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EDGEWORTH-ALLOCATIONS IN AN  
EXCHANGE ECONOMY WITH  
MANY TRADERS\*

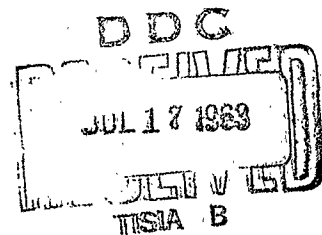
BY  
KARL VIND

TECHNICAL REPORT NO. 13  
JUNE, 1963

PREPARED UNDER CONTRACT Nonr-222(77)  
(NR-047-029)

FOR  
OFFICE OF NAVAL RESEARCH

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EDGEWORTH-ALLOCATIONS IN AN EXCHANGE  
ECONOMY WITH MANY TRADERS<sup>1</sup>

by

Karl Vind

1. INTRODUCTION

In recent papers, Scarf [5], Debreu [2], Debreu and Scarf [3], and Aumann [1], have defined and characterized the core of an economy. The economy is a pure trade economy, i.e., an economy where a set of consumers possesses initially certain quantities of the commodities. The trade is simply a reallocation of the commodities.<sup>2</sup> We assume that the consumers have preference relations. An allocation is by definition in the core or an Edgeworth-allocation if no group of consumers can combine and reallocate their initial allocation in such a way that all consumers in the group prefer the new allocation.

When an allocation is an Edgeworth-allocation, it means in particular that none of the individual consumers prefer their initial allocation. At the other extreme it means that the set of all consumers cannot reallocate in such a way that

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<sup>1</sup>This paper was written while the author was a Rockefeller fellow at the University of California, Berkeley. Professor Debreu has made many useful comments to earlier versions. Preparation of the manuscript was supported by the Office of Naval Research under Contract ONR 222(77) with the University of California.

<sup>2</sup>Debreu and Scarf [3] also treat a case where production is possible.

everybody is made better off. An Edgeworth-allocation is thus a special case of a Pareto-optimal allocation. Given an Edgeworth-allocation, it is not only impossible for the individual consumer or the set of all consumers to get a better allocation, it is also impossible for any subset of consumers to combine and get a better allocation by reallocating this subset's initial allocation.

Scarf has "rediscovered" the problem after the original treatment by Edgeworth and has proved [5] that if we let the number of consumers increase in a very special way, then the set of Edgeworth-allocations will in the limit be the competitive allocations. Debreu [2] and Debreu and Scarf [3] have simplified the proof and extended the applicability of this result. Aumann [1] has shown that, if we have a continuum of consumers and all the individual consumers are unimportant, then Edgeworth-allocations will, under certain conditions, be competitive allocations.

In this paper we define a non-atomic exchange economy; in intuitive terms this is an economy with no production and many small traders. The set of values of the net trade for all possible subsets of consumers in such an economy has some very useful properties (Theorem 1); and the set of values of the net trade for all possible subsets of consumers with allocations preferred to any given allocation is a convex set (Theorem 2). Theorem 2 is a consequence of a general mathematical lemma, which may have independent interest. This lemma is stated and proved in an appendix.

These surprisingly simple and general results are applied in Theorem 3 to give a result comparable to Aumann's theorems.

In order to simplify the reading, there is on page 19 a summary of the definitions and interpretations of the concepts used.

## II. THE ECONOMY

The economy consists of a set  $C$  of consumers and a finite set of  $\ell$  commodities. Some subsets of  $C$  will be coalitions. Let  $\mathcal{C}$  denote the set of all possible coalitions. We assume that this system of subsets  $\mathcal{C}$  has the following properties:

- (a)  $C \in \mathcal{C}$  ; (the set of all consumers form a coalition).
- (b)  $A \in \mathcal{C}$  and  $B \in \mathcal{C}$  implies that  $A \setminus B \in \mathcal{C}$  (if  $A$  and  $B$  are two coalitions, the set of consumers in coalition  $A$  but not in coalition  $B$  is also a coalition).
- (c)  $A \in \mathcal{C}$  and  $B \in \mathcal{C}$  implies that  $A \cap B \in \mathcal{C}$ ; (if  $A$  and  $B$  are coalitions, the set of consumers in both  $A$  and  $B$  also forms a coalition).
- (d)  $A \in \mathcal{C}$  and  $B \in \mathcal{C}$  implies that  $A \cup B \in \mathcal{C}$ . (The union of two coalitions is a coalition.)
- (e)  $A_j \in \mathcal{C}$  for  $j = 1, 2, \dots$ , implies that  $\bigcup_{j=1}^{\infty} A_j \in \mathcal{C}$ ; (the union of denumerably many coalitions is a coalition).

When a system of subsets has the properties (a)-(d), it is called a field. If it also has the property (e), it is called a  $\sigma$ -field. We will in the following assume that  $\mathcal{C}$  is a  $\sigma$ -field of subsets of  $C$ . (The properties listed do not form a minimal set of axioms for fields or  $\sigma$ -fields; e.g., property (c) is implied by the other properties.)

$\ell$  is the number of commodities, and  $\Omega$  is the non-negative orthant of  $R^\ell$ , the  $\ell$ -dimensional Euclidean space.  $\Omega$  will be called the commodity space.

An allocation is a function defined on  $\mathcal{C}$  with values in  $\Omega$ . The function  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_\ell)$  gives to any coalition  $A \in \mathcal{C}$  an allocation  $\alpha(A)$ , which is a point in the commodity space. We will include the following properties in the definition of an allocation:

- (1)  $\alpha$  is defined on  $\mathcal{C}$  and has values in  $\Omega$ .
- (2)  $\alpha(A \cup B) = \alpha(A) + \alpha(B)$  for  $A \cap B = \phi$ . (For two coalitions with no consumers in common, the amount of commodities in the allocation for the union is the sum of the allocations.)
- (3)  $\alpha(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \alpha(A_i)$  for  $A_i \cap A_j = \phi$ ,  $i \neq j$ .  
(This is condition (2) extended to the union of denumerably many coalitions.)

