $ux \leq (x^T Cx)^{\frac{1}{2}} \leq 1$, a contradiction.

<div align="right">Q.E.D.</div>

BIBLIOGRAPHY

[1] R. Bellman, Introduction to Matrix Analysis, McGraw-Hill, New York, 1960.

[2] A. Charnes and W.W. Cooper, "The Strong Minkowski-Farkas-Weyl Theorem for Vector Spaces over Ordered Fields," Proc. Natl. Acad. Sci., Vol. 44, (1958) p. 914-916.

[3] G.B. Dantzig, Linear Programming and Extensions, Princeton University Press, Princeton, N.J. (In press).

[4] E. Eisenberg, "A Note on Semi-Definite Matrices," Research Report No. 9, Operations Research Center, University of California, Berkeley, July 20, 1961.

[5] E. Eisenberg, "Supports of a Convex Function," Bull. Am. Math. Soc., Vol. 68, No. 3, (May 1962) p. 192-195.

[6] Ky Fan, "Minimax Theorems," Proc. Natl. Acad. Sci., Vol. 39, No. 1, (January 1953) p. 42-47.

[7] M. Frank and P. Wolfe, "An Algorithm for Quadratic Programming," Nav. Res. Log. Quart., Vol. 3, (March and June 1956) p. 95-110.

[8] S. Karlin, Mathematical Methods and Theory in Games, Programming and Economics, Addison-Wesley Press, Cambridge, Massachusetts, 1959.

[9] H. Markowitz, Portfolio Selection — An Efficient Diversification of Investments, Wiley Press, New York, 1959.

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