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MATHEMATICAL METHODS FOR CALCULATING SHIP HULL FORMS OF DECREASED WAVE-MAKING RESISTANCE

BY

S. W. W. SHOR

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DEPARTMENT OF THE NAVY
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S. W. W. Shor

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ABSTRACT

The development of ship hull forms of decreased wave-making resistance by mathematical methods has received some investigation in the past few years but little application. However, it appears that much more application can be obtained by use of improved mathematical methods recently developed. The theory and effectiveness of work done by other authors is reviewed. In addition, the mathematical machinery necessary to calculate hull forms by a method of steep descent is developed and applied to a simple example. The necessary resistance equations are formulated. A closed form solution is obtained for the smoothly varying portion of the wave-making resistance of an assemblage of sources and sinks traveling near the surface of the water. An integral for the interference terms which comprise the fluctuating portion of the wave-making resistance is also presented in the shape of a Laplace transform. It is concluded that much more extensive use may be made of mathematical methods to improve hull forms than has been the case heretofore.

I. Introduction

The purpose of this paper is to examine what can be done to calculate the shapes of ship hulls which will have less wave-making resistance than those developed by ordinary drafting techniques. The work done in this field will be reviewed and it will be shown to contain evidence that the mathematical approach has a large potential for making improvements. This has not, however, been exploited for two reasons. First, the mathematical techniques used so far are not flexible enough to be applied to many practical cases. Second, the wave-making resistance formula on which these techniques are based has not produced precise predictions of experimental results, and this has cast doubt on the validity of computations derived from it. It will be pointed out in this paper that a way of bringing theory and experiment into agreement which has been worked out by Inui for certain special cases can probably be extended for more general application. In addition, a mathematical method of more general applicability than those used until now will be discussed -- the method of steep descent -- and the mathematical machinery and reformulation of the wave-making resistance equations necessary to permit its use will be outlined. At this point it will be concluded that the way is now open to more extensive use of mathematical techniques for the calculation of practical ship forms of decreased wave-making resistance.

A few words on the history of the wave-making resistance problem may serve to explain these comments, and show how the stage has been set for an advance. In the past, although there has been a great deal of experimental measurement of the wave-making resistance of ships, there has been little use of mathematical methods to find ways to decrease this resistance. Since William Froude demonstrated by his experiments the general correctness of the modeling law for wave-making resistance in the 1870's, it has been possible for naval architects to predict from model tests the wave-making resistance of the ships corresponding to the particular models. This prediction added to an estimate of the "frictional resistance" has been accurate enough to establish the horsepower required to drive the full-sized ships. Many thousands of models have been towed and good predictions made. Naval architects have also understood the relationship between the overall dimensions of a ship and its wave-making resistance. It has been recognized, for example, that if a given amount of displacement is placed in a long ship, this ship will probably have less wave-making resistance than a shorter ship of the same displacement. Until recently, however, there has been little application of detailed understanding of the theory of wave production by ships' hulls to the improvement of those hulls. So little systematic improvement was made over many years that a certain set of hull designs developed early in the twentieth century, Taylor's Standard Series, was long taken as a standard of goodness. If a hull design had as little wave resistance as Taylor's models, it was considered a very good design. Most designs had more.

Despite the lack of progress in application of theory to improving hull designs, a considerable amount of theoretical understanding of wave resistance was developed by a few investigators, starting with J. Michell, who published a classic paper on the subject in 1898 [1]. Michell derived an expression for the wave-making resistance of a thin hull moving on the surface of an ideal fluid. Sir Thomas Havelock followed with a long series of papers in which this basic theory was applied, expanded, and reformulated in a more tractable form, which has been shown [2] to be equivalent to the original theory of Michell. He considered the hull to be the set of closed streamlines generated by a set of moving sources and sinks. Unfortunately, two factors caused naval architects to disregard the Michell-Havelock theory. First, the results of experiment, although qualitatively the same as the predictions of the theory, never agreed with it exactly. Second, the application of the theory to practical hull forms required a great deal of calculation.

Several developments in the past ten years appear likely to permit far more use of theoretical analysis in the development of hull forms of less wave resistance. Not the least important is the general availability of high speed computers. In addition to this, however, Takao Inui in Japan has laid the groundwork for a more accurate calculation of the actual wave resistance. First, he has found that by describing the hull by the streamlines generated by the set of sources and sinks employed in Havelock's theory rather than using a simpler approximation to the hull, he can get much better agreement with experiment than before. Second, by using a small number of semi-empirical parameters to account for the sheltering effect of the hull, the motion of the wake, and the effect of linearizing approximations, he can get precise agreement between experiment and theory [3] . In addition, he has demonstrated theoretically and experimentally that it is possible in the case of certain hull shapes to add a spherical appendage at bow and stern in such a location as nearly to cancel the bow waves and stern waves produced by the hull [4] . In this country Weinblum developed tables which have permitted the use of Ritz's method to find improvements in hulls which can be described by a limited set of Polynomials [5] . Even though the calculated resistance does not agree exactly with the experimental resistance, several hulls developed in this way have been found experimentally to be considerably better than those of the same dimensions developed in the ordinary way.

Using a different approach, Karp, Kotik, and Lurye have succeeded in applying the classical calculus of variations to the problem of finding the strut of minimum wave-making resistance [6] . The particular strut is one which can be described by a distribution of dipoles along a plane of finite length but infinite depth, the distribution being independent of the depth and described between the ends of the plane by functions integrable when multiplied by a certain Green's function. While the results are of limited practical importance, they do demonstrate that a solution does exist in this case -- a matter concerning which there had been some question.

Despite these encouraging developments there has not yet been developed a general approach which will permit the systematic improvement of practical hull forms. It is not always possible, for example, to adopt a form which can be expressed as a polynomial, or to install a sphere beneath the bow. In any case most of the shape of the ship will be dictated by what it has to carry, stability, and other considerations, so that features to reduce wave resistance must be made compatible. There seems to be no reason why an approach cannot be developed which will do this. This paper will first review the work which has been done toward finding ways to reduce the wave-making resistance of hulls, and will then suggest how a more general approach, using a method of steep descent, may be applied to improve any hull form whose major characteristics have already been set.

II. Derivation and Discussion of Wave Resistance Equations

Both theory and experiment show wave-making resistance of a ship results from the system of waves which the ship leaves behind as it travels over the surface of the water. Although at a constant ship speed the waves made by the ship form a pattern which appears to move with the ship, this pattern actually extends farther and farther aft with time.

The energy required to increase its size is provided by the ship. Since the equations which describe both the pattern and the energy required to produce it have been developed by others, and the development is quite difficult, only as much of the derivation will be included here as to make clear the limitations in their use.

1. ASSUMPTIONS AND EQUATIONS OF MOTION

The assumptions in the first part of the following discussion are those [7] employed by Lunde. They will be modified later in certain cases to account for some semi-empirical corrections made by Inui. The notation is similar to that used by Lunde, although not exactly the same. It will be used throughout this paper for the sake of consistency, even though it differs considerably from that of many of the authors referred to. The assumptions are these:

a. The fluid is incompressible, homogeneous, and of zero viscosity.

b. The motion is irrotational. It can therefore be described by a velocity potential ϕ such that the fluid velocity vector is given by $\vec{q} = -\nabla\phi$. Hence if $\vec{q} = \vec{i}\,u + \vec{j}\,v + \vec{k}\,w$, where $\vec{i}$, $\vec{j}$, and $\vec{k}$ are unit vectors parallel to the cartesian coordinates x, y, z, the fluid velocity has the components

$$u = -\phi_x, \quad v = -\phi_y, \quad w = -\phi_z.$$

c. The wave height is small in comparison with the wave length, so that the wave slope is a small quantity.

d. The ship has been moving in a straight line on calm, infinitely deep water for an infinitely long time.

e. The motion set up by the ship can be approximated by the motion produced by a set of sources and sinks or doublets.

f. The wave-making resistance is independent of the frictional and eddy resistance of the ship and can be calculated separately.

The notation will employ cartesian coordinates throughout. The following conventions will be employed:

The axis of x will be in the mean surface of the water, and it will be oriented so that x is positive in the direction in which the ship is moving, y is positive to starboard, and z is positive upward. The origin of the x-coordinates is taken in the ship and the x-axis moves with the ship.

The steady speed of the ship in the x direction will be c.

The acceleration of gravity will be g.

The parameter $K_o = g/c^2$.

The elevation of the surface of the water will be ζ, and this will be positive in the up direction. It will take the value zero at z = 0 (the mean position of the surface).

For differential equations we have the following:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \qquad \text{...(2.1)}$$

This is the equation of continuity.

$$\frac{\partial^2 \phi}{\partial x^2} + K_o \frac{\partial \phi}{\partial z} = 0 \quad \text{on } z = 0. \qquad \text{...(2.2)}$$

This provides the equation of wave motion on the surface.

In addition, it is possible to relate the elevation of the surface to the velocity potential by the following equation:

$$\zeta = -(c/g)\frac{\partial \phi}{\partial x}. \qquad \text{...(2.3)}$$

2. THE VELOCITY POTENTIAL OF A MOVING SOURCE

Using the assumptions discussed above, an expression for the velocity potential has been derived both by Peters and Stoker and by Lunde, using different approaches. In this section their results will be shown to be equivalent.

It is necessary that the expression for ϕ be a solution of both (2.1) and (2.2), that it approach cx at infinity, that it give a pattern which in the vicinity of the ship moves with the ship, and that the water be undisturbed at a relatively small distance ahead of the ship. All these requirements are met by solving (2.1) and (2.2) as an initial value problem and then letting the time go to infinity. Peters and Stoker used this device of letting the time go to infinity rather than the technique of adding an artificial frictional resistance used by Lunde. In this way Peters and Stoker obtained a Green's function which can be interpreted as the velocity potential of an isolated moving source [8]. Let m be the strength of the source, and let (h, k, -f) be its location in cartesian coordinates. Then if we convert the Green's function into Lunde's notation we get the following expression for the velocity potential:

$$\phi = \frac{m}{\sqrt{(x-h)^2 + (y-k)^2 + (z+f)^2}} - \frac{m}{\sqrt{(x-h)^2 + (y-k)^2 + (z-f)^2}} +$$

$$+ \frac{4mg}{\pi} \, \mathrm{Re} \int_0^{\pi/2} \int_L \frac{e^{K(z-f)} \, e^{iK(x-h)\cos\theta} \cos\left[K(y-k)\sin\theta\right] dK \, d\theta}{g - K c^2 \cos^2\theta} =$$

$$= \frac{m}{\sqrt{(x-h)^2 + (y-k)^2 + (z+f)^2}} - \frac{m}{\sqrt{(x-h)^2 + (y-k)^2 + (z-f)^2}} -$$

$$- \frac{4mK_o}{\pi} \, \mathrm{Re} \int_0^{\pi/2} \int_L \frac{e^{K(z-f)} \, e^{-iK(x-h)\cos\theta} \cos\left[K(y-k)\sin\theta\right] \sec^2\theta \, dK \, d\theta}{K - K_o\sec^2\theta}$$

...(2.4)

The contour $\bar{L}$ goes from the origin to infinity along the real axis but is deformed above the real axis in the vicinity of $K = K_o \sec^2\theta$.

In the case of a ship there will always be symmetry about the x-z plane, so that for each source m at (h, k, -f) there will be another at (h, -k, -f). It follows that the velocity potential will also be symmetrical in the x-z plane so that

$$\frac{\partial \phi}{\partial y}(x,0,z) = 0 \qquad \ldots(2.5)$$

From this it follows that if we describe a ship by a set of sources distributed symmetrically with respect to the x-z plane we will get a velocity potential corresponding to any symmetrical pair which is the sum of two velocity potentials. This is simply

$$\phi^* = \phi(x,y,z;\ h,k,-f) + \phi(x,y,z;\ h,-k,-f) \qquad \ldots(2.6)$$

If we perform the addition indicated in equation (2.6) we obtain the velocity potential explicitly.

$$\phi^* = \frac{m}{\sqrt{(x-h)^2 + (y-k)^2 + (z+f)^2}} - \frac{m}{\sqrt{(x-h)^2 + (y-k)^2 + (z-f)^2}} +$$

$$+ \frac{m}{\sqrt{(x-h)^2 + (y+k)^2 + (z+f)^2}} - \frac{m}{\sqrt{(x-h)^2 + (y+k)^2 + (z-f)^2}} -$$

$$- \frac{4mK_o}{\pi} \operatorname{Re} \int_0^{\pi/2} \int_{\bar{L}} \frac{e^{K(z-f)}\, e^{-iK(x-h)\cos\theta}(2)\cos(Ky\sin\theta)\cos(Kk\sin\theta)\, \sec^2\theta\, dK\, d\theta}{K - K_o\sec^2\theta}$$

$$\ldots(2.7)$$

The contour integration of (2.7) gives for the last term the value $2\pi i\, \operatorname{Res}(K_o\sec^2\theta)$. Hence if we call the denominators of the first four terms of (2.7) r_1, r_2, r_3, and r_4, we have the following results:

$$\phi^* = m\left(\frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r_4}\right) -$$

$$- 16mK_o \int_0^{\pi/2} e^{K_o(z-f)\sec^2\theta} \sin\left[K_o(x-h)\sec\theta\right] \cos\left[K_o y \sin\theta \sec^2\theta\right] \times$$

$$\times \cos\left[K_o k \sin\theta \sec^2\theta\right] \sec^2\theta \, d\theta \qquad \ldots(2.8)$$

This is the wave pattern produced by a symmetrical pair of sources, one to port and one to starboard of the centerline of a ship. We observe that at $z = 0$ the terms in r_1 through r_4 cancel out and leave us with the integral term only.

We may compare this with Lunde's expression for the elevation of the surface at a great distance aft of the ship. To do this we take

$$\zeta = -(c/g)\frac{\partial \phi^*}{\partial x} = 16\ m\ K_o^2(c/g) \int_0^{\pi/2} e^{K_o(z-f)\sec^2\theta} \cos\left[K_o(x-h)\sec\theta\right] \times$$

$$\times \cos\left[K_o y \sin\theta \sec^2\theta\right] \cos\left[K_o k \sin\theta \sec^2\theta\right] \sec^3\theta \, d\theta$$

$$\ldots(2.9)$$

For $k = 0$ this is equivalent to his expression (7.10):

$$\zeta = \frac{4K_o}{c} \int_{-\pi/2}^{\pi/2} \sum_{s=1}^{n} m_s \cos\left[K_o(x-h_s) \sec\theta\right] \cos\left[K_o(y-k_s)\sec^2\theta \sin\theta\right] \times$$

$$\times \exp(-K_o f \sec^2\theta) \sec^3\theta \, d\theta$$

$$\ldots(2.9a)$$

The depth $-z$ will be zero in both equations. The change in range of integration from $-\pi/2$ to $\pi/2$ down to 0 to $\pi/2$ together with the fact that equation (2.9) assumes two sources -- one port and one starboard -- each of strength m, while Lunde's equation refers to only one source, explains the difference of a factor of 4 between the equations. The identity

$$K_o^2(c/g) = K_o(g/c^2)(c/g) = K_o/c$$

completes the demonstration.

Since Peters and Stoker obtained their result without recourse to artificial friction laws, the fact that their equation is the same as Lunde's shows that Lunde's later conclusions following this equation do not depend on the existence of such friction in water*.

3. THE RESISTANCE INTEGRAL

If a ship travels into quiet water it will cause an increase in the wave energy in the region into which it advances. By calculating the rate of change of the wave energy in a large region containing the ship together with the amount of wave energy which crosses the boundary of the region it is possible to calculate the amount of energy which the ship converts into waves. Since we know the motion of the water everywhere once we have the velocity potential, it is certainly possible to calculate the energy put into the water by the ship in this manner.

Let there be a fixed large area of the water surface into which the ship is advancing. It is sufficient to bound this by two infinite planes, one well forward of the ship and one well aft of it, each at right angles to the direction of the ship's motion. We will call these respectively plane A and plane B. If we let E(A) and E(B) be the rate at which wave energy crosses into the area across boundaries A and B, W(A) and W(B) be the rate at which work is done on the fluid within these boundaries at the boundaries, R the wave resistance and c the speed of the ship, then we can write the energy balance on the large region:

*In discussing such comparisons as these, it is worth remembering that the definitions of m used by some authors (Inui, for example) differ from the one used by Lunde by a factor $4\pi/c$. This results in a corresponding factor $c/4\pi$ in the coefficient of the expression for the wave elevation ζ.

$$R c - E(A) - E(B) + W(A) + W(B) = 0 \qquad \ldots(2.10)$$

Since the plane far ahead of the ship is undisturbed by the ship's motion, we can set E(A) = W(A) = 0. We then can write the wave resistance of the ship as

$$R = (1/c)\left[E(B) - W(B)\right] \qquad \ldots(2.11)$$

By substituting for E(B) and W(B) their equivalent in terms of the velocity potential and going through some extensive manipulation, Lunde obtains finally an expression for the wave resistance [7].

$$R = 16\pi\rho K_o^2 \int_0^{\pi/2} (P_e^2 + P_o^2 + Q_e^2 + Q_o^2) \sec^3\theta \, d\theta \qquad \ldots(2.12)$$

$$\left.\begin{aligned}
P_e &= \int_s \sigma e^{-K_o f \sec^2\theta} \cos(K_o h \sec\theta) \cos(K_o k \sin\theta \sec^2\theta)\, ds \\
P_o &= \int_s \sigma e^{-K_o f \sec^2\theta} \sin(K_o h \sec\theta) \sin(K_o k \sin\theta \sec^2\theta)\, ds \\
Q_e &= \int_s \sigma e^{-K_o f \sec^2\theta} \sin(K_o h \sec\theta) \cos(K_o k \sin\theta \sec^2\theta)\, ds \\
Q_o &= \int_s \sigma e^{-K_o f \sec^2\theta} \cos(K_o h \sec\theta) \sin(K_o k \sin\theta \sec^2\theta)\, ds
\end{aligned}\right\} \qquad \ldots(2.13)$$

Here σ is the source density at any point (h, k, -f) on the hull or within its boundary and s indicates integration over the volume of the hull. The quantity ρ is the density of water.

It is possible to simplify this expression even further in the case that the hull is symmetrical about the centerline plane. Then the following result can be used:

$$R = 16\pi\rho K_o^2 \int_0^{\pi/2} (P^2 + Q^2) \sec^3\theta \, d\theta \qquad \ldots(2.14)$$

$$\left.\begin{matrix} P \\ Q \end{matrix}\right\} = \int_s \begin{matrix} \cos \\ \sin \end{matrix} \left[K_o(h\cos\theta + k\sin\theta)\sec^2\theta\right] \exp(-K_o f \sec^2\theta)\, ds \qquad \ldots(2.15)$$

It should be emphasized that in this case it is necessary to carry out the integration over both halves of the ship, since the simplification has made the result correct only when the two halves are added. This should be apparent from an examination of equation (2.15), which is not an even function of k.

Another relationship which makes the interactions more obvious can be derived from equations (2.12) and (2.13). Here we will substitute a finite sum for the integral.

$$6(s) \longrightarrow \sum_{r=1}^{n} m_r$$

If we substitute this in equations (2.12) and (2.13), combine terms and simplify, we finally get the following expression:

$$R = 16\pi\rho K_o^2 \left\{ \sum_{r=1}^{n} m^2 \int_0^{\pi/2} e^{-2K_o f_r \sec^2\theta} \sec^3\theta \, d\theta \right. +$$

$$+ 2 \sum_{s=r+1}^{n} \sum_{r=1}^{n-1} m_r m_s \int_0^{\pi/2} e^{-K_o(f_r+f_s)\sec^2\theta} \cos\left[K_o(h_r-h_s)\sec\theta\right] \times$$

$$\left. \times \cos\left[K_o(k_r-k_s) \quad \sin\theta\sec^2\theta\right] \sec^3\theta \, d\theta \right\} \qquad \ldots(2.16)$$

Equation (2.16) may be put in the form of a Laplace transform. We will let $t+1 = \sec^2\theta$. Then equation (2.16 can be written in the following form:

$$R = 16\pi\rho K_o^2 \left\{ \sum_{r=1}^{n} m_r^2 \frac{e}{2}^{-p_{rr}} \int_0^{\infty} e^{-p_{rr}t} (1+t)^{1/2} t^{-1/2} dt \right. +$$

$$\left. + 2\sum_{s=r+1}^{n} \sum_{r=1}^{n-1} m_r m_s \frac{e}{2}^{-p_{rs}} \int_0^{\infty} e^{-p_{rs}t} (1+t)^{1/2} t^{-1/2} \cos\left[q_{rs}(1+t)^{1/2}\right] \cos\left[q'_{rs} t^{1/2}(1+t)^{1/2}\right] dt \right\}$$
$$\ldots(2.17)$$

Here we have set

$p_{rr} = 2K_o f_r$; $p_{rs} = K_o(f_r+f_s)$; $q_{rs} = K_o(h_r-h_s)$; and $q'_{rs} = K_o(k_r-k_s)$.

It should be observed that the expression is an even function of k, and therefore it provides the same value for the interaction between a source on the centerline and either of two symmetric sources off-center on either side. Equations (2.14) and (2.15) provide a different value for the interaction term between a source on the centerline and a source on the port side than they do for the interaction term between the source on the centerline and the symmetrical source on the starboard side, and provide a correct result only when all the interaction terms are summed. This complicates interpretation. On the other hand, equations (2.16) and (2.17), which are actually more general, provide terms which can be interpreted directly as interactions between the sources m_r and m_s. In consequence, equation (2.17) will be used in much of the discussion later.