The set of all allocations will be denoted  $\mathcal{A}$ . Property 1 means that  $\alpha$  is a vector-valued set function, 2 means that  $\alpha$  is additive, and 3 that  $\alpha$  is  $\sigma$ -additive.  $\mathcal{A}$  is then the set of all  $\sigma$ -additive set functions defined on  $\mathcal{C}$  with values



in  $\Omega$ . A  $\sigma$ -additive set function with values in  $\Omega$  is a finite vector-measure, so any  $\alpha \in \mathcal{A}$  is a finite vector measure. We will also need the set of all  $\sigma$ -finite vector measures  $\bar{\mathcal{A}}$ . For an  $\alpha \in \bar{\mathcal{A}}$  we may have  $\alpha_1(A) = +\infty$ , but in that case there will exist a partitioning of  $A$ ;  $\bigcup A_j = A$  such that  $\alpha_1(A_j) < +\infty$ . ( $A \in \bar{\mathcal{A}}$ .)

The initial allocation  $\iota$  is an element in  $\mathcal{A}$ . In order to avoid trivial cases we will assume  $\iota_1(C) > 0$  for  $i = 1, 2, \dots, l$ . The economy we want to treat is a pure trade economy so the final allocation must be a reallocation of the initial allocation; i.e., if  $\alpha$  is a final allocation, we must have  $\alpha(C) = \iota(C)$ . The set of allocations  $\alpha$  with  $\alpha(C) = \iota(C)$  will be denoted  $\mathcal{B}$  and will be called the set of possible allocations.

A subset  $A \in \mathcal{C}$  is an atom with respect to  $\alpha$  if  $\alpha(A) \neq 0$  and if  $A \supseteq B \in \mathcal{C}$  implies that  $\alpha(B) = 0$  or  $\alpha(B) = \alpha(A)$ . The subsets of  $A$  have either no allocation or the same allocation as  $A$ . One way of expressing the intuitive idea that the economy consists of many small consumers is to assume that  $\mathcal{C}$  does not contain any atoms with respect to  $\iota$ . This assumption implies that any coalition  $A$  with  $\iota(A) \neq 0$  can be divided into smaller coalitions with an initial allocation between  $\iota(A)$  and 0. If  $\mathcal{C}$  does not contain any atoms with respect to  $\alpha$ , we call  $\alpha$  non-atomic; the set of all non-atomic allocations will be denoted  $\mathcal{A}_0$ . The set of all non-atomic  $\sigma$ -finite vector measures will be denoted  $\bar{\mathcal{A}}_0$ . ( $\mathcal{A}_0 \subset \bar{\mathcal{A}}_0 \subset \bar{\mathcal{A}}$ .) The

assumption that  $\iota \in \mathcal{A}_0$ , will be part of the definition of a non-atomic economy. An allocation  $\alpha$  in both  $\mathcal{A}_0$  and  $\mathcal{B}$  will be non-atomic and have  $\alpha(C) = \iota(C)$ ; the set of all such allocations will be denoted  $\mathcal{B}_0$ .

From the definitions, we immediately get  $\mathcal{B}_0 = \mathcal{A}_0 \cap \mathcal{B}$ ;  $\mathcal{A} \supset \mathcal{A}_0 \supset \mathcal{B}_0$  and  $\mathcal{A} \supset \mathcal{B} \supset \mathcal{B}_0$ . For any  $\alpha \in \mathcal{B}$  we have  $\alpha(A) \in U = \{x \mid 0 \leq x_1 \leq \iota_1(C)\}$  for all  $A \in \mathcal{C}$ . For any  $\alpha \in \mathcal{A}$  we will denote the range of  $\alpha$  by  $\mathcal{R}(\alpha)$ .  $\mathcal{R}(\alpha) = \{x \mid x = \alpha(A); A \in \mathcal{C}\}$  is for  $\alpha \in \mathcal{B}$  a subset of  $U$ . When  $\alpha$  and  $\beta$  are elements in  $\mathcal{A}$ ,  $(\alpha, \beta)$  is a vector measure with values in  $\mathbb{R}^{2\ell}$ . The range of this measure will be denoted  $\mathcal{R}(\alpha, \beta)$ . The difference  $\alpha(A) - \iota(A)$  is a vector in  $\mathbb{R}^\ell$ , where the coordinates indicate the net trade of coalition  $A$  when the allocation is  $\alpha$ . The range of the net trade is of great importance, so we define  $\mathcal{R}(\alpha - \iota) = \{x \mid x = \alpha(A) - \iota(A); A \in \mathcal{C}\}$ .

The following lemma gives some properties of the sets  $\mathcal{R}(\alpha)$ ,  $\mathcal{R}(\alpha, \beta)$  and  $\mathcal{R}(\alpha - \iota)$  for non-atomic allocations.

LEMMA 1. Let  $\alpha$ ,  $\beta$  and  $\iota$  be elements in  $\mathcal{A}_0$ , then  $\mathcal{R}(\alpha)$ ,  $\mathcal{R}(\alpha, \beta)$  and  $\mathcal{R}(\alpha - \iota)$  are closed, convex and symmetric around  $\frac{1}{2} \alpha(C)$ ,  $\frac{1}{2} (\alpha(C), \beta(C))$  and  $\frac{1}{2} (\alpha(C) - \iota(C))$ , respectively.  $\mathcal{R}(\alpha)$  contains the closed segment  $[0, \alpha(C)]$ ,  $\mathcal{R}(\alpha, \beta)$  contains the closed segment  $[0, (\alpha(C), \beta(C))]$ ,  $\mathcal{R}(\alpha - \iota)$  contains the closed segment  $[0, \alpha(C) - \iota(C)]$ .

