63-3-4

GIMRADA Research Note No. 8

FORMULAS FOR COMPUTING ATMOSPHERIC REFRACTION FOR OBJECTS INSIDE OR OUTSIDE THE ATMOSPHERE

By Angel A. Baldini

9 January 1963

U.S. ARMY
GEODESY, INTELLIGENCE AND MAPPING
RESEARCH AND DEVELOPMENT AGENCY
CORPS OF ENGINEERS

FORT BELVOIR VA

U. S. ARMY ENGINEER
GEODESY, INTELLIGENCE AND MAPPING RESEARCH AND DEVELOPMENT AGENCY

Research Note No. 8

FORMULAS FOR COMPUTING ATMOSPHERIC REFRACTION FOR OBJECTS INSIDE OR OUTSIDE THE ATMOSPHERE

Task 8T35-12-001-01

9 January 1963

Distributed by

The Director
U. S. Army Engineer
Geodesy, Intelligence and Mapping Research and Development Agency

Prepared by

Angel A. Baldini
Research and Analysis Division
U. S. Army Engineer
Geodesy, Intelligence and Mapping Research and Development Agency
Fort Belvoir, Virginia

The views contained herein represent only the views of the preparing agency and have not been approved by the Department of the Army.

GIMRADA Research Notes are published to record and disseminate significant results of research tasks or portions of overall programs during the course of the work.

CONTENTS

SUMMARY

This report presents the derivation of new equations for determining the changes in the direction of a ray of light as it passes through the atmosphere from an object to the observer. The equations are applicable to objects inside or outside the atmosphere.

Equations are also derived for obtaining the topocentric distance of the object as a function of the object's height and the observed zenith distance.

FORMULAS FOR COMPUTING ATMOSPHERIC REFRACTION FOR OBJECTS INSIDE OR OUTSIDE THE ATMOSPHERE

I. INTRODUCTION

Existing astronomic refraction equations cannot be satisfactorily applied to objects at distances up to a few thousands of miles from the earth because they were developed for a special use in which the object is at an infinite distance. In this paper, equations are derived that are applicable to objects both inside and outside the atmosphere.

II. INVESTIGATION

1. Fundamental Concepts. Refer to Fig. 1 and let

A = observing station

S = position of the object

h = height of the atmosphere over the station

C = earth's center

Δ = distance from the satellite to the station

Z_o = observed zenith distance

AS = curve of the ray path

S_1 = any point on the curve AS

ϵ = astronomical refraction

AZ = vertical of the station through C

R = refraction to be determined

x,y = rectangular coordinate system

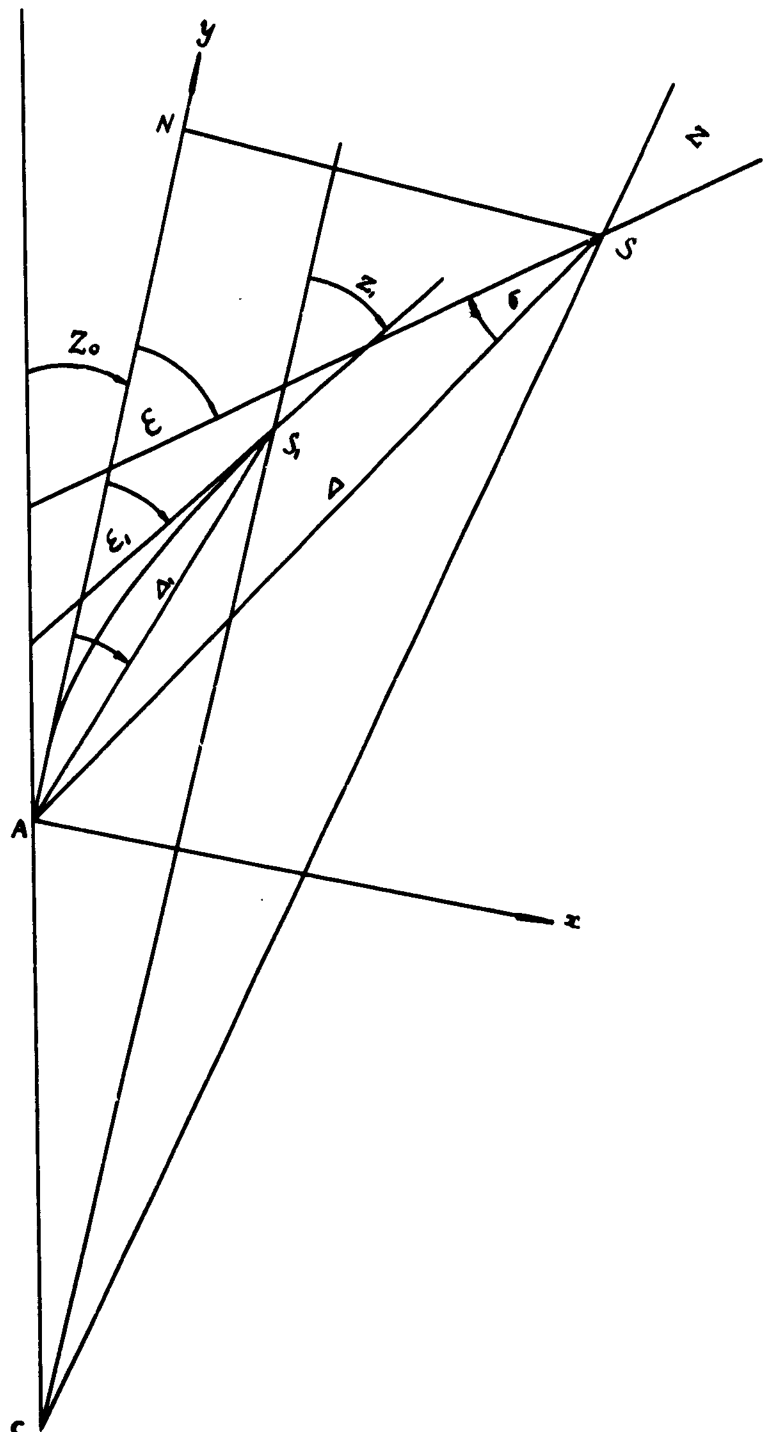

Fig. 1. Fundamental Concepts.

The observed zenith distance Z_o is the apparent zenith distance of the object when the ray reaches the observer. The true zenith distance is the angle $Z_o + R$, which the ray makes with the vertical of the observer's station before it enters the atmosphere.

The direction in which the observer sees the object is along the tangent to the curve at A. The origin of the x - y coordinate system is at A, the y-axis is oriented tangent to the ray path at A, and the x-axis is oriented 90° to the increasing zenith distances.

Let N be the point where the normal through S intersects the y-axis. The desired refraction is the angle NAS. It is obtained by equation

$$\sin R = \frac{SN}{\Delta} \tag{1}$$

We need to find SN, in order to obtain the refraction correction, R.

Because here we consider zenith distances less than 75°, R is always less than three minutes so it can be expressed as follows:

$$R = \frac{x}{\Delta \sin 1''}$$

in which

$$x = SN \tag{2}$$

We find x from

$$x = \int \varepsilon \, dy \tag{3}$$

so that

$$R = \frac{1}{\Delta \sin 1''} \int \varepsilon \, dy \tag{4}$$

in which ε is the angle between the y-axis and the tangent at any point of the ray path.

Assume a spherical earth with the atmosphere arranged in spherical layers. If n indicates the index of refraction of one layer, n + dn is the refraction of the next layer.

If in the first layer we have a zenith distance Z, the next layer is $Z + d\varepsilon$.

From the law of refraction we have

$$\frac{\sin (Z + d\epsilon)}{\sin Z} = \frac{n}{n + dn} \tag{5}$$

which we can transform as follows

$$1 + d\epsilon \cot Z = 1 - \frac{dn}{n}$$

so we obtain for $d\epsilon$

$$d\epsilon = - \tan Z \frac{dn}{n} \tag{6}$$

Then

$$\epsilon = - \int \tan Z \frac{dn}{n} \tag{7}$$

Insert equation (7) into equation (4) and we obtain:

$$R = - \frac{1}{\sin 1''} \int \int \tan Z \frac{dn}{n} \cdot dy \tag{8}$$

To solve equation (8) we need an expression for tn Z and another for $\frac{dn}{n}$.

To develop these we must first find expressions for the density of the atmosphere (on which refraction depends) as a function of the height.

2. Density of the Atmosphere As a Function of Height. The following symbols are used:

ρ, the density of the atmosphere at any height

ρ_o, the density at the earth's surface where the observer is located

h, altitude above the observer's station

n_o, index of refraction at the observer's station

n, index of refraction at height h.

In order to determine the refraction it is necessary to have an expression for the density of the air as a function of the height.

We derive an empirical law of diminution of density from observations and take into consideration the fact that the power of reflecting light ceases at about 60 kilometers.*

From the observed values of the density at different heights, we find that ρ decreases exponentially with altitude following the equation

$$\frac{\rho}{\rho_o} = e^{-\frac{h}{h_o}} \tag{9}$$

in which h_o is a constant.

The values of density ρ up to 20 kilometers follow equation (9) with accuracy. For heights over 20 km the equation is less accurate, but still sufficiently accurate because the refraction is small and the power of reflected light decreases rapidly at increasing heights (Fig. 2).

The constant h_o, was computed by weighting the observations proportionately to the power of reflected light at the height of observation. The value

$$h_o = 9.240 \text{ km}$$

and

$$\frac{1}{h_o} = \frac{0.1082}{\text{km}} \tag{10}$$

Introduce this value into equation (9) and we obtain for the density ρ

$$\rho = \rho_o \, e^{-0.1082 \, h \, (\text{km})} \tag{11}$$

*The basic material for this development was obtained from The Handbook of Geophysics for Air Force Designers, Geophysics Research Directorate, Air Force Cambridge Research Center, Air Research and Development Command, Cambridge, Massachusetts 1957, and from available balloon observation data.

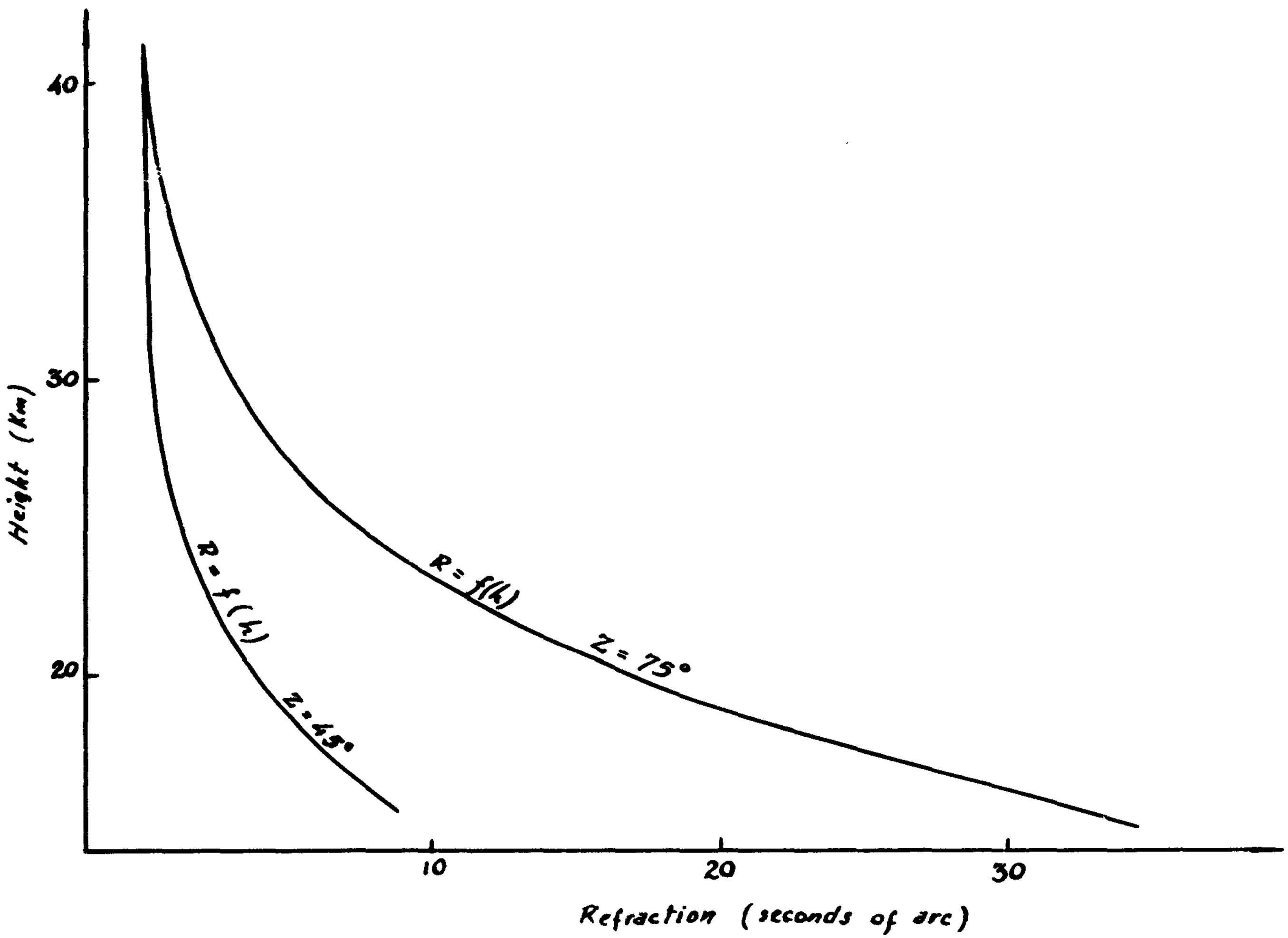

Fig. 2. Variation of refraction with height for Z = 75° and Z = 45°.