The first integral of equation (2.17) can be evaluated in closed form. Let us call the resistance corresponding to this first term $R^{(1)}$, and that corresponding to the second term $R^{(2)}$, and let R be their sum:

$$R = R^{(1)} + R^{(2)} \qquad \ldots(2.18)$$

We may now write $R^{(1)}$ in closed form. We will let $R^{(1)}$ be described by a sum.

$$R^{(1)} = \sum_{r=1}^{n} R_{rr} \qquad \ldots(2.19)$$

$$R_{rr} = 4\pi^2 \rho K_o^2 \, p_{rr}^{-1} \, m_r^2 e^{-3p_{rr}/2} k_1(p_{rr}/2) \qquad \ldots(2.20)$$

The function $k_1(p_{rr}/2)$ is Bateman's function. Derivation of equation (2.20) is outlined in appendix 1. For very small K_o, which corresponds to very large c, the value of R_{rr} is small. As K_o increases, R_{rr} increases, reaches a maximum, and then decreases. The term $R^{(1)}$, which is the smoothly varying part of the wave-making resistance, is a function only of the depth of the sources, not of their position along the hull or their distance outboard of the centerline.

There seems to be no closed form value for the terms which comprise $R^{(2)}$. On the other hand, they are closely related in form to the terms which comprise $R^{(1)}$. Let us write $R^{(2)}$ as a sum.

$$R^{(2)} = 2 \sum_{s=r+1}^{n} \sum_{r=1}^{n-1} R_{rs} . \qquad \ldots(2.21)$$

Then we can write

$$R_{rs} = 16\pi\rho K_o{}^2 m_r m_s \frac{e}{2}^{-p_{rs}} \int_0^{\infty} e^{-p_{rs}t}(1+t)^{1/2}t^{-1/2}\cos\left[q_{rs}(1+t)^{1/2}\right] \times$$
$$\times \cos\left[q'_{rs}\, t^{1/2}(1+t)^{1/2}\right] dt \qquad \ldots(2.22)$$

It is possible to show if $q'_{rs} = 0$ that if the integral is taken over a finite interval from the origin to any zero of cos $\left[q_{rs}(1+t)^{1/2}\right]$, the error will be less than the value of the integral from that zero to the next one. The same can be shown if $q_{rs} = 0$, $q'_{rs} \neq 0$, except that the zeros referred to are those of cos $\left[q'_{rs}t^{1/2}(1+t)^{1/2}\right]$. The demonstration of this is shown in appendix 1. When both q_{rs} and q'_{rs} differ from zero, the estimation of the error incurred by terminating the integration is less simple but it is possible to get an upper limit by observing that for all $t > t_o$, $1<(1+t)^{1/2}t^{-1/2} <(1+1/t_o)^{1/2}$. Consequently the error will be less than

$$\left| \int_{t_o}^{\infty} e^{-p_{rs}t}(1+1/t_o)^{1/2}\, dt \right| = \left| (1/p_{rs})\, e^{-p_{rs}t_o}(1+1/t_o)^{1/2} \right| .$$

Also it is apparent from examination of equations (2.17) through (2.22) that the magnitude of the interaction term between any pair of sources m_r and m_s must be less than or equal to the magnitude of the sum of the resistance of m_r and m_s taken separately.

$$\left| 2R_{rs} \right| \leq R_{rr} + R_{ss} \qquad \ldots(2.23)$$

The equality sign can hold only if the two sources are superposed. It is, of course, possible to have a negative value for R_{rs}. The inequality shows that no pair of sources can interact with each other in such fashion as to produce a negative total resistance.

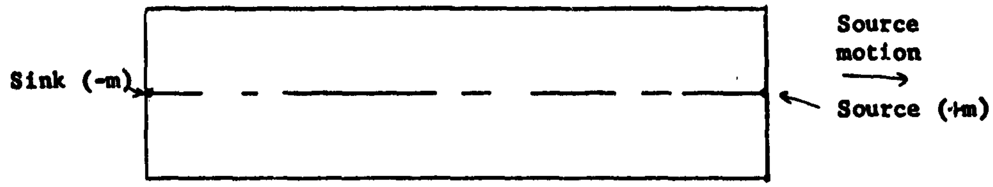
4. EMPIRICAL CORRECTIONS

Comparison of the wave-making resistance as calculated by equation (2.14) with the resistance as measured in model basins has produced good qualitative agreement, but in general has not produced exact quantitative agreement. In particular it has been found that the humps and hollows of the curve of resistance vs. speed, although of approximately the right separation in speed are somewhat displaced from their proper position, and the calculated humps and hollows appear exaggerated when compared with the experimental ones. In addition, there have been some systematic differences in the magnitude of the resistance. In view of the approximations which have been made in deriving the resistance equations (no viscosity, squares of velocities other than ship speed negligible, small slope of waves, and so forth) these differences are not altogether surprising. Fortunately, an explanation of the major differences has been produced by Takao Inui. In the case of a number of models whose source distributions could be described by simple continuous functions he has found it possible to correct for these differences with a relatively small number of semi-empirical parameters [3]. It appears possible to extend these techniques to other distributions.

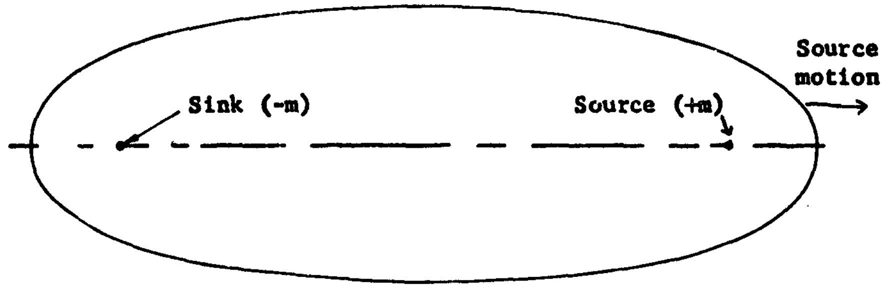
Inui's first discovery was that the agreement between theory and experiment could be significantly improved if the hull form were found by plotting the closed streamlines produced by the set of sources and sinks employed to describe the hull, rather than using the approximation employed previously. The approximation used by earlier investigators was this:

$$\sigma = - \frac{c}{2\pi} \frac{\partial k}{\partial h} \qquad \ldots(2.24)$$

Here σ is the source density, and h and k are the x and y coordinates of the hull boundary. This approximation is only satisfactory for <u>very</u> thin ships, but Inui showed [3] that it is unsatisfactory where the beam is as great as one-twentieth the length -- a ship still thinner than any used in practice. An example of extreme differences in results between using approximation (2.24) and the actual plotting of the streamlines can be obtained by an illustration of the results for a two-dimensional source forward of a two-dimensional sink. In the approximation (2.24) this amounts to an infinitely deep rectangular barge; however, if the streamlines are traced a somewhat longer, more fair shape is produced -- still, of course, in the form of an infinitely deep strut.



Two-dimensional source and sink by Michell's approximation (2.24).



Two-dimensional source and sink by tracing streamlines.

His next discovery [3] was that in the case of hulls which have both a well-defined bow wave and a well-defined stern wave, with no waves originating between bow and stern, he could bring the calculated and observed humps in the resistance curve together. For purposes of calculation of the fluctuating part of the resistance he assumed that the stern wave originated a small distance aft of its originally calculated position. That this should apply to the fluctuating part of the resistance is obvious from equation (2.20), which shows that the smoothly varying part of the resistance is a function of depth, but not of horizontal position. The extension of Inui's result to the resistance equations discussed above would require that for purposes of calculating the interference terms the positions of the negative sources (sinks) at the after end of the hull would be shifted further aft by a small amount. This would affect the coefficient q_{rs} in equation (2.22). While this has not been demonstrated by experiment, it appears to be a reasonable extrapolation from Inui's findings.

In addition, to correct the scale of the resistance (slightly in error because of the finite height of the waves), Inui multiplied all wave heights as computed by a correction factor $\gamma < 1$. This would translate in the case of our equations to a multiplication of each m_r and m_s by the factor γ. He found further that the wave height of the stern waves was reduced by an additional factor β', and this would require that all sources and sinks at the after end of the ship be multiplied by a factor $\beta' < 1$. Finally, for purposes of calculating the interference terms only he reduced the amplitude of the bow wave by another factor $\alpha' < 1$. This would be equivalent to multiplying the amplitudes of sources and sinks at the forward end of the ship by the factor α' in the calculation of the interference terms by equation (2.22). This is to account for the fact that the resistance equations were derived on the assumption that the waves could propagate over the entire surface of the water, but they are actually prevented from moving aft from the bow through the water occupied by the ship. Therefore some part of the bow wave is less efficient than theory would predict in interfering with the stern wave.

While the theoretical derivation of the four parameters discussed here is sketchy, they have worked when applied to bow wave and stern wave amplitudes. Since any ship can be described by sources placed toward the bow and sinks placed toward the stern, it appears reasonable to try to transfer the results obtained with wave amplitudes to the formulation using sources. Certainly the case of a hull described by a single source placed forward and a single sink placed aft should provide a satisfactory candidate for the transfer: the bow wave starts at the source, and is proportional to its intensity; the stern wave starts at the sink, and is also proportional to its intensity. Application of the correction factors Inui derived for the bow and stern wave amplitudes would therefore apply directly to the strength of the source and sink respectively, and the correction for the location of the stern wave would apply to the location of the sink, and the correspondence would be one-to-one.

Since the calculated curve of wave resistance vs. speed can be made identical with the observed curve over the entire range of speed by the introduction of only four parameters (three if we consider that γ^2 only provides a proportionality correction between the curves and does not affect the shape), it seems reasonable to use these parameters in investigating the results of small changes in the shape of a known ship. That is, if we can correct the theoretical curve for a known set of sources and sinks by the use of these parameters so that it is identical with the experimental curve, then it is probable that the change in wave resistance we calculate for small changes in the magnitudes of these sources and sinks will be correct.

5. RESISTANCE OF A SEMI-INFINITE STRUT DESCRIBED BY A DISTRIBUTION OF DIPOLES ON THE CENTERLINE PLANE

One form which has been given a great deal of investigation is the semi-infinite strut which can be described by a distribution of dipoles over the centerline plane of the hull. This form has a particularly simple wave-making resistance formula, and therefore is attractive for investigation. In addition, a plot of the density of dipole moment along the axis of the ship looks like a plan view of one side of a ship and so permits easy visualization of the meaning of a particular distribution. The dipole distribution is usually a reasonable equivalent to the hull shape. However, there are exceptions; for example, Karp, Kotik and Lurye found a case where the dipole density became infinite at the end of a hull, but the actual width of the hull remained finite and the shape smooth [6].

If we start with equations (2.14) and (2.15) and substitute $\sec^2\theta = \cosh^2 u$, we get the following expressions for the case k = 0 with discrete sources m_r:

$$R = 16\pi\rho K_o^2 \int_0^\infty (I^2 + J^2) \cosh^2 u \, du \qquad \ldots(2.25)$$

$$\left.\begin{aligned} I &= \sum_{r=1}^{n} m_r \cos(K_o h_r \cosh u)\, e^{-K_o f_r \cosh^2 u} \\ J &= \sum_{r=1}^{n} m_r \sin(K_o h_r \cosh u)\, e^{-K_o f_r \cosh^2 u} \end{aligned}\right\} \qquad \ldots(2.26)$$

We may now take any source and an equal sink (i.e. a source of negative sign), the sink aft of the source by a distance Δh, and combine them to provide a dipole moment $m_r \Delta h$. The value of this product we can call M. If the hull is described by a large number of such sources and sinks, it is possible to combine all the sources and sinks at any one depth into a set of dipoles with axes oriented in the direction of motion, that is in the x-direction.

To make this explicit, let us assume that we have a set of n discrete sources at a depth f beneath the surface, positioned along the centerline plane of the ship. Since the x-direction is oriented so that it is positive in the direction of motion of the ship, we will assign the index r so that the value r=1 corresponds to the most forward source, that is, the one with the largest value of h for its x-coordinate. The index will increase by one unit as we come to each source aft of this one. Since the set of sources must describe a closed body, it is necessary that the sum of their strengths be zero.

$$\sum_{r=1}^{n} m_r = 0 \; . \qquad \ldots(2.27)$$

We will define the distance between the r^{th} source and the $r+1^{st}$ source so that this distance will always be positive:

$$\Delta h_r = h_r - h_{r+1} \qquad \ldots(2.28)$$

It is now possible to arrange the n sources m_r, r=1,...,n, into n-1 dipoles $M_r = M(h_r')$, r = 1,...,n-1.

$$\begin{aligned}
M_1 &= M(h_1') = m_1 \Delta h_1 \;, & h_1' &= h_1 - \Delta h_1/2 \\
M_2 &= M(h_2') = (m_1+m_2)\Delta h_2 \;, & h_2' &= h_2 - \Delta h_2/2 \\
M_3 &= M(h_3') = (m_1+m_2+m_3)\Delta h_3 \;, & h_3' &= h_3 - \Delta h_3/2 \qquad \ldots(2.29) \\
&\vdots & &\vdots \\
M_{n-1} &= M(h'_{n-1}) = (m_1+\ldots+m_{n-1})\,\Delta h_{n-1} \;, & h'_{n-1} &= h_{n-1} - \Delta h_{n-1}/2
\end{aligned}$$

The term M_{n-1} terminates the sequence, since the next term would contain the sum (2.27), equal to zero, as a factor. The coordinate h_r' is the x-coordinate of the r^{th} dipole of strength $M(h_r')$. The arrangement just made depends on taking the source m_1 as the forward member of the first dipole, and then replacing the source m_2 by the sum $\left[-m_1 + (m_1 + m_2)\right]$. The portion $-m_1$ becomes the after member of the first dipole, and the sum $(m_1 + m_2)$ becomes the forward member of the second dipole. Observe that the dipole M_{n-1} is actually determined by the strength of the last source (which will have negative strength), m_n. This is clear since

$$(m_1 + m_2 + \dots + m_{n-1}) = - m_n$$

as a consequence of the requirement (2.27) that the set of sources describe a closed body. From this it also follows that the set of dipoles $\left\{M_r\right\}$ is the same whichever end of the distribution one starts with in making up the source-sink pairs which define the dipoles.

We may now investigate the special case where instead of discrete sources m_s we have source density σ along the x-axis, and the function $\sigma(h)$ is integrable. The variable h is the x-coordinate of a point on the centerline plane of the hull. We will let the number n in equation (2 29) become infinite in such a fashion that $\max \Delta h_r \rightarrow 0$. Then we can define a new quantity, $\mu(h)$, which we will call the dipole density.

$$\mu(h) = \lim_{\Delta h \rightarrow 0} \frac{M(h')}{\Delta h} \quad . \qquad \dots(2.30)$$

In the limit of large n the quantity h' will coincide with the quantity h. Then we obtain the following formula:

$$\mu(h) = \int^{h} \sigma \, dh \quad . \qquad \dots(2.31)$$

The lower limit of the integral in (2.31) is the forward end of the source distribution.

In order to find the wave resistance of such a distribution we must return to the definitions (2.25) through (2.29). These give us the following expressions in place of equations (2.26):

$$I = \sum_{r=1}^{n-1} \frac{M_r}{\Delta h_r'} \left\{ \cos\left[K_o(h_r' + \Delta h_r'/2) \cosh u\right] - \cos\left[K_o(h_r' - \Delta h_r'/2) \cosh u\right] \right\} e^{-K_o f_r \cosh^2 u} =$$

$$= \sum_{r=1}^{n-1} \frac{2M_r}{\Delta h_r'} \sin(K_o h_r' \cosh u) \sin\left[K_o \Delta h_r'/2) \cosh u\right] e^{-K_o f_r \cosh^2 u} \approx$$

$$\approx -K_o \cosh u \sum_{r=1}^{n-1} M_r \sin(K_o h_r' \cosh u) e^{-K_o f_r \cosh^2 u}, \text{ for } \Delta h_r' \text{ small. } \Delta h_r' = \Delta h_r.$$

$$J = \sum_{r=1}^{n-1} \frac{M_r}{\Delta h_r'} \left\{ \sin\left[K_o(h_r' + \Delta h_r'/2) \cosh u\right] - \sin\left[K_o(h_r' - \Delta h_r'/2) \cosh u\right] \right\} e^{-K_o f_r \cosh^2 u} =$$

$$= \sum_{r=1}^{n-1} \frac{2M_r}{\Delta h_r'} \cos(K_o h_r' \cosh u) \sin\left[(K_o \Delta h_r'/2) \cosh u\right] e^{-K_o f_r \cosh^2 u} \approx$$

$$\approx K_o \cosh u \sum_{r=1}^{n-1} M_r \cos(K_o h_r' \cosh u) e^{-K_o f_r \cosh^2 u}, \ \Delta h_r' \text{ small.} \qquad \ldots(2.32)$$

We now assume that there are many layers of dipoles of identical distribution but different depths, so that we can separate the index of the variable f, the vertical position on the centerline plane, from the variable h, the fore-and-aft position on the centerline plane. Then we can rewrite equation (2.32) as a double sum. We assume that the depth of the hull is infinite.

$$\left.\begin{aligned} I &\approx -K_o \cosh u \sum_{s=1}^{n-1} \sum_{r=1}^{\infty} M_s \sin(K_o h_s' \cosh u) e^{-K_o f_r \cosh^2 u} \\ J &\approx K_o \cosh u \sum_{s=1}^{n-1} \sum_{r=1}^{\infty} M_s \cos(K_o h_s' \cosh u) e^{-K_o f_r \cosh^2 u} \end{aligned}\right\} \qquad \ldots(2.33)$$

We now replace the M_s with $\mu(h,f)dh\ df = \mu(h)\ dh\ df$ and replace the sums with integrals. Then we get the following expressions for I and J:

$$\left.\begin{aligned}
I &= -K_o \cosh u \int_L \int_o^\infty \mu\,(h) \sin(K_o h \cosh u)\, e^{-K_o f \cosh^2 u}\, dh\, df = \\
&= -(\cosh u)^{-1} \int_L \mu\,(h) \sin(K_o h \cosh u)\, dh \\
J &= K_o \cosh u \int_L \int_o^\infty \mu\,(h) \cos(K_o h \cosh u)\, e^{-K_o f \cosh^2 u}\, dh\, df = \\
&= (\cosh u)^{-1} \int_L \mu\,(h) \cos(K_o h \cosh u)\, dh
\end{aligned}\right\} \quad \ldots(2.34)$$

The prime has been dropped from the variable h because in passing to the limit of infinitely small Δh_s, h and h' become coincident. The range L for the integral is over the distribution in length from the extreme forward source to the extreme after sink. In the limit of continuous dipole moment distribution the range of source positions and dipole moment positions is the same.

It is now possible to calculate the resistance of this dipole distribution. We square the two expressions.

$$\left.\begin{aligned}
I^2 &= (\cosh u)^{-2} \int_L \int_L \mu(h)\, \mu(h') \sin\,(K_o h \cosh u) \sin(K_o h' \cosh u)\, dh\, dh' \\
J^2 &= (\cosh u)^{-2} \int_L \int_L \mu(h)\, \mu(h') \cos\,(K_o h \cosh u) \cos(K_o h' \cosh u)\, dh\, dh'
\end{aligned}\right\} \ldots(2.35)$$

If we now add and simplify the trigonometric functions we get the following result:

$$I^2 + J^2 = (\cosh u)^{-2} \int_L \int_L \mu(h)\, \mu(h') \cos\left[K_o (h-h') \cosh u\right] dh\, dh' \qquad \ldots(2.36)$$

This result may now be substituted in equation (2.25) to obtain the resistance.

$$R = 16\pi\rho K_o{}^2 \int_o^\infty \int_L \int_L \mu(h)\, \mu(h') \cos\left[K_o (h-h') \cosh u\right] dh\, dh'\, du \qquad \ldots(2.37)$$

We may eliminate one integration by use of the identity

$$\int_0^\infty \cos\left[K_o(h-h')\cosh u\right]du = -(\pi/2)\; Y_o(\left|K_o(h-h')\right|) = -(\pi/2)\; Y_o(K_o\left|h-h'\right|) \qquad \ldots(2.38)$$

Then we get

$$R = -8\,\pi^2\,\rho\,K_o^2 \int_L\int_L \mu(h)\;\mu(h')\;Y_o(K_o\left|h-h'\right|)\;dh\;dh'. \qquad \ldots(2.39)$$

The function $Y_o(x)$ is Bessel's function of the second kind. Although the function Y_o has a singularity at the origin, its integral is a bounded function. For positive argument Y_o is bounded and oscillates from positive to negative, gradually diminishing in amplitude with increasing argument. The dipole density $\mu(h)$ must be integrable but not necessarily continuous.

6. RELATIONSHIP BETWEEN HULL SHAPE AND SOURCE DISTRIBUTION

Since the resistance equations discussed in the preceding paragraphs are all based on calculating the resistance of a collection of sources and sinks whose closed streamlines outline a hull, it is necessary to find how to relate the hull shape to the source distribution. In principle it is a simple matter to find the hull shape corresponding to a given source distribution, but it is significantly more difficult to find the source distribution for a given hull shape.