Proof: That  $\mathcal{R}(\alpha)$ ,  $\mathcal{R}(\alpha, \beta)$  and  $\mathcal{R}(\alpha - \iota)$  are closed and convex follows from a theorem proved by Lyapunov (1940); see Halmos [4], p. 421. The theorem can be formulated: The range of a non-atomic (signed) vector-measure is convex and closed.  $\alpha - \iota$  is the difference between two finite measures and therefore a signed measure. When  $\alpha$ ,  $\beta$  and  $\iota$  are non-atomic,  $(\alpha, \beta)$  and  $\alpha - \iota$  are also non-atomic. (If  $A$  is an atom for  $(\alpha, \beta)$ , it is an atom for  $\alpha$  or  $\beta$ ; if  $A$  is an atom for  $\alpha - \iota$ , we can apply Lemma B in the appendix with  $(\alpha, \iota) = \mu$  and  $\alpha - \iota = \nu$  to get a  $B \subseteq A$  with  $\alpha(B) - \iota(B) = \alpha(A) - \iota(A)$  and  $(\alpha(B), \iota(B)) = (0, 0)$ , and this is clearly a contradiction, so  $\alpha - \iota$  is non-atomic.) Therefore Lyapunov's theorem can be applied.  $\phi$  and  $C$  are elements in  $\mathcal{L}$  and  $\mathcal{R}(\alpha)$ ,  $\mathcal{R}(\alpha, \beta)$  and  $\mathcal{R}(\alpha - \iota)$  are convex; therefore, the closed segments in the lemma are contained in  $\mathcal{R}(\alpha)$ ,  $\mathcal{R}(\alpha, \beta)$  and  $\mathcal{R}(\alpha - \iota)$ , respectively.

If  $x = \alpha(A) \in \mathcal{R}(\alpha)$ , then  $\alpha(C \setminus A) = \alpha(C) - x = y \in \mathcal{R}(\alpha)$ ;  $x + y = \alpha(C)$  and  $\mathcal{R}(\alpha)$  is therefore symmetric around  $\frac{1}{2} \alpha(C)$ . The proofs for  $\mathcal{R}(\alpha, \beta)$  and  $\mathcal{R}(\alpha - \iota)$  are analogous.

For any measure defined on  $\mathcal{L}$  we can define the restriction to A of the measure; if for instance  $\alpha$  is an allocation, then  $\alpha^A$  — the restriction of  $\alpha$  to  $A$  — is defined by

$$\alpha^A(B) = \begin{cases} \alpha(B) & \text{for } B \subseteq A \text{ and } B \in \mathcal{L} \\ \text{not defined} & \text{otherwise.} \end{cases}$$

In some of the later proofs we need the following

Corollary. Lemma 1 applies also to the restrictions of  $\alpha$ ,  $(\alpha, \beta)$  and  $\alpha - \iota$  to a coalition  $A$  (with  $C$  changed to  $A$ ).

Proof. The restriction of a finite non-atomic (signed) measure is also a finite non-atomic (signed) measure, so the same proof applies.

It will be useful to have explicitly the properties of  $\mathcal{R}(\alpha-\iota)$  for  $\alpha \in \mathcal{B}_0$ .

THEOREM 1. When  $\alpha \in \mathcal{B}_0$ ,  $\mathcal{R}(\alpha-\iota)$  is closed, convex and symmetric around 0.  $\mathcal{R}(\alpha-\iota)$  is contained in a proper subvector-space if and only if 0 is not an interior point in  $\mathcal{R}(\alpha-\iota)$ .

Proof. The first part is a repetition of Lemma 1 with  $\alpha(C)-\iota(C) = 0$ . If  $x$  is interior in  $\mathcal{R}(\alpha-\iota)$ , then  $-x$  is interior (symmetry) and  $0 = \frac{1}{2}(x-x)$  is interior (convexity). 0 not interior implies therefore that no points are interior in the convex set  $\mathcal{R}(\alpha-\iota)$ . This implies that  $\mathcal{R}(\alpha-\iota)$  is contained in a proper subvector space. The opposite implication is obvious.

### III. PREFERENCES

Usually preferences are expressed as a relation on  $\Omega$  for each consumer. What we will do is to define a function  $S$  on  $\bar{\mathcal{A}} \times \bar{\mathcal{A}}$  with values in  $\mathcal{C}$ . To every ordered pair  $\alpha, \beta$ , we have a coalition  $A = S(\alpha, \beta)$ . In the interpretation  $S(\alpha, \beta)$  is the set of consumers who prefer  $\alpha$  to  $\beta$  or are indifferent between them. The set  $S(\alpha, \beta)$  will be the union of the disjoint sets  $P(\alpha, \beta)$  and  $I(\alpha, \beta)$ , where  $P(\alpha, \beta) = S(\alpha, \beta) \setminus S(\beta, \alpha)$

and  $I(\alpha, \beta) = S(\alpha, \beta) \cap S(\beta, \alpha)$ .  $P(\alpha, \beta)$  is the set of consumers who prefer  $\alpha$  to  $\beta$ , and  $I(\alpha, \beta)$  is the set of indifferent consumers. We will also use the following notation

$$\alpha \underset{A}{\gg} \beta \text{ if } A \subseteq P(\alpha, \beta) \text{ and } A \neq \emptyset.$$

The interpretation of this notation should be obvious.

#### IV. PREFERRED ALLOCATIONS

We will now define the functions  $P$  and  $P_0$ .  $P$  is defined on  $\bar{\mathcal{A}} \times \mathcal{C}$  and has values in the set of subsets of  $\bar{R}^l$ . ( $\bar{R} = R \cup \{+\infty\}$ .) Precisely,

$$P(\alpha, A) = \{x \in \bar{R}^l \mid x = \beta(A) - \iota(A); \beta \in \bar{\mathcal{A}}; \beta \underset{A}{\gg} \alpha\} \text{ for } A \neq \emptyset$$

and  $P(\alpha, \emptyset) = \{0\}$ .

$P(\alpha, A)$  is the range of the net trade of  $A$ , when we let the allocation  $\beta$  vary over the set of allocations preferred to  $\alpha$  by coalition  $A$ .  $P(\alpha, A) + \{\iota(A)\}$  is for the given coalition  $A$  "the set of points above the coalition indifference surface through  $\alpha(A)$ ."