From Glandstone and Dale's law the values of the indexes of refraction, in terms of the atmospheric density, are given by equation

$$\frac{n - 1}{\rho} = c = \text{constant} \tag{12}$$

in which

$$c = 0.226$$

Use for ρ the value given by equation (11), and the index of refraction from equation (12) can be expressed as follows:

$$n = 1 + 0.226\, \rho_o\, e^{-0.1082\, h} \tag{13}$$

in which the height h must be expressed in km.

From equation (13) we obtain

$$\frac{dn}{n} = - \frac{0.226\, \rho_o\, e^{-0.1082\, h}}{1 + 0.226\, \rho_o\, e^{-1082}}\, 0.1082\, dh \tag{14}$$

but from equation (12)

$$0.226\, \rho_o = n_o - 1 \tag{15}$$

and because

$$0.1082 = \frac{1}{h_o}$$

We can rewrite equation (14) as follows:

$$\frac{dn}{n} = \frac{(n_o - 1)\, e^{-\frac{h}{h_o}}\, dh}{h_o\, [1 + (n_o - 1)\, e^{-\frac{h}{h_o}}]} \tag{16}$$

3. Expression for tn Z. To find an expression for tn Z we must consider:

a. The maximum height of the stratus over which the reflecting power can be assumed zero.

b. The maximum zenith distance Z_o.

For (a) we find

$$h < 64 \text{ km} \tag{17}$$

and for (b) we adopt

$$Z_o \le 75^o \tag{18}$$

With these values for h and Z_o, the maximum value of Z should be:

$$Z = Z_o - \xi \tag{19}$$

in which

$$\xi < 2\overset{o}{.}2$$

We can then express tn Z as follows:

$$\text{tn } Z = \text{tn } Z_o + \Delta \tag{20}$$

in which

$$\Delta = f(x) \tag{21}$$

Expand f(x) using the McLaurin's series. Then, we have

$$\Delta = f(0) + x f'(0) + \frac{x^2}{2} f''(0) + \frac{x^3}{6} f'''(0) + .. \tag{22}$$

Differentiating equation (20), we have

$$(1 + \text{tn}^2 Z_o) \frac{dZ}{dx} = f'(x) \tag{23}$$

After the second differentiation

$$\frac{d^2Z}{dx^2} (1 + \text{tn}^2 Z) + 2 \left(\frac{dZ}{dx}\right)^2 \text{tn } Z (1 + \text{tn}^2 Z) = f''(x) \tag{24}$$

and from the third differentiation

$$\frac{d^3Z}{dx^3}\left[1 + tn^2 Z\right] + 6 \, tn \, Z \, (1 + tn^2 Z) \frac{d^2Z}{dx^2} \frac{dZ}{dx} + 2\left(\frac{dZ}{dx}\right)^3 (1 + tn^2 Z)^2 (1 + 3 \, tn^2 Z) = f'''(x) \tag{25}$$

To find the $\frac{dZ}{dx}$, $\frac{d^2Z}{dx^2}$, $\frac{d^3Z}{dx^3}$, we use Snell's law

$$nr \sin Z = n_o \, r_o \sin Z_o = \text{constant}$$

Because we can express r as follows:

$$r = r_o \left(1 + \frac{h}{r_o}\right) \tag{26}$$

and for n

$$n = 1 + (n_o - 1) \, e^{-\frac{h}{h_o}}$$

Snell's law can be rewritten in the form

$$\sin Z = \frac{n_o \sin Z_o}{\left(1 + \frac{h}{r_o}\right)\left[(n_o - 1) \, e^{-\frac{h}{h_o}} + 1\right]} \tag{27}$$

Place

$$x = \frac{h}{n_o r_o} + (n_o - 1)\left[\frac{e^{-\frac{h}{h_o}}}{n_o}\left(1 + \frac{h}{r_o}\right) - 1\right] \text{ and} \tag{28}$$

we obtain

$$\sin Z = \sin Z_o \, (1 + x)^{-1} \tag{29}$$

It follows from equation (26) when $h = 0$

$$r = r_o$$

$$n = n_o$$

then, from equation (28)

$$x = 0$$

and from equation (29)

$$Z = Z_o$$

from which $f'(0) = 0$

From equation (29) we obtain for $x = 0$

$$\left(\frac{dZ}{dx}\right)_o = -\mathrm{tn}\, Z_o \qquad (30)$$

$$\left(\frac{d^2Z}{dx^2}\right)_o = 2\,\mathrm{tn}\, Z_o + \mathrm{tn}^3\, Z_o \qquad (31)$$

$$\left(\frac{d^3Z}{dx^3}\right)_o = -\,3\,\mathrm{tn}\, Z_o\,(1 + \mathrm{tn}^2\, Z_o)\,(2 + \mathrm{tn}^2\, Z_o) + 2\,\mathrm{tn}^3\, Z_o \qquad (32)$$

For $x = 0$, equations (23),(24), and (25) become

$$f'(0) = (1 + \mathrm{tn}^2\, Z_o)\left(\frac{dZ}{dx}\right)_o$$

$$f''(0) = \left(\frac{d^2Z}{dx^2}\right)_o (1 + \mathrm{tn}^2\, Z_o) + 2\left(\frac{dZ}{dx}\right)_o^2 \mathrm{tn}\, Z_o\,(1 + \mathrm{tn}^2\, Z_o) \qquad (33)$$

$$f'''(0) = \left[1 + tn^2 Z_o\right] \left[\left(\frac{d^3Z}{dx^3}\right)_o + 6 \, tn \, Z_o \left(\frac{d^2Z}{dx^2}\right)_o \left(\frac{dZ}{dx}\right)_o \right.$$

$$\left. + 2 \left(\frac{dZ}{dx}\right)_o^3 (1 + 3 \, tn^2 Z_o) \right]$$

Insert the values of equations (30), (31), and (32) in the groups of equations (33) and we obtain finally

$$f'(0) = -tn \, Z_o \, (1 + tn^2 Z_o)$$

$$f''(0) = (2 \, tn \, Z_o + 3 \, tn^2 Z_o) \, (1 + tn^2 Z_o) \tag{34}$$

$$f'''(0) = - 3 \, tn \, Z_o \, (1 + tn^2 Z_o) \left[2 + tn^2 Z_o \, (7 + 5 \, tn^2 Z_o)\right]$$

Now, introduce the values of equation (34) into equation (22) and equation (20) becomes

$$tn \, Z = tn \, Z_o - x \, tn \, Z_o \, (1 + tn^2 Z_o) + \frac{x^2}{2} tn \, Z_o \, (1 + tn^2 Z_o) \, (2 + 3 \, tn^2 Z_o)$$

$$- \frac{x^3}{2} tn \, Z_o \, (1 + tn^2 Z_o) \left[2 + tn^2 Z_o \, (7 + 5 \, tn^2 Z_o)\right] + \ldots \tag{35}$$

If we introduce

$$A = tn \, Z_o \, (1 + tn^2 Z_o)$$

$$B = \tfrac{1}{2} A \, (2 + 3 \, tn^2 Z_o) \tag{36}$$

$$C = A \left[1 + \tfrac{1}{2} tn^2 Z_o \, (7 + 5 \, tn^2 Z_o)\right]$$

equation (35) can then be rewritten as follows:

$$tn \, Z = tn \, Z_o - x \, A + x^2 \, B - x^3 \, C + \ldots \tag{37}$$

This equation (37) and the equation (16) which can also be written in the form

$$\frac{dn}{n} = (n_o - 1)\, e^{-\frac{h}{h_o}} \left[1 - (n_o - 1)\, e^{-\frac{h}{h_o}}\right] \frac{dh}{h_o} \qquad (38)$$

are the two expressions which must be inserted into equation #(8).

dy remains unknown but can be obtained from

$$dy = \frac{dh}{\cos Z}$$

Its value is found later. Let us first solve equation (7)

4. Solution for $\int \tan Z \frac{dn}{n}$. Introduce into equation (7) the values given by equations (37) and (38), and we have

$$\epsilon = - \tan Z_o \int \frac{dn}{n} - (n_o - 1)\, A \int x\, e^{-\frac{h}{h_o}} \left[1 - (n_o - 1)\, e^{-\frac{h}{h_o}}\right] \frac{dh}{h_o}$$

$$+ (n_o - 1)\, B \int x^2\, e^{-\frac{h}{h_o}} \left[1 - (n_o - 1)\, e^{-\frac{h}{h_o}}\right] \frac{dh}{h_o} \qquad (39)$$

$$- (n_o - 1)\, C \int x^3\, e^{-\frac{h}{h_o}} \left[1 - (n_o - 1)\, e^{-\frac{h}{h_o}}\right] \frac{dh}{h_o}$$

The solution of the first integral is

$$- \int_{n_o}^{n} \frac{dn}{n} = \left[\frac{1}{2\, n^2} \right]_{n_o}^{n} = \frac{n_o^2 - n^2}{2\, n_o^2\, n^2} \qquad (40)$$

The indexes of refraction differ from unity by a small quantity

$$n < 1.0003$$

Indicate the small quantity .0003, by γ and we have

$$n = 1 + \gamma \qquad (41)$$

in which γ has the values given by equation (26)

$$\gamma = (n_o - 1)\, e^{-\frac{h}{h_o}} \tag{42}$$

so we can rewrite equation (40) as follows:

$$\frac{n_o^2 - n^2}{2\, n_o^2\, n^2} = \left[\gamma_o - \gamma\right] \left[1 + \frac{\gamma_o + \gamma}{2}\right] \left[1 - 2\,(\gamma_o + \gamma)\right]$$

Neglect terms of the second power and we have

$$\frac{n_o^2 - n^2}{2\, n_o^2\, n^2} = (\gamma_o - \gamma) \left[1 - \frac{3}{2}\,(\gamma_o + \gamma)\right] \tag{43}$$

γ_o is the value for $h = 0$. From equation (41)

$$\gamma_o = n_o - 1 \tag{44}$$

Introduce the values of γ given by equations (44) and (42), and equation (43) becomes

$$\frac{n_o^2 - n^2}{2\, n_o^2\, n^2} = (n_o - 1)\,(1 - e^{-\frac{h}{h_o}}) \left[1 - \frac{3}{2}\,(n_o - 1)\,(1 + e^{-\frac{h}{h_o}})\right] \tag{45}$$

With this value the first integral of equation (39) is

$$-\text{tn}\, Z \int \frac{dn}{n} = (n_o - 1)\, \text{tn}\, Z_o\, (1 - e^{-\frac{h}{h_o}}) \left[1 - \frac{3}{2}\,(n_o - 1)\,(1 + e^{-\frac{h}{h_o}})\right] \tag{46}$$

Solve the second integral

$$II = \int x\, e^{-\frac{h}{h_o}} \left[1 - (n_o - 1)\, e^{-\frac{h}{h_o}}\right] \frac{dh}{h_o} \tag{47}$$

in which x is determined by equation (28)

$$x = -(n_o - 1) + \frac{(n_o - 1)}{n_o} \; e^{-\frac{h}{h_o}} + \frac{h}{n_o r_o} + (n_o - 1) \frac{h}{n_o r_o} e^{-\frac{h}{h_o}}$$

Introduce the value of x into this integral (47) and we have

$$\begin{aligned} II = & -(n_o - 1) \int e^{-\frac{h}{h_o}} \left[1 - (n_o - 1) e^{-\frac{h}{h_o}}\right] \frac{dh}{h_o} \\ & + \frac{(n_o - 1)}{n_o} \int e^{-2\frac{h}{h_o}} \left[1 - (n_o - 1) e^{-\frac{h}{h_o}}\right] \frac{dh}{h_o} \\ & + \frac{1}{n_o r_o} \int h \, e^{-\frac{h}{h_o}} \left[1 - (n_o - 1) e^{-\frac{h}{h_o}}\right] \frac{dh}{h_o} \\ & + \frac{n_o - 1}{n_o r_o} \int h \, e^{-\frac{h}{h_o}} \left[1 - (n_o - 1) e^{-\frac{h}{h_o}}\right] \frac{dh}{h_o} \end{aligned} \qquad (48)$$