One of the assumptions on which the resistance equations are based is that the wave height is small. This implies that the flow about the hull is not significantly affected by the existence of the waves. In such a case the hull shape can be described by the closed streamlines on either side of the plane of symmetry between a set of sources and their mirror image in the free surface, the double hull so described moving through the water far below the free surface. Inui [3] found that such an approximation worked for his models up to a Froude number of about 0.7. We can therefore write the velocity potential:

$$\phi = \sum_{s=1}^{n} m_s \left(\frac{1}{r_{1s}} + \frac{1}{r_{2s}} \right) \qquad \ldots(2.40)$$

$$\left.\begin{aligned} r_{1s} &= \left[(x-h)^2 + (y-k)^2 + (z+f)^2\right]^{1/2} \\ r_{2s} &= \left[(x-h)^2 + (y-k)^2 + (z-f)^2\right]^{1/2} \end{aligned}\right\} \qquad \ldots(2.41)$$

It is clear that this velocity potential will provide symmetry in the plane $z=0$, so this plane may be considered to be a dividing boundary across which there is no flow. This also follows because $\left.\frac{\partial\phi}{\partial z}\right|_{z=0} = 0$. To find the hull form corresponding to a given source distribution we need only calculate ϕ from equations (2.40) and (2.41) and then calculate the velocity components from ϕ.

$$u = -\frac{\partial\phi}{\partial x} \; ; \quad v = -\frac{\partial\phi}{\partial y} \; ; \quad w = -\frac{\partial\phi}{\partial z} \; .$$

We may now find the streamlines by a numerical integration of the following differential equation:

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \; . \qquad \ldots(2.42)$$

Since the coordinates move with the ship, this equation will give the shape of the hull.

<u>Source distribution for a given hull form</u>

The calculation of the source distribution corresponding to a given hull form is a much more difficult matter. John L. Hess and A.M.O. Smith have worked out a way of doing this if the sources are distributed on the surface of the hull [9]. With such a technique it should be simple to provide a good estimate of the wave-making resistance with the formulas worked out earlier in this paper. Unfortunately, an attempt to vary the strength of a set of sources on the surface of the hull to find a hull of improved wave-making resistance will probably run into difficulty if any decreases in hull volume are permitted. This is because it will not be possible to define a hull whose surface runs inside of that defined by these sources without decreasing their strength to zero. Otherwise we will calculate a hull with sources outside its surface -- a possible arrangement of sources, certainly, but one which will generate isolated appendages outside the main hull.

Notwithstanding these difficulties, the method of Hess and Smith seems important enough to require its description here. We may assume that the given hull form, described as a double hull in an infinite fluid, is known and given by the following equation:

$$F(x,y,z) = 0. \qquad \text{...(2.43)}$$

We can further assume that the velocity of the ship is c and that we can write the velocity potential as the sum of a term cx and a term ϕ' which vanishes at $x^2 + y^2 + z^2 \rightarrow \infty$.

$$\phi = \phi' + cx \;. \qquad \text{...(2.44)}$$

We may also describe the unit normal vector directed outward from the hull by $\vec{n}$.

$$\vec{n} = \pm \left[\frac{\text{grad } F}{|\text{grad } F|} \right]_{F=0} . \qquad \text{...(2.45)}$$

Now the normal velocity to the hull surface, described in coordinates which move with the ship, must be zero at the hull surface.

$$-\frac{\partial \phi}{\partial n}\bigg|_S = -\vec{n} \cdot \text{grad } \phi \bigg|_{F=0} = 0. \qquad \text{...(2.46)}$$

We may substitute the relation (2.44) in (2.46).

$$\left[-\frac{\partial}{\partial n} (\phi' + cx) \right]_S = -\left[\vec{n} \cdot \text{grad}(\phi' + cx) \right]_{F=0} = 0$$

$$\frac{\partial \phi}{\partial n}\bigg|_S = \left[n \cdot \text{grad } \phi' \right]_{F=0} = -\bar{n} \cdot \bar{i}\, c \;. \qquad \text{...(2.47)}$$

We may now assume that the surface of the double hull is covered by a surface source density distribution σ. Then we can write the velocity potential as follows:

$$\phi'(x,y,z) = \iint_S \frac{\sigma(q)}{r(P,q)} \, dS \qquad \ldots(2.48)$$

where r(P,q) is the distance from the integration point q on the surface to the field point P with coordinates (x,y,z) where the potential ϕ' is being evaluated. Kellogg [10] has shown that the normal derivative of ϕ' at the point p on the surface of the distribution can be written in the following way:

$$\left.\frac{\partial \phi'}{\partial n}\right|_S = -2\pi \sigma(p) + \iint \frac{\partial}{\partial n}\left[\frac{1}{r(p,q)}\right] \sigma(q) \, dS \qquad \ldots(2.49)$$

We can substitute this into (2.47) to obtain an integral equation for the distribution σ.

$$2\pi\sigma(p) - \iint_S \frac{\partial}{\partial n}\left[\frac{1}{r(p,q)}\right] \sigma(q) \, dS = \vec{n} \cdot \vec{i} \, c \qquad \ldots(2.50)$$

Observe that $\vec{n}(p)$ is the unit normal vector at F = 0 and the variable r(p,q) is the straight line distance between two points p and q on the surface of the hull.

Equation (2.50) has been integrated numerically to provide not a source density but the source strengths at a finite number of points on a double hull. This set of sources provides a close approximation to the flow about that hull. With this result it is possible to calculate the wave-making resistance directly from equations (2.18) to (2.22) -- using, of course, only those sources which are on the half of the double hull which is submerged when it operates as a surface ship.

III. Methods Which Have Been Used To Find Forms Of Decreased Wave Resistance

It is possible to draw a number of conclusions from the form of the resistance equations and from the integrals which describe forms produced by a ship. Beyond this, Karp, Kotik, and Lurye [6] found the solution to the problem of the form of that semi-infinite strut of minimum wave-making resistance which, between the ends of its distribution of sources and sinks, can be described by an integrable distribution of dipole moment. The first step toward a method of steep descent was made by Hogner [11] in 1936, but never carried through to completion. Several successful attempts to find the best form among a family which can be described by certain polynomials subjected to various constraints have been made -- all of these using some variation of the Ritz method, and all based on some integrals calculated by Weinblum [5] , [12] , [13]. A more striking result has been obtained by Takao Inui [4] by determining the form of the waves produced by a ship and then adding to the hull appendages which will produce roughly similar waves of opposite phase. Finally, certain conclusions have been drawn by Inui as to the effect of discontinuities in the source distribution or any of its derivatives on the waves produced by a semi-infinite strut. Some further conclusions can be drawn from these as to what kind of forms cannot have very small wave-making resistance without being very small themselves. All of these matters will be treated in turn in this section.

1. CONCLUSIONS WHICH CAN BE DRAWN FROM THE FORM OF THE RESISTANCE EQUATIONS AND OF THE INTEGRALS DESCRIBING THE WAVE FORMS

Effect of Depth on Resistance

One of the more obvious conclusions which may be drawn from the form of the equations is that if the sources describing a hull can be submerged deeply enough the wave-making resistance can be made as small as we please. This follows from equation (2.13) which contains the term $\exp(-K_o f \sec^2\theta)$, where $K_o = g/c^2$ and f is the depth. On the other hand, the resistance is certainly non-negative, since equation (2.12) is the integral of a sum of squares times a positive definite trigonometric function.

A further conclusion of the same sort can be drawn by examining equations (2.19) and (2.20). These equations show that the smoothly varying portion of the wave-making resistance is small for very low K_o, increases to a maximum, and then decreases again. For a proof see appendix 1. Since $K_o = g/c^2$, it is clear that the largest value of the resistance occurs at lower speeds for small f (small depth) and at higher speeds for large f(large depth). This means that for high enough speeds the contribution to the smoothly varying part $R^{(1)}$ of the wave-making resistance from parts of the hull corresponding to sources very close to the surface will approach zero. The interaction term $R^{(2)}$ displays a similar behavior, although it is clear that the interaction between a shallow source and a deep one will reach its maximum absolute value at a speed which is intermediate between the speed for the maximum of the smoothly varying term for the shallow source and that for the deep source. The general conclusion which can be drawn from these facts is that the part of the hull near the waterline can be designed for relatively low resistance at low speeds, and the part of the hull deep in the water for relatively low resistance at high speeds, and the interaction terms between shallow and deep sources can be used to improve the behavior at high speeds.

Effect of Symmetry Fore and Aft

Another conclusion, this an old and frequently misstated one, is that which relates to the effects of fore-and-aft symmetry. Suppose that we have a dipole moment density distribution such that the after half of the distribution is a reflection of the forward half (i.e., if the origin of x is at the midships section, $\mu(x)$ is even). Then the function I in equation (2.34) is zero and the resistance of (2.25) reduces to

$$R = 16\pi\rho K_0^2 \int_0^\infty J^2 \cosh^2 u \, du \qquad \ldots(3.1)$$

Now if we add to $\mu(x)$, which we have assumed to be even, any odd function of x there will be no change in J but I will become different from zero. Then we will have to add to J^2 in (3.1) a positive definite quantity I^2, and so will increase the resistance. Since the odd function added to $\mu(x)$ will not change the total dipole moment (which is roughly proportional to volume), it follows that the hull described by an even function u(x) has less resistance than the hull of roughly the same volume described by that even function plus an odd function of x. This is not the same thing as a statement that a hull symmetrical fore-and-aft has less resistance than one which is asymmetrical -- it is extremely easy to devise a symmetrical hull with more resistance than any given asymmetrical hull of the same volume. The demonstration of this paragraph can be generalized to hulls described by an arbitrary symmetrical fore-and-aft distribution of dipoles plus an arbitrary antisymmetrical distribution. We need only use equation (2.32) instead of equation (2.34).

It is possible to argue that the experience of practical ship designers does not bear out the conclusion that the hull of least resistance is symmetrical fore-and-aft. This can also be demonstrated by appealing to the semi-empirical parameters used by Inui to correct his theoretical resistance curves to coincide with those found from experiment. For example, he found that he had to reduce the effect of the sources at the extreme after end of the ship by a factor $\beta' < 1$ in order to bring the calculations into accord with observation. This is because the after end of the ship is relatively less effective in making waves than the bow. We can do this in our formulation by multiplying each source at the after end of the ship by β' and then repeating the derivation of equation (2.32). Then the source distribution which makes $I^2 = 0$ will be symmetrical fore and aft in the moments derived from sources multiplied (at the after end) by β'; and this will certainly not be symmetrical fore-and-aft in the actual dipole moments M_r.

Effect of Length

It has long been known empirically that the resistance per unit of volume tends to decrease as the ship is made longer. This is not true for all small changes in length: for example, if the length is such that the transverse bow and stern waves tend to cancel each other, increasing the length a small amount may cause them to reinforce and increase the resistance. On the other hand, for large changes in length the general tendency will be determined by the smoothly varying portion of the resistance expression. But we showed in equation (2.20) that the smoothly varying portion of the resistance is a function only of the strength of the sources and of their depth, not of their horizontal position. On the other hand, if we have a single source forward of an equal sink moving through the water at a constant speed, the set of closed streamlines which they generate will increase in volume monotonically as they are moved farther and farther apart. It follows from this that the longer is a hull described by a single source and a single sink of constant intensity, the smaller will be the resistance per unit volume if we consider only the part of the resistance which varies smoothly with speed. The same result can be obtained if any assembly of sources and sinks which produces a closed set of streamlines is moved farther apart in the x-direction.

Effect of Breadth

For a very thin hull it has been accepted that the source density can be approximated by equation (2.24):

$$\sigma = -\frac{c}{2\pi}\frac{\partial k}{\partial h} \qquad \ldots(2.24)$$

It follows that if we multiply the source density σ by the parameter a we multiply the breadth of the hull by a:

$$k = -\frac{2\pi a}{c}\int_{bow}^{h}\sigma\, dh \qquad \ldots(3.2)$$

The minus sign results from the fact that the axis of x is positive forward. Now if we substitute $a\sigma'$ for σ in equation (2.15), we get

$$P(a\sigma') = aP(\sigma')$$

$$R(a\sigma') = 16\pi\rho K_o^2 \int_0^{\pi/2}\left[P^2(a\sigma') + Q^2(a\sigma')\right] \sec^3\theta\, d\theta =$$

$$= 16\pi\rho K_o^2 \int_0^{\pi/2} a^2 \left[P^2(\sigma') + Q^2(\sigma')\right] \sec^3\theta\, d\theta = a^2 R(\sigma'). \qquad \ldots(3.3)$$

From (3.2) and (3.3) it is clear that the resistance of very thin hulls is proportional to the square of the breadth. Inui [3] and Hess and Smith [9] have shown, however, that for hulls of finite breadth this relationship is incorrect. No general relationship has been worked out for such hulls, and it is clear from the plots that have been obtained that any such relationship would be very complicated.

Effect of Bow Shape on Resistance

If the source distribution describing the hull consists of a distribution over the centerline plane of the ship in such a fashion that it is not a function of depth but is a continuous function of length and has continuous derivatives of all orders, it is possible to calculate the effect of the shape of the bow on resistance. Let us consider equation (2.9a) and substitute for the assemblage of discrete sources a source distribution $\sigma(h,0,-f) = \sigma(h)$

on the centerline plane. Then we may integrate with respect to f and take f to infinity (infinite depth).

$$\zeta = \frac{4K_o}{c} \int_{-\pi/2}^{\pi/2} \int_{h=0}^{h=-L} \int_{f=0}^{\infty} \sigma(h) \cos\left[K_o(x-h) \sec\theta\right] \cos\left[K_o y \sec^2\theta \sin\theta\right] \times$$

$$\times \exp(-K_o f \sec^2\theta) \sec^3\theta \, d\theta \, dh \, df =$$

$$= \frac{4}{c} \int_{-\pi/2}^{\pi/2} \int_{h=0}^{h=-L} \sigma(h) \cos\left[K_o(x-h) \sec\theta\right] \cos\left[K_o y \sec^2\theta \sin\theta\right] \sec\theta \, d\theta \, dh \qquad \ldots(3.4)$$

It is now possible to integrate by parts with respect to h so that we obtain a series in $\sigma(h)$ and its successive derivatives, evaluated at $h = 0$ (the bow) and $h = -L$ (the stern).

$$\int_{h=0}^{h=-L} \sigma(h) \cos\left[K_o(x-h) \sec\theta\right] \sec\theta \, dh = -\frac{1}{K_o}\left\{\sigma(h) \sin\left[K_o(x-h) \sec\theta\right]\right\}_{h=0}^{h=-L} +$$

$$+ \frac{1}{K_o} \int_{h=0}^{h=-L} \sigma'(h) \sin\left[K_o(x-h) \sec\theta\right] dh =$$

$$= \frac{1}{K_o} \sigma(0) \sin(K_o x \sec\theta) - \frac{1}{K_o} \sigma(-L) \sin\left[K_o(L+x) \sec\theta\right] +$$

$$+ \frac{1}{K_o^2 \sec\theta}\left\{\sigma'(h) \cos\left[K_o(x-h)\sec\theta\right]\right\}_{h=0}^{h=-L} - \frac{1}{K_o^2 \sec\theta} \int_{h=0}^{h=-L} \sigma''(h) \cos\left[K_o(x-h)\sec\theta\right] dh =$$

$$= \frac{1}{K_o}\left\{\sigma(0) - \frac{\sigma''(0)}{K_o^2 \sec^2\theta} + \frac{\sigma^{iv}(0)}{K_o^4 \sec^4\theta} - \ldots\right\} \sin(K_o x \sec\theta) -$$

$$- \frac{1}{K_o}\left\{\frac{\sigma'(0)}{K_o \sec\theta} - \frac{\sigma'''(0)}{K_o^3 \sec^3\theta} + \ldots\right\} \cos(K_o x \sec\theta) -$$

$$- \frac{1}{K_o}\left\{\sigma(-L) - \frac{\sigma''(-L)}{K_o^2 \sec^2\theta} + \frac{\sigma^{iv}(-L)}{K_o^4 \sec^4\theta} - \ldots\right\} \sin\left[K_o(L+x)\sec\theta\right] +$$

$$+ \frac{1}{K_o}\left\{\frac{\sigma'(-L)}{K_o \sec\theta} - \frac{\sigma'''(-L)}{K_o^3 \sec^3\theta} + \ldots\right\} \cos\left[K_o(L+x)\sec\theta\right] =$$

$$= S(0,\theta) \sin(K_o x \sec\theta) + C(0,\theta) \cos(K_o x \sec\theta)$$

$$-\left\{S(-L,\theta) \sin\left[K_o(x+L)\sec\theta\right] + C(-L,\theta) \cos\left[K_o(x+L)\sec\theta\right]\right\} \qquad \ldots(3.5)$$

where

$$S(h,\theta) = \frac{1}{K_o}\left\{ \sigma(h) - \frac{\sigma''(h)}{K_o^2 \sec^2\theta} + \frac{\sigma^{iv}(h)}{K_o^4 \sec^4\theta} - \dots \right\}$$

$$C(h,\theta) = -\frac{1}{K_o}\left\{ \frac{\sigma'(h)}{K_o \sec\theta} - \frac{\sigma'''(h)}{K_o^3 \sec^3\theta} + \dots \right\} \qquad \dots(3.6)$$

We can now write equation (3.4) in terms of S and C:

$$\zeta = \frac{4}{c} \int_{-\pi/2}^{\pi/2} S(0,\theta) \sin(K_o x \sec^2\theta\cos\theta)\cos(K_o y \sec^2\theta\sin\theta)\, d\theta \; +$$

$$+ \frac{4}{c} \int_{-\pi/2}^{\pi/2} C(0,\theta) \cos(K_o x \sec^2\theta\cos\theta)\cos(K_o y\sec^2\theta \sin\theta)\, d\theta \; -$$

$$- \frac{4}{c} \int_{-\pi/2}^{\pi/2} S(-L,\theta) \sin\left[K_o(x+L)\sec^2\theta\cos\theta\right] \cos(K_o y\sec^2\theta \sin\theta)\, d\theta \; -$$

$$- \frac{4}{c} \int_{-\pi/2}^{\pi/2} C(-L,\theta) \cos\left[K_o(x+L)\sec^2\theta\cos\theta\right] \cos(K_o y\sec^2\theta \sin\theta)\, d\theta \qquad \dots(3.7)$$

It is now possible to write the wave-making resistance corresponding to this wave form. For this purpose we will examine only the first two terms of the expression (3.7), which are the waves generated at the bow of the ship. If we disregard the interference between bow and stern wave, the resistance produced by the bow waves can be written as follows [3]:

$$R = \text{const} \int_{0}^{\pi/2} \left[\left\{S(0,\theta)\right\}^2 + \left\{C(0,\theta)\right\}^2\right] \cos^3\theta \, d\theta \qquad \dots(3.8)$$

For very low velocities K_o is very large, and so the dominant term in (3.8) is the first term in the expression for $S(0,\theta)$ in (3.6) -- that is, the source density at the bow. Therefore the smaller this source density the smaller the resistance for very low speeds. The source density at the bow is proportional to the angle of entrance at all points except at the extreme bow, so it is clear that for very small speeds the wave-resistance produced by the bow waves is determined by the entrance angle of the bow -- the larger the angle, the larger the resistance. However, as the speed increases K_o becomes smaller and the later terms in the expansion for S and C become important, and when K_o is of the order of unity they become of nearly the same order of importance as the source density itself. For K_o small enough, for example, it is possible to have the first and second terms of $S(0,\theta)$ become of equal magnitude and opposite sign so that they cancel for some values of θ. Then if all other derivatives $6^{(k)}(0,\theta)$, k=1,3,4,... are zero or small in magnitude we may find that a ship of small resistance will have a large source density at the bow and a small density farther aft. This implies a bluff bow with large entrance angle.

Non-Zero Resistance of a Continuous Symmetric Source Distribution

One further conclusion may be drawn from equations (3.4) to (3.8). We may take advantage of the fact that $S(-L,\theta) = -S(0,\theta)$ and $C(-L,\theta) = C(0,\theta)$ for a hull which is symmetrical about its midship section (i.e., fore and aft). Then we have the following expression for the wave height ζ :

$$\left.\begin{aligned}\zeta = \frac{4}{c}\int_{-\pi/2}^{\pi/2} S(0,\theta)\left\{\sin(K_o x\sec^2\theta\cos\theta) + \sin\left[K_o(x+L)\sec^2\theta\cos\theta\right]\right\}\cos(K_o y\sec^2\theta\sin\theta)\,d\theta\ + \\ + \frac{4}{c}\int_{-\pi/2}^{\pi/2} C(0,\theta)\left\{\cos(K_o x\sec^2\theta\cos\theta) - \cos\left[K_o(x+L)\sec^2\theta\cos\theta\right]\right\}\cos(K_o y\sec^2\theta\sin\theta)\,d\theta = \\ = \frac{8}{c}\int_{-\pi/2}^{\pi/2} S(0,\theta)\sin\left[K_o(x+L/2)\sec^2\theta\cos\theta\right]\cos\left[K_o(L/2)\sec^2\theta\cos\theta\right]\cos(K_o y\sec^2\theta\sin\theta)\,d\theta\ + \\ + \frac{8}{c}\int_{-\pi/2}^{\pi/2} C(0,\theta)\sin\left[K_o(x+L/2)\sec^2\theta\cos\theta\right]\sin\left[K_o(L/2)\sec^2\theta\cos\theta\right]\cos(K_o y\sec^2\theta\sin\theta)\,d\theta\end{aligned}\right\}\quad\ldots(3.9)$$

We may combine the factors in $\left[K_o(L/2)\sec^2\theta\cos\theta\right]$ with $S(0,\theta)$ in the first term and with $C(0,\theta)$ in the second term. We can write then

$$\begin{aligned}S_1(-\tfrac{L}{2},\theta) &= C(0,\theta)\sin\left[K_o(L/2)\sec^2\theta\cos\theta\right]\\ S_2(-\tfrac{L}{2},\theta) &= S(0,\theta)\cos\left[K_o(L/2)\sec^2\theta\cos\theta\right]\end{aligned}\qquad\ldots(3.10)$$

It follows that the resistance of the ship is given by the following expression:

$$R = \text{const} \int_0^{\pi/2} \left[S_1(-\frac{L}{2},\theta) + S_2(-\frac{L}{2},\theta)\right]^2 \cos^3\theta \, d\theta \qquad \ldots(3.11)$$

Since $\cos\theta$ is real and non-negative for the entire range of integration, and the squared sum in the integrand is also real and non-negative, $S_1 \neq -S_2$ for $S_1, S_2 \neq 0$, R can be made zero only if $S_1(-\frac{L}{2},\theta)$ and $S_2(-\frac{L}{2},\theta)$ are separately zero for all θ in the range $0 \leq \theta \leq \pi/2$. But this can be true only if $\sigma(0)$, $\sigma'(0)$, $\sigma''(0)$,... are separately zero, since the series in (3.6) cannot otherwise be zero for all θ in this range. It follows that the wave-making resistance of an infinitely deep strut which is symmetrical fore-and-aft, which is described by a continuous distribution of source density on the centerline plane, which is not a function of depth, and which has continuous derivatives of all orders cannot be zero. Further, it follows from a previous result that if a continuous fore-and-aft anti-symmetrical distribution of dipole moment density is added so that the hull loses its fore-and-aft symmetry, the resistance can only be increased. This result says nothing concerning the resistance of an assemblage of discrete sources and sinks.