The only difference between  $P(\alpha, A)$  and  $P_0(\alpha, A)$  is that in the definition of  $P_0$  we only allow  $\beta$  to vary over the set  $\bar{\mathcal{A}}_0$  of non-atomic  $\sigma$ -finite vector-measures. Thus

$$P_0(\alpha, A) = \{x \in \bar{R} \mid x = \beta(A) - \iota(A); \beta \in \bar{\mathcal{A}}_0; \beta \underset{A}{\gg} \alpha\}$$

for  $A \neq \emptyset$  and  $P_0(\alpha, \emptyset) = \{0\}$ .

We will also need the following concepts:

$$P(\alpha) = \bigcup_{\substack{A \in \mathcal{L} \\ A \neq \emptyset}} P(\alpha, A) = \{x \mid x = \beta(A) - \iota(A); \beta \in \bar{\mathcal{A}}; \beta \succ_A \alpha; \alpha \in \mathcal{L}\}$$

$$\text{and } P_o(\alpha) = \bigcup_{\substack{A \in \mathcal{L} \\ A \neq \emptyset}} P_o(\alpha, A) = \{x \mid x = \beta(A) - \iota(A); \beta \in \bar{\mathcal{A}}_o; \beta \succ_A \alpha; A \in \mathcal{L}\}.$$

$P(\alpha)$  is the set of points preferred to  $\alpha$ , in the sense that any point in  $P(\alpha)$  is the net trade for a coalition  $A$  with an allocation  $\beta$  preferred to  $\alpha$  by  $A$ .

$P_o(\alpha)$  is the same as  $P(\alpha)$  except that we restrict the allocations to the set of non-atomic allocations. If  $x \in P_o(\alpha)$ , there exists a non-atomic  $\beta$  and an  $A \in \mathcal{L}$  such that  $x = \beta(A) - \iota(A)$  and  $\beta \succ_A \alpha$ .

#### V. ASSUMPTIONS ON THE PREFERENCES

In the following theorems we need one or more of the following assumptions.

I  $\alpha_1^A = \alpha_2^A$  implies that

(a)  $S(\alpha_1, \beta) \cap A = S(\alpha_2, \beta) \cap A$  and

(b)  $S(\beta, \alpha_1) \cap A = S(\beta, \alpha_2) \cap A$  for all  $\beta \in \bar{\mathcal{A}}$ .

$\alpha_1$  and  $\alpha_2$  are two different allocations but  $\alpha_1^A = \alpha_2^A$  means that they give the same allocation to every sub-coalition of  $A$ . The assumption means that the preferences of  $A$  are independent of the values of the allocations outside  $A$ . The sets  $P(\alpha, A)$  and  $P_o(\alpha, A)$  are independent of the allocations  $\alpha$  gives to coalitions in the complement of  $A$ .

II If  $\gamma(B) > \beta(B)$  for all  $B \subseteq A$  with  $\beta(B) \neq 0$  and  $\beta \gg_A \alpha$ ,  
then  $\gamma \gg_A \alpha$  and  $\gamma \gg_A \beta$ .

$\gamma(B) > \beta(B)$  means  $\gamma_1(B) \geq \beta_1(B)$  and  $\gamma(B) \neq \beta(B)$ . The assumption corresponds to the usual assumption of monotonicity of preferences. If allocation  $\gamma$  gives at least the same amount as  $\beta$  to all subcoalitions of  $A$  and more of at least one commodity to all subcoalitions of  $A$  with  $\beta(B) \neq 0$ , then  $A$  prefers  $\gamma$  to  $\beta$  and to any allocation worse than  $\beta$ . If  $x \in P(\alpha, A)$  and  $y > 0$ , then  $x+y \in P(\alpha, A)$  and  $\alpha(A) - \iota(A) + y \in P(\alpha, A)$ , ( $A \neq \emptyset$ ).

III  $P(\alpha, A) + \{\iota(A)\}$  for  $A \neq \emptyset$  is open in  $\Omega$ .

This assumption corresponds to the assumption that the set of points preferred to any given allocation is an open set. The assumption means that if  $\beta$  is preferred to  $\alpha$  by  $A$ , then any commodity bundle in a neighborhood of  $\beta(A)$  can be allocated to  $A$  in such a way that the new allocation still is preferred to  $\alpha$  by  $A$ .

The assumptions have some useful implications for the concepts we have just introduced.

LEMMA 2. Let  $\alpha \in B_0$ , and let Assumption II hold, then  
 $R(\alpha - \iota) \subseteq \overline{P(\alpha)}$ .

Proof. We have to prove that  $P(\alpha - \iota)$  is contained in the adherence  $\overline{P(\alpha)}$  of  $P(\alpha)$ . But this is obvious, when we make

the monotonicity assumption; if  $\alpha(A) - \iota(A)$  is a point in  $\mathcal{R}(\alpha - \iota)$ , then all points  $\alpha(A) - \iota(A) + y$ , where  $0 \neq y \in \Omega$  is the value of a net trade preferred by A to  $\alpha$ .

LEMMA 3. Let  $\alpha \in \mathcal{B}_0$ , and let Assumptions I and II hold, then  $\mathcal{P}(\alpha) = \mathcal{P}_0(\alpha)$ .

Proof.  $\mathcal{P}_0(\alpha) \subseteq \mathcal{P}(\alpha)$  is obvious.