After rearrangement and because $(n_o - 1)$ is less than 0.0003,

$$\frac{1}{n_o} + (n_o + 1) = \frac{1}{(n_o - 1) + 1} + (n_o - 1) = 1$$

and

$$\frac{1}{n_o} = 1 - (n_o - 1)$$

we obtain

$$II = -\frac{(n_o - 1)}{h_o} \int_o^h e^{-\frac{h}{h_o}} dh + \frac{(n_o - 1)}{h_o} \int_o^h e^{-2\frac{h}{h_o}} dh$$

$$+ \frac{1}{n_o r_o h_o} \int_o^h h\, e^{-\frac{h}{h_o}}\, dh - \frac{(n_o - 1)^2}{n_o h_o} \int_o^h e^{-3\frac{h}{h_o}}\, dh$$

$$- \frac{(n_o - 1)^2}{n_o h_o} \int_o^h h\, e^{-3\frac{h}{h_o}}\, dh$$

Integrate to obtain

$$II = -(n_o - 1)\left[1 - e^{-\frac{h}{h_o}}\right] + \frac{(n_o - 1)}{2}\left(1 - e^{-2\frac{h}{h_o}}\right)$$

$$+ \frac{h_o}{n_o r_o}\left[1 - e^{-\frac{h}{h_o}}\left(\frac{h}{h_o} + 1\right)\right] - \frac{(n_o - 1)^2}{3\, n_o}\left[1 - e^{-3\frac{h}{h_o}}\right] \quad (49)$$

$$- \frac{(n_o - 1)^2 h_o}{9\, n_o}\left[1 - e^{-3\frac{h}{h_o}}\left(1 + 3\frac{h}{h_o}\right)\right]$$

which is the solution of the integral of equation (47)

Solve the third integral of equation (39)

$$\int x^2 e^{-\frac{h}{h_o}}\left[1 - (n_o - 1)\, e^{-\frac{h}{h_o}}\right]\frac{dh}{h_o} = III \quad (50)$$

From equation (36), for the maximum zenith distance $Z = 75$, $\tan Z = 3.7$, so taking $\tan Z = 4$, we have from equation (36)

$$A = \tan Z_o\,(1 + \tan^2 Z_o) = 68$$

$$B = \tfrac{1}{2} A\,(2 + 3\tan^2 Z_o) = 1630$$

but because

$$(n_o - 1) < 0.0003$$

all terms containing $(n_o - 1)^3$ can be neglected, and we have for the maximum value

$$B\ (n_o - 1)^3 < 0''.003$$

$$x^2 = -2\ \frac{(n_o - 1)h}{n_o r_o} + \frac{h^2}{n_o^2\ r_o^2} \tag{51}$$

Substitute equation (51) into (50) and we obtain

$$III = -\ \frac{2(n_o - 1)}{n_o\ r_o\ h_o} \int h\ e^{-\frac{h}{h_o}}\ dh + \frac{1}{n_o^2\ r_o^2\ h_o} \int h^2\ e^{-\frac{h}{h_o}}\ dh$$

$$-\ \frac{(n_o - 1)}{n_o\ r_o^2\ h_o} \int h^2\ e^{-2\frac{h}{h_o}}\ dh \tag{52}$$

The solution of these integrals is

$$\int_o^h h\ e^{-\frac{h}{h_o}}\ dh = h_o^2 \left[1 - e^{-\frac{h}{h_o}} \left(\frac{h}{h_o} - 1\right) \right] \tag{53}$$

$$\int_o^h h^2\ e^{-\frac{h}{h_o}}\ dh = h_o^3 \left[2 - e^{-\frac{h}{h_o}} \left(\frac{h^2}{h_o^2} + 2\ \frac{h}{h_o} + 2\right)\right] \tag{54}$$

$$\int_o^h h^2\ e^{-\frac{2h}{h_o}}\ dh = \frac{h_o^3}{4} \left[1 - e^{-2\frac{h}{h_o}} \left(2\ \frac{h^2}{h_o^2} + 2\ \frac{h}{h_o} + 1\right)\right] \tag{55}$$

Introduce equations (53), (54) and (55) into equation (52), and we have

$$III = -\frac{2(n_o - 1)\ h_o}{n_o\ r_o} \left[1 - e^{-\frac{h}{h_o}} \left(\frac{h}{h_o} + 1\right) \right]$$

$$+ \frac{1}{n_o^2} \frac{h_o^2}{r_o^2} \left[2 - e^{-\frac{h}{h_o}} \left(\frac{h^2}{h_o^2} + 2 \frac{h}{h_o} + 2 \right) \right] \tag{56}$$

$$- \frac{(n_o - 1) h_o^2}{4 n_o r_o^2} \left[1 - e^{-\frac{2h}{h_o}} \left(2 \frac{h^2}{h_o^2} + 2 \frac{h}{h_o} + 1 \right) \right]$$

The last term equation of (56) has no influence in the refraction correction. By introducing this term into equation (39) its influence is given by

$$dR_3 = \frac{B (n_o - 1)^2}{4 n_o} \frac{h_o^2}{r_o^2} \left[1 - e^{-2 \frac{h}{h_o}} \left(2 \frac{h^2}{h_o^2} + 2 \frac{h}{h_o} + 1 \right) \right] \tag{57}$$

The maximum value corresponds to a maximum value of $h = h_o$.

It follows that

$$\left[1 - e^{-\frac{2h}{h_o}} \left(2 \frac{h^2}{h_o^2} + 2 \frac{h}{h_o} + 1 \right) \right]_{max.} = 0.14$$

For $\tan Z = 6$ $\quad (Z = 80^\circ.5)$

$B = 4218$

Retaining

$$\frac{h_o}{n_o r_o} = 0.01$$

$$(n_o - 1) < 0.0003$$

we obtain

$$dR_3 < 0''001$$

Similar computations for the second term give

$$\text{for } Z = 75^{\circ} \qquad dR_2 < 0''06$$
$$\text{for } Z = 81^{\circ}5 \qquad dR_2 < 3''8 \tag{58}$$

For the third integral we make

$$x^3 = \frac{h^3}{n_o^3\, r_o^3} \tag{59}$$

$$IV = \int x^3\, e^{-\frac{h}{h_o}} \left[1 - (n_o - 1)\, e^{-\frac{h}{h_o}}\right] \frac{dh}{h_o}$$

$$IV = \frac{1}{n_o^3\, h_o\, r_o^3} \int h^3\, e^{-\frac{h}{h_o}}\, dh \tag{60}$$

neglecting the second term. The solution is

$$IV = -\frac{h_o^3}{n_o r_o^3}\, e^{-\frac{h}{h_o}} + 3\,\frac{h_o^3}{n_o r_o^3} \left[2 - e^{-\frac{h}{h_o}} \left(\frac{h^2}{h_o^2} + 2\,\frac{h}{h_o} + 2\right)\right] \tag{61}$$

Multiply equation (61) by $-C(n_o - 1)$, equation (49) by $-A(n_o - 1)$, the first two terms of equation (56) by $B(n_o - 1)$, and add to equation (46). Then, equation (39) can be written as follows:

$$\epsilon = (n_o - 1)\, \tan Z_o \left(1 - e^{-\frac{h}{h_o}}\right) \left[1 - \frac{3}{2}(n_o - 1)\left(1 + e^{-\frac{h}{h_o}}\right)\right]$$
$$+ (n_o - 1)^2 A \left[1 - e^{-\frac{h}{h_o}}\right] - \frac{(n_o - 1)^2}{2} A \left(1 - e^{-2\frac{h}{h_o}}\right)$$
$$- \frac{(n_o - 1)\, A\, h_o}{n_o r_o} \left[1 - e^{-\frac{h}{h_o}} \left(\frac{h}{h_o} + 1\right)\right] + A\,\frac{(n_o - 1)^3}{3 n_o} \left[1 - e^{-3\frac{h}{h_o}}\right]$$

$$+ \frac{h_o (n_o - 1)^3}{9 n_o} A \left[1 + \frac{1}{r_o}\right] \left[1 - e^{-3 \frac{h}{h_o}} (1 + 3 \frac{h}{h_o})\right]$$

$$- \frac{2 (n_o - 1)^2 h_o}{n_o r_o} B \left[1 - e^{-\frac{h}{h_o}} (\frac{h}{h_o} + 1)\right] \tag{62}$$

$$+ \frac{(n_o - 1)}{n_o^2} B \frac{h_o^2}{r_o^2} \left[2 - e^{-\frac{h}{h_o}} \left(\frac{h^2}{h_o^2} + 2 \frac{h}{h_o} + 2\right)\right]$$

$$+ \frac{(n_o - 1) C h_o^3}{n_o r_o^3} e^{-\frac{h}{h_o}}$$

$$- \frac{3(n_o - 1) C h_o^3}{n_o r_o^3} \left[2 - e^{-\frac{h}{h_o}} \left(\frac{h^2}{h_o^2} + 2 \frac{h}{h_o} + 2\right)\right]$$

Collect the constant terms in A, and we have

$$C_A = (n_o - 1) \left[\frac{n_o - 1}{n_o} - \frac{n_o - 1}{2} - \frac{h_o}{n_o r_o} + \frac{h_o}{9} \frac{(n_o - 1)^2}{n_o}\right] \tag{63}$$

Let C_B be the constant terms of B. We then find

$$C_B = 2(n_o - 1) \left[\frac{h_o^2}{n_o^2 r_o^2} - (n_o - 1) \frac{h_o}{n_o r_o}\right] \tag{64}$$

$$C_C = 6 \frac{(n_o - 1)}{n_o} \frac{h_o^3}{r_o^3}$$

Let E_1, be the summation of terms in $e^{-\frac{h}{h_o}}$; E_2 of the $e^{-\frac{2h}{h_o}}$, and E_3 be of the $e^{-\frac{3h}{h_o}}$. We then obtain

$$E_1 = -(n_o - 1)\ \text{tn}\ Z_o + A\left[(n_o - 1)\ \frac{h_o}{n_o r_o} - \frac{(n_o - 1)^2}{n_o}\right]$$

$$+ \frac{6\ C\ (n_o - 1)\ h_o^3}{n_o\ r_o^3} \tag{65}$$

$$+ 2\ B\left[(n_o - 1)^2\ \frac{h_o}{n_o r_o} - \frac{(n_o - 1)\ h_o^2}{n_o^2\ r_o^2}\right]$$

$$E_2 = \frac{(n_o - 1)^2}{2}\ A + \frac{3}{2}\ (n_o - 1)^2\ \text{tn}\ Z_o \tag{66}$$

$$E_3 = A\ \frac{h_o\ (n_o - 1)^3}{9\ n_o}\ (1 + \frac{1}{r_o}) \tag{67}$$

Let M_1 be the sum of terms of $\frac{h}{h_o}\ e^{-\frac{h}{h_o}}$ and M_3 the sum of terms of $\frac{h}{h_o}\ e^{-3\frac{h}{h_o}}$. We then have

$$M_1 = (n_o - 1)\ \frac{h_o}{n_o r_o}\ A + 2\ B\left[(n_o - 1)^2\ \frac{h_o}{n_o r_o} + \frac{(n_o - 1)\ h_o^2}{n_o^2\ r_o^2}\right] \tag{68}$$

$$M_3 = \frac{1}{9}\ A\ \frac{h_o\ (n_o - 1)^2}{n_o}\ (1 + \frac{1}{r_o}) \tag{69}$$

and letting N be

$$N = \frac{3\,(n_o - 1)\,h^3}{n_o\,r_o^3}\,C - \frac{(n_o - 1)\,h_o^2\,B}{n_o^2\,r_o^2} \tag{70}$$

Substitute equations (63) to (70) in (62)

$$\begin{aligned} \epsilon = {} & (n_o - 1)\,\tan Z_o \left[1 - \frac{3}{2}(n_o - 1)\right] A\,C_A + B\,C_B - C\,C_C \\ & + E_1\,e^{-\frac{h}{h_o}} + E_2\,e^{-2\frac{h}{h_o}} - E_3\,e^{-3\frac{h}{h_o}} \\ & + M_1\,\frac{h}{h_o}\,e^{-\frac{h}{h_o}} - M_3\,\frac{h}{h_o}\,e^{-3\frac{h}{h_o}} \\ & + N\,\frac{h^2}{h_o^2}\,e^{-\frac{h}{h_o}} \end{aligned} \tag{71}$$

This equation (71) gives ϵ as a function of the height h, which we must introduce in equation (3) to find x. Then

$$\begin{aligned} x = {} & \left\{ (n_o - 1)\,\tan Z_o \left[1 - \frac{3}{2}(n_o - 1)\right] + A.C_A + B.C_B - C.C_C \right\} \int_o^y dy \\ & + E_1 \int_o^y e^{-\frac{h}{h_o}}\,dy + E_2 \int_o^y e^{-2\frac{h}{h_o}}\,dy - E_3 \int_o^y e^{-3\frac{h}{h_o}}\,dy \\ & + M_1 \int_o^y \frac{h}{h_o}\,e^{-\frac{h}{h_o}}\,dy - M_3 \int_o^y \frac{h}{h_o}\,e^{-3\frac{h}{h_o}}\,dy \\ & + N \int_o^y \frac{h^2}{h_o^2}\,e^{-\frac{h}{h_o}}\,dy \end{aligned} \tag{72}$$

The first integral is

$$\int_0^y dy = y$$

To solve the other integrals we need an expression for dy. We find

$$dy = \frac{dh}{\cos Z} \qquad (73)$$

To find $\frac{1}{\cos Z}$ we use equations (37) and (29).

$$\text{tn } Z = \text{tn } Z_0 - x A + B x^2 - C x^3$$

$$\sin Z = \sin Z_0 (1 + x)^{-1}$$

By division we obtain

$$\frac{1}{\cos Z} = \frac{1 + x}{\cos Z_0} - \frac{x (1 + x) A}{\sin Z_0} + \frac{(1 + x) x^2 B}{\sin Z_0} \cdots \qquad (74)$$

Because in all the integrals the exponential $e^{-m \frac{h}{h_0}}$ $(m = 1,2,3)$ decreases with increasing h we consider only terms containing x^2 and neglect the other terms containing x^3 and so on.