Effect of a Discontinuity in Source Distribution or Any Derivative

Let us assume in the derivation of equation (3.5) that the k^{th} derivative has a discontinuity at $h=h_k$. This will give rise to a term of the form

$$\frac{\sigma^{(k)}(h_k+) - \sigma^{(k)}(h_k-)}{K_o^k \sec^k\theta},$$

so that in addition to the wave pattern described by equation (3.7) we have a wave described by the following equation:

$$\zeta = \pm \frac{4}{c} \int_{-\pi/2}^{\pi/2} \frac{1}{K_o} \frac{\sigma^{(k)}(h_k+) - \sigma^{(k)}(h_k-)}{K_o^k \sec^k\theta} \begin{bmatrix}\sin \\ \cos\end{bmatrix} K_o(x+h_k)\sec^2\theta\cos\theta \, \cos(K_o y \sec^2\theta \sin\theta)\, d\theta \qquad \ldots(3.12)$$

where the sine or cosine is taken depending on whether the integer k is even or odd, and the sign is positive for $k = 4n$, $k = 4n-1$, $n = 0,1,2,\ldots$, and negative otherwise. This explains the existence of the so-called shoulder waves of a hull, which start at such discontinuities as the connection between the fair bow section and the parallel middle body. Since these additional waves occur at a point where they cannot in general be fully canceled by waves originating at the bow and stern, they will add to the wave resistance.

2. APPLICATION OF THE CALCULUS OF VARIATIONS TO FINDING HULL FORMS OF MINIMUM RESISTANCE

An apparently reasonable approach to the problem of finding a ship of minimum wave resistance is to apply the classical calculus of variations. Unfortunately, the failure to distinguish between the hull form and the dipole density which generates it together with the general difficulty of the problem prevented a solution being obtained for many years. Sretenskij [14] published a paper in 1935 in which he demonstrated that in certain cases, at least, there was no solution. Even today the problem has only been solved for the case of a strut of infinite depth.

Karp, Kotik, and Lurye [15] solved the problem in 1959. They first considered a dipole moment distribution uniform in depth on an infinitely deep centerline plane of length L, and assumed that the dipole moment density vanished at the bow and stern and had an integrable first derivative with respect to x. They then were able to prove that if an additional requirement were imposed that the dipole moment per unit depth be held constant, the integral equation resulting from the application of the classical calculus of variations had no solution. They reasoned that the cause was that the class of functions chosen for the dipole moment density was too restrictive, and then tried a more general set of functions. This choice of functions required that the velocity potential not become infinite at finite points of the (x,y) plane other than the centerline plane on which the dipoles are distributed and that the integral of the dipole moment over the length L remain constant per unit depth. With this extended choice of functions they were able to find a solution for the dipole moment distribution which provided a form of minimum wave-making resistance.

In order to sketch the method employed by Karp, Kotik, and Lurye their notation will be converted to that used elsewhere in this paper. We will start by assuming that the velocity potential, as before, can be written as the sum of a term corresponding to the flow in the absence of the ship and a second term, which vanishes at infinity, corresponding to the disturbance caused by the distribution of dipoles along the centerline plane of the hull. The axis of x will be positive in the direction of ship motion.

$$\phi = \phi' + cx \qquad \ldots(2.44)$$

The dipole distribution will be taken along a strip of infinite depth and length L. We will lose no generality if we assume that the length L of the distribution is unity. We can always correct to other lengths by Froude's law of similitude. Since we have already demonstrated that if a form symmetrical fore-and-aft can be found whose resistance is minimized any addition of an anti-symmetrical form to it will only increase the resistance, we may also assume that the hull form is symmetrical about the midships section. We will write the dipole moment distribution in terms of a new variable

$$\xi = h + 1/2 \qquad \ldots(3.13)$$

Then we will choose a value for the integral of the dipole moment over the length of the distribution. If this is held constant, then the volume per unit depth, which is approximately proportional to the dipole moment, will be held constant to first order. We will let this integral be

$$\int_{-1/2}^{1/2} \mu(\xi)\, d\xi = A > 0. \qquad \ldots(3.14)$$

The quantity $\mu(\xi)$ is the dipole moment per unit depth and unit length. The other restriction to be placed on $\mu(\xi)$ is that the following integral for the velocity potential be well-behaved:

$$\phi' = \text{const} \int_{0}^{\infty} \int_{-1/2}^{1/2} \mu(\xi) \frac{\partial G(x,y,z;\xi,0,-f)}{\partial \xi}\, d\xi\, df \qquad \ldots(3.15)$$

The function $G(x,y,z;\xi,0,-f)$ is the velocity potential of a unit source and can be derived from the first term of equation (2.4) by letting $m = 1$ and substituting equation (3.13).

$$G(x,y,z;\xi,0,-f) = \frac{1}{\sqrt{(x+\frac{1}{2}-\xi)^2 + y^2 + (z+f)^2}} \qquad \ldots(3.16)$$

The partial derivative with respect to ξ converts G into the Green's function for a dipole with axis oriented along the x-axis. The restriction placed on $\mu(\xi)$ in addition to (3.14) is that ϕ' be finite everywhere except on the plane on which the dipoles are distributed, and that it go to zero as $x^2 + y^2 + z^2 \rightarrow \infty$. The quantity x is still reckoned from the forward end of the dipole distribution.

In part II of this paper we proved that the resistance of an assemblage of dipoles can be written in the following way (see equation (2.39)):

$$R = -8\pi^2 \rho K_o^2 \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \mu(\xi)\, \mu(\xi')\, Y_o(K_o|\xi - \xi'|)\, d\xi\, d\xi' . \qquad \ldots(3.17)$$

We may regard the coefficient outside the integral as a constant and combine this equation with (3.14) to write the variational equation:

$$\Phi = C \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \left[\mu(\xi) + \epsilon\zeta(\xi)\right] Y_o(K_o|\xi - \xi'|)\left[\mu(\xi') + \epsilon\zeta(\xi')\right] d\xi\, d\xi' +$$

$$+ \lambda \left\{ \int_{-1/2}^{1/2} \left[\mu(\xi) + \epsilon\zeta(\xi)\right] d\xi - A \right\} \qquad \text{...(3.18)}$$

We may now find the Euler equation by taking $\frac{\partial\Phi}{\partial\epsilon}$ and letting $\epsilon \rightarrow 0$.

$$\left.\frac{\partial\Phi}{\partial\epsilon}\right|_{\epsilon\rightarrow 0} = 2C \int_{-1/2}^{1/2} \zeta(\xi') \int_{-1/2}^{1/2} \mu(\xi) Y_o(K_o|\xi - \xi'|)\, d\xi\, d\xi' +$$

$$+ \lambda \int_{-1/2}^{1/2} \zeta(\xi')\, d\xi' = 0 \qquad \text{...(3.18a)}$$

From this we can write

$$\int_{-1/2}^{1/2} \zeta(\xi') \left\{ 2C \int_{-1/2}^{1/2} \mu(\xi)\, Y_o(K_o|\xi - \xi'|)\, d\xi + \lambda \right\} d\xi' = 0\,. \qquad \text{...(3.18b)}$$

But the function $\zeta(\xi')$ is arbitrary, so it follows that the quantity in braces is zero for all ξ'. This is the Euler equation for the problem. The linear nature of the equation in braces makes it possible to solve the equation for arbitrary λ and then normalize the solution by multiplying by the proper factor to satisfy equation (3.14), which we regain from (3.18a) by taking $\left.\frac{\partial\Phi}{\partial\lambda}\right|_{\epsilon\rightarrow 0} = 0$. Hence, we may write without loss of generality,

$$\int_{-1/2}^{1/2} \mu(\xi)\, Y_o(K_o|\xi - \xi'|)\, d\xi = \lambda' \qquad \text{...(3.19)}$$

and solve for $\lambda' = 1$ and then normalize the solution to (3.14). Equation (3.19) has been shown [15] to have solutions of the form

$$\mu(\xi) = g(\xi) \frac{1}{\sqrt{(1/4) - \xi^2}} \qquad \text{...(3.20)}$$

where $g(\xi)$ is regular in the range $-1/2 \leq \xi \leq 1/2$, and not zero at the endpoints. In consequence the density of the dipole moment goes to infinity at each end of the distribution. It turns out, however, that this does not result in the velocity potential becoming infinite. We may show this by substituting (3.20) in (3.15). Then we may examine the portion of the velocity potential which results from the singularity at $\xi = 1/2$. We will call this $\Delta \phi'$.

$$\Delta \phi' \approx \text{const} \int_0^\infty df \int_{1/2-\epsilon}^{1/2} g(\xi) \frac{\partial G}{\partial \xi} \frac{d\xi}{\sqrt{(1/4) - \xi^2}} =$$

$$\approx \text{const} \int_0^\infty df\ g(1/2-\epsilon\theta)\frac{\partial G}{\partial \xi}(1/2-\epsilon\theta) \int_{1-2\epsilon}^{1} \frac{d\xi'}{\sqrt{1-\xi'^2}} =$$

$$= \text{const} \int_0^\infty g(1/2-\epsilon\theta)\frac{\partial G}{\partial \xi}(1/2-\epsilon\theta,0,-f) \left[\sin^{-1}1 - \sin^{-1}(1-2\epsilon)\right] df$$

But $\sin^{-1}(1) - \sin^{-1}(1-2\epsilon) = \cos^{-1}(1-2\epsilon) \approx 2\sqrt{\epsilon}$; and so

$$\Delta \phi' \approx \text{const} \int_0^\infty g(1/2-\epsilon\theta)\frac{\partial G}{d\xi}(1/2-\epsilon\theta,0,-f)(2\sqrt{\epsilon})df,\ 0<\theta<1. \qquad \ldots(3.21)$$

Equation (3.21) is still bounded after integration with respect to f, and so the contribution of the singularity to ϕ' is also bounded and the condition on $\mu(\xi)$ is met.

Karp, Kotik, and Lurye have solved equation (3.19) numerically for numerous cases and have plotted the results as streamlines [6]. The closed streamlines which define the boundary of the strut extend farther forward and farther aft than the source distribution. For large values of the speed they have found that the strut of minimum resistance has a cross-section shaped rather like a dumbbell -- rounded at the leading edge, then narrow, then expanded and rounded at the trailing edge. For very large speeds (Froude numbers greater than about 0.65) the function $g(\xi)$ in equation (3.20) becomes a constant. For small speeds the function $g(\xi)$ is a minimum at the end points and a maximum at the midships section.

There is a simple interpretation which will make the infinities at the ends of the dipole distribution appear more reasonable than they might seem at first. If, for example, we started at the forward end of the distribution with an isolated dipole of strength M, and a source of strength m, while at the after end of the distribution we had another isolated dipole of strength M and a sink of strength -m, we would have a set of singularities which are well known to generate a reasonable set of streamlines in two-dimensional flow. If, further, we identified the isolated dipole with a delta-function multiplied by a strength M, then we would have a condition where the dipole density became infinite within the bounds of the delta-function, but the total moment remained finite and equal to M. The distribution obtained by Karp, Kotik, and Lurye for very high speeds is not too different from the condition just described.

3. USE OF THE RITZ METHOD FOR FINDING HULL FORMS OF REDUCED WAVE-MAKING RESISTANCE

The bilinear nature of the resistance integral leads in a natural way to the use of the Ritz method for finding ways to develop improved hull forms. It has been used successfully for this purpose, although its success has been limited both by the restricted set of functions employed for the description of the ship and by the (to date) consistent failure to distinguish between the distribution of dipole moment and the actual shape of the hull. One of the more striking successes was the design of a hull which, when towed at a Froude number of 0.5, had about 13 per cent less wave-making resistance than a destroyer of similar dimensions, and which, even after the other components of resistance were included, had about 8 per cent less total resistance than the destroyer [12].

In order to see the problem in simple terms, we will start with equations (2.14) and (2.15). Here we have a source density σ which generates the hull form. If we add to this source density another distribution $a\sigma'$, then we have a resistance formula which is a quadratic in the parameter a.

$$R = 16\pi\rho K_o^2 \int_0^{\pi/2} \left[P(\sigma)^2 + 2aP(\sigma)P(\sigma') + a^2P(\sigma')^2 + Q(\sigma)^2 + 2aQ(\sigma)Q(\sigma') + a^2Q(\sigma')^2\right]\sec^3\theta \, d \qquad \ldots(3.22)$$

This can be rewritten as follows:

$$R = R(\sigma,\sigma) + 2aR(\sigma,\sigma') + a^2R(\sigma',\sigma') \qquad \ldots(3.23)$$

where the functional $R(\sigma,\sigma')$ is defined by

$$R(\sigma,\sigma') = 16\pi\rho K_o^2 \int_0^{\pi/2} \left[P(\sigma)P(\sigma') + Q(\sigma)Q(\sigma')\right] \sec^3\theta \, d\theta \qquad \ldots(3.24)$$

It is now a simple matter to take the derivative of R in equation (3.23) with respect to the parameter a and find the optimum amount of the distribution σ' to add to the given distribution σ by setting the derivative $\frac{dR}{da} = 0$.

To generalize this result, we need only follow the same pattern but give each of arbitrarily many linearly independent source distributions $\sigma_1,\sigma_2,\ldots,\sigma_n$ a coefficient $a_1,a_2,\ldots,a_n$ and write

$$R = \sum_{r=1}^{n} a_r^2 R(\sigma_r,\sigma_r) + 2 \sum_{s=r+1}^{n} \sum_{r=1}^{n-1} a_r a_s R(\sigma_r,\sigma_s) \qquad \ldots(3.25)$$

If the set of functions $\{\sigma_1,\sigma_2,\ldots\}$ is also a complete set, then its members may be used to describe an arbitrary dipole moment distribution on the interval on which they are defined. It is a simple matter also to add constraints. We merely write them in the form

$$f_i(a_j, a_k, \ldots) = 0 \qquad \ldots(3.26)$$

and then apply the method of Lagrange multipliers to find a stationary value of R subject to these constraints. To do this we write

$$\Phi = R + \sum_i \lambda_i f_i \quad , i = 1, \ldots \qquad \ldots(3.27)$$

where R is defined in equation (3.25), and then take the partial derivative of Φ with respect to each a_r and λ_i in turn and set it equal to zero.

$$\left.\begin{aligned} \frac{\partial \Phi}{\partial a_r} &= 2a_r R(\sigma_r, \sigma_r) + 2\sum_{s \neq r} a_s R(\sigma_r, \sigma_s) + \sum_i \lambda_i \frac{\partial f_i}{\partial a_r} = 0, \ r = 1, 2, \ldots, n \\ \frac{\partial \Phi}{\partial \lambda_i} &= f_i(a_j, a_k, \ldots) = 0, \ i = 1, \ldots \end{aligned}\right\} \qquad \ldots(3.28)$$

The functions σ_r which are admissible are not completely arbitrary, but it is clear from the experience of Karp, Kotik, and Lurye that certain singularities may be permitted. The analogy to the requirement which they found necessary, when translated from a dipole distribution to a source distribution, is the requirement that the potential ϕ' remain bounded when it is described by

$$\phi' = \text{const} \iiint_V \sigma_r G(x,y,z;\ h,k,-f)\ dV \qquad \ldots(3.29)$$

where

$$\phi = \phi' + cx \ . \qquad \ldots(2.44)$$

The integration V is over the volume in which the source distribution is non-zero and ϕ' must be bounded only outside this volume.

Exactly the same treatment as outlined above may be applied to distributions of dipole moment density $\mu(h,k,-f)$ rather than source density. This requires only that we substitute equation (2.33) or its equivalent in terms of an integral into equation (2.25) to get the resistance in terms of dipole density rather than source density. Then we get a result equivalent to (3.28) except that the resistance components must be written $R(\mu_r, \mu_s)$ rather than $R(\sigma_r, \sigma_s)$. In such a case the restrictions on the singularities in μ_r

are probably given by equation (3.15) extended to include the case $k \neq 0$. It is also possible to use assemblages of discrete sources or discrete dipoles in place of the densities discussed above.

The treatments using this concept, despite their simplicity, have been limited mainly to work which was started by G. P. Weinblum at the David Taylor Model Basin, and carried further at the towing tank at Stevens Institute of Technology and at the University of California. Weinblum described his dipole distribution (which he did not initially distinguish from the offsets of the hull) by the product of a polynomial in the variable f describing the distance below the surface and a polynomial in the variable h describing the longitudinal position. After converting these to non-dimensional form he could write the offset η in the form

$$\eta = X(\xi)\, Z(\zeta) = (1 - \sum_n a_n \xi^n - \sum_m b_m \xi^m)(1 - e\zeta^r) \qquad \ldots(3.30)$$

where the half-length of the hull is 1 so that $-1 \leqq \xi \leqq 1$ and the draft of the hull is 1 so that $0 \leqq \zeta \leqq 1$. The signs of the a_n are so arranged as to make them describe functions symmetric fore-and-aft, while the b_m are for anti-symmetric functions. The quantity $e \leqq 1$ is a positive parameter which is used to permit the hull to have a flat bottom (if $e < 1$). In his first report [5] Weinblum used $r = 4$ and limited the calculations to $n = 2,3,4,6,8,10,12$. He then calculated the resistance terms which enter into a sum of the form (3.25), explicitly differentiating the dimension (or dipole moment distribution) to convert it to the form 6 which enters into that equation. He also used another distribution in the direction of depth which gave him the equivalent of a V-bottom.

In Weinblum's initial paper the integrals corresponding to the values of n and r mentioned above are given. Later the calculation is carried through to the point where optimum forms are obtained by a technique similar to that described earlier [16]. These forms are restricted in that the dipole moment density is made zero at the ends of the ship, although Karp, Kotik and Lurye [6] later found that it should be permitted to become infinite to minimize resistance. The separability of the functions $X(\xi)$ and $Z(\zeta)$ severely limits the forms which can be investigated. Nevertheless, even these forms show the possibility of significant improvement over conventional forms. For example, Martin and White [12] in 1961 selected only the exponents n = 2,3,4, and 6, added the constraint that the value of η be zero at bow and stern, constrained the integral $\int_0^1 \eta(\xi)\, d\xi$ = constant, and then found the form of minimum resistance within the very limited family of shapes which resulted. Since the result they obtained is properly interpreted as a dipole density but was used as the hull offsets in building a model, it is clear that the model towed was probably not as good as it might have been had the streamlines been calculated directly from the dipole density. Nonetheless, the model, which was optimized for a Froude number of 0.5, had a wave-making resistance about 13 per cent less than that of the model of a good destroyer hull of similar dimensions intended to run at a comparable speed. The success of this limited effort shows that there is much to be gained by a mathematical approach even within the confined of conventional hull shapes.

In an effort to see what could be done by recognizing that the form of the afterbody of a ship is largely determined by the requirements for good flow of water to the propellers, J. Richard Gauthey published in 1961 the results of some calculations using a similar method specifying the afterbody of the ship completely and permitting only the forward half to vary [13]. His results bring out two major points: (1) the form which resulted was quite different from the forebody which had been designed by conventional methods to go with the particular afterbody; and (2) small changes in shape can have large effects. Gauthey's results, while limited in much the same fashion as those of Weinblum and of Martin and White, add to the confidence that much can be done to find improved hulls of conventional form by mathematical methods.

4. CANCELING WAVES BY ADDING APPENDAGES WHICH PRODUCE SIMILAR WAVES OF OPPOSITE PHASE

In a series of important papers starting in the summer of 1960, Takao Inui and his collaborators at the University of Tokyo described the results of intentionally adding large bulbs to hulls in such a position and of such a size as to produce waves about of the same form but of opposite phase to those produced by the bow and stern of the ship [17], [3]. The ability to do this might be inferred from equations (3.5) to (3.8). Those equations show that a hull form whose equivalent source distribution is a continuous function of position along the hull and which has continuous derivatives of all orders produces two well-defined wave systems, one starting at the bow and the other at the stern. Each system has a so-called "sine component" and a "cosine component". To see this, we may write the first term of (3.7).

$$\zeta_{bow,sine} = \frac{4}{c} \int_{-\pi/2}^{\pi/2} S(0,\theta)\ \sin(K_o x\ \sec^2\theta\ \cos\theta)\ \cos(K_o y\ \sec^2\theta\ \sin\theta)\ d\theta \qquad \ldots(3.31)$$

It is possible to assume that $S(0,\theta)$ is an even function of θ for a ship which is symmetrical on the centerline plane, as nearly all ships are. Then since $\sin(K_o y\sec^2\theta\ \sin\theta)$ is an odd function of θ, it follows that

$$\frac{4}{c}\int_{-\pi/2}^{\pi/2} S(0,\theta)\cos(K_o x\sec^2\theta\cos\theta)\sin(K_o y\sec^2\theta\sin\theta)\,d\theta = 0. \qquad \ldots(3.32)$$

We may now add equations (3.31) and (3.32) and get

$$\zeta_{bow,sine} = \frac{4}{c}\int_{-\pi/2}^{\pi/2} S(0,\theta)\sin\left[K_o(x\cos\theta + y\sin\theta)\sec^2\theta\right] d\theta \qquad \ldots(3.33)$$

A similar manipulation can be made with the other terms to provide a term in $\cos\left[K_o(x\cos\theta+y\sin\theta)\sec^2\theta\right]$ and a pair of terms with $x \longrightarrow x+L$.