$\mathcal{P}_0(\alpha) \supseteq \mathcal{P}(\alpha)$ . Assume that  $\beta \in \bar{\mathcal{A}} \setminus \bar{\mathcal{A}}_0$ , with  $\beta(A) - \iota(A) \in \mathcal{P}(\alpha)$  and  $\beta \gg \alpha$ ; we have to prove that  $\beta(A) - \iota(A) \in \mathcal{P}_0(\alpha)$ .  $\beta$  has atom(s) <sup>A</sup> in A and by Lemma B in the appendix it has therefore an atom  $B_0 \subseteq A$  with  $\alpha(B_0) = \iota(B_0) = 0$ ; now  $\beta(B_0) \in \Omega$  and can be "added" to the allocation  $\beta$  for the coalition  $A \setminus B_0$  to give a new allocation  $\gamma$  with  $\gamma(A \setminus B_0) = \beta(A \setminus B_0) + \beta(B_0) = \beta(A)$  and  $\gamma(B) > \beta(B)$  for all  $B \subseteq A \setminus B_0$  with  $\beta(B) \neq 0$ . We know that  $\beta \gg \alpha$ , and we can therefore  <sup>$A \setminus B_0$</sup>  apply Assumption II and get  $\gamma \gg \alpha$ ; but  $\gamma(A \setminus B_0) - \iota(A \setminus B_0) = \beta(A) - \iota(A) \in \mathcal{P}(\alpha)$ .  <sup>$A \setminus B_0$</sup>  The same procedure can be applied to all the atoms for  $\beta$  and we get finally a non-atomic allocation with a preferred net trade equal to  $\beta(A) - \iota(A)$ ;  $\beta(A) - \iota(A)$  is therefore also an element in  $\mathcal{P}_0(\alpha)$ . In intuitive terms, we have distributed the allocation for the atoms, over the non-atomic part of the coalition, and this gives the non-atomic part a preferred allocation.

For the sets  $\mathcal{P}_0(\alpha, A)$  and  $\mathcal{P}_0(\alpha)$ , we are now able to prove the basic

THEOREM 2. Assumption I implies that  $\mathcal{P}_0(\alpha, A)$  and  $\mathcal{P}_0(\alpha)$  are convex. ( $\alpha \in \mathcal{B}_0$ ,  $A \in \mathcal{L}$ )



Proof. We just have to check that  $P_o(\alpha, A)$  as a function of  $A$  has the properties of the function  $F$  in lemma A in the appendix.

- 1)  $P_o(\alpha, \phi) = \{0\}$  is clear.
- 2)  $P_o(\alpha, A \cup B) = P_o(\alpha, A) + P_o(\alpha, B)$  for  $A \cap B = \phi$ .

This is a consequence of Assumption I.

- 3)  $P_o(\alpha, \bigcup_{i=1}^{\infty} A_i) = \sum P_o(\alpha, A_i)$ ,  $A_i \cap A_j = \phi$  for  $i \neq j$ .

The only extra problem here is to make sure that the addition on the right side is possible.  $P_o(\alpha, A)$  is bounded from below in each coordinate by  $-v(C)$ , so this point is also obvious.

- 4) This follows from the  $\sigma$ -additivity of the  $\beta$ 's in the definition of  $P_o(\alpha, A)$ .

The non-atomicity of  $P_o$  is a consequence of the non-atomicity of  $\bar{A}_o$  in the definition of  $P_o$ .

## VI. CLASSES OF ALLOCATIONS IN A NON-ATOMIC EXCHANGE ECONOMY

All the concepts defined and used can be derived from the basic concepts  $C, \Omega, \mathcal{C}, \mathcal{A}, v$  and  $S$ .  $\mathcal{E} = (C, \Omega, \mathcal{C}, \mathcal{A}, v, S)$  is a non-atomic exchange economy with a preference function, if  $v$  is non-atomic.  $\mathcal{B}$  is the set of possible states of the economy (an  $\alpha \in \mathcal{B}$  is an allocation with  $\alpha(C) = v(C)$ ).

In this section we will define two classes of allocations for the economy  $\mathcal{E}$  and prove a theorem giving relations between these classes.

A coalition A is blocking for an allocation  $\alpha \in \mathcal{A}$  if there exist an allocation  $\beta \in \mathcal{A}$  such that  $\beta \succ_A \alpha$  and  $\beta(A) = \iota(A)$ . The meaning of this is that A can get  $\beta$  by re-allocating their own initial allocation and that they prefer  $\beta$  to  $\alpha$ . The well-known concept of a Pareto-optimal allocation is related to our definition in the following way: An allocation is Pareto-optimal if C is not a blocking coalition.

A possible allocation is an Edgeworth-allocation if no coalitions are blocking. The set of all Edgeworth-allocations is called  $\mathcal{A}_E$ . By definition,  $\mathcal{A}_E \subset \mathcal{B}$ .

If  $0 \in \mathcal{P}(\alpha)$ , there exists an A and a  $\beta$  such that  $\beta(A) - \iota(A) = 0$  and  $\beta \succ_A \alpha$ . There exists therefore a blocking allocation if and only if  $0 \in \mathcal{P}(\alpha)$ . This means that we can characterize  $\mathcal{A}_E$  as  $\{\alpha \in \mathcal{B} \mid 0 \notin \mathcal{P}(\alpha)\}$ .

An allocation  $\alpha \in \mathcal{B}$  is a competitive allocation or a Walras-allocation, if there exists a price-vector  $p \in \mathbb{R}^b$  such that  $p\alpha(A) = p\iota(A)$  for all  $A \in \mathcal{C}$  and  $p x > 0$  for all  $x \in \mathcal{P}(\alpha)$ . The set of all Walras-allocations will be denoted  $\mathcal{A}_W$ .  $\mathcal{A}_W$  may, of course, be empty, and we will always have  $\mathcal{A}_W \subset \mathcal{B}$ .  $p\alpha(A) = p\iota(A)$  means in the interpretation that coalition A can buy the amount  $\alpha(A)$  for the money A gets by selling  $\iota(A)$ .  $p x > 0$  for all  $x \in \mathcal{P}(\alpha)$  means that no coalitions can buy an allocation preferred to  $\alpha$  if the price is p.

We will call  $\mathcal{E}$  a positive exchange economy if  $\iota(A) \neq 0$  implies  $\iota_i(A) \neq 0$  for all i. If a coalition has an initial allocation, it has a positive initial allocation of all commodities.

THEOREM 3. In an exchange economy we have  $\mathcal{A}_W \subseteq \mathcal{A}_E$ . In a positive, non-atomic exchange economy, where Assumptions I, II and III hold for the preference function, we have  $\mathcal{A}_W = \mathcal{A}_E$ .