From equations (36) we obtain

$$\frac{A}{\sin Z_0} = \frac{1 + \text{tn}^2 Z_0}{\cos Z_0}$$

$$\frac{B}{\sin Z_0} = \frac{1}{2} \frac{(1 + \text{tn}^2 Z_0)}{\cos Z_0} (2 + 3 \text{ tn}^2 Z_0) \qquad (75)$$

By introducing (75) into (74) we have

$$\frac{1}{\cos Z} = \frac{1}{\cos Z_o} - x \frac{tn^2 Z_o}{\cos Z_o} + \frac{3}{2} \frac{x^2}{\cos Z_o} tn^2 Z_o (1 + tn^2 Z_o) + \quad (76)$$

Multiply this equation (76) by dh, and equation (73) can be written as follows:

$$dy = \frac{dh}{\cos Z_o} - x \frac{tn^2 Z_o}{\cos Z_o} dh + \frac{3}{2} x^2 \frac{tn^2 Z_o (1 + tn^2 Z_o)}{\cos Z_o} dh \quad (77)$$

Because all the corrections are small ones and decrease with increasing height, it is sufficiently accurate to retain

$$x = \frac{h}{n_o r_o} - (n_o - 1)$$

$$x^2 = \frac{h^2}{n_o^2 r_o^2} \quad (78)$$

Substitute equation (78) into (77) and we have

$$dy = \frac{dh}{\cos Z_o} + (n_o - 1) \frac{tn^2 Z_o}{\cos Z_o} dh - \frac{tn^2 Z_o}{\cos Z_o} \cdot \frac{h}{n_o r_o} dh$$
$$+ \frac{3}{2} tn^2 Z_o \frac{(1 + tn^2 Z_o)}{\cos Z_o} \cdot \frac{h^2}{n_o^2 r_o^2} dh \quad (79)$$

Introduce equation (79) into each integral of equation (72) and we have for the second one

$$E_1 \int_0^y e^{-\frac{h}{h_o}} dy = \frac{E_1}{\cos Z_o} \left[1 + (n_o - 1) \operatorname{tn}^2 Z_o\right] \int_0^h e^{-\frac{h}{h_o}} dh$$

$$- \frac{E_1 \operatorname{tn}^2 Z_o}{n_o r_o \cos Z_o} \int_0^h h \, e^{-\frac{h}{h_o}} dh \qquad (80)$$

$$+ \frac{3}{2} \frac{E_1 \operatorname{tn}^2 Z_o (1 + \operatorname{tn}^2 Z_o)}{n_o^2 r_o^2 \cos Z_o} \int_0^h h^2 e^{-\frac{h}{h_o}} dh$$

Integrate and we have

$$E_1 \int_0^y e^{-\frac{h}{h_o}} dy = \frac{E_1 h_o}{\cos Z_o} \left[1 + (n_o - 1) \operatorname{tn}^2 Z_o\right] \left[1 - e^{-\frac{h}{h_o}}\right]$$

$$+ \frac{E_1 \operatorname{tn}^2 Z_o h_o^2}{n_o r_o \cos Z_o} \left[e^{-\frac{h}{h_o}} \left(\frac{h}{h_o} + 1\right) - 1\right] \qquad (81)$$

$$+ \frac{3E_1 \operatorname{tn}^2 Z_o (1 + \operatorname{tn}^2 Z_o)}{n_o^2 r_o^2 \cos Z_o} h_o^3 \left[1 - e^{-\frac{h}{h_o}} \left(\frac{h^2}{2 h_o^2} + \frac{h}{h_o} + 1\right)\right]$$

For the third integral we have

$$E_2 \int_0^y e^{-\frac{2h}{h_o}} dy = E_2 \frac{\left[1 + (n_o - 1) \operatorname{tn}^2 Z_o\right]}{\cos Z_o} \int_0^h e^{-\frac{2h}{h_o}} dh$$

$$- E_2 \frac{\operatorname{tn}^2 Z_o}{\cos Z_o} \frac{1}{n_o r_o} \int_0^h e^{-2\frac{h}{h_o}} h \, dh \qquad (82)$$

$$+ \frac{3}{2} E_2 \operatorname{tn}^2 Z_o \frac{(1 + \operatorname{tn}^2 Z_o)}{\cos Z_o} \frac{1}{n_o^2 r_o^2} \int_0^h h^2 e^{-2\frac{h}{h_o}} dh$$

the solution of which is

$$E_2 \int_0^y e^{-2\frac{h}{h_0}} dy = E_2 \frac{\left[1 + (n_0 - 1)\, tn^2 Z_0\right]}{\cos Z_0} \frac{h_0}{2} \left[-e^{-2\frac{h}{h_0}} + 1\right]$$

$$+ E_2 \frac{tn^2 Z_0}{\cos Z_0} \frac{h_0^2}{4\, r_0 n_0^2} \left[e^{-2\frac{h}{h_0}} \left(2\frac{h}{h_0} + 1\right) - 1\right] \qquad (83)$$

$$+ \frac{3}{8} E_2\, tn^2 Z_0 \frac{(1 + tn^2 Z_0)}{\cos Z_0} \frac{h_0^3}{n_0 r_0^2} \left[1 - e^{-2\frac{h}{h_0}} \left(2\frac{h^2}{h_0^2} + 2\frac{h}{h_0} + 1\right)\right]$$

Solve for the fourth integral and we have

$$E_3 \int_0^y e^{-3\frac{h}{h_0}} dy = -\frac{E_3}{\cos Z_0} \left[1 + (n_0 - 1)\, tn^2 Z_0\right] \int_0^h e^{-3\frac{h}{h_0}} dh$$

$$+ \frac{E_3\, tn^2 Z_0}{n_0 r_0 \cos Z_0} \int_0^h e^{-\frac{3h}{h_0}} \cdot h\, dh \qquad (84)$$

$$- \frac{3}{2} E_3\, tn^2 Z_0 \frac{(1 + tn^2 Z_0)}{\cos Z_0} \frac{1}{n_0^2\, r_0^2} \int_0^h e^{-3\frac{h}{h_0}} h^2\, dh$$

the solution of which is:

$$-E_3 \int_0^y e^{-3\frac{h}{h_0}} dy = \frac{E_3 h_0}{3\cos Z_0} \left[1 + (n_0 - 1)\, tn^2 Z_0\right] \left[e^{-3\frac{h}{h_0}} - 1\right]$$

$$+ \frac{E_3\, tn^2 Z_0\, h_0^2}{9\, n_0 r_0 \cos Z_0} \left[1 - e^{-\frac{3h}{h_0}} \left(3\frac{h}{h_0} + 1\right)\right] \qquad (85)$$

$$- \frac{3E_3\, tn^2 Z_0\, (1 + tn^2 Z_0)}{54\, n_0 r_0^2 \cos Z_0} h_0^3 \left[2 - e^{-3\frac{h}{h_0}} \left(9\frac{h^2}{h_0^2} + 6\frac{h}{h_0} + 2\right)\right]$$

For the fifth integral we have

$$M_1 \int_o^y \frac{h}{h_o} e^{-\frac{h}{h_o}} dy = M_1 \frac{\left[1 + (n_o - 1) \, tn^2 Z_o\right]}{\cos Z_o} \int_o^h \frac{h}{h_o} e^{-\frac{h}{h_o}} dh$$

$$- \frac{M_1 \, tn^2 Z_o}{n_o r_o \cos Z_o} \int_o^h \frac{h^2}{h_o} e^{-\frac{h}{h_o}} dh \qquad (86)$$

$$+ \frac{3 M_1 \, tn^2 Z_o (1 + tn^2 Z_o)}{2 \; n_o^2 \, r_o^2 \; \cos Z_o} \int_o^h \frac{h^3}{h_o} e^{-\frac{h}{h_o}} dh$$

the solution of which is:

$$M_1 \int_o^y \frac{h}{h_o} e^{-\frac{h}{h_o}} dh = \frac{M_1 \cdot h_o}{\cos Z_o} \left[1 + (n_o - 1) \, tn^2 Z_o\right]\left[1 - e^{-\frac{h}{h_o}}\left(\frac{h}{h_o} + 1\right)\right]$$

$$- \frac{M_1 \, tn^2 Z_o}{n_o r_o \cos Z_o} h_o^2 \left[2 - e^{-\frac{h}{h_o}} \left(\frac{h^2}{h_o^2} + 2 \frac{h}{h_o} + 2\right)\right] \qquad (87)$$

$$+ \frac{3M_1 \, tn^2 Z_o (1 + tn^2 Z_o)}{2 \; n_o^2 \, r_o^2 \; \cos Z_o} h_o^3 \left[6 - e^{-\frac{h}{h_o}}\left(\frac{h^3}{h_o^3} + 3 \frac{h^2}{h_o} + 6 \frac{h}{h_o} + 6\right)\right]$$

For the sixth integral we have

$$- M_3 \int_o^y \frac{h}{h_o} e^{-3 \frac{h}{h_o}} dh = - \frac{M_3}{\cos Z_o} \left[1 + (n_o - 1) \, tn^2 Z_o\right] \int_o^h \frac{h}{h_o} e^{-3 \frac{h}{h_o}} dh$$

$$+\frac{M_3\ tn^2\ Z_o}{n_o r_o\ \cos Z_o}\int_o^h \frac{h^2}{h_o}\ e^{-3\frac{h}{h_o}}\ dh \tag{88}$$

$$-\frac{3\ M_3\ tn^2\ Z_o(1+tn^2\ Z_o)}{2\ n_o^2\ r_o^2\ \cos Z_o}\int_o^h \frac{h^3}{h_o}\ e^{-3\frac{h}{h_o}}\ dh$$

the solution of which is:

$$-M_3\int_o^y \frac{h}{h_o}\ e^{-\frac{3h}{h_o}}\ dy = \frac{h_o\ M_3}{9\ \cos Z_o}\left[1+(n_o-1)\ tn^2\ Z_o\right]\left[e^{-3\frac{h}{h_o}}(3\frac{h}{h_o}+1)-1\right]$$

$$+\frac{M_3\ tn^2\ Z_o\ .\ h_o^2}{27\ n_o r_o\ \cos Z_o}\left[2-e^{-3\frac{h}{h_o}}\ (9\frac{h^2}{h_o^2}+6\frac{h}{h_o}+2)\right] \tag{89}$$

$$+\frac{3M_3\ tn^2Z_o(1+tn^2Z_o)}{162\ n_o^2\ r_o^2\ \cos Z_o}\ h_o^3\left[e^{-3\frac{h}{h_o}}(27\frac{h^3}{h_o^3}+27\frac{h^2}{h_o^2}\right.$$

$$\left.+18\frac{h}{h_o}+6)-6\right]$$

and finally, for the last integral we have

$$N\int_o^y \frac{h^2}{h_o^2}\ e^{-\frac{h}{h_o}}\ dy = \frac{N}{\cos Z_o}\left[1+(n_o-1)\ tn^2 Z_o\right]\int_o^h \frac{h^2}{h_o^2}\ e^{-\frac{h}{h_o}}\ dh$$

$$-\frac{N\ tn^2\ Z_o}{n_o r_o\ \cos Z_o}\int_o^h \frac{h^3}{h_o^3}\ e^{-\frac{h}{h_o}}\ dh \tag{90}$$

$$+ \frac{3 N \, tn^2 Z_o (1 + tn^2 Z_o)}{2 \; n_o^2 \; r_o^2 \; \cos Z_o} \int_o^h \frac{h^4}{h_o^2} \; e^{-\frac{h}{h_o}} \, dh$$

the solution of which is:

$$N \int_o^y \frac{h^2}{h_o^2} e^{-\frac{h}{h_o}} dy = \frac{h_o N}{\cos Z_o} \left[1 + (n_o - 1) \; tn^2 Z_o\right] \left[2 - e^{-\frac{h}{h_o}} \left(\frac{h^2}{h_o^2} + 2\frac{h}{h_o} + 2\right)\right]$$

$$+ \frac{N \; tn^2 Z_o \; h_o^2}{n_o r_o \; \cos Z_o} \left[e^{-\frac{h}{h_o}} \left(\frac{h^3}{h_o^3} + 3 \frac{h^2}{h_o^2} + 6 \frac{h}{h_o} + 6\right) - 6 \right] \qquad (91)$$

$$+ \frac{3N \; h_o^3 \; tn^2 Z_o (1+tn^2 Z_o)}{2 \; n_o^2 \; r_o^2 \; \cos Z_o} \left[24 - e^{-\frac{h}{h_o}} \left(\frac{h^4}{h_o^4} + 4 \frac{h^3}{h_o^3} + 12 \frac{h^2}{h_o^2} + 24 \frac{h}{h_o} + 24\right)\right]$$