Lunde [7] has shown that quite generally the waves produced by a ship which is symmetrical on its centerline can be described at points a long way from the ship by a sum of a sine component and a cosine component as follows:

$$\zeta_{ship} = \frac{4}{c}\int_{-\pi/2}^{\pi/2} S(h_o,\theta)\sin\left\{K_o\left[(x-h_o)\cos\theta + y\sin\theta\right]\sec^2\theta\right\} d\theta \; +$$

$$+ \frac{4}{c}\int_{-\pi/2}^{\pi/2} C(h_o,\theta)\cos\left\{K_o\left[(x-h_o)\cos\theta + y\sin\theta\right]\sec^2\theta\right\} d\theta \qquad \ldots(3.34)$$

Where the functions $S(h_o,\theta)$ and $C(h_o,\theta)$ are the results of combining a well-defined bow wave and a well-defined stern wave, they are not likely to be a form which is easily canceled by an added appendage such as a bulb. On the other hand, Inui has found that for ships of certain forms moving at relatively low speeds, the wave originating at the bow and the wave originating at the stern may each be described by a sine component whose amplitude function $S(h,\theta)$ is positive and does not change rapidly with increasing θ. That this is likely can be seen from equation (3.6), which clearly has as its dominant term for large K_o (small speed) the value

$$\underset{K_o\,\text{large}}{S(h,\theta) \longrightarrow} \frac{1}{K_o}\,6(h) \qquad \ldots(3.35)$$

and it follows that the cosine term goes to zero:

$$\underset{K_o\,\text{large}}{C(h,\theta) \longrightarrow} 0.$$

The success of Inui's technique is based on this observation. He uses the well-known result that the waves produced by a doublet can be described entirely by a sine-wave component. The amplitude function of that component, $S(\theta)$, is not really very close in form to that of a ship's bow wave, but it is opposite in sign. By matching amplitudes at $\theta = 0$, and counting on the factor $\cos^3\theta$ in equation (3.8) to minimize the effects of the differences for large θ, he gets virtually complete canceling of the transverse wave system and some decrease in strength of the diverging system of waves. Since most of the energy in a ship wave system for slow and intermediate speed ships is in the transverse waves, this results in a striking decrease in wave-making resistance for such ships whose bow waves can be described by a sine wave system. The resistance of the combination of the sine wave bow waves and the spherical bulb can be written in the following fashion for any given speed:

$$R = \text{const} \int_0^{\pi/2} \left\{ A_F(h_F,\theta) - B\ (h_F,\theta) \right\}^2 \cos^3\theta \, d\theta \qquad \ldots(3.36)$$

Here we have let the sine wave component of the amplitude function for the bow be written $A_F(h_F,\theta)$, where h_F is the longitudinal position at which the bow waves originate. The function $-B(h_F,\theta)$ is the amplitude function of the waves produced by the bulb, and h_F the longitudinal position of its center, which has been made coincident with the longitudinal position where the bow waves originate. The amplitude function for the bulb can be written [4]

$$-B(h_F,\theta) = -\text{const}\ M \sec^4\theta \exp(-K_o f \sec^2\theta) \qquad \ldots(3.37)$$

The magnitude of the constant depends on the convention used in writing source strength. M is the dipole moment of the spherical bulb. If we compare (3.37) with (3.35) where we have for a very deep hull at low speed, $S(\theta) = A_F(h_F, \theta) =$ const, we see that to minimize R in (3.36) we must not only make the bulb such a size that the quantities A_F and B have roughly the same absolute value at $\theta = 0$, but we must make the product $\sec^4\theta \exp(-K_o f \sec^2\theta)$ as nearly constant as possible over the range of θ for which $\cos^3\theta$ has a significant value. The only way this can be done is to adjust the value of f, the depth of immersion of the center of the bulb, assuming the speed and so K_o to be fixed. If we set $K_o f \sim .55$, for example, the value of B will be about the same for $\theta = 35^o$ as for $\theta = 0$, and for $\theta > 35^o$ the magnitude of B will fall off rapidly. In such a case a deep ship traveling at a speed of 32 ft/sec (slightly less than 19 knots) would have to have the bulb immersed $.55 \times 32^2/g = 17.5$ ft to its center.

Although Inui has only used spheres to provide cancellation of the "sine wave" bow waves and stern waves, it is easy to show that a source-sink pair, the source forward of the sink, separated by a little less than a half wave-length, will provide a wave which is much like that produced by a spherical bulb. To see this consider a source-sink pair, the source forward of the sink a distance δ and at the same depth below the surface. The wave pattern far aft of this source which is produced by a traveling source is described by the following equation:

$$\zeta \sim \int_{-\pi/2}^{\pi/2} C(\theta) \cos (K_o p \sec^2\theta) \, d\theta \qquad \ldots(3.38)$$

$$C(\theta) = \text{const } m \sec^3\theta \exp(-K_o f \sec^2\theta) \qquad \ldots(3.39)$$

where

$$p = x \cos \theta + y \sin \theta \qquad \ldots(3.40)$$

If we use (3.38) to write the wave-pattern produced by the source-sink pair, we get

$$\zeta \sim \int_{-\pi/2}^{\pi/2} C(\theta) \cos (K_o p_1 \sec^2\theta)\, d\theta - \int_{-\pi/2}^{\pi/2} C(\theta) \cos(K_o p_2 \sec^2\theta)\, d\theta \qquad \ldots(3.41)$$

where

$$p_1 = p + \frac{\delta}{2} \cos \theta \quad \text{and} \quad p_2 = p - \frac{\delta}{2} \cos \theta \qquad \ldots(3.42)$$

We may simplify equation (3.41) by using (3.42) and we get

$$\zeta \sim -2 \int_{-\pi/2}^{\pi/2} C(\theta) \sin(K_o p \sec^2\theta) \sin(K_o \frac{\delta}{2} \sec \theta)\, d\theta \qquad \ldots(3.43)$$

We may now combine the portions of the amplitude function and write

$$\zeta \sim \int_{-\pi/2}^{\pi/2} C_1(\theta) \sin(K_o p \sec^2\theta)\, d\theta \qquad \ldots(3.44)$$

where

$$C_1(\theta) = -2 \text{ const } m \sec^3\theta \sin(K_o \frac{\delta}{2} \sec \theta) \exp(-K_o f \sec^2\theta) \qquad \ldots(3.45)$$

We may compare the amplitude function for the source-sink pair with the amplitude function for a sphere, that is, equation (3.45) with (3.37). It is clear that they are of the same form except that in place of one of the terms sec θ for the sphere there is the term $\sin(K_o \frac{\delta}{2} \sec \theta)$ for the source-sink pair. This in one sense is an advantage, since it gives us the quantity δ to manipulate in order to improve the match between the amplitude function for the bulb and the amplitude function for the ship's bow wave or stern wave. On the other hand, it makes the bulb's performance more sensitive to speed, since it also contains the factor K_o. The "bulb" described by the source-sink pair in this instance is an elongated body of revolution, rather like a blimp. If this kind of bulb can be used for a propeller shaft housing or faired into the hull it may be an improvement over a sphere. On the other hand, if it merely sticks out forward like a ram, it is no improvement at all.

So far all the applications of Inui's method have consisted of adding bulbs to cancel the "sine wave" component of the bow waves and stern waves of relatively slow ships. Because the sphere, which is the form generated by a concentrated dipole, produces a wave form which is of form similar to these "sine wave" components of the bow and stern waves, but of opposite phase, it is natural to exploit its use. However, the cosine wave component becomes important in ships of higher speed than cargo liners -- destroyers, for example. Since these ships combine high speed with shallow draft, a spherical bulb is an inappropriate appendage to add in any case. (At high speed it would have to be deep -- see discussion following equation (3.37).) Therefore further investigation with a view to finding some other way of canceling the waves of cosine form is in order.

A first observation on this matter is that the waves produced by an isolated source or sink are waves which have the phase of a cosine component. The waves produced by a sink are opposite in phase to those of a sou ce. Hence if a ship produces waves of cosine form, the proper appendage to add, in theory, is an isolated sink. Unfortunately an isolated sink is no more a real entity than an isolated magnetic pole; it must always have associated with it a source. However, it is easy to see that if the hull can reasonably be described by an isolated source forward and an isolated sink aft, its wave-making resistance can be greatly decreased in the following way: add a sink of strength equal to half the forward source at the location of that source and add another source of strength equal to the source at a distance about a half wave-length forward of it. This is equivalent to splitting the source in two and moving half of it a half wave-length forward of its initial position. Then the resistance of the forward system (disregarding its interaction with the stern wave) will certainly be reduced to less than half of what it was. This follows from equation (2.17). Suppose we let m be the strength of the original source and f its depth. Then its resistance will be

$$R = 16\pi\rho K_o^2 \, m^2 \, \frac{e}{2}^{-2K_o f} \int_0^\infty e^{-2K_o ft}(1+t)^{1/2} \, t^{-1/2} \, dt \quad . \qquad \ldots(3.46)$$

If we cut the source in two, and place half of it a distance $(h_r - h_s)$ forward of the original source such that $K_o(h_r - h_s) = \pi$, then the resistance of the combination will be

$$R = 16\pi\rho K_o^2 \, \frac{m}{2}^2 \, \frac{c}{2}^{-2K_o f} \int_0^\infty e^{-2K_o ft}(1+t)^{1/2} \, t^{-1/2} \left\{ 1 + \cos\left[\pi(1+t)^{1/2}\right] \right\} dt \qquad \ldots(3.47)$$

The quantity in brackets for $0 \leq t < 5/4$ is less than unity, and since for any reasonable value of $K_o f$ very nearly all the value of the first integral will be attained in this range, R will be much less than half its original value. The same result follows from equation (2.23) without calculation, since the interference part of the resistance will certainly be negative in this case.

Unfortunately, the technique outlined here is not practical for small, high speed ships for the reason that a half wave-length at 35 knots is several hundred feet, and it is not reasonable to add so much length to a small ship. It therefore appears that Inui's method or any obvious variation is not likely to provide the answer to finding very high-speed hull forms of low resistance.

5. USE OF THE GRADIENT OF THE WAVE RESISTANCE FORMULA TO FIND LOCATIONS WHERE CHANGES IN HULL FORM CAN BRING IMPROVEMENT

In 1936 Hogner made use of the bilinear character of the wave resistance formula to determine where changes might be made in the sectional area of a ship to decrease the wave-making resistance [11]. In his one published article on the subject, he derived the function which may be regarded as the gradient of the functional describing the resistance and promised further development of the idea in a second article which unfortunately was never published. This concept leads in a natural way to the method of steep descent for the calculation of improved hull forms. It is therefore possible that failure to follow up on Hogner's paper has resulted in a twenty-five year delay in using this simple method to find forms of decreased wave-making resistance.

To understand Hogner's method, we may consider a quadratic functional with a symmetric kernel $K(h,h')$:

$$R(\mu) = \iint \mu(h)K(h,h')\mu(h')\,dh\,dh'$$

If we now add a small quantity ϵ times a delta-function to $\mu(h)$ at the point h_1, we can write

$$\begin{aligned} R(\mu + \epsilon\delta(h_1)) &= \iint \left[\mu(h) + \epsilon\delta(h)\right] K(h,h') \left[\mu(h') + \epsilon\delta(h_1')\right] dh\,dh' = \\ &= \iint \mu(h)\,K(h,h')\,\mu(h')\,dh\,dh' + \\ &+ \epsilon \int \mu(h)K(h,h_1')\,dh + \\ &+ \epsilon \int K(h_1,h')\mu(h')\,dh + \\ &+ \epsilon^2\,K(h_1,h_1) \quad . \end{aligned} \qquad \ldots(3.48)$$

Because of the symmetry of $K(h,h')$, and the fact that ϵ is a small number, we can write

$$R(\mu + \epsilon\delta(h_1)) - R(\mu) = 2\epsilon \int \mu(h)\,K(h,h_1)\,dh + \epsilon^2\,K(h_1,h_1) \qquad \ldots(3.49)$$

But the difference in the two values of R is just the change which results from the addition of a small increment $\epsilon\delta(h_1)$ to the given function $\mu(h)$; and if we divide by ϵ we have what may reasonably be described as the gradient of the functional R if we let $\epsilon \longrightarrow 0$.

$$\Phi(h_1) = \text{grad } R = 2 \int \mu(h)K(h,h_1)\,dh \qquad \ldots(3.50)$$

This may be related to the problem of finding an improved strut of infinite depth by identifying the kernel K(h,h') with the kernel $Y_o(K_o|h-h'|)$, which is clearly symmetrical, and identifying the function $\mu(h)$ with the dipole moment distributed on the centerline plane. It follows that the function Φ gives the effect of a change in dipole moment density on resistance as a function of the coordinate h, the longitudinal position on the hull. To make a change to the hull form which will decrease the resistance while holding the total dipole moment constant (thereby holding the volume per unit depth nearly constant), we may make a small change in the dipole moment density by adding a quantity $\epsilon \Phi$ and then subtracting a quantity $\epsilon (1/L) \int_L \Phi \, dh$:

$$\mu^{(1)}(h) = \mu^{(0)}(h) + \epsilon \Phi(h) - \frac{\epsilon}{L} \int_L \Phi \, dh \qquad \ldots(3.51)$$

It is evident that

$$\int_L \mu^{(1)}(h)\, dh = \int_L \mu^{(0)}(h)\, dh + \epsilon \int_L \left[\Phi(h) - \frac{1}{L} \int_L \Phi(h')dh'\right] dh = \int_L \mu^{(0)}(h)\, dh \; . \qquad \ldots(3.52)$$

Hence we have found the condition for an iteration to find an improved hull form while holding the volume constant. It is clear that no further improvement can be made when the condition is reached that $\Phi(h)$ = constant. But this is just the condition for minimum resistance which was found by use of the calculus of variations in the problem as solved by Karp, Kotik, and Lurye, namely equation (3.19) in different notation:

$$\int_L \mu(h) Y_o(K_o \left| h-h' \right|)\, dh = \lambda \qquad \ldots(3.53)$$

From this it is evident that the end result is equivalent in the two approaches.

In his paper Hogner proceeded as far as deriving the function $\mathfrak{J}$ (h) for several infinitely deep hull forms which could be described by elementary functions, but failed to carry the calculation further. In a later portion of this paper the calculation of an infinitely deep strut of reduced wave-making resistance will be carried through by this method, and will be shown to be roughly equivalent to the result of using the calculus of variations on the same problem. An extension of the method to more complicated problems, with hull forms varying in three dimensions, will also be outlined.

IV. A Method of Steep Descent for Developing Improved Hull Forms

It appears reasonable to conclude from the work done to date in finding hull forms of decreased wave-making resistance by mathematical methods that significant improvements can be made over conventional hull forms. However, all the applications so far have been limited in their scope. The classical calculus of variations has only been applied to the case of infinitely deep struts. Moreover it seems too difficult for shapes of hull which vary in three dimensions. The Ritz method has been applied only in finding the best of a very limited family of polynomials, but it does produce improvement. The method of Inui, which consists of canceling the ship waves with roughly similar waves of opposite phase produced by a submerged sphere, has considerable potential for intermediate speed ships which can handle the addition of a large sphere, but it does not seem suitable for very high speed ships like destroyers. Further, it has the appearance of "fixing" a poor hull design rather than calculating a good one. It is clear from the limitations of the work already done that there is room for a method which can work improvements in hull forms which are partly constrained in their shape (for example, by the shape of what they must carry) but which can still be varied in some respects so as to decrease their wave-making resistance.

1. REASONS FOR CHOICE OF A METHOD OF STEEP DESCENT, AND A DESCRIPTION OF THE METHOD

The choice of a method should not be determined by the fact that there is no proof that the form found by it will have an absolute minimum of resistance. The problem appears to be that any hull shape with a stationary value of wave-making resistance will only be optimum compared with other shapes of a restricted set. Other sets may contain shapes which cause even less resistance than the stationary value already found. Techniques for identifying such other sets are therefore most desirable. It follows that any method which not only finds improved forms in a given set but points to other sets which may be better is worth investigating. In this respect the so-called direct methods of the calculus of variations seem to have an advantage over the classical indirect methods. They are attractive for hull form calculations for several other reasons as well:

(1) The description of a ship's hull is limited in practice to the definition of its coordinates at a finite number of points. Since the direct methods of the calculus of variations deal in general with a finite number of variables initially (although they may allow the number ultimately to become infinite), there is a natural relationship between the normal description of a ship and the language of the direct methods.

(2) If it is not possible to permit the number of variables to become infinite in the limit, then there is still much to be gained by dealing with a finite number. With only a finite number the question of convergence is greatly simplified. If the result obtained with a finite number of variables is as accurate as the description of a ship can be in practice, then the practical limit of success has been reached anyway.

(3) The mathematics of the direct methods are relatively simple.

(4) The direct methods are susceptible to simple applications of constraints.

Although the direct methods used so far (by Weinblum and others) do not have all these advantages, there are other methods available by no means as limited as those already applied. Further, the mathematical difficulties involved in the classical calculus of variations, when applied to any but the simplest ship shapes, make some alternative essential for the more complicated problems of practical ship forms. The choice therefore should be made among the available direct methods.

The direct methods are not in fact so far removed in concept from the classical indirect ones. Richard Courant has pointed out [18] that there is a close relationship between the Euler differential expression of the classical calculus of variations and the gradient of a function in a finite-dimensional vector space. In fact, Euler's differential expression may be considered the gradient of a functional in function space. In view of this, a reasonable approach to the ship resistance problem is to write the resistance as a function of a finite number of variables, which we will consider as the elements of a vector in a finite-dimensional vector space, and then examine the behavior of the gradient of the resistance in that vector space. If we start with the variables so valued as to represent a ship and vary them along a trajectory opposite to the gradient of the resistance (or as closely to that direction as the applied constraints permit), then we should produce an improvement in the hull form if one can be produced by continuous variation of its defining variables. If, in addition, we follow the trajectory far enough, we may approach a stationary value of the resistance functional.

A Theorem on the Method of Steep Descent

This discussion leads us to the employment of a method of steep descent for the calculation of improved hull forms. In order to justify this for the particular application, we must prove the following theorem:

Theorem: If C is a non-negative continuous functional of the elements of a finite-dimensional vector space W, and the elements $m = (m_1, m_2, \ldots, m_n)$ of the space are differentiable functions of a parameter t, then if $\dot{m}$ is unrestricted or if $\dot{m}$ is only restricted in that certain elements $m_h, \ldots, m_k$ are not functions of t or that $\dot{m}$ must be orthogonal to some given vector a, then the vector m(t) will, within the constraints imposed, have some direction which it can follow which will provide the most rapid decrease or least rapid increase in the functional C. This $\dot{m}$ will be a function of m. If the direction provides a decrease and the vector m follows the trajectory $\dot{m}(t)$, then C will reach or will approach asymptotically a stationary value, and this stationary value will be a relative minimum value.

Lemma 1: Let W be a finite-dimensional vector space, and let C(W) be a finite non-negative continuous real functional of the elements of W. Then if we write grad C as the gradient of the functional C with respect to the elements of W, and if all the elements $m_1, m_2, \ldots, m_n$ of the vector space W are differentiable functions of a parameter t, then $C(m_1, m_2, \ldots, m_n)$ goes over into a function of t, and we can write

$$\dot{C}(t) = \sum^{n} \dot{m}_i C_{m_i} = \left(\frac{dm}{dt}, \text{ grad } C\right), \text{ where } \frac{dm}{dt} = (\dot{m}_1, \dot{m}_2, \ldots, \dot{m}_n) \qquad \ldots(4.1)$$

Now if we choose $\frac{dm}{dt}$ such that its absolute value is fixed and that its direction is subject to certain constraints, but within these constraints is such as to give $\dot{C}(t)$ as small a real value as possible, then as t increases C(t) will ultimately reach or asymptotically approach a stationary value.

$$C(t) = \int_0^t \left(\frac{dm}{dt}, \text{ grad } C\right) dt$$

Proof: If the smallest value of $\dot{C}(t)$ which we can obtain subject to the constraints of the problem is positive, then C(t) is a (minimum) stationary value and the condition of the lemma is satisfied. If, on the other hand, $\dot{C}(t)$ is negative, then C(t) will decrease as t increases until $\dot{C}(t)$ has increased to zero or until C(t) has decreased to zero, whichever occurs first; and in either case, since both C and $\dot{C}$ are continuous functions of t, a stationary (minimum) value of C will have been reached. If C = 0, this is a stationary value because of the continuity of C together with the fact that C is non-negative. If, instead, $\dot{C}$ asymptotically approaches zero, then C will approach a stationary value as closely as we please; and although it will not reach a true stationary value, it will be such for all engineering purposes. Notice that this proof is independent of whether the functional C(W) is a bilinear functional of the elements of W. It is enough that it be non-negative. Of course the functional C(W) is, to first order, a quadratic functional of a vector $\beta \in W$; but although this is a necessary condition if, for example, a method such as Weinblum's method is used, it is <u>not</u> necessary here. It is evident from the physical conditions of the problem that C(W) cannot become negative: this would be equivalent to a negative wave-making resistance, and in turn would imply the addition of energy to our ship hull from a previously undisturbed ocean. It is also clear from the equations themselves. Hence a stationary value of the resistance can be reached by allowing C to vary as a function of increasing t, and our lemma is proved.