Proof.  $\mathcal{A}_W \subseteq \mathcal{A}_E$ . Assume  $\alpha \in \mathcal{A}_W$ ;  $px > 0$  for  $x \in \mathcal{P}(\alpha)$  implies  $0 \notin \mathcal{P}(\alpha)$  and therefore  $\alpha \in \mathcal{A}_E$ .

Remark. This is a very general result; we do not use any assumptions on the preference function, and we do not use the non-atomicity of  $\iota$ .

$\mathcal{A}_W \supseteq \mathcal{A}_E$ . Assume  $\alpha \in \mathcal{A}_E$ . We will first prove that  $\alpha \in \mathcal{B}_0$ . By lemma B in the appendix, there exists an atom B with  $\iota(B)=0$  if  $\alpha$  is not non-atomic. But because of Assumption II,  $C \setminus B$  can get and would prefer to get the amount  $\alpha(B)$  extra. Therefore  $C \setminus B$ , where B is any atom for  $\alpha$  with  $\iota(B) = 0$ , is a blocking coalition. This means that  $\alpha \in \mathcal{B}_0$ .  $0 \notin \mathcal{P}(\alpha)$ ,  $\mathcal{P}(\alpha) = \mathcal{P}_0(\alpha)$  (Lemma 3) and  $\mathcal{P}_0(\alpha)$  is convex (Theorem 2); there exists therefore a hyperplane H through 0 with normal  $p \neq 0$ , such that  $px \geq 0$  for  $x \in \mathcal{P}(\alpha)$ . Lemma 2 implies that  $\mathcal{R}(\alpha-\iota) \subseteq \overline{\mathcal{P}(\alpha)}$  and therefore  $px \geq 0$  for  $x \in \mathcal{R}(\alpha-\iota)$ . The symmetry of  $\mathcal{R}(\alpha-\iota)$  (Theorem 1) gives  $px = 0$  for  $x \in \mathcal{R}(\alpha-\iota)$ . If  $px > 0$  for all  $x \in \mathcal{P}(\alpha)$ , we know that  $\alpha \in \mathcal{A}_W$ , and the proof would be finished. Assume therefore  $px = 0$  for some  $x = \beta(A) - \iota(A)$  with  $\beta \succ_A \alpha$ . We know from the assumption that the initial allocation is positive [ $\iota(A) \neq 0$  implies that  $\iota_1(A) \neq 0$ ], that  $p\beta(A) = p\iota(A) \neq 0$ . At least one  $p_1\beta_1(A)$  is therefore non-zero; by Assumption III any point in an  $\varepsilon$ -neighborhood of  $\beta(A)$  is the value of a preferred allocation for A; and this gives the wanted contradiction with the fact that  $px \geq 0$  for all  $x \in \mathcal{P}(\alpha)$ .

# APPENDIX

## Proof of Lemma A and Lemma B

In this appendix we will state and prove two lemmas used in some of the proofs. Lemma A gives some properties of "non-atomic set-measures." This lemma appears to be extremely important in the investigation of the mathematical properties of non-atomic economies. It can be applied to "preferred net trade" sets, as in this paper, and also to consumption sets and production possibility sets. Lemma B is more technical and compares two vector-measures defined on the same  $\sigma$ -field, one of them non-atomic and the other with atoms.

Definitions.  $C$  is an arbitrary set with subsets  $A, B, \dots$ .  $\mathcal{C}$  is a  $\sigma$ -field of subsets of  $C$ .  $R$  is the real line.  $\bar{R} = R \cup \{+\infty\}$ .  $F$  is a set-valued set function or a set-measure, when  $F$  is a function defined on  $\mathcal{C}$  with values in the set of subsets of  $\bar{R}^k$  with the properties

- 1)  $F(\phi) = \{0\}$ .
- 2)  $F(A \cup B) = F(A) + F(B)$  for  $A \cap B = \phi$ .
- 3)  $F(\bigcup_1^\infty A_i) = \Sigma F(A_i)$  for  $A_i \cap A_j = \phi, i \neq j$ .
- 4) If  $+\infty \in F_v(A)$ , there exist  $x_i$  and  $A_i$  ( $i = 1, 2, \dots$ ) with  $+\infty > x_i \in F_v(A_i)$ ;  $\bigcup A_i = A$ ,  $A_i \cap A_j = \phi$  for  $i \neq j$  and  $\Sigma x_i = +\infty$ .

We will first prove some relations between a set-measure and sets of vector-measures. Let  $\mathcal{M}$  be the set of all signed  $\sigma$ -finite vector-measures with the property  $\mu(A) \in F(A)$  for all  $A \in \mathcal{C}$ . ( $\mathcal{M} = \{\mu \mid \mu(A) \in F(A); A \in \mathcal{C}\}$ .) We will prove

first some relations between a set-measure and sets of vector-measures. Let  $\mathcal{M}$  be the set of all signed  $\sigma$ -finite vector-measures with the property  $\mu(A) \in F(A)$  for all  $A \in \mathcal{L}$ .

( $\mathcal{M} = \{\mu \mid \mu(A) \in F(A); A \in \mathcal{L}\}$ ). The set  $\mathcal{M}$  has the property that if  $\mu_1 \in \mathcal{M}$ , then  $\mu \in \mathcal{M}$ , where  $\mu(B) = \sum \mu_1(A_i \cap B)$ ; ( $C = \bigcup A_i$ ,  $A_i \cap A_j = \emptyset$  for  $i \neq j$ ). On the other hand, it is easily seen that any set with this property defines an  $F$  with the properties 1), 2), 3) and 4).