Use the abbreviation

$$t = tn \; Z_o$$

and add the equations (81), (83), (85), (87), (89), and (91). Then equation (72) becomes:

$$x = \int \epsilon \; dy = \left\{(n_o - 1) \; t \left[1 - \tfrac{3}{2} (n_o - 1)\right] + A.C_A + B.C_B + C.C_C\right\} y$$

$$+ E_1 \frac{h_o}{\cos Z_o} \left[1 + (n_o - 1) t^2\right] \left[1 - e^{-\frac{h}{h_o}}\right] + \frac{E_1 t^2 \; h_o^2}{n_o r_o \cos Z_o} \left[e^{-\frac{h}{h_o}} \left(\frac{h}{h_o} + 1\right) - 1\right]$$

$$+ 3 \frac{E_1 t^2(1+t^2)\, h_o^3}{n_o^2\, r_o^2 \cos Z_o} \left[1 - e^{-\frac{h}{h_o}} \left(\frac{1}{2} \frac{h^2}{h_o^2} + \frac{h}{h_o} + 1 \right) \right]$$

$$+ \frac{1}{2} E_2 \frac{h_o \left[1 + (n_o - 1) t^2\right]}{\cos Z_o} \left(1 - e^{-2\frac{h}{h_o}}\right) + \frac{1}{4} E_2 \frac{t^2\, h_o^2}{n_o r_o \cos Z_o} \left[e^{-2\frac{h}{h_o}} \left(2\frac{h}{h_o} - 1\right)\right]$$

$$+ \frac{3}{8} E_2 \frac{t^2(1 + t^2)\, h_o^3}{\cos Z_o \cdot n_o r_o^2} \left[1 - e^{-2\frac{h}{h_o}} \left(2 \frac{h^2}{h_o^2} + 2 \frac{h}{h_o} + 1\right)\right]$$

$$+ \frac{1}{3} E_3 \frac{h_o}{\cos Z_o} \left[1 + (n_o - 1)\, t^2 \right] \left[e^{-3\frac{h}{h_o}} - 1\right]$$

$$+ \frac{1}{9} E_3 \frac{t^2\, h_o^2}{n_o r_o \cos Z_o} \left[1 - e^{\frac{3h}{h_o}} \left(3 \frac{h}{h_o} + 1\right)\right]$$

$$+ \frac{1}{18} E_3 \frac{t^2(1 + t^2)}{\cos Z_o} \cdot \frac{h_o^3}{n_o r_o^2} \left[e^{-3\frac{h}{h_o}} \left(9 \frac{h^2}{h_o^2} + 6 \frac{h}{h_o} + 2\right) - 2\right]$$

$$+ M_1 \frac{h_o}{\cos Z_o} \left[1 + (n_o - 1)\, t^2\right] \left[1 - e^{-\frac{h}{h_o}} \left(\frac{h}{h_o} + 1\right)\right] \qquad (92)$$

$$+ \frac{M_1\, t^2\, h_o^2}{\cos Z_o\, n_o r_o} \left[e^{-\frac{h}{h_o}} \left(\frac{h^2}{h_o^2} + 2 \frac{h}{h_o} + 2\right) - 2\right]$$

$$+ \frac{3}{2} M_1 \frac{t^2\, (1 + t^2)\, h_o^3}{n_o^2\, r_o^2 \cos Z_o} \left[6 - e^{-\frac{h}{h_o}} \left(\frac{h^3}{h_o^3} + 3 \frac{h^2}{h_o^2} + 6 \frac{h}{h_o} + 6\right]\right.$$

$$+ \frac{M_3}{9} \frac{h_o}{\cos Z_o} \left[1 + (n_o - 1)\, t^2\right] \left[e^{-3 \frac{h}{h_o}} (3 \frac{h}{h_o} + 1) - 1\right]$$

$$+ \frac{M_3}{27} \frac{t^2 \quad h_o^2}{n_o r_o \cos Z_o} \left[2 - e^{-3 \frac{h}{h_o}} (9 \frac{h^2}{h_o^2} + 6 \frac{h}{h_o} + 2)\right]$$

$$+ \frac{3}{162} M_3 \frac{t^2 (1 + t^2)\, h_o^3}{n_o^2\, r_o^2 \cos Z_o} \left[e^{-3 \frac{h}{h_o}} (27 \frac{h^3}{h_o^3} + 27 \frac{h^2}{h_o^2} + 18 \frac{h}{h_o} + 6) - 6\right]$$

$$+ \frac{N\ h_o}{\cos Z_o} \left[1 + (n_o - 1)\, t^2\right] \left[2 - e^{- \frac{h}{h_o}} (\frac{h^2}{h_o^2} + 2 \frac{h}{h_o} + 2)\right]$$

$$+ \frac{N\ t^2\ h_o^2}{n_o r_o \cos Z_o} \left[e^{- \frac{h}{h_o}} (\frac{h^3}{h_o^3} + 3 \frac{h^2}{h_o^2} + 6 \frac{h}{h_o} + 6) - 6\right]$$

$$+ \frac{3}{2} N \frac{h_o^3\, t^2 (1 + t^2)}{n_o^2\, r_o^2 \cos Z_o} \left[24 - e^{- \frac{h}{h_o}} (\frac{h^4}{h_o^4} + 4 \frac{h^3}{h_o^3} + 12 \frac{h^2}{h_o^2} + 24 \frac{h}{h_o} + 24\right]$$

To obtain the refraction correction we divide the value of x by the distance Λ, as was indicated in equation (4).

In order to consider the terms necessary for $Z = 75°$ we must compute the values of E_1, E_2, E_3, M_1, M_3, and N. We find for $Z = 75°$

$$E_1 = 203''$$

$$E_2 = 0.6$$

$$E_3 = 0$$

$$M_1 = 5.0$$

$$M_3 = 1.0$$

$$N = -0.05$$

The term in the right hand of equation (65) can be neglected because its maximum value reaches

$$6\,C\,(n_o - 1)\,\frac{h_o^3}{n_o r_o^3} < 0\overset{''}{.}03$$

and each coefficient of E_1 is less than unity.

Since we hold as maximum zenith distance $Z_o = 75^o$, R_{max} does not reach more than 3 minutes which permits us to retain

$$\Lambda = y$$

with an error less than 0.1 meter for the maximum value of Λ (when the object is at the limit of the atmosphere).

Then, the final formula for computing the refraction for an object inside the atmospherc is as follows:

$$R = A_o\,(n_o - 1)\,\mathrm{tn}\,Z_o + A_1\,(n_o - 1)\,\mathrm{tn}^3\,Z_o + A_2\,(n_o - 1)\,\mathrm{tn}^5\,Z_o$$

$$+ \frac{E_1\,h_o}{\Lambda\cos Z_o}\left[(1 + (n_o - 1)\,\mathrm{tn}^2\,Z_o)\,(1 - e^{-\frac{h}{h_o}})\right]$$

$$+ \frac{E_1\,h_o^2\,\mathrm{tn}^2\,Z_o}{n_o r_o\,\Lambda\cos Z_o}\left[e^{-\frac{h}{h_o}}\left(\frac{h}{h_o} + 1\right) - 1\right]$$

$$+ \frac{1}{2} \frac{h_o E_2}{\Delta \cos Z_o} \left[1 - e^{-\frac{h}{h_o}} \right]$$

$$+ \frac{M_1 h_o}{\Delta \cos Z_o} \left[1 - e^{-\frac{h}{h_o}} \left(\frac{h}{h_o} + 1 \right) \right] \tag{93}$$

$$+ \frac{M_3 h_o}{9 \Delta \cos Z_o} \left[e^{-3 \frac{h}{h_o}} \left(3 \frac{h}{h_o} + 1 \right) - 1 \right]$$

$$+ N \frac{h_o}{\Delta \cos Z_o} \left[1 + (n_o - 1) \tan^2 Z_o \right] \left[2 - e^{-\frac{h}{h_o}} \left(\frac{h^2}{h_o^2} + 2 \frac{h}{h_o} + 2 \right) \right]$$

in which

$$A_o = +0.99827$$

$$A_1 = -0.00130$$

$$A_2 = +0.000006$$

If we express Δ in kilometers, and holding

$$h_o = 9.24 \text{ km}$$

$$\frac{h_o^2}{n_o r_o} = 0.0134 \text{ km}$$

$$\frac{1}{2} h_o = 4.62 \text{ km}$$

$$\frac{1}{9} h_o = 1.027 \text{ km}$$

The height of the station h_a, does not affect the results except for extreme heights. If this is so, we replace h and r_o as follows:

$$h = h_s - h_a$$

$$n_o r_o = 6372 + h_a$$

where

h_s = height of the object

h_a = height of the station

both h_s and h_a are expressed in kilometers.

Equation (93) can be rewritten as follows:

$$
\begin{aligned}
R = {} & A_o\,(n_o - 1)\,\mathrm{tn}\,Z_o + A_1\,(n_o - 1)\,\mathrm{tn}^3\,Z_o + A_2\,(n_o - 1)\,\mathrm{tn}^5\,Z_o \\
& + \frac{9.24\,E_1}{\Delta\cos Z_o}\left[1 + (n_o - 1)\,\mathrm{tn}^2\,Z_o\right]\left[1 - e^{-\frac{h}{h_o}}\right] \\
& + 0.0134\,\frac{E_1\,\mathrm{tn}^2\,Z_o}{\Delta\cos Z_o}\left[e^{-\frac{h}{h_o}}\left(\frac{h}{h_o} + 1\right) - 1\right] \\
& + \frac{4.620\,E_2}{\Delta\cos Z_o}\left[1 - e^{-\frac{h}{h_o}}\right] \qquad (94) \\
& + \frac{9.240\,M_1}{\Delta\cos Z_o}\left[1 + (n_o - 1)\,\mathrm{tn}^2\,Z_o\right]\left[1 - e^{-\frac{h}{h_o}}\left(\frac{h}{h_o} + 1\right)\right] \\
& + \frac{1.026\,M_3}{\Delta\cos Z_o}\left[1 + (n_o - 1)\,\mathrm{tn}^2\,Z_o\right]\left[e^{-3\frac{h}{h_o}}\left(3\,\frac{h}{h_o} + 1\right) - 1\right]
\end{aligned}
$$

$$+ \frac{9.240\ N}{\Delta \cos Z_o} \left[1 + (n_o - 1)\ tn^2\ Z_o\right] \left[2 - e^{-\frac{h}{h_o}} \left(\frac{h^2}{h_o^2} + 2\frac{h}{h_o} + 2\right)\right]$$

$$A_o = +0.99827$$

$$A_1 = -0.00130$$

$$A_2 = +0.000006$$

5. <u>Refraction Viewed from an Object Inside the Atmosphere.</u> The refraction viewed from the object is the angle

$$\sigma = NSA$$

of Figure 1. We see that

$$\sigma = \varepsilon - R \tag{95}$$

The angle ε can be computed by using equation (71).