Lemma 2: If certain of the $(m_1, m_2, \ldots, m_n)$, say $m_h, \ldots m_k$ are not functions of t, then Lemma 1 holds except that the definitions of $\dot{C}$ and $\frac{dm}{dt}$ are altered to eliminate terms with these subscripts.

Proof: This follows from the fact that $\dot{m}_h, \ldots, \dot{m}_k = 0$.

The interpretation of this Lemma is that if, for example, it is desired to hold constant, to first order, certain dimensions of a ship (that is, to hold constant the values of certain sources and sinks) then this can be done and the trajectory will provide a stationary value of C(t) subject to these constraints. It also follows that if we write m in terms of a basis in which $m_h, \ldots, m_k$ are orthogonal to the remainder, then we may find $\dot{C}(t)$ in terms of the derivatives of the remaining basis vectors and so find a trajectory which is orthogonal to the vector components $m_h, \ldots, m_k$.

Lemma 3: The direction of $\dot{m}(t)$ which for a given magnitude of $\dot{m}(t)$ will produce the most rapid change in $C(t)$ is that $\dot{m}(t)$ which is parallel to grad C.

Proof: $\dot{C}(t) = (\dot{m}(t), \text{grad } C)$. If we set $\dot{m}(t) = \frac{\dot{m}(t)}{|\dot{m}(t)|} |\dot{m}(t)|$, then

$$\|\dot{C}(t)\| = \|\dot{m}(t)\| \, \|\text{grad } C\| \, \frac{(\dot{m}(t), \text{grad } C)}{\|\dot{m}(t)\| \, \|\text{grad } C\|} \leq \|\dot{m}(t)\| \, \|\text{grad } C\| , \qquad \ldots(4.2)$$

the equal sign holding only if $\dot{m}(t)$ is parallel or anti-parallel to grad C.

Lemma 4: The direction of $\dot{m}(t)$ which for a given magnitude of $\dot{m}(t)$ will produce the most rapid change in $C(t)$ if $\dot{m}(t)$ is constrained to be orthogonal to some vector, say grad V, is that direction which lies in the plane (two-dimensional vector space) which is spanned by grad C and grad V, and is orthogonal to grad V.

Proof: By the Gram-Schmidt orthogonalization process we may construct an orthogonal basis for the two-dimensional subspace spanned by grad V and grad C. We take grad V as the first of our basis vectors. Then we obtain a second basis vector orthogonal to grad V by writing

$$\alpha = \text{grad } C - \frac{(\text{grad } C, \text{grad } V)}{(\text{grad } V, \text{grad } V)} \text{grad } V \qquad \ldots(4.3)$$

We can show that α is orthogonal to grad V simply by writing out (α, grad V). In addition we can show that α is not orthogonal to grad C by writing

$$(\alpha, \text{grad } C) = (\text{grad } C, \text{grad } C) - \frac{(\text{grad } C, \text{grad } V)^2}{(\text{grad } V, \text{grad } V)} \qquad \ldots(4.4)$$

If grad C $\neq 0$, which we showed in lemma 3 must be the case if we have not already found a stationary value of C, then

$$\frac{(\alpha, \text{grad } C)}{(\text{grad } C, \text{grad } C)} = 1 - \frac{(\text{grad } C, \text{grad } V)^2}{(\text{grad } C, \text{grad } C)\ (\text{grad } V, \text{grad } V)} \qquad \ldots(4.5)$$

But the second term of this expression is always non-negative and less than unity unless grad C is parallel to grad V, in which case α will be orthogonal both to grad V (which it is by definition) and to grad C. But this means that unless grad C and grad V coincide in direction, α will not be orthogonal to grad C. Now, since α is both in the two-dimensional subspace spanned by grad C and grad V, and is also orthogonal to grad V, it must be parallel to that vector $\dot{m}$ which is orthogonal to grad V and which we have stated will, of all vectors of that magnitude orthogonal to grad V, produce the largest value of $|\dot{C}|$.

That this is the case can be shown as follows:

Let

$$\text{grad } C = a(\text{grad } V) + b\alpha$$

but since $\dot{m}$ is parallel to α, we can write, with $\alpha = c\dot{m}$,

$$\text{grad } C = a(\text{grad } V) + b(c\dot{m}).$$

But

$$\dot{C} = (\text{grad } C, \dot{m}) = a(\text{grad } V, \dot{m}) + bc(\dot{m},\dot{m}) = bc(\dot{m},\dot{m}) .$$

Now let us assume that there is some other vector $\dot{m}'$ of the same magnitude as $\dot{m}$ so that $(\dot{m}',\dot{m}') = (\dot{m},\dot{m})$ but such that $|(\text{grad } C, \dot{m}')| > |(\text{grad } C,\dot{m})|$.

Then

$$\dot{m}' = b'\alpha + \beta,$$

where $(\text{grad } V,\beta) = 0$ and $(\beta,\alpha) = 0$

so that $(\beta,\dot{m}) = 0$. It follows that if $\dot{C}'$ is the value of $\dot{C}$ corresponding to this vector $\dot{m}'$, then

$$\dot{C}' = (\text{grad } C, \dot{m}') = (a \text{ grad } V + bc\dot{m}, b'c\dot{m} + \beta) = bb'c^2(\dot{m},\dot{m}).$$

But

$$(\dot{m}',\dot{m}') = b'^2c^2(\dot{m},\dot{m}) + (\beta, \beta) = (\dot{m},\dot{m}) \text{ by hypothesis; so}$$

$$b'c = \sqrt{1 - (\beta,\beta)/(\dot{m},\dot{m})} < 1. \text{ Then since}$$

$$\dot{C}' = (b'c)bc(\dot{m},\dot{m}) = (b'c)\,\dot{C} ,$$

it follows that

$$|\dot{C}'| < |\dot{C}| .$$

But this is contrary to hypothesis, so the Lemma is proved.

Proof of theorem: By Lemma 1 we showed that so long as C(W) is a finite, non-negative continuous real functional of the elements of W, there will be a stationary relative minimum value of C(W). We also showed that this value can be found or approached asymptotically by allowing each of the elements of W to become a function of some parameter t, and then varying the elements as a function of t so that C decreases as rapidly as possible for a given absolute value of $\frac{dm}{dt}$. By Lemma 2 we showed that if certain elements of W were held constant, the theorem still held subject to this condition. By Lemma 3 we showed that the direction in which the function $\frac{dm}{dt}$ should be varied to produce the required most rapid possible decrease in C subject to the restriction of Lemma 2 could be found. By Lemma 4 we showed that even if the restriction were imposed that the trajectory of m(t) be such that some other functional of the elements of the vector space be held constant, a direction of $\frac{dm}{dt}$ could be found so that for a given absolute value $\left|\frac{dm}{dt}\right|$ the most rapid possible decrease would result in C(W). But since this is so, our Theorem is proved.

It is possible to derive the direction which $\frac{dm}{dt}$ must take in terms of the given basis vectors if two constraints of the sort described in Lemma 4 are applied. However, it results in complicated algebra. In consequence it is simpler to rotate coordinates so that the basis for the vector space includes the gradient of each of the functionals to be held constant. Then the rest of the basis vectors are made orthogonal to each of these gradients, and the gradient vectors of the functionals to be held constant are disregarded in the calculation of the gradient of C(W). This follows because in the expression $\dot{C} = (\text{grad } C, \dot{m})$ there is no contribution from any of the elements of grad C which are parallel to the basis vectors grad V_1, grad V_2, ..., where the grad V_i are the gradients of the functionals V_i which are to be held constant. Observe that if the V_i are linear functionals of the m_i then their gradients will be independent of the value of m and so constant in direction throughout the vector space W.

2. PRINCIPLES OF APPLICATION OF THE METHOD OF STEEP DESCENT TO THE SHIP WAVE PROBLEM

We may now identify the variables of the theorem of the preceding section with the variables of the problem of reducing wave-making resistance.

The variable C may be identified either with the drag coefficient C_w or the wave-making resistance R of a ship, since either is a non-negative functional of the elements of the finite-dimensional vector space in which the hull is defined. We will choose to identify C with R, the wave-making resistance. The elements of W we may identify with the sources which are used to generate a set of closed streamlines which outline the hull. The ship itself then is described by a vector in the vector space W. Since any n orthonormal linear combinations of the sources also comprise an orthonormal basis for our vector space W, the elements of such a basis may also be identified with the m_i of the preceding section. It will be shown later that linearly independent combinations of the sources(with certain restrictions) may also be used instead of orthonormal combinations, since they too comprise a basis for the vector space W. One such linearly independent basis is just the summation

$$\eta_j = \sum_{i=1}^{j} m_i \qquad \ldots(4.6)$$

for the case of a set of sources and sinks evenly spaced along the axis of the hull. Notice here that the m_i in equation (4.6) are just the sources themselves, and the η_j are to be identified with the m_i of the preceding section. The η_j may be interpreted as the first-order approximation to the offsets of the hull. The vector defining the ship may then be written as $\eta = (\eta_1, \eta_2, \ldots, \eta_n)$. As Inui has shown, this is a poor approximation to the hull form, but it is just the approximation which is obtained by a literal interpretation of Michell's classic paper [1].

Another combination is a linear combination of linear combinations, and this can be written

$$\nu_j = \sum_{k=1}^{n} a_k \sum_{i=1}^{j} m_{ik} \quad . \qquad \ldots(4.7)$$

Equation (4.7) can be interpreted as describing (still to first order) the shape of the ship's hull as the sum of a set of n other first order approximations to ship's hulls. Each of these other hulls is taken as a basis vector for our vector space W, and the hull described by the ν_j is then a vector described in terms of these other ships as basis vectors. The method used by Weinblum can also be interpreted in the same manner, except that in his work he used the terms of a polynomial as the basis vectors for the description of his hull.

The fact that the wave-making resistance R is non-negative follows from the physical considerations of the problem. For the case of an ideal fluid it is also obvious from one form of the equation for the wave-making resistance of a continuous distribution of x-directed dipoles in a half-strip.

$$R = 16\pi\rho K_o^2 \int_0^{\infty} (I^2 + J^2) \cosh^2 u \, du \qquad \ldots(2.25)$$

Here the functions I and J are real so the integrand and the integral are clearly non-negative.

The functionals V_i may be identified with any of the quantities which must be constrained to be constant in order to make the statement of the wave-resistance problem meaningful. For example, it is necessary that the sum of the source strengths be zero, and we may identify V_1 with the sum of the source strengths. A possible constraint is that the volume of the hull be constant, and we may identify the functional V_2 with the volume. (In practice it will probably be simpler to identify it instead with the sum of the first moments of the source strengths in the direction of ship motion, which is a first approximation to the hull volume. This sum is linear in the source strengths and so has a gradient which is constant everywhere.)

In addition to the constraints discussed above, there is one other set which is inherent in the way the problem is set up. This is the location of the sources. They are located as part of the initial information in the definition of the problem, and the number of linearly independent vectors in our finite-dimensional vector space W must always be less than the number of source locations. This follows from the fact that the sum of the source strengths must be zero if the sources are to generate a closed hull form. There is, however, no limitation on where we place the sources. For example, if we wish to find the effect of extending a hull beyond its original length we may postulate a source of zero strength at some point forward of the bow. Then we may calculate the effect on the ship resistance of increasing the strength of this source by an infinitesimal amount. If this results in a decrease in the resistance, we may conclude that a bow extending out to this point may be of some value. It is this ability to test the effect of sources placed arbitrarily which permits the method of steep descent to be used to explore changes to the hull beyond the original framework of its definition.

Approach to the Minimizing Problem

The outline of the method to be followed now becomes clear. First we choose a hull form as the starting point for the variation. Then we describe it in terms of a set of sources and sinks. Next, we add in further sources, but of zero strength, at all locations where we think it might be reasonable to have them in the final ship design. In our next step we select those sources, or functionals of source strength, which are to be held constant in intensity. Then we calculate the gradient of the wave-making resistance. Then we find that vector direction which, subject to the applied constraints, is most nearly parallel to the negative of the gradient. Following this direction will produce the most rapid decrease in the value of the wave-making resistance. Then we vary the sources and sinks in strength in this vector direction, recalculating the direction to be followed at such short intervals that the direction changes only a small amount between calculations. In this manner we follow a trajectory in the n-dimensional vector space in which the ship's hull is defined until we reach or make an asymptotic approach to a stationary value of the wave-making resistance.

This simple concept need not be confined to the minimization of wave-making resistance. There is no reason why it cannot be extended to minimizing the sum of the wave-making resistance together with the other components of resistance, subject only to the proviso that the total resistance be written as a non-negative functional of the source strengths. There is also no reason why we cannot use any formulation of the resistance, whether it provides precisely correct total resistance or not, so long as it furnishes a correct or nearly correct description of the direction of the gradient of the resistance in the n-dimensional space in which the hull shape is defined. This means that it may be feasible to use a partially theoretical, partially empirical formulation of the resistance -- and even change it as the vector defining the hull form follows its trajectory through its n-dimensional vector space -- so long as the direction of the gradient is correctly described.

Demonstration that the Resulting Ship Shape Will be Smooth

One requirement of any method for finding hull forms of decreased wave-making resistance is that it not introduce unwanted discontinuities into the hull surface. This is necessary to prevent increase in other components of the resistance than the wave-making resistance. The description of a ship in terms of the streamlines of the potential flow arising from a distribution of sources and sinks inside the closed streamlines insures that the ship shape will be fair and continuous. This follows from the fact that the potential arising from each source or sink is a solution of Laplace's equation, and the potential corresponding to the entire array of sources and sinks which describes the flow around the ship is simply the sum of the potentials of the individual sources and sinks. But since we have restricted the distribution of sources to within the streamline which defines the ship's hull surface and the medium outside the hull has a constant density, the solutions of Laplace's equation outside the hull are everywhere continuous. From this it follows that the sum of the potentials arising from a finite number of sources and sinks within the hull boundary is also continuous, and since Laplace's equation is linear, the sum is also a solution of Laplace's equation. But if this is so, then each of its derivatives is also a solution of Laplace's equation and is also continuous, since, for example,

$$\nabla^2 \left[\frac{\partial \phi}{\partial x}\right] = \frac{\partial}{\partial x}\left[\nabla^2 \phi\right] = 0$$

and so on for all higher derivatives. Since the direction of any streamline is defined by

$$\nabla \phi = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$$

where $\vec{i}$, $\vec{j}$, and $\vec{k}$ are unit vectors in the directions of the x,y,z axes respectively, and since $\frac{\partial \phi}{\partial x}$, $\frac{\partial \phi}{\partial y}$, and $\frac{\partial \phi}{\partial z}$ are continuous, it is clear that there can be no sudden changes in direction of streamlines which result from the flow from a finite number of sources and sinks located away from the streamlines.

With this assurance that the ship forms defined by a finite number of sources and sinks distributed inside the streamline which bounds the hull will be continuous and fair, we may proceed with confidence to use the distribution of a finite number of sources and sinks to define our ship forms. They will at least be reasonable forms, and not the polygonal approximations which would result from applying the same procedure to the offsets of the hull.

Change of Basis

It was pointed out earlier that if the quantities to be held constant are linear functionals of the elements of W, it may be desirable to change the basis of the vector space W so that it includes among its elements the gradients of these quantities. Then all the other basis elements are made orthogonal to these gradients, and the trajectory through W of the vector describing the hull is calculated disregarding the basis elements which are parallel to gradients of the quantities to be held constant. That is, any linear combination whatever of the new basis elements other than those parallel to these gradients may be added to a given vector without changing the quantities to be held constant. This will continue to work along the entire trajectory if the directions of the gradients of the functionals to be held constant are independent of position in the vector space. That independence is assured by the requirement that these functionals be linear.

The process of changing the basis may sound like a complicated one. In fact it is not, since the only use to be made of the new basis vectors is to find what relation must exist between the quantities added to the several sources and sinks which define the hull. All calculations involving the resistance equations themselves are made in terms of the original set of sources and sinks, changed in magnitude but not in position. This means that the calculations of the resistance integrals need be done only once, and may even be entered into the problem as part of the initial information.

Since the natural description of the hull is in terms of the intensities of sources at predetermined locations, the change in basis will in general make the new basis elements linear combinations of the natural ones. We may designate the old basis elements as $\alpha_1,\ldots,\alpha_n$, each corresponding to unit values of sources with strength $m_1,\ldots,m_n$. Then we may write the vector which defines the hull in terms of this basis as $m = m_1\alpha_1 + \ldots + m_n\alpha_n$. We may write the new basis elements as $\beta_1,\ldots,\beta_n$. Then each one must be multiplied by an appropriate scalar $\eta_1,\ldots,\eta_n$, so that the vector which defines the hull will become $\eta_1\beta_1 + \ldots + \eta_n\beta_n$. Then it follows that the new basis is related to the old by equations of the form

$$\beta_i = \sum_k c_{ik}\alpha_k \qquad \ldots(4.8)$$

Since in the original basis it was possible to write the gradient of the functional C as

$$\operatorname{grad} C\Big|_\alpha = \frac{\partial C}{\partial m_1}\alpha_1 + \frac{\partial C}{\partial m_2}\alpha_2 + \ldots + \frac{\partial C}{\partial m_n}\alpha_n$$

we may define a gradient in the new basis as

$$\text{grad } C\Big|_\beta = \frac{\partial C}{\partial \eta_1}\beta_1 + \frac{\partial C}{\partial \eta_2}\beta_2 + \cdots + \frac{\partial C}{\partial \eta_n}\beta_n .$$

If the basis vectors $\beta_1,\ldots,\beta_n$ are orthonormal like the basis vectors $\alpha_1,\ldots,\alpha_n$, then it follows from the results of ordinary vector calculus that the direction of the gradient vector is constant regardless of what orthonormal basis is used to write it. On the other hand, if the basis vectors $\beta_1,\ldots,\beta_n$ are not orthonormal it is not necessarily the case that the gradient vector $\text{grad } C\big|_\beta$ is in the same direction as $\text{grad } C\big|_\alpha$. In particular, if the basis vectors $\beta_1,\ldots,\beta_n$ are not orthogonal but only linearly independent, then it is easy to find examples where $\text{grad } C\big|_\beta$ is not parallel to $\text{grad } C\big|_\alpha$. Nevertheless, for a given magnitude of $\sqrt{\Delta\eta_1^{\,2} + \Delta\eta_2^{\,2} + \ldots + \Delta\eta_n^{\,2}}$ a vector motion parallel to $\text{grad } C\big|_\beta$ will provide the largest change in C.

To find a relation between the gradients in the two bases we can examine the effect of making a small change $d\eta_i$ in the coefficient of β_i.

$$\beta_i d\eta_i = (\sum_k c_{ik}\alpha_k)\, d\eta_i = \sum_k \alpha_k (c_{ik} d\eta_i).$$

This shows that a small change $d\eta_i$ in the coefficient of β_i corresponds to a change $c_{ik} d\eta_i$ in the coefficient of each of the α_k. But then we can write

$$dC = \sum_k \frac{\partial C}{\partial m_k}(c_{ik} d\eta_i) = (\sum_k \frac{\partial C}{\partial m_k} c_{ik})\, d\eta_i ,$$

where dC is the small change in the functional C resulting from the small change $d\eta_i$ in the coefficient of α_i. But by definition,

$$dC = \frac{\partial C}{\partial \eta_i}\, d\eta_i ,$$

and by equating coefficients of $d\eta_i$ in this and the preceding equation we have

$$\frac{\partial C}{\partial \eta_i} = \sum_k c_{ik} \frac{\partial C}{\partial m_k} . \qquad \ldots(4.9)$$

We make use of the new basis in the following fashion: We write the gradient grad $C\Big|_\beta$ by use of equation (4.9) and then write the motion vector

$$\dot{\eta} = \dot{\eta}_1\beta_1 + \dot{\eta}_2\beta_2 + \ldots + \dot{\eta}_{n-j}\beta_{n-j} + 0\,\beta_{n-j+1} + \ldots + 0\,\beta_n$$

where the last j basis vectors are given zero coefficients because they are parallel with the gradients of the functionals to be held constant. For the non-zero coefficients we substitute the corresponding coefficients of grad $C\Big|_\beta$ and then we are able to write

$$\Delta\eta = \left(\frac{\partial C}{\partial \eta_1}\Delta t\right)\beta_1 + \left(\frac{\partial C}{\partial \eta_2}\Delta t\right)\beta_2 + \ldots + \left(\frac{\partial C}{\partial \eta_{n-j}}\Delta t\right)\beta_{n-j}. \qquad \ldots(4.10)$$

The Δt is a negative real number chosen to make the step size in the steep descent correct. It must be negative, of course, to provide a decrease in C. We now have a vector $\Delta\eta$ in a subspace $V \subset W$ which is orthogonal to the gradients of the functionals which are to be held constant. This vector $\Delta\eta$ is parallel to that portion of the gradient grad $C\Big|_\beta$ which is included in V. But for a given magnitude of $\Delta\eta$ this is the direction which will produce the greatest change in C, since

$$\Delta C = \left(\text{grad } C\Big|_\beta, \Delta\eta\right)$$

for $\Delta\eta$ small enough.

Hence we have found the direction of steepest descent within the given constraints.

We now use equation (4.9) to apply the step change $\Delta\eta$ to the source strengths $m_1, \ldots, m_n$. We write

$$\left(\frac{\partial C}{\partial \eta_i} \Delta t\right)\beta_i = \left(\frac{\partial C}{\partial \eta_i} \Delta t\right)\sum_k c_{ik}\alpha_k = \sum_k \left(\frac{\partial C}{\partial \eta_i} \Delta t\, c_{ik}\right)\alpha_k$$

Then we sum the coefficients for all i of each α_k and add the result to the coefficient of α_k in the expression $m = m_1\alpha_1 + \ldots + m_k\alpha_k \ldots + m_n e_n$.

$$m_k^{(1)} = m_k^{(0)} + \sum_{i=1}^{n-j} \left(\frac{\partial C}{\partial \eta_i}^{(0)} \Delta t\, c_{ik}\right) . \qquad \ldots(4.11)$$

This iteration continues with $1 \rightarrow 2$, $2 \rightarrow 3, \ldots$ until a minimum is reached or the calculation is terminated for some other reason.