We will assume that  $F(A) = \{x \in \bar{R}^k \mid x = \mu(A); \mu \in \mathcal{M}\}$ , where  $\mathcal{M}$  is as defined above. This will be the case if  $F$  is defined by  $F(A) = \{\mu(A) \mid \mu \in \mathcal{M}\}$ .<sup>1</sup>

Define  $\mathcal{M}_0$  as  $\mathcal{M}_0 = \{\mu \mid \mu \in \mathcal{M}, \mu \text{ non-atomic}\}$ . We can not be sure that  $F(A) = \{x \in \bar{R}^k \mid x = \mu(A); \mu \in \mathcal{M}_0\}$ ; i.e., we cannot be sure that for any  $A$  and any  $x \in F(A)$  there exists a non-atomic  $\mu \in \mathcal{M}$  with  $\mu(A) = x$ . If  $F$  is such that in fact  $F(A) = \{x \in \bar{R}^k \mid x = \mu(A); \mu \in \mathcal{M}_0\}$ , we will call  $F$  non-atomic. Expressed in other words:  $A$  is an atom for  $F$  if there exists an  $x \in F(A)$  such that  $x = \mu(A)$  and  $\mu \in \mathcal{M}$  implies that  $A$  is an atom for  $\mu$ ; and  $F$  is non-atomic if there are no atoms; or  $F$  is non-atomic if any  $x \in F(A)$  ( $A \in \mathcal{L}$ ,  $x \neq 0$ ) can be expressed as the sum of  $x_1 \in F(A_1)$  and  $x_2 \in F(A_2)$ , where  $x_1 \neq 0 \neq x_2$ ,  $A_1 \cup A_2 = A$  and  $A_1 \cap A_2 = \emptyset$ .

LEMMA A. Let  $F$  be a non-atomic set-measure with  $F(A) = \{\mu(A) \mid \mu \in \mathcal{M}_0\}$ , then  $\bigcup_{A \in \mathcal{L}} F(A)$  is convex and  $F(A)$  is convex for all  $A \in \mathcal{L}$ .

<sup>1</sup>This assumption will always hold if  $\mathcal{L}$  is a separable  $\sigma$ -field. A  $\sigma$ -field is separable if it is generated by a countable family of subsets.

I. Proof that  $\bigcup F(A)$  is convex. Let  $x_1$  and  $x_2$  be points in  $\bigcup F(A)$ ; property 4) implies that is enough to treat the case  $x_{1v} < +\infty$ ;  $x_{2v} < +\infty$ , for  $v = 1, 2, \dots, l$ . We must then prove that  $\lambda x_1 + (1-\lambda)x_2 \in \bigcup F(A)$  for  $0 < \lambda < 1$ . From the relation between non-atomic set-measures and sets of non-atomic vector-measures, we know that  $x_1 = \mu_1(A_1)$  and  $x_2 = \mu_2(A_2)$  where  $\mu_1$  and  $\mu_2 \in \mathcal{M}_0$ . We know from Lyapunov's theorem that the range of a non-atomic measure is convex. We apply this theorem to  $\mu_1$ ,  $\mu_2$  and  $(\mu_1, \mu_2)$  and this gives us a  $B_1 \subset A_1 \setminus A_2$ , a  $B_2 \subset A_2 \setminus A_1$ , and a  $B_3 \subset A_1 \cap A_2$ , with  $\mu_1(B_1) = \lambda\mu_1(A_1 \setminus A_2)$ ,  $\mu_2(B_2) = (1-\lambda)\mu_2(A_2 \setminus A_1)$ , and  $\mu_1(B_3) = \lambda\mu_1(A_1 \cap A_2)$  and  $\mu_2(B_3) = (1-\lambda)\mu_2(A_1 \cap A_2)$ . By defining  $B_4 = (A_1 \cap A_2) \setminus B_3$  and  $B = B_1 \cup B_2 \cup B_3 \cup B_4$ , we get  $\mu_2(B_4) = (1-\lambda)\mu_2(A_1 \cap A_2)$ . We can now define  $\mu \in \mathcal{M}_0$  by  $\mu(A) = \mu_1(A \cap (B_1 \cup B_3)) + \mu_2(A \cap (B_2 \cup B_4)) + \mu_3(A \setminus B)$ , where  $\mu_3$  is any  $\mu_3 \in \mathcal{M}_0$ . Due to the property of  $\mathcal{M}_0$  mentioned earlier,  $\mu \in \mathcal{M}_0$ , because  $\mu_1$ ,  $\mu_2$  and  $\mu_3$  are.  $B$  is an element of  $\mathcal{C}$ , and  $\mu(B) = \mu_1(B_1 \cup B_3) + \mu_2(B_2 \cup B_4) = \lambda\mu_1(A_1) + (1-\lambda)\mu_2(A_2) = \lambda x_1 + (1-\lambda)x_2$ , and the proof of the convexity of  $\bigcup F(A)$  is finished.

II. Proof that  $F(A)$  is convex. If  $0 \in F(A)$  for all  $A$ , we will have that  $F(A) = \bigcup_{B \in \mathcal{C}} F(A \cap B)$  ( $F(A \cap A) = F(A)$  and  $F(A \cap B) \subseteq F(A)$ ).  $\bigcup_{B \in \mathcal{C}} F(A \cap B)$  is convex by I. ( $F(A \cap B)$  as a function of  $B$  is a non-atomic set-measure.) Therefore  $F(A)$  is convex for all  $A \in \mathcal{C}$ .

If  $0 \notin F(A)$  for some  $A$ , we choose any element  $\mu_0 \in \mathcal{M}_0$  and define  $G(A) = \{v(A) \mid v = \mu - \mu_0; \mu \in \mathcal{M}_0\}$ . We can now apply the first argument to  $G(A)$  and as  $F(A) = G(A) + \{\mu_0(A)\}$  we will also in this case have that  $F(A)$  is convex for all  $A$ .

LEMMA B. Let  $\mu$  and  $v$  be two signed vector-measures defined on  $\mathcal{C}$  and assume that  $\mu$  is non-atomic; then any atom for  $v$  will contain an atom  $B_0$  with  $\mu(B_0) = 0$ .