Then the refraction σ can be computed as follows:

$$\sigma = E_1\ e^{-\frac{h}{h_o}} - \frac{9.240\ E_1}{\Delta \cos Z_o} \left[1 + (n_o - 1)\ tn^2\ Z_o\right] \left[1 - e^{-\frac{h}{h_o}}\right]$$

$$+ 0.0134 \frac{E_1\ tn^2\ Z_o}{\Delta \cos Z_o} \left[1 - e^{-\frac{h}{h_o}} \left(\frac{h}{h_o} + 1\right)\right]$$

$$+ E_2\ e^{-2\frac{h}{h_o}} - \frac{4.620}{\Delta \cos Z_o}\ E_2 \left[1 - e^{-\frac{h}{h_o}}\right] \tag{96}$$

$$+ M_1 \frac{h}{h_o} e^{-\frac{h}{h_o}} \cdot \frac{9.240\, M_1}{\Delta \cos Z_o} \left[1 + (n_o - 1)\, tn^2 Z_o\right]\left[1 - e^{-\frac{h}{h_o}}\left(\frac{h}{h_o} + 1\right)\right]$$

$$- M_3 \frac{h}{h_o} e^{-3\frac{h}{h_o}} - \frac{1.026\, M_3}{\Lambda \cos Z_o} \left[1 + (n_o - 1)\, tn^2 Z_o\right]\left[e^{-3\frac{h}{h_o}}\left(3\frac{h}{h_o} + 1\right) - 1\right]$$

$$+ N \frac{h^2}{h_o^2} e^{-\frac{h}{h_o}} - \frac{9.240}{\Delta \cos Z_o} \left[1 + (n_o - 1) tn^2 Z_o\right]\left[2 - e^{-\frac{h}{h_o}}\left(\frac{h^2}{h_o^2} + 2\frac{h}{h_o} + 1\right)\right]$$

As always Λ must be expressed in kilometers. The refraction σ is always less than R, consequently

$$\epsilon > R > \frac{\epsilon}{Z}$$

6. Computation of the Distance Δ of an object inside the Atmosphere. When the distance Λ is unknown it can be obtained as follows:

$$dy = \frac{dh}{\cos Z}$$

From

$$\frac{1}{\cos Z} = \frac{tn\ Z}{\sin Z}$$

we have found

$$\frac{1}{\cos Z} = \frac{1}{\cos Z_o} - x \frac{tn^2 Z_o}{\cos Z_o} + \frac{3}{2} x^2 \frac{tn^2 Z_o}{\cos Z_o}$$

$$+ \frac{3}{2} x^2 \frac{tn^4}{\cos Z_0} + x^3 \frac{tn^2 Z_0}{\cos Z_0} (1 + tn^2 Z_0) + \ldots.$$

where

$$x = \frac{h}{n_0 r_0} + (n_0 - 1) e^{-\frac{h}{h_0}} (1 + \frac{h}{r_0})$$

$$x^m = \left(\frac{h}{n_0 r_0} \right)^m \qquad m = 2,3 \ldots.$$

After these values are replaced and integrated we find

$$\begin{aligned}
\Delta = & \frac{h}{\cos Z_0} - \frac{1}{2} \frac{h^2}{n_0 r_0} \frac{tn^2 Z_0}{\cos Z_0} + (n_0 - 1) h \frac{tn^2 Z_0}{\cos Z_0} \\
& - \frac{(n_0 - 1) h_0}{n_0} \frac{tn^2 Z_0}{\cos Z_0} - (n_0 - 1) \frac{h_0^2 \, tn^2 Z_0}{n_0 r_0 \cos Z_0} \\
& - \frac{3}{2} \frac{(n_0 - 1)}{n_0 r_0} h^2 \frac{tn^2 Z_0}{\cos Z_0} (1 + tn^2 Z_0) \\
& + 3 \frac{(n_0 - 1)}{n_0 r_0} h_0^2 \frac{tn^2 Z_0}{\cos Z_0} (1 + tn^2 Z_0) \\
& + \frac{1}{2} \frac{h^3}{n_0^2 r_0^2} \frac{tn^2 Z_0}{\cos Z_0} (1 + tn^2 Z_0) \\
& + (n_0 - 1) \frac{h_0}{n_0} \frac{tn^2 Z_0}{\cos Z_0} e^{-\frac{h}{h_0}}
\end{aligned} \qquad (97)$$

$$+ \frac{3}{2} \frac{h^2 (n_o - 1)}{n_o r_o} \frac{tn^2 Z_o}{\cos Z_o} e^{-\frac{h}{h_o}} \left(\frac{h}{h_o} + 1 \right)$$

$$- 3 \frac{(n_o - 1)}{n_o r_o} h_o^2 e^{-\frac{h}{h_o}} \left(\frac{h}{h_o} + 1 \right) \frac{tn^2 Z_o}{\cos Z_o} (1 + tn^2 Z_o)$$

7. Working Equations to Obtain Coefficients E_1, E_2, E_3, M_1, M_3, and N. These coefficients can be computed from the following working equations:

$$E_1 = - (n_o - 1) \; tn \; Z_o + \left[0''0680 - 0''00010 \; tn^2 Z_o \right] tn \; Z_o \; (1 + tn^2 Z_o)$$

$$E_2 = + 0''033 \; tn \; Z_o + 0''0081 \; tn^3 Z_o$$

$$E_3 = + 0''000005 \; tn \; Z_o \; (1 + tn^2 Z_o)$$

$$M_1 = + \left[0''0849 + 0''00044 \; tn^2 Z_o \right] tn \; Z_o \; (1 + tn^2 Z_o) \qquad (98)$$

$$M_3 = + 0''0168 \; tn \; Z_o \; (1 + tn^2 Z_o)$$

$$N = - 0''000006 \; tn \; Z_o \; (1 + tn^2 Z_o) \; (2 + 3 \; tn^2 Z_o)$$

8. Computation of $(n_o - 1)$. The refractive index (n) of standard air at optical frequencies can be obtained from that given by Barrel and Sears

$$(n_s - 1) \, 10^7 = 2876.04 + \frac{16.288}{\lambda^2} + \frac{0.136}{\lambda^4} \qquad (99)$$

where

λ = the light group wavelength in microns. This equation agrees to 1 in 10^8 over the visible spectrum.

$$(n_o - 1) = \frac{n_G - 1}{1 + \alpha t} \frac{P}{760} - \frac{0.000000055\, e}{1 + \alpha t} \text{ where} \tag{100}$$

n_o = refractive index under ambient conditions

n_G = refractive index in dry air with 0.03 % CO_2 at NTP (0° C, 760 mm Hg) for light of the group wavelength employed, as calculated here

t = temperature in centigrade

P = atmospheric pressure in mm Hg

α = coefficient of expansion of air (α = 0.00367)

e = partial vapor pressure in mm Hg

Table I gives the vapor pressure corresponding to saturation at various temperatures

Table I. Pressure of Saturated Water Vapor

Temp. in 0° C	mm of Hg
-5	3.02
0	4.58
5	6.54
10	9.21
15	12.79
20	17.55
25	23.78
30	31.86
35	42.23
40	55.40

Let us compute an example

The following conditions are chosen as being in effect at the observing site:

φ	= latitude	38°
P	= atmosphere pressure	760 mm Hg
t	= temperature	$+10^\circ$ C
RH	= relative humidity =	60%
λ	= effective wave length	0.578
hs	= height of the object =	13.96 km
Z_o	= observed zenith distance =	70°
h_a	= height of the station above sea level	0.1 km

Computation of the Refractive Index

$$(n_s - 1)\, 10^7 = 2876.04 + \frac{16.288}{\lambda^2} + \frac{0.136}{\lambda^4}$$

$$(n_o - 1) = \frac{(n_s - 1)}{1 + \alpha t} \frac{P}{760} - \frac{0.00000055\, e}{1 + \alpha t}$$

$\lambda = 0.578$

$\lambda^2 = 0.330625$

$\lambda^4 = 0.109313$

$\frac{16.288}{\lambda^2}$ 49.264

$\frac{0.136}{\lambda^4}$ 1.244

$(n_s - 1)\, 10^7 = \frac{2876.04}{2926.55}$

$$n_s - 1 = 0.000292655$$

$$\alpha = 0.00367$$

From Table I, for 60 % R. H. and temperature = 10°

$$e = 5.53$$

so

$$n_o - 1 = 0.00028180$$

$$(n_o - 1)'' = 58''1254$$

<u>Computation of the Distance Δ</u>

$h = hs - h_a$	13.86	km
$s_o = \dfrac{h_o}{n_o r_o}$	0.001450	"
$s = \dfrac{h}{n_o r_o}$	0.002176	"
$\dfrac{h}{h_o} =$	1.5	"
$\dfrac{h}{\cos Z_o}$	40.5239	"
$- \dfrac{1}{2} s\, h \dfrac{tn^2 Z_o}{\cos Z_o}$	-0.3328	"
$+ (n_o - 1)\, h \dfrac{tn^2 Z_o}{\cos Z_o}$	+0.0862	"
$- (n_o - 1)\, h_o \dfrac{tn^2 Z_o}{\cos Z_o}$	-0.0575	"
$- s_o . h_o (n_o - 1) \dfrac{tn^2 Z_o}{\cos Z_o}$	-0.0001	"
$- \dfrac{3}{2} s\, h (n_o - 1) \dfrac{tn^2 Z_o}{\cos Z_o} (1 + tn^2 Z_o)$	-0.0024	"

$$+ 3\ s_o h_o\ (n_o - 1)\ \frac{tn^2\ Z_o}{\cos\ Z_o}\ (1 + tn^2\ Z_o) \qquad +0.0022 \text{ km}$$

$$+ \frac{1}{2}\ h\ s^2\ \frac{tn^2\ Z_o}{\cos\ Z_o}\ (1 + tn^2\ Z_o) \qquad +0.0062 \text{ "}$$

$$+ (n_o - 1)\ h_o\ \frac{tn^2\ Z_o}{\cos\ Z_o}\ e^{-\frac{h}{h_o}} \qquad +0.0128 \text{ "}$$

$$+ \frac{3}{2}\ (n_o - 1)\ h_o s_o\ \frac{tn^2\ Z_o}{\cos\ Z_o}\ e^{-\frac{h}{h_o}}\ \left(\frac{h}{h_o} + 1\right) \qquad +0.0001 \text{ "}$$

$$- 3\ (n_o - 1)\ h_o s_o\ e^{-\frac{h}{h_o}}\ \left(\frac{h}{h_o} + 1\right)\ (1 + tn^2\ Z_o)\ \frac{tn^2\ Z_o}{\cos\ Z_o} \qquad -0.0012 \text{ "}$$

$$\Delta = 40.2374 \text{ km}$$

Computation of the Coefficients E_1, E_2, M_1, M_3, and N

Equation (98)

$tn\ Z_o = 2.747477$

$tn^2 Z_o = 7.5486$

$tn^3 Z_o = 20.74$

$2 + 3\ tn^2 Z_o = 24.65$

$tn\ Z_o\ (1 + tn^2 Z_o) = 23.49$

$$E_1 = -(n_o - 1)\ tn\ Z_o + [0''0680 - 0''00010\ tn^2 Z_o]\ tn\ Z_o\ (1 + tn^2 Z_o)$$

$$\underline{E_1 = -157''.118}$$

$$E_2 = +0''033\ tn\ Z_o + 0''0081\ tn^3\ Z_o$$

$$\underline{E_2 = 0''259}$$

$$M_1 = +[0''0168 + 0''00044 \text{ tn}^2 Z_o] \text{ tn } Z_o \ (1 + \text{tn}^2 Z_o)$$

$$\underline{M_1 = +\ 2''072}$$

$$M_3 = +0''0168 \text{ tn } Z_o \ (1 + \text{tn}^2 Z_o)$$

$$\underline{M_3 = +\ 0''395}$$

$$N = -0''00006 \text{ tn } (1 + \text{tn}^2 Z_o)\ (2 + 3 \text{ tn}^2 Z_o)$$

$$\underline{N = -0''035}$$

Computation of Refraction R

Equation (94)

$A_o = +0.99827$		$\cos Z_o$	$= 0.34202$
$A_1 = -0.00130$		$\text{tn } Z_o$	$= 2.747477$
$A_2 = +0.000006$		$\text{tn}^3 Z_o$	$= 20.74$
$h = 13.86$		$\text{tn}^5 Z_o$	$= 156.6$
$h_o = 9.24$		$e^{-\frac{h}{h_o}}$	$= 0.2231$
$\Delta = 40.2374$		$e^{-3\frac{h}{h_o}}$	$= 0.0111$
$\Delta \cos Z_o = 13.7620$		$1+(n_o-1)\text{tn}^2 Z_o$	$= 1.0021$

$$E_1 = -157''.118$$

$$E_2 = +0.259$$

$$M_1 = +2.072$$

$$M_3 = +0.395$$

$$N = -0.035$$

$$1 - e^{-\frac{h}{h_o}} = 0.7769$$

$$e^{-h/h_o}\left(\frac{h}{h_o} + 1\right) - 1 = -0.4422$$

$$e^{-3\,h/h_o}\left(3\frac{h}{h_o} + 1\right) - 1 = -0.9390$$

$$2 - e^{-\frac{h}{h_o}}\left(\frac{h^2}{h_o^2} + 2\frac{h}{h_o} + 2\right) = +0.3825$$