It is easy to show that the direction of grad $C\big|_\beta$ is different from that of grad $C\big|_\alpha$ for a particular case. Suppose that all the basis vectors $\beta_1, \beta_2, \ldots, \beta_n$ are identical with $\alpha_1, \alpha_2, \ldots, \alpha_n$ except that $\beta_{k+1} = \alpha_k + e\alpha_{k+1}$, $0 < e < 1$. Then suppose $\frac{\partial C}{\partial m_k} = \frac{\partial C}{\partial m_{k+1}} = 1$, all other $\frac{\partial C}{\partial m_j} = 0$. Then from equation (4.9) we have $\frac{\partial C}{\partial \eta_k} = 1$, $\frac{\partial C}{\partial \eta_{k+1}} = 1 + e$. But from equation (4.8) the component of grad $C\big|_\beta$ in the direction of α_{k+1} will be $(1+e)e = e + e^2$, which will be much less than 1 for e small enough. But for any $e \neq 0$ the basis $\beta_1, \ldots, \beta_n$ spans the vector space and is admissible, so it is clear that we may have grad $C\big|_\beta$ not parallel to grad $C\big|_\alpha$.

It is clear that if we have a basis β such as the one just described and make e small enough, the path of steep descent calculated by the method described above will be essentially orthogonal to the direction α_{k+1} until a stationary value of C with respect to all other allowable directions in the vector space is approached. Then the path will acquire a significant component in the direction α_{k+1} and proceed more directly toward a true stationary value with respect to all motions. It is therefore evident that the path of steep descent is not independent of the basis vectors which we use to describe the hull. Nonetheless, it does not appear that the ultimate destination of the path is necessarily affected. On the other hand, since there is no proof available that the solution to be obtained is unique except in the case where the resistance functional is bilinear, it may well be that two different paths will result in two separate stationary values, just as a traveler going down a hill may end up in any of several separate valleys in the bottom.

It is possible to select a set of linearly independent basis vectors for the description of the hull in many different ways. It should be observed first that if we use n sources to describe a hull there are really only n-1 degrees of freedom in the choice of their values, since their sum must be zero. If, further, we wish to hold constant the hull volume to first order, then we must hold constant the first moment of the source strength in the x-direction. This provides another constraint, leaving only n-2 degrees of freedom. If we choose n-2 basis vectors which are orthogonal both to the gradient of the total source strength and to the gradient of the first moment of the sources in the x-direction, and linearly independent of each other, we will have a basis for a subspace V included in W in which every vector is orthogonal to these gradients. It will, moreover, contain every such vector. It follows that any trajectory through this subspace will leave the volume of the hull unchanged, at least to first order. We may find the best such trajectory of C(W) through the subspace V by taking the gradient of C with respect to the basis elements of the subspace V and following a trajectory parallel to this gradient.

It may sound as if it is a difficult problem to specify a basis which is orthogonal to the gradient of the sum of the source strengths and to the gradient of the first moment of the source strengths in the x-direction. In practice it turns out to be quite simple. A few examples will make this clear.

Suppose we have a hull which we have described by eight sources, equally spaced along its centerline plane. We may describe their strength by multiples of eight unit sources $(\alpha_1,\ldots,\alpha_8)$ which comprise a basis for the description of any ship which can be described in terms of sources placed at these locations. We may have, for example,

$$2\alpha_1 \quad 4\alpha_2 \quad 2\alpha_3 \quad \alpha_4 \quad -\alpha_5 \quad -2\alpha_6 \quad -4\alpha_7 \quad -2\alpha_8 \; .$$

Since the sum of the source strengths is zero, this set generates a closed set of streamlines which could be a ship. The anti-symmetrical form makes this generate a set of streamlines which are symmetrical fore-and-aft.

We may provide a basis $(\beta_1,\ldots,\beta_6)$, such that any vector described in terms of its elements can be added to our given hull without changing either the sum of its elements (which is zero) or its first moment, which is roughly proportional to the volume of the hull. The basis might be the following:

$$\beta_1 = \alpha_1/2 - \alpha_2 + \alpha_3/2 + 0\alpha_4 + 0\alpha_5 + 0\alpha_6 + 0\alpha_7 + 0\alpha_8$$

$$\beta_2 = 0\alpha_1 + \alpha_2/2 - \alpha_3 + \alpha_4/2 + 0\alpha_5 + 0\alpha_6 + 0\alpha_7 + 0\alpha_8$$

$$\beta_3 = 0\alpha_1 + 0\alpha_2 + \alpha_3/2 - \alpha_4 + \alpha_5/2 + 0\alpha_6 + 0\alpha_7 + 0\alpha_8$$

$$\beta_4 = 0\alpha_1 + 0\alpha_2 + 0\alpha_3 + \alpha_4/2 - \alpha_5 + \alpha_6/2 + 0\alpha_7 + 0\alpha_8$$

$$\beta_5 = 0\alpha_1 + 0\alpha_2 + 0\alpha_3 + 0\alpha_4 + \alpha_5/2 - \alpha_6 + \alpha_7/2 + 0\alpha_8$$

$$\beta_6 = 0\alpha_1 + 0\alpha_2 + 0\alpha_3 + 0\alpha_4 + 0\alpha_5 + \alpha_6/2 - \alpha_7 + \alpha_8/2$$

The six vectors $\beta_1,\ldots,\beta_6$ are linearly independent of each other. When any multiple of any of them is added to the set of vectors which describe the ship, neither the sum of the source strengths nor their first moment is changed.

Hence these six vectors meet the requirements we have set.

We may also define vectors which hold constant other features: for example, we may require that an appendage which can only be produced by a set of three sources in a particular magnitude relation to one another retain its shape although not its size. To do this we may make one of our basis vectors, or one term of a basis vector, the sum of the three sources each multiplied by its appropriate magnitude.

We can generalize our basis vectors which maintain the volume of a ship constant (to first order) to the case where the ship is described by a number of rows of sources distributed over the centerline plane of the ship. A single basis vector might be described by the following array of numbers (the multiplied basis vectors in terms of which the hull is described are omitted).

$$\beta_1 = \begin{matrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . \end{matrix}$$

We provide n-2 vectors of this sort. To do this easily we may hold the pair of numbers in the upper left-hand corner constant and associate with it in turn n-2 pairs like the one shown in the second row, each with its axis parallel to the direction of motion of the ship and with the order of signs reversed from the order of the reference pair in the upper left-hand corner. Observe that some of the additional pairs will have one member in each of two rows. Also we must include one vector which looks like this (we will call it β_2):

$$\beta_2 = \begin{matrix} 1 & \begin{matrix}-1\\-1\end{matrix} & 1 & 0 & 0 & 0 & 0 & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ . & . & . & . & . & . & . & . & . & . \end{matrix} = \begin{matrix} 1 & -2 & 1 & 0 & 0 & 0 & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ . & . & . & . & . & . & . & . & . \end{matrix}$$

If we wish to hold the sectional area curve constant we may use a set of basis vectors like these:

$$\beta_1 = \begin{array}{ccccccc} 1 & -1 & 0 & 0 & 0 & 0 & . \\ -1 & 1 & 0 & 0 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & 0 & 0 & . \\ . & . & . & . & . & . & . \end{array} \qquad \beta_2 = \begin{array}{ccccccc} 1 & -1 & 0 & 0 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & 0 & 0 & . \\ -1 & 1 & 0 & 0 & 0 & 0 & . \\ . & . & . & . & . & . & . \end{array}$$

$$\beta_3 = \begin{array}{ccccccc} 0 & 1 & -1 & 0 & 0 & 0 & . \\ 0 & -1 & 1 & 0 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & 0 & 0 & . \\ . & . & . & . & . & . & . \end{array} \qquad \beta_4 = \begin{array}{ccccccc} 0 & 1 & -1 & 0 & 0 & 0 & . \\ 0 & 0 & 1 & 0 & 0 & 0 & . \\ 0 & -1 & 0 & 0 & 0 & 0 & . \\ . & . & . & . & . & . & . \end{array}$$

Observe that if there are m columns in the last array that there will be a total of n-m-1 linearly independent basis vectors. Hence if the sectional area curve is kept constant there will be fewer degrees of freedom than if only the hull volume is held constant. It is, of course, possible to provide sets of vectors which hold the volume constant on each waterline or which observe other constraints.

Iteration Sequence

Once we have described the ship's hull in terms of sources and sinks we may now calculate the resistance by one of the formulae developed in section II. Then we determine what is to be held constant and develop a set of basis vectors for the description of changes in the ship's hull with this in mind, as described in the preceding paragraphs. Then, assuming we have decided to identify the wave-making resistance with the variable C, we calculate the partial derivatives $\frac{\partial R}{\partial m_k}$. Then we calculate the partial derivatives of R with respect to the magnitude η_i of the new vectors $\eta_i \beta_i$ (except those identified with gradients of vectors to be held constant) using equation (4.9), substituting R for C. If there are 2 constraints we will have n-2 such partial derivatives. Finally we can write

$$\Delta \eta^{(j)} = \Delta t \sum_i \beta_i \frac{\partial R^{(j)}}{\partial \eta_i} = \Delta t \sum_i \sum_k c_{ik} \alpha_k \frac{\partial R^{(j)}}{\partial m_k}, \quad \Delta t < 0. \qquad \ldots(4.12)$$

$$\eta^{(j+1)} = \eta^{(j)} + \Delta \eta^{(j)} \qquad \ldots(4.13)$$

This expresses in vector form the same result as is given in equation (4.11). The simple iterative formula (4.13) serves to lead us to the desired form of relative minimum resistance by a path of steep descent down from $\eta^{(0)}$. It is necessary to insure in the iteration that the steps are kept short enough so that the minimum point is not overshot; or that the change in direction of the actual gradient over the length of the step is not so great that a step which makes for a decrease in resistance in its initial portion results finally in an increase. The simplest solution to such difficulties is to proceed until such are found, and then return to the preceding step and try again with the step length Δt cut in half. The magnitude of successive values of the double sum in (4.12) can be used as an indication of whether progress is being made toward a stationary value of R. If, for example, the direction of the vector $\Delta\eta^{(j)}$ changes but its magnitude does not ultimately decrease, then it is conceivable that the iteration is resulting in a circular path instead of following the path of steepest descent; but if this occurs, then the size of the steps may be cut in half in order to make the actual path followed stay closer to the path of steepest descent.

For the case where all derivatives higher than the second are zero it is possible to calculate a value of Δt which will provide the largest possible decrease in the functional C. Since the resistance functional is bilinear this will be the case in the wave-making resistance problem. Suppose that we can write ΔC in the form of a Taylor Series of only two terms:

$$\Delta C = \sum_{r=1}^{n} \frac{\partial C}{\partial \eta_r}\left(\Delta t \frac{\partial C}{\partial \eta_r}\right) + \frac{1}{2!} \sum_{s=1}^{n} \sum_{r=1}^{n} \frac{\partial^2 C}{\partial \eta_r \, \partial \eta_s}\left(\Delta t \frac{\partial C}{\partial \eta_r}\right)\left(\Delta t \frac{\partial C}{\partial \eta_s}\right).$$

Since $\Delta t < 0$, we may differentiate with respect to Δt and set the result equal to zero as follows:

$$\frac{d(\Delta C)}{d(\Delta t)} = \sum_{r=1}^{n}\left(\frac{\partial C}{\partial \eta_r}\right)^2 - \frac{2}{2!}\left|\Delta t\right| \sum_{s=1}^{n} \sum_{r=1}^{n} \frac{\partial^2 C}{\partial \eta_r \partial \eta_s} \frac{\partial C}{\partial \eta_r} \frac{\partial C}{\partial \eta_s} = 0$$

and so we have for the optimum size of the step Δt

$$\left|\Delta t\right|_{\text{optimum}} = \frac{\sum_{r=1}^{n}\left(\frac{\partial C}{\partial \eta_r}\right)^2}{\sum_{s=1}^{n} \sum_{r=1}^{n} \frac{\partial^2 C}{\partial \eta_r \partial \eta_s} \frac{\partial C}{\partial \eta_r} \frac{\partial C}{\partial \eta_s}}$$

A short discussion of the computational techniques involved can be found in a chapter by Charles B. Tompkins in "Modern Mathematics for the Engineer" [19].

It is also possible to use instead of the rotation of coordinates the constraint of equation (4.3) to provide a trajectory which is orthogonal to the gradient of a function which is to be held constant. This has the disadvantage that it is really simple only if there is a single constraint -- not a common situation.

It may be asked, "Why not just differentiate the resistance with respect to each of the source strengths in turn and set each to zero, then solve the set of resulting linear equations?" The answer is that it may be undesirable to permit the hull shape to change all the way to the point of minimum resistance. Since the gradient will decrease as the relative minimum of resistance is approached, a larger and larger change in hull shape will be required to obtain a given decrease in resistance as the iteration proceeds.

Presumably the iteration will start with a conventional hull form and show how it may be changed to decrease its resistance. There will probably be some hull form found whose wave-making resistance is less than that of the original and acceptable from other standpoints, and it may well be that iteration beyond that point will produce a form which is unacceptable. It is also true that the path of steep descent used is not required to be the path of steepest descent. That is, certain of the basis vectors may be varied less than required by equation (4.12), or not varied at all, so long as the result of the variation is a decrease in total resistance. This may well produce an acceptable form in cases where the strict method of steepest descent produces an unacceptable one.

3. THE RESULTS OF A SIMPLE APPLICATION OF THE METHOD OF STEEP DESCENT TO THE PROBLEM OF IMPROVING AN INFINITELY DEEP PRISMATIC HULL

In order to try out the method of steep descent, a sample calculation was made using a slide rule. This calculation started with a hull which could be described by a uniform distribution of dipoles with axes oriented in the direction of motion over a centerline plane of unit length and infinite depth. This is equivalent to a two-dimensional source near the forward end of a hull and a two-dimensional sink near the after end. A Froude number of 0.316 was assumed for the calculation. The calculation was constrained to hold the total dipole moment constant. After six iterations the shape of the dipole distribution approached that which Karp, Kotik, and Lurye [6] calculated for the same problem with Froude number 0.38 using the classical calculus of variations, and differed from it in the direction to be expected from extrapolation from their results for higher Froude numbers.

This problem has only one constraint, so the method of equation (4.3) can be used to handle it. That is, with the resistance R and the sum of the dipole moment V, a vector α parallel to the desired direction of change $\dot{m}(t)$ could be calculated by

$$\alpha = \text{grad } R - \frac{(\text{grad } R, \text{ grad } V)}{(\text{grad } V, \text{ grad } V)} \text{ grad } V \qquad \ldots(4.3a)$$

The equation used for the resistance was equation (3.17),

$$R = -8\pi^2 \rho K_o^2 \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \mu(\xi)\, \mu(\xi')\, Y_o(K_o|\xi - \xi'|)\, d\xi\, d\xi' \qquad \ldots(3.17)$$

A form symmetrical fore-and-aft was assumed at the start, and since this provided symmetrical changes the symmetry persisted through the calculation. This nearly halved the amount of calculation required over what an asymmetrical distribution would have required. The length of the dipole distribution was divided into ten intervals, and with one exception the dipole moment over the whole interval and the value of Y_o was assumed to be that for the argument at the mid-point of the interval. The exception was the case where this produced the result $Y_o(0) = -\infty$. Here it was necessary to use a more precise calculation, since the function $Y_o(x)$ has a singularity at $x = 0$ but the integral $\int_0^x Y_o(x)dx$ remains bounded for all $x > 0$. The expression for the integral is available in terms of other tabulated functions as follows:

$$\int_0^x Y_o(x)dx = x\, Y_o(x) + \frac{\pi}{2}\, x \left\{ Y_1(x)\, H_o(x) - Y_o(x)\, H_1(x) \right\} \qquad \ldots(4.14)$$

Here $Y_1(x)$ is a Bessel's function of the second kind, and $H_o(x)$ and $H_1(x)$ are Hankel functions.

The length of each interval of the distribution was taken as $\Delta\xi_j$ and the intervals were numbered from 1 to 10 with midpoints ξ_j, $j = 1,2,\ldots,10$. The dipole density at midpoint was written M_j. Then the approximation to the resistance was written as

$$R = -8\pi^2 \rho K_o^2 \sum_j \left[\sum_{i \neq j} 2M_i M_j Y_o(K_o|\xi_i - \xi_j|)\Delta\xi_i + M_j^2 \int_{-\Delta\xi_j/2}^{\Delta\xi_j/2} Y_o(K_o|x|)dx \right] \Delta\xi_j .$$

...(4.15)

Then by differentiation, and changing the limits on the integral,

$$\frac{\partial R}{\partial M_j} = -16\pi^2 \rho K_o^2 \left[\sum_{i \neq j} M_i Y_o(K_o|\xi_i - \xi_j|)\Delta\xi_i + 2M_j \int_0^{\Delta\xi_j/2} Y_o(K_o|x|)\, dx \right] \Delta\xi_j .$$

...(4.16)

It was then feasible to write

$$\text{grad } R = \left\{ \frac{\partial R}{\partial M_1}, \frac{\partial R}{\partial M_2}, \ldots, \frac{\partial R}{\partial M_n} \right\} \qquad \ldots(4.17)$$

Since α in equation (4.3a) is linear in grad R, and we are interested only in its direction, we may disregard the magnitude of the constants in equation (4.16) in calculating α. Further, since grad V enters both denominator and numerator to the same power in (4.3a) we may multiply grad V by any convenient multiplier. If we take $\Delta\xi_j$ = constant for all j, then we may write $(\text{grad } V)_j =$ = 1 for all j. Here $(\text{grad } V)_j$ is the j^{th} component of grad V. With this set of simplifications we can now write the equation for α.

Observe that $(\text{grad } V, \text{grad } V) = \sum_{j=1}^{n} (\text{grad } V)_j^2 = n$, and

$$(\text{grad } R, \text{grad } V) = \sum_{j=1}^{n} \frac{\partial R}{\partial M_j} \cdot 1 = \sum_{j=1}^{n} \frac{\partial R}{\partial M_j} .$$

It follows that

$$\alpha_j = \frac{\partial R}{\partial M_j} - \frac{1}{n} \sum_{j=1}^{n} \frac{\partial R}{\partial M_j} \qquad \ldots(4.18)$$

where

$$\alpha = \left\{ \alpha_1, \alpha_2, \ldots, \alpha_n \right\} \qquad \ldots(4.19)$$

and by disregarding the multiplicative constants in (4.16) we get

$$\alpha_j' = \frac{\partial R'}{\partial M_j} - \frac{1}{n} \sum_{j=1}^{n} \frac{\partial R'}{\partial M_j} \qquad \ldots(4.20)$$

where c_j' and R' are the results obtained by dropping the constants.

The procedure used is to calculate the $\frac{\partial R'}{\partial M_j}$ from equation (4.16), then the α_j' from (4.20), and then find a new value for the m_j by taking

$$M_j^{(1)} = M_j^{(0)} + \Delta t \; \alpha_j'^{(0)} \qquad \ldots(4.21)$$

Since the multiplicative constants were negative in (4.16), we require $\Delta t > 0$ for this iteration. It continues, of course, with $0 \rightarrow k$, $1 \rightarrow k+1$ in (4.21). The limitation on selecting the value of Δt was to make sure that it was not so big as to make any of the m_j become negative. Otherwise, the size was an estimate by the calculator that it might produce rapid convergence of the calculation. As a measure of the progress, it is possible to use the magnitude of $|\alpha'_\beta| = \sqrt{(\alpha', \alpha')}$. In the test calculation the values of $|\alpha'|$ went in the ratio .82: 3.74: 2.02: 1.04: .69: .41: .24. The initial increase was the result of the first step being too large. Later decreases resulted from using smaller steps -- from a little experience on the part of the calculator. The last value in the sequence is the value of $|\alpha'|$ computed from the values of $M_j^{(6)}$. (These are the final values of the calculation.) This means that the dipole density distribution reached in the particular calculation was not an actual stationary value, but was certainly a much better shape than the one with which the calculation was started. In order to see how it compares with others developed by the calculus of variations, it is plotted together with two distributions for higher values of the Froude number calculated by Karp, Kotik, and Lurye [6].

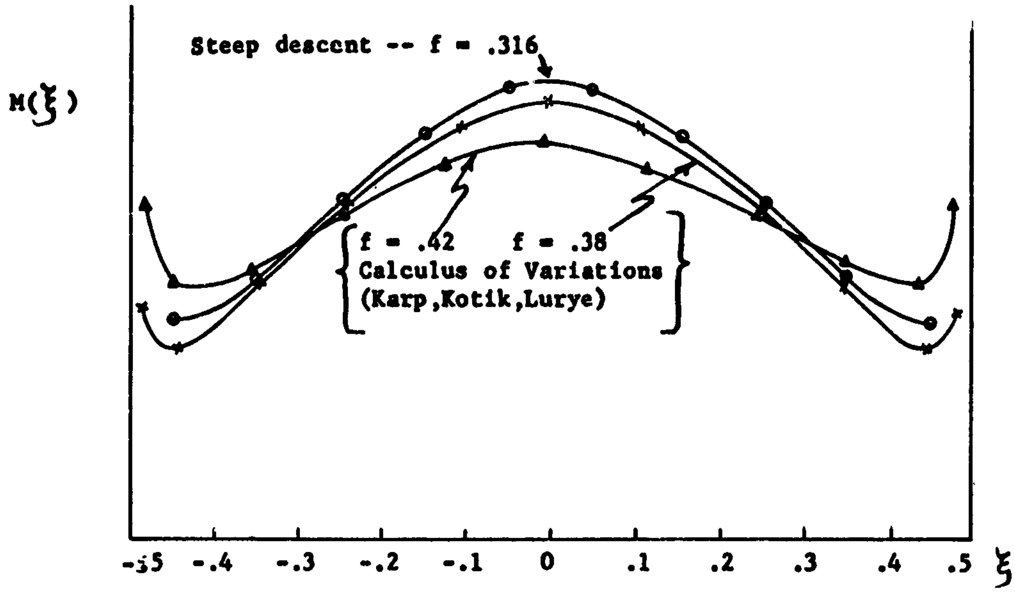


Figure IV-1. Dipole Density for a Strut.