Proof. Assume that  $B_1$  is an atom for  $v$  and that  $\mu(B_1) = x \neq 0$ . Use Lyapunov's theorem to partition  $B_1 = B_2 \cup B_2'$  where  $\mu(B_2') = \mu(B_2) = \frac{1}{2}x$ ,  $v(B_2) = v(B_1)$  and  $v(B_2') = 0$ . Repeat the same procedure with  $B_2$  to get a  $B_3, \dots, B_i, \dots$  with  $\mu(B_3) = \frac{1}{4}x$  and  $v(B_3) = v(B_1)$  and with  $\mu(B_i) = 1/2^{i-1}x$  and  $v(B_i) = v(B_1)$ . Now  $B_0 = \bigcap_{i=1}^{\infty} B_i$  is the wanted atom.  $B_0 \in \mathcal{C}$ ,  $\mu(B_0) = \lim_{n \rightarrow \infty} 2^{-n}x = 0$  and  $v(B_0) = v(B_1)$ .

### Summary of Definitions and Interpretations

| Notation      | Definition                                                       | Interpretation     |
|---------------|------------------------------------------------------------------|--------------------|
| $C$           | Arbitrary set                                                    | Set of consumers   |
| $R^l$         | $l$ -dimensional Euclidean space                                 |                    |
| $\Omega$      | Non-negative orthant of $R^l$                                    | Commodity space    |
| $\mathcal{C}$ | $\sigma$ -field of subsets of $C$                                | Set of coalitions  |
| $\mathcal{A}$ | Set of finite vector-measures defined on $\mathcal{C}$           | Set of allocations |
| $\mathcal{F}$ | Set of $\sigma$ -finite vector-measures defined on $\mathcal{C}$ |                    |

| Notation                      | Definition                                                                                                                                                                                                  | Interpretation                                                            |
|-------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------|
| $\iota$                       | $\iota \in \mathcal{A}$ and $\iota_i(C) > 0$ for $i = 1, 2, \dots, l$                                                                                                                                       | Initial allocation                                                        |
| $\mathcal{B}$                 | $\{\alpha \in \mathcal{A} \mid \alpha(C) = \iota(C)\}$                                                                                                                                                      | Possible allocations                                                      |
| $\mathcal{A}_0$               | $\{\alpha \in \mathcal{A} \mid \alpha \text{ non-atomic}\}$                                                                                                                                                 |                                                                           |
| $\bar{\mathcal{A}}_0$         | $\{\alpha \in \bar{\mathcal{A}} \mid \alpha \text{ non-atomic}\}$                                                                                                                                           |                                                                           |
| $\mathcal{B}_0$               | $\mathcal{A}_0 \cap \mathcal{B}$                                                                                                                                                                            |                                                                           |
| $U$                           | $\{x \in \Omega \mid x_i \leq \iota_i(C)\}$                                                                                                                                                                 | Range of possible allocations                                             |
| $\mathcal{R}(\alpha)$         | $\{x \in \Omega \mid x = \alpha(A); A \in \mathcal{C}\}$                                                                                                                                                    | Range of allocation $\alpha$                                              |
| $\mathcal{R}(\alpha, \beta)$  | $\{x \in \Omega \times \Omega \mid x = (\alpha(A), \beta(A)); A \in \mathcal{C}\}$                                                                                                                          |                                                                           |
| $\mathcal{R}(\alpha - \iota)$ | $\{x \in \mathbb{R}^l \mid x = \alpha(A) - \iota(A); A \in \mathcal{C}\}$                                                                                                                                   | Range of net trade                                                        |
| $S(\alpha, \beta)$            | $S: \bar{\mathcal{A}} \times \bar{\mathcal{A}} \rightarrow \mathcal{C}$                                                                                                                                     | The set of consumers which prefers $\alpha$ to $\beta$ or is indifferent. |
| $P(\alpha, \beta)$            | $S(\alpha, \beta) \setminus S(\beta, \alpha)$                                                                                                                                                               | The set of consumers which prefers $\alpha$ to $\beta$ .                  |
| $I(\alpha, \beta)$            | $S(\alpha, \beta) \cap S(\beta, \alpha)$                                                                                                                                                                    | The set of consumers which is indifferent.                                |
| $\alpha \succ_A \beta$        | $A \subseteq P(\alpha, \beta); A \neq \emptyset$                                                                                                                                                            | All consumers in $A$ prefer $\alpha$ to $\beta$ .                         |
| $\mathcal{P}(\alpha, A)$      | $\begin{cases} \{x \in \mathbb{R}^l \mid x = \beta(A) - \iota(A); \\ \beta \succ_A \alpha; \beta \in \bar{\mathcal{A}}\} & \text{for } A \neq \emptyset \\ \{0\} & \text{for } A = \emptyset \end{cases}$   | Range of net trade for $A$ with preferred allocations.                    |
| $\mathcal{P}_0(\alpha, A)$    | $\begin{cases} \{x \in \mathbb{R}^l \mid x = \beta(A) - \iota(A); \\ \beta \succ_A \alpha; \beta \in \bar{\mathcal{A}}_0\} & \text{for } A \neq \emptyset \\ \{0\} & \text{for } A = \emptyset \end{cases}$ | Range of net trade for $A$ with preferred non-atomic allocations.         |
| $\mathcal{P}(\alpha)$         | $\bigcup_{A \in \mathcal{C}; A \neq \emptyset} \mathcal{P}(\alpha, A)$                                                                                                                                      | Range of net trade with allocation preferred to $\alpha$ .                |
| $\mathcal{P}_0(\alpha)$       | $\bigcup_{A \in \mathcal{C}; A \neq \emptyset} \mathcal{P}_0(\alpha, A)$                                                                                                                                    | Range of net trade with non-atomic allocation preferred to $\alpha$ .     |



| Notation | Definition                                                                                                                                                 | Interpretation               |
|----------|------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------|
| $A_E$    | $\{\alpha \in B \mid 0 \notin P(\alpha)\}$                                                                                                                 | Set of Edgeworth-allocations |
| $A_W$    | $\{\alpha \in B \mid px = 0 \text{ for } x \in R(\alpha-1);$<br>$px > 0 \text{ for } x \in P(\alpha), \text{ for}$<br>$\text{some } p \in R^l; p \neq 0\}$ | Set of Walras-allocations    |

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