$(n_o - 1)\,A_o\,\mathrm{tn}\,Z_o$	$159''.422$
$+\,(n_o - 1)\,A_1\,\mathrm{tn}^3 Z_o$	-1.568
$+\,(n_o - 1)\,A_2\,\mathrm{tn}^5 Z_o$	$+0.055$
$+\,\frac{9.24\,E_1}{\Delta\cos Z_o}\left[1 + (n_o - 1)\,\mathrm{tn}^2 Z_o\right]\left[1 - e^{-\frac{h}{h_o}}\right]$	-82.126
$+\,0.0134\,E_1\,\frac{\mathrm{tn}^2 Z_o}{\Delta\cos Z_o}\left[e^{-\frac{h}{h_o}}\left(\frac{h}{h_o} + 1\right) - 1\right]$	$+0.511$
$+\,\frac{4.620\,E_2}{\Delta\cos Z_o}\left[1 - e^{-\frac{h}{h_o}}\right]$	$+0.068$

$$+ \frac{9.24\ M_1}{\Delta \cos Z_0}\left[1 + (n_0 - 1)\ tn^2\ Z_0\right]\left[1 - e^{-\frac{h}{h_0}}\left(\frac{h}{h_0} + 1\right)\right] \qquad +0.616$$

$$+ \frac{1.026\ M_3}{\Delta \cos Z_0}\left[1 + (n_0 - 1) tn^2\ Z_0\right]\left[e^{-3\frac{h}{h_0}}\left(3\frac{h}{h_0} + 1\right) - 1\right] \qquad +0.028$$

$$+ \frac{9.24\ N}{\Delta \cos Z_0}\left[1 + (n_0 - 1) tn^2\ Z_0\right]\left[2 - e^{-\frac{h}{h_0}}\left(\frac{h^2}{h_0^2} + 2\frac{h}{h_0} + 2\right)\right] \qquad +0.009$$

77″.02

R = 77″.02

Computation of the Refraction ϵ

Equation (71)

$\frac{h}{h_0} = 1.5$

$e^{-\frac{h}{h_0}} = 0.2231$

$e^{-2\frac{h}{h_0}} = 0.050$

$e^{-3\frac{h}{h_0}} = 0.011$

$(n_0 - 1)\ A_0\ tn\ Z_0$	159″.422
$+ (n_0 - 1)\ A_1\ tn^3 Z_0$	- 1.568
$+ (n_0 - 1)\ A_2\ tn^5 Z_0$	+ 0.065
$+ E_1\ e^{-\frac{h}{h_0}}$	-35.053
$+ E_2\ e^{-2\frac{h}{h_0}}$	+ 0.013

$$- E_3 \, e^{-3 \frac{h}{h_o}} \qquad 0.000$$

$$+ M_1 \frac{h}{h_o} \, e^{- \frac{h}{h_o}} \qquad + 0.693$$

$$- M_3 \frac{h}{h_o} \, e^{-3 \frac{h}{h_o}} \qquad - 0.007$$

$$+ N \left(\frac{h}{h_o} \right)^2 e^{- \frac{h}{h_o}} \qquad - 0.018$$

$$123''.537$$

$$\epsilon = 2' \, 03''.54$$

Computation of the Refraction σ

$$E_1 \, e^{- \frac{h}{h_o}} \qquad - 35.053$$

$$- \frac{9.240}{\Delta \cos Z_o} \left[1 + (n_o - 1) \operatorname{tn}^2 Z_o \right] \left[1 - e^{- \frac{h}{h_o}} \right] \qquad + 82.126$$

$$+ 1.0134 \frac{E_1 \operatorname{tn}^2 Z_o}{\Delta \cos Z_o} \left[1 - e^{- \frac{h}{h_o}} \left(\frac{h}{h_o} + 1 \right) \right] \qquad - 0.511$$

$$+ E_2 \, e^{-2 \frac{h}{h_o}} \qquad + 0.013$$

$$- \frac{4.620 \, E_2}{\Delta \cos Z_o} \left[1 - e^{- \frac{h}{h_o}} \right] \qquad - 0.068$$

$$+ M_1 \frac{h}{h_o} \, e^{- \frac{h}{h_o}} \qquad + 0.693$$

$$- \frac{9.240\ M_1}{\Delta \cos Z_o} \left[1 + (n_o - 1)\ tn^2 Z_o\right] \left[1 - e^{-\frac{h}{h_o}} \left(\frac{h}{h_o} + 1\right)\right] \qquad -0.616$$

$$- M_1 \frac{h}{h_o} e^{-3\frac{h}{h_o}} \qquad -0.007$$

$$- \frac{1.026\ M_3}{\Delta \cos Z_o} \left[1 + (n_o - 1)\ tn^2 Z_o\right] \left[e^{-3\frac{h}{h_o}} \left(3\frac{h}{h_o} + 1\right) - 1\right] \qquad -0.028$$

$$+ N \left(\frac{h}{h_o}\right)^2 e^{-\frac{h}{h_o}} \qquad -0.018$$

$$- \frac{9.240}{\Delta \cos Z_o} \left[1 + (n_o - 1)\ tn^2 Z_o\right] \left[2 - e^{-\frac{h}{h_o}} \left(\frac{h^2}{h_o^2} + 2\frac{h}{h_o} + 1\right)\right] \qquad -0.009$$

46".52

$$\sigma = 46''.52$$

$$\sigma = \epsilon - R$$

$$\sigma = 46''52$$

Astronomical Refraction

$$\epsilon_a = (n_o - 1)\ A_o\ tn\ Z_o + (n_o - 1)\ A_1\ tn^3 Z_o + (n_o - 1) A_2 tn^5 Z_o$$

$$\epsilon_a = 157''91$$

The set of values obtained are as follows:

$$\epsilon_a = 2'37''91$$

$$\epsilon = 2'03''54$$

$$R = 1\ 17.02$$

$$\sigma = 46.52$$

$$R > \frac{1}{2}\epsilon$$

$$1'17''.02 > 1'01''.8$$

9. <u>Refraction Viewed from an Object Outside the Atmosphere.</u> The foregoing equations are correctly applied to an object inside the atmosphere for zenith distances less than 75° and with the height of the atmosphere of 65 kilometers, which may be regarded as the limit beyond which the air does not produce any appreciable refraction. For such conditions we have expanded tn Z in series. An object outside the atmosphere can be treated as follows:

In Figure 3 let S be the object. There is a point S_o at which the ray coming from S reaches the upper layer of the atmosphere after which following the curve S_oA, it reaches the point A.

The x and y coordinates have the same meaning as before.

At the limit of the atmosphere, formula (94) gives R_o.

ϵ_a = TNS is only the astronomic refraction.

The refraction we are looking for is R_s = TAS.

From Figure 3 it follows that

$$\sin R_s = \frac{x_s}{\Delta_s}$$

(101)

$$\sin R_o = \frac{x_o}{R_o}$$

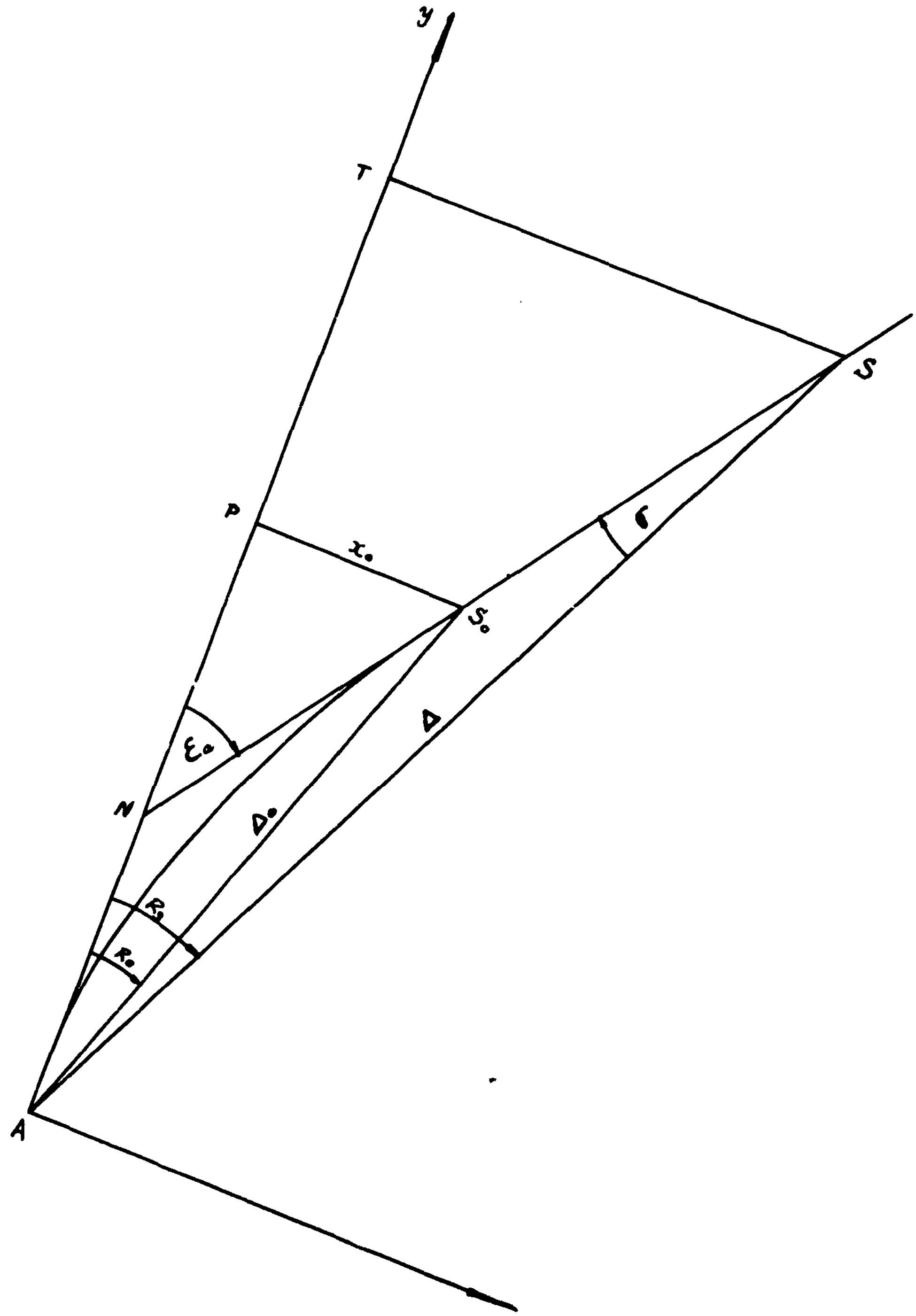

Fig. 3. Fundamental concepts in relation to Fig. 1.

but

$$x_s = NT \operatorname{tn} \epsilon_a$$

$$NT = NP + PT$$

and

$$NP = \frac{x_o}{\operatorname{tn} \epsilon_a}$$

so

$$x_s = x_o + PT \operatorname{tn} \epsilon_a$$

then

$$\sin R_s = \frac{x_o}{\Delta_s} + \frac{PT}{\Delta_s} \operatorname{tn} \epsilon_a \qquad (102)$$

Because R_s or R_o is a small angle of about 3 minutes for $Z_o = 75^\circ$

$$PT = \Delta_s - \Delta_o$$

so

$$\sin R_s = \frac{x_o}{\Delta_s} + \frac{\Delta_s - \Delta_o}{\Delta_s} \operatorname{tn} \epsilon_a \qquad (103)$$

The refraction R_o can be obtained from equation (94) without considering the terms which contain $e^{-\frac{h}{h_o}}$ and $e^{-3\frac{h}{h_o}}$, because at the height of S_o, they become extremely small. Since we can write

$$\epsilon_a = (n_o - 1) \operatorname{tn} \left[1 - \frac{3}{2}(n_o - 1)\right] + A.C_A + B.C_B + C.C_C$$

We can write equation (93) as follows:

$$R_o = \epsilon_a + \frac{E_1}{\Delta_o}\left[\frac{h_o}{\cos Z_o}\left(1 + (n_o - 1) \operatorname{tn}^2 Z_o - \frac{h_o \operatorname{tn}^2 Z_o}{n_o r_o}\right)\right]$$

but

$$x_s = NT \tan \epsilon_a$$

$$NT = NP + PT$$

and

$$NP = \frac{x_o}{\tan \epsilon_a}$$

so

$$x_s = x_o + PT \tan \epsilon_a$$

then

$$\sin R_s = \frac{x_o}{\Delta_s} + \frac{PT}{\Delta_s} \tan \epsilon_a \qquad (102)$$

Because R_s or R_o is a small angle of about 3 minutes for $Z_o = 75^\circ$

$$PT = \Delta_s - \Delta_o$$

so

$$\sin R_s = \frac{x_o}{\Delta_s} + \frac{\Delta_s - \Delta_o}{\Delta_s} \tan \epsilon_a \qquad (103)$$