4. USE OF THE SOURCE DISTRIBUTION CALCULATED BY THE METHOD OF HESS AND SMITH TO FIND IMPROVED HULL FORMS

It was pointed out in part II of this paper that one of the more difficult parts of the calculation of hull forms of reduced wave-making resistance is to find a source distribution equal to a given hull. The most useful such distribution is one in which the sources are placed on the centerline plane of the hull. However, there is no simple general method for calculating a centerline plane distribution corresponding to a given hull, although there are many special cases available. On the other hand, there is a general method available for describing an arbitrary hull form by sources placed on its surface, and this method will provide a basis (both figuratively and literally) for exploring in a general fashion the desirability of appendages to add to a given hull. Such appendages are not limited to spheres, but can be virtually any shape which can be described by sources and sinks.

The method of Hess and Smith [9], which is described in part II of this paper, provides a description of an arbitrary hull in terms of the strengths of sources and sinks distributed in a regular network over the surface of a hull. It is possible to describe the theoretical wave-making resistance of such a set of sources and sinks by equation (2.17). (If a model of the ship has been towed, it is probable that the theoretical resistance curves can be brought into precise conformance with the experimental curves by use of Inui's semi-empirical parameters.) With such a description it is possible to find the effect of additions to the hull, whether incorporated into the hull shape or installed as exterior appendages.

To explore the effect of appendages, all that need be done is to add to the set of sources describing the hull a set of sources of zero strength distributed in the locations in which it is desired to explore the effect of additions. A spherical appendage can be described by a source-sink pair, the source a very short distance forward of the sink. Other appendages may be described either by sets of simple, independent sources placed in the proper locations, or sets of sources linked by definite amplitude relationship, depending on the restrictions to be placed on the form of the appendage. In either case, the method of steep descent is then applied, permitting only the sources being investigated to vary. If the sources are linked by a definite amplitude relationship, then it will be necessary to apply equations (4.9) and (4.10) to find the particular basis vector for the change; otherwise they are handled without rotation of coordinates, which corresponds to $c_{ik} = 1$ for $k = i$, $c_{ik} = 0$ for $k \neq i$ in equation (4.9). In any case, we proceed to find the values of $\frac{\partial R}{\partial \eta_i}$ using (2.17), (4.9) and (4.10) and then use the method of steepest descent in which we vary only the sources and sinks which are added. It will, of course, be necessary to constrain these added sources and sinks to have a total source strength of zero in order to retain closure of the hull form. It will also be necessary to insure that the variation is such that at no point is a net negative change made in the first moment of the sources in the direction of motion. This would probably result in the envelope of the main hull moving inside the set of sources originally used to describe it, and leave them to describe an isolated appendage instead. Also, an isolated source-sink pair with the sink ahead of the source is a meaningless result, and it will be necessary to constrain the calculation to prevent it. With these restrictions, however, the procedure outlined permits us to use the source distributions provided by the method of Hess and Smith to explore the effect of additions to given hulls of any shape.

It will also be possible to calculate the effect of additions to the volume in a somewhat different fashion. This is by distributing source-sink pairs arranged as dipoles of zero strength along the centerline plane, and then using the method of steep descent, permitting them to increase but not to decrease in strength. This is equivalent to permitting increases in volume, but no decreases. Since no designer has been known to decrease the volume of his hull after first laying it out, this is probably an eminently practical technique.

5. USE OF A METHOD OF STEEP DESCENT TO FIT A SOURCE DISTRIBUTION TO A HULL FORM

As pointed out earlier, it is difficult to find a source distribution on a centerline plane which describes accurately a given hull form. However, it is certainly possible to do so by the method of steep descent, although it may not be an economical method of calculation. Suppose we have a hull that is described by a set of n "offsets", that is, by n values of the coordinate y where the hull is described by the equation $y = \pm y(x,z)$. Suppose further that we have a total of N sources placed on the centerline plane which are assigned strengths m_k, and we use these to provide an approximation to the shape of the hull. If we can write the y-coordinate of the closed streamline corresponding to this approximation as Y_s, and the corresponding value for the actual hull as y_s, we may define a function

$$S = \sum_{s=1}^{n} (Y_s - y_s)^2 \qquad \text{...(4.22)}$$

The function S is a measure of the deviation of the approximation from the actual hull. If we can find the relationship $\frac{\partial Y_s}{\partial m_k}$, $k = 1,\ldots,N$ for each of the Y_s, we can write

$$\frac{\partial S}{\partial m_k} = \sum 2(Y_s - y_s) \frac{\partial Y_s}{\partial m_k}, \qquad k=1,\ldots,N. \qquad \text{...(4.23)}$$

We may now follow the method of steep descent to find the set of values of the m_k which will minimize S and so make the approximation best fit the actual hull in the sense of least squares. It is necessary, of course, that $\sum_{k=1}^{N} m_k = 0$ in order to insure closure of the streamlines. Therefore the descent must be in a direction orthogonal to the gradient of $\sum_{k=1}^{N} m_k$. When we have minimized S we have found the best fit for the set of sources chosen. It is always possible to add more sources if the fit is not good enough, or to provide weights to the terms in (4.22) and (4.23) if it is felt that certain places on the hull are more important than others in getting a good fit.

The method as outlined sounds simple. Unfortunately the relationship between Y_s and m_k implied by $\frac{\partial Y_s}{\partial m_k}$ is not an easy one to write down, since the streamlines are frequently found by integrating along a streamline. However, the method is possible in principle and may be worth using if nothing else can be found. A rough approximation to $\frac{\partial Y_s}{\partial m_k}$ may be found by using a variation of Michell's formula -- i.e. assume that the <u>addition</u> of width to the hull is correctly expressed by equation (3.2) if the addition is small enough.

V. Summary and Conclusions

Although some knowledge of the theory of wave-making resistance of ships has been available for about eighty years and thousands of tests have been made to measure this resistance, there has been little application of this theoretical knowledge to decreasing the resistance of ships. The main advances in application of the theory have come in the past ten years, and all of these have been limited in their usefulness. The principal contributions to this advance have been these:

1. The observation by Inui that the agreement between theoretical and measured resistance could be greatly improved if the hull form were defined by the closed streamlines generated by its definition in terms of sources and sinks, rather than by the simpler approximation used earlier.

2. A discovery by Inui that the waves of some ship forms could be largely canceled by installation of a large sphere beneath the bow and another beneath the stern. Wigley investigated this problem and narrowly missed the discovery some twenty-five years earlier [20].

3. A series of applications of the Ritz method by Weinblum and others, and an employment of the calculus of variations by Karp, Kotik, and Lurye to find hull forms of decreased wave-making resistance. These applications have served principally to dispel the belief that nothing could be done.

In addition, there have been several advances in related fields which have not yet been fully exploited, but which appear to have the potential to make a decisive contribution to the ability to calculate hull forms of decreased wave-making resistance. Perhaps the most important is the general availability of high-speed computers, which make possible the employment of mathematical methods which would be impractical without them. There has also been developed the mathematical method of steep descent, which appears to be ideally suited for finding hull forms of decreased wave-making resistance. An example is calculated in part IV and compared with earlier results by the calculus of variations. The method of steep descent can be used in a hand calculation in simple cases, as in the example cited in part IV, but a high-speed computer is needed in more complicated problems of ships of a practical shape. In addition the ability to describe an arbitrary hull form in terms of sources and sinks has been developed by Smith and Hess in the past three years, and the conversion of the wave-making resistance equations into a form which will permit exploitation of this ability with the method of steep descent is shown in part II of this paper. In consequence, it appears that the mathematical machinery necessary to obtain much more improvement than has been possible heretofore is now available. The only obstacle to a completely general method of finding improved hull forms is the lack of a simple way to find a source distribution within the confines of the hull, rather than on its surface, which generates an arbitrar, hull form. However, the method of Smith and Hess for finding a source distribution on the surface of the hull equivalent to an arbitrary hull permits the easy calculation of any change which adds to a hull volume, and in view of the normal method used for designing ships, this should nearly always be sufficient. A method for carrying out such a calculation is outlined in part IV.

In summary, it appears that although little practical application has yet been made of mathematical methods for finding hull forms of decreased wave-making resistance, the techniques are now available which will permit this to be generally done.

Appendix 1.

Integration of the Resistance Equations.

In this section estimates will be made of the value of the smoothly varying portion of the wave-making resistance of an assemblage of sources and sinks, and then a closed form solution will be obtained for this portion of the resistance. After this a limit will be derived for the error incurred by terminating the numerical integration of the expression for the fluctuating part of the resistance at some zero of the integrand.

The discussion here will be based on equations (2.17), (2.18), (2.19), (2.21), and (2.22). For convenience they are reproduced here:

$$R = 16\pi\rho K_o^2 \left\{ \sum_{r=1}^{n} m_r^2 \frac{e}{2}^{-p_{rr}} \int_0^\infty e^{-p_{rr}t}(1+t)^{1/2} t^{-1/2} dt \; + \right.$$

$$\left. + \; 2 \sum_{s=r+1}^{n} \sum_{r=1}^{n-1} m_r m_s \frac{e}{2}^{-p_{rs}} \int_0^\infty e^{-p_{rs}t}(1+t)^{1/2} t^{-1/2} \cos\left[q_{rs}(1+t)^{1/2}\right] \cos\left[q'_{rs} t^{1/2}(1+t)^{1/2}\right] dt \right\} \quad \ldots(2.17)$$

Here we have set

$p_{rr} = 2K_o f_r$; $p_{rs} = K_o(f_r+f_s)$; $q_{rs} = K_o(h_r-h_s)$; and $q'_{rs} = K_o(k_r-k_s)$.

$$R = R^{(1)} + R^{(2)} \quad \ldots(2.18)$$

$$R^{(1)} = \sum_{r=1}^{n} R_{rr} \quad \ldots(2.19)$$

where

$$R_{rr} = 16\pi\rho K_o^2 m_r^2 \frac{e}{2}^{-p_{rr}} \int_0^\infty e^{-p_{rr}t}(1+t)^{1/2} t^{-1/2} dt \quad \ldots(2.19a)$$

$$R^{(2)} = 2 \sum_{s=r+1}^{n} \sum_{r=1}^{n-1} R_{rs} \quad \ldots(2.21)$$

$$R_{rs} = 16\pi\rho K_o^2 m_r m_s \frac{e}{2}^{-p_{rs}} \int_0^\infty e^{-p_{rs}t}(1+t)^{1/2} t^{-1/2} \cos\left[q_{rs}(1+t)^{1/2}\right] \cos\left[q'_{rs} t^{1/2}(1+t)^{1/2}\right] dt$$

The expression $R^{(1)}$ is the smoothly varying part of the resistance, while $R^{(2)}$ is the fluctuating part. Each term R_{rs} of the fluctuating part will become alternately positive and negative as K_o increases monotonically. Each term R_{rr} of the smoothly varying part $R^{(1)}$ will start small for small K_o, increase monotonically to a maximum, and then decrease as K_o becomes still larger.

We can estimate the behavior of R_{rr} as follows:

$$\left| \int_0^\infty e^{-p_{rr}t}(1+t)^{1/2}t^{-1/2}\,dt \right| \leq \left| \int_0^3 (1+t)^{1/2}t^{-1/2}dt + \int_3^\infty e^{-p_{rr}t}(1+t)^{1/2}t^{-1/2}dt \right| \leq$$

$$\leq \left| 2\int_0^3 t^{-1/2}dt + \int_3^\infty e^{-p_{rr}t}\sqrt{4/3}\,dt \right| = 4\sqrt{3} + \sqrt{4/3}\,\frac{e^{-3p_{rr}}}{p_{rr}}$$

It follows that

$$R_{rr} \leq 16\pi\rho K_o^2 m_r^2 \frac{e^{-p_{rr}}}{2}\left[4\sqrt{3} + \sqrt{4/3}\,\frac{e^{-3p_{rr}}}{p_{rr}} \right] =$$

$$= 16\pi\rho K_o^2 m_r^2 \frac{e^{-2K_of_r}}{2}\left[4\sqrt{3} + \sqrt{4/3}\,\frac{e^{-6K_of_r}}{2K_of_r} \right] \qquad \ldots(A.1)$$

For any $f_r > 0$, this function is small for small K_o, increases smoothly to a maximum, and then decreases to zero as K_o becomes infinite. Since our derivation of the velocity potential has assumed that $f_r > 0$ so that the flow can be described by a source and its mirror image in the surface, the restriction on f_r adds no restriction not already implicit in the formulas for the resistance. The relation $K_o = g/c^2$ means that zero speed corresponds to infinite K_o and vice versa. The maximum value of R_{rr} as estimated in equation (A.1) is reached between the values

$$4 < 2K_of_r < 6 .$$

This corresponds to

$$\sqrt{\frac{g\,f_r}{2}} > c > \sqrt{\frac{g\,f_r}{3}} . \qquad \ldots(A.2)$$

However, this approximate result really applies to the coefficient of resistance defined by $C_{rr} = R_{rr}c^{-2}$, rather than the resistance itself. This is because the definition of source strength used here includes a factor c so that for a given hull source strength increases with increasing speed. See footnote to page 10.

Equations (2.19) and (2.19a) show that $R^{(1)}$, the smoothly varying portion of the resistance, is for a given speed and strength of a source a function only of the depth of the source. On the other hand, equations (2.21) and (2.22) show that the fluctuating part of the wave-making resistance is a function not only of the sum of the depths of each pair of interacting sources, but also of their fore-and-aft and athwartships displacement from each other. This arises through the quantity q_{rs} which is a function of $\left| h_r - h_s \right|$, and the quantity q'_{rs} which is a function of $\left| k_r - k_s \right|$, the fore-and-aft and athwartships separations respectively. These results provide a simple explanation of a phenomenon apparently first noticed by Inui.* He found that to bring the calculated wave resistance of a ship into conformity with the observed wave resistance he had to adjust the position of the waves originating at the stern of the ship aft of their actual position in the fluctuating term of his formula for wave resistance, but he needed to make no such adjustment in the smoothly varying portion. It appears that the smoothly varying portion is not a function of horizontal position, so it is insensitive to corrections made in that position.

*Inui, Takao, "Study on Wave-Making Resistance of Ships", The Society of Naval Architects of Japan, 60th Anniversary Series, Vol, 2, pp. 172-355. The discussion starts on page 207.

The integral of equation (2.19a) may be evaluated in closed form as a Laplace transform. From Tables of Integral Transforms, Vol I, Bateman Manuscript Project, p. 139, Section 4.3, transform (17), we may find its value. The transform as tabulated is this:

$$\text{If } f(t) = 0, \qquad 0 < t < 2b$$
$$= (t+2a)^{\nu} \quad (t-2b)^{-\nu}, \qquad t > 2b,$$

and

$$\arg(a+b) < \pi, \quad \mathrm{Re}\,\nu < 1,$$

then the Laplace transform is

$$L\left\{f(t)\right\} = g(p) = \nu\pi\left[\csc(\nu\pi)\right] \; p^{-1} e^{-(a+b)p} k_{2\nu}\left[(a+b)p\right] .$$

We can put the integral of equation (2.19a) in this form if we set $\nu = 1/2$, $a = 1/2$, $b=0$, $\arg(a+b) = 0$. Then, since $\csc(\pi/2) = 1$, it follows that

$$\int_0^\infty e^{-p_{rr}t}(1+t)^{1/2}t^{-1/2}\,dt = (\pi/2)\,p_{rr}^{-1}\,e^{-p_{rr}/2}\,k_1(p_{rr}/2). \quad \ldots(A.3)$$

If we put this result into equation (2.19a) we get

$$R_{rr} = 4\pi^2 \rho K_o^{\,2} p_{rr}^{\,-1}\, m_r^{\,2}\, e^{-3p_{rr}/2}\, k_1(p_{rr}/2) . \qquad \ldots(2.20)$$

The function $k_1(p_{rr}/2)$ is Bateman's function. It is a composite of known hypergeometric functions and the gamma function, and can certainly be tabulated without difficulty.

There does not appear to be a tabulated Laplace transform for the fluctuating terms of the wave-resistance R_{rs}. However, we can integrate the function numerically, and it is possible to show that if $q'_{rs} = 0$ and we integrate up to any zero of the factor $\cos q_{rs}\sqrt{1+t}$ except the first one the error is less than the value of the integral between that zero and the next one. To see this, let $1+t = v^2$ in equation (2.22). Then we have for the integral

$$R_{rs} = \text{const} \int_1^{\infty} e^{-v^2 p_{rs}} (v^2-1)^{-1/2} \cos(q_{rs} v)\, dv. \qquad \ldots(A.4)$$

We may write this as an alternating series:

$$R_{rs} = \text{const}\left\{ \left|\int_1^{v_1}\right| - \left|\int_{v_1}^{v_2}\right| + \left|\int_{v_2}^{v_3}\right| - \left|\int_{v_3}^{v_4}\right| + \ldots \right\} e^{-v^2 p_{rs}} (v^2-1)^{-1/2} \cos(q_{rs} v)\, dv \qquad \ldots(A.5)$$

The limits v_1, v_2, v_3, ... are the successive zeros of $\cos(q_{rs}v)$. If we can show that the terms after some point become progressively smaller and tend to zero, then we will have shown not only that the integral of equation (A.4) and hence of (2.22) converges but that the error made by dropping all terms beyond some point is less than the magnitude of the first term dropped. This requires that the first term dropped be smaller in magnitude than the previous one. We observe that if we break up two successive terms in (A.5) into small intervals of equal argument of the cosine, then not only will the intervals Δt of the later term be no greater for equal argument than the corresponding intervals of the earlier term, but the absolute value of the integrand will be smaller in every case for a small interval in the later term than for the corresponding interval in the earlier term. From this it follows that every term of (A.5) except perhaps the first one is certainly followed by a term which is smaller in absolute value. But if this is so, then we may terminate the integration of (2.22) when $q'_{rs} = 0$ at any zero of the cosine, except perhaps the first one, with the knowledge that the error is less than the value of the integral between that zero and the next one.

If we set $q_{rs} = 0$, $q'_{rs} \neq 0$, then

$$R_{rs} = \text{const} \int_1^{\infty} e^{-v^2 p_{rs}} (v^2-1)^{-1/2} \cos(q'_{rs} v \sqrt{v^2-1})\, dv \qquad \ldots(A.6)$$

and the identical argument holds as for $q'_{rs} = 0$, $q_{rs} \neq 0$. For the case where neither q_{rs} nor q'_{rs} is zero, the estimate of the error to the precision obtained above is not easy to obtain, but a somewhat less precise upper limit of the error is worked out in the text.

It is possible to draw from equations (2.19a) to (2.22) some simple deductions which show that they provide known results in limiting cases. For example, if we let $q_{rs} = q'_{rs} = 0$, which means two sources superposed, we see that for $m_r = -m_s$ we get

$$R = R_{rr} + 2R_{rs} + R_{ss} = \text{const} \times (m_r^2 - 2m_r^2 + m_r^2) \times \text{integral} = 0.$$

Similarly if we have two positive equal sources superposed we get a resistance four times as great as for a single one -- which would follow from the fact that superposing two equal sources is the same as doubling the strength of a single source, and the smoothly varying portion of the resistance varies as the square of the source strength.

It is also possible to derive from these equations the effect of placing a source and an equal sink near to and below a given source. With proper separation and sufficient depth below the surface of both the given source and the added source-sink pair, we can get as good cancellation as we please of the given source's wave pattern. This follows because with large p_{rs}, which results from large depth f_r of the given source and f_s of the added sources, very nearly all the value of integrals in equations (2.19a) and (2.22) is obtained for very small t. We need only choose q_{rs} such that $\cos q_{rs} = 1$ and adjust the value of m_s to get virtually complete cancellation if f_r and f_s are large enough.

It may be desirable in some cases to use dipole moments rather than sources and sinks to describe the ship. If this is done, then the resistance equation and all that follows from it can be placed in a form which is parallel to that used for sources and sinks in equation (2.17). If the dipole moment is M_r, then the resistance becomes

$$R = 16\pi\rho K_o^4 \left\{ \sum_{r=1}^{n} M_r^2 \frac{e}{2}^{-p_{rr}} \int_o^\infty e^{-p_{rr}t}(1+t)^{3/2}t^{-1/2} \, dt \right. +$$

$$\left. + 2 \sum_{s=r+1}^{n} \sum_{r=1}^{n-1} M_r M_s \frac{e}{2}^{-p_{rs}} \int_o^\infty e^{-p_{rs}t}(1+t)^{3/2}t^{-1/2} \cos\left[q_{rs}(1+t)^{1/2}\right] \cos\left[q'_{rs} \; t^{1/2}(1+t)^{1/2}\right] dt \right\}$$

...(A.7)

The argument concerning the error incurred by terminating the integral is a little more complicated than for the case of sources and sinks, but similar in nature: it is clear that there is some value of t beyond which the error incurred by terminating the integral (for $q'_{rs} = 0$) at any zero of $\cos\left[q_{rs}(1+t)^{1/2}\right]$ is less than the value of the integral between that zero and the next one.

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