The refraction R_o can be obtained from equation (94) without considering the terms which contain $e^{-\frac{h}{h_o}}$ and $e^{-3\frac{h}{h_o}}$, because at the height of S_o, they become extremely small. Since we can write

$$\epsilon_a = (n_o - 1) \tan \left[1 - \frac{3}{2} (n_o - 1)\right] + A.C_A + B.C_B + C.C_C$$

We can write equation (93) as follows:

$$R_o = \epsilon_a + \frac{E_1}{\Delta_o} \left[\frac{h_o}{\cos Z_o} \left(1 + (n_o - 1) \tan^2 Z_o - \frac{h_o \tan^2 Z_o}{n_o r_o}\right)\right]$$

$$+\frac{1}{2}\frac{E_2}{\Delta_0}\frac{h_0}{\cos Z_0}+\frac{M_1}{\Delta_0}\frac{h_0}{\cos Z_0}-\frac{M_3}{\Delta_0}\frac{h_0}{9\cos Z_0} \qquad (104)$$

Because

$$x_0 = \Delta_0 \cdot \sin R_0 = \Delta \cdot R_0$$

it follows that

$$x_0 = \epsilon \cdot \Delta_0 + E_1 \left[\frac{h_0}{\cos Z_0} (1 + (n_0 - 1) \ tn^2 Z_0 - \frac{h_0 \ tn^2 Z_0}{n_0 r_0}) \right]$$

$$(105)$$

$$+\frac{1}{2} E_2 \frac{h_0}{\cos Z_0} + M_1 \frac{h_0}{\cos Z_0} - M_3 \frac{h_0}{9 \cos Z_0}$$

Indicate by Σ all the terms of the right hand free of Δ_0 and we have

$$x_0 = \epsilon \cdot \Delta_0 + \Sigma \qquad (106)$$

Now, introduce this value x_0 into equation (103) and we obtain

$$R_s = \epsilon_a \frac{\Delta_0}{\Delta_s} + \frac{\Sigma}{\Delta_s} + \frac{\Delta_s - \Delta_0}{\Delta_s} \epsilon_a \qquad (107)$$

which simplifies to

$$R_s = \epsilon_a + \frac{\Sigma}{\Delta_s} \qquad (108)$$

because

$$\epsilon = \epsilon_a$$

Equation (108) shows that the refraction outside the atmosphere is obtained by introducing a correction to the astronomical refraction, by an amount given by $\frac{\Sigma}{\Delta_s}$.

Since Δ_s can be any distance, for an object at infinity

$$\frac{\Sigma}{\Delta_s} = 0$$

consequently,

$$R_s = \epsilon_a$$

which is the astronomical refraction.

10. Computation of the Distance Δ of an Object Outside the Atmosphere. Equation (97) is not good now for computing the distance of an object outside the atmosphere because it was developed to be used inside the atmosphere. For a height over 65 kilometers the following equation must be used:

$$\Delta = \frac{1}{2}\, s\, r_o \frac{1 + \cos\theta}{\cos\frac{1}{2}\theta\, \cos(\gamma - \sigma)} \tag{109}$$

where

$$s = \frac{h_s}{r_o}$$

$$\theta = Z_o + \epsilon_a - Z$$

ϵ_a = astronomical refraction

$$Z = \text{arc sin}\,\left(\frac{n_o}{1+s}\sin Z_o\right)$$

$$\epsilon_a = (n_o - 1)\, A_o\, \text{tn}\, Z_o + (n_o - 1) A_1 \text{tn}^3 Z_o + (n_o - 1) A_2 \text{tn}^5 Z_o$$

$$\gamma = \frac{1}{2}(Z_o + \epsilon_a + Z)$$

σ always is a small angle (a few seconds) so we can assume $\sigma = 0$ for computing the distance Δ, with sufficient accuracy for obtaining R or σ. To obtain an error dR = 0″01 the error $\delta\Delta$ in Δ reaches the following values for a zenith distance $Z_0 = 70^\circ$:

$$H = 100 \text{ km} \quad \text{is } \delta\Delta = 200 \text{ meters}$$

$$H = 1000 \text{ km} \quad \text{is } \delta\Delta = 10{,}000 \text{ meters}$$

If it is required to obtain Λ with higher accuracy, in equation (109) the value of σ computed from equation (111) must be introduced.

Then, the refraction R can be obtained by using the following equation:

$$\begin{aligned} R = {} & (n_0 - 1)\, A_0 \operatorname{tn} Z_0 + (n_0 - 1)\, A_1 \operatorname{tn}^3 Z_0 + (n_0 - 1)\, A_2 \operatorname{tn}^5 Z_0 \\ & + \frac{9.24\, E_1}{\Delta \cos Z_0} \left[1 + (n_0 - 1) \operatorname{tn}^2 Z_0\right] \\ & - 0.0134 \frac{E_1 \operatorname{tn}^2 Z_0}{\Lambda \cos Z_0} + \frac{4.62\, E_2}{\Lambda \cos Z_0} \\ & + \frac{9.24\, M_1}{\Delta \cos Z_0} \left[1 + (n_0 - 1) \operatorname{tn}^2 Z_0\right] \\ & - 1.026 \frac{M_3 \left[1 + (n_0 - 1) \operatorname{tn}^2 Z_0\right]}{\Delta \cos Z_0} + \frac{18.48\, N}{\Lambda \cos Z_0} \end{aligned} \tag{110}$$

11. <u>Computation of the Refraction Angle σ of an Object outside the Atmosphere</u>. This refraction is the angle σ. It can be obtained by using the following equation:

$$\sigma = -\frac{9.24\, E_1}{\Lambda \cos Z_0} \left[1 + (n_0 - 1) \operatorname{tn}^2 Z_0\right] + 0.0134 \frac{E_1 \operatorname{tn}^2 Z_0}{\Lambda \cos Z_0}$$

$$- \frac{4.62 E_2}{\Delta \cos Z_0} - 9.24 \frac{M_1 \left[1 + (n_0 - 1) \operatorname{tn}^2 Z_0\right]}{\Delta \cos Z_0} \qquad (111)$$

$$+ 1.026 \frac{M_3 \left[1 + (n_0 - 1) \operatorname{tn}^2 Z_0\right]}{\Delta \cos Z_0} - \frac{18.48 N}{\Delta \cos Z_0}$$

Example of Computation

We assume the same data as used for computing R and σ for an object inside the atmosphere, and change h_s to:

$$h_s = 100 \text{ kilometers}$$

$$h_s = 1{,}000 \text{ kilometers}$$

as before

$$Z_0 = 70^\circ$$

and assuming

$$h_a = 0$$

the constants used are:

$$E_1 = - 157''.118$$

$$E_2 = + \quad 0.259$$

$$M_1 = + \quad 2.072$$

$$M_3 = + \quad 0.395$$

$$N = - \quad 0.035$$

$$\gamma_0 = 6370.06 \text{ km}$$

Computation of the Distance

$$\sin Z = \frac{n_0}{(1 + s)} \sin Z_0$$

	H = 100 km	H = 1,000 km
s	0.0156984	0.1569840
$\sin Z_0$	0.9396926	
n_0	1.0002818	
$\sin Z$	0.9254296	0.8124204
Z	67° 43′ 59″.6	54° 19′ 59.0
Z_0	70°	70°
ϵ_a	2′ 37.9	2 37.9
γ	68 53 18.7	62 11 18.4
θ	2 18 38.3	15 42 38.9
$\frac{1}{2}\theta$	1 09 19.2	7 51 19.4
$1 + \cos\theta$	1.999 1868	1.962 6407
$\cos\gamma$	0.36018 36	0.466 5650
Δ	277.579	2123.206
$\Lambda . \cos Z_0$	94.938	
$\gamma - \sigma$	58° 53′03″6	62° 11′16″3
$\cos(\gamma - \sigma)$	0.360 2519	0.466 5741
Λ	277.5264	2123.1698
$\Delta \cos Z_0$	94.920	726.167

Computation of the Refraction σ

	H = 100	H = 1,000
$-\ 9.24\ \frac{E_1\left[1 + (n_o - 1)tn^2\ Z_o\right]}{\Delta \cos Z_o}$	+15.422	+2.016
$+\ 0.0134\ \frac{E_1\ tn^2\ Z_o}{\Delta \cos Z_o}$	-0.167	-0.022
$-\ \frac{4.62\ E_2}{\Delta \cos Z_o}$	-0.013	-0.002
$-\ \frac{9.24\ M_1\left[1 + (n_o - 1)\ tn^2\ Z_o\right]}{\Delta \cos Z_o}$	-0.202	-0.026
$+\ 1.026\ \frac{M_3\left[1 + (n_o - 1)\ tn^2\ Z_o\right]}{\Delta \cos Z_o}$	+0.043	+0.001
$-\ \frac{18.48\ N}{\Delta \cos Z_o}$	+0.006	+0.001
σ =	+15″083	1.968

Now, introduce these values of σ, and the new values of Δ are:

H = 100 km: Δ = 277.5264 km

H = 1000 km: Δ = 2123.1698 km

These new values do not alter the preceding value.

Then, the results are as follows:

H = 100 km H = 1000 km

$$R = \epsilon_a - R$$

$\epsilon_a = 2'37''.9$	$\epsilon_a = 2'37''.9$
$\sigma = 15''.1$	$\sigma = 2''.0$
$R = 2'22''.8$	$R = 2'35''.9$

AD ______________________ Accession No. ____________

U. S. Army Engineer Geodesy, Intelligence and Mapping Research and Development Agency, Fort Belvoir, Virginia - FORMULAS FOR COMPUTING ATMOSPHERIC REFRACTION FOR OBJECTS INSIDE OR OUTSIDE THE ATMOSPHERE - Angel A. Baldini

Research Note No. 8, 9 January 1963, 56 pp, 3 illus, 1 table
DA Task 8T35-12-001-01 Unclassified Report

Report presents derivation of new equations for determining changes in direction of a ray of light as it passes through the atmosphere from an object to observer. Equations here are applicable to objects inside or outside the atmosphere. Equations are also derived for obtaining topocentric distance of object as function of object's height and observed zenith distance.

UNCLASSIFIED

1. Mapping, Charting & Geodesy - Mapping & Geodetic Research

2. Contract - None

AD ______________________ Accession No. ____________

U. S. Army Engineer Geodesy, Intelligence and Mapping Research and Development Agency, Fort Belvoir, Virginia - FORMULAS FOR COMPUTING ATMOSPHERIC REFRACTION FOR OBJECTS INSIDE OR OUTSIDE THE ATMOSPHERE - Angel A. Baldini

Research Note No. 8, 9 January 1963, 56 pp, 3 illus, 1 table
DA Task 8T35-12-001-01 Unclassified Report

Report presents derivation of new equations for determining changes in direction of a ray of light as it passes through the atmosphere from an object to observer. Equations here are applicable to objects inside or outside the atmosphere. Equations are also derived for obtaining topocentric distance of object as function of object's height and observed zenith distance.

UNCLASSIFIED

1. Mapping, Charting & Geodesy - Mapping & Geodetic Research

2. Contract - None

AD ______________________ Accession No. ____________

U. S. Army Engineer Geodesy, Intelligence and Mapping Research and Development Agency, Fort Belvoir, Virginia - FORMULAS FOR COMPUTING ATMOSPHERIC REFRACTION FOR OBJECTS INSIDE OR OUTSIDE THE ATMOSPHERE - Angel A. Baldini

Research Note No. 8, 9 January 1963, 56 pp, 3 illus, 1 table
DA Task 8T35-12-001-01 Unclassified Report

Report presents derivation of new equations for determining changes in direction of a ray of light as it passes through the atmosphere from an object to observer. Equations here are applicable to objects inside or outside the atmosphere. Equations are also derived for obtaining topocentric distance of object as function of object's height and observed zenith distance.

UNCLASSIFIED

1. Mapping, Charting & Geodesy - Mapping & Geodetic Research

2. Contract - None

AD ______________________ Accession No ____________

U. S. Army Engineer Geodesy, Intelligence and Mapping Research and Development Agency, Fort Belvoir, Virginia - FORMULAS FOR COMPUTING ATMOSPHERIC REFRACTION FOR OBJECTS INSIDE OR OUTSIDE THE ATMOSPHERE - Angel A. Baldini

Research Note No. 8, 9 January 1963, 56 pp, 3 illus, 1 table
DA Task 8T35-12-001-01 Unclassified Report

Report presents derivation of new equations for determining changes in direction of a ray of light as it passes through the atmosphere from an object to observer. Equations here are applicable to objects inside or outside the atmosphere. Equations are also derived for obtaining topocentric distance of object as function of object's height and observed zenith distance.

UNCLASSIFIED

1. Mapping, Charting & Geodesy - Mapping & Geodetic Research

2. Contract - None