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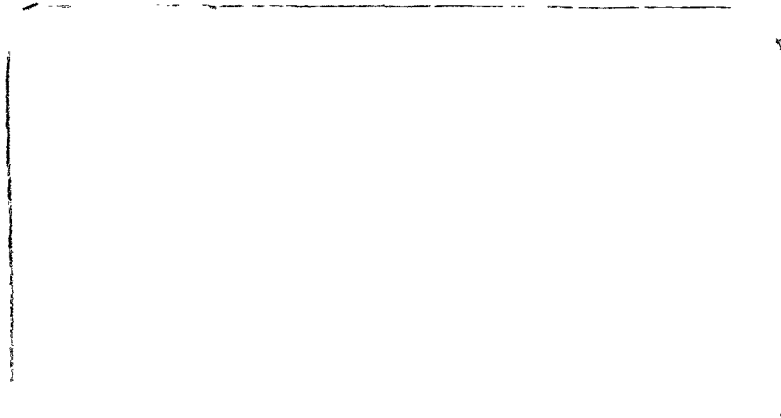
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JOBSHOP-LIKE QUEUEING SYSTEMS

by

James R. Jackson

Summary. The equilibrium joint probability distribution of queue lengths is obtained for a broad class of jobshop-like "networks of waiting lines," where the mean arrival rate of customers depends almost arbitrarily upon the number already present, and the mean service rate at each service center depends almost arbitrarily upon the queue length there. This extension of the author's earlier work is motivated by the observation that real production systems are usually subject to influences which make for increased stability by tending, as the amount of work-in-process grows, to reduce the rate at which work is injected or to increase the rate at which processing takes place.

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JOBSHOP-LIKE QUEUEING SYSTEMS

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Summary. The equilibrium joint probability distribution of queue lengths is obtained for a broad class of jobshop-like "networks of waiting lines," where the mean arrival rate of customers depends almost arbitrarily upon the number already present, and the mean service rate at each service center depends almost arbitrarily upon the queue length there. This extension of the author's earlier work is motivated by the observation that real production systems are usually subject to influences which make for increased stability by tending, as the amount of work-in-process grows, to reduce the rate at which new work is injected or to increase the rate at which processing takes place.

1. Introduction

This paper provides the equilibrium joint probability distribution of queue lengths for a broad class of queueing-theoretical models representing multipurpose production systems composed of special-purpose service centers (e.g, manufacturing jobshops). In the systems modelled, customers arrive from time to time, each with a routing, which is an ordered list of service centers where the customer must be served. An arriving customer joins the queue at the first center on his routing, remains there until his service there is completed, then goes to the second center on his routing (if any) and remains there until served, and so forth, until all his required services are completed, at which time he leaves the system.

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1. This work was supported by the Office of Naval Research under Task 047-003 and by the Western Management Science Institute Ford Foundation grant. Reproduction in whole or in part is permitted for any purpose of the United States Government.
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The customer arrivals are modelled as a generalized Poisson process, whose parameter (mean arrival rate) varies almost arbitrarily with the total number of customers already in the system. Service completions at each center are also modelled as generalized Poisson processes, the parameter (mean service rate) at each center varying almost arbitrarily with the queue length there (number of customers in the queue, waiting and being served).

The generation of routings is modelled as a random walk among the names of the service centers, which with probability one reaches a terminal state marking the ends of routings. The mechanism for generating routings is sufficiently general to include, for instance, cases where every conceivable routing may occur, the possibility that all customers require services at all centers in the same order, and instances in which each customer requires service at just one of the centers.

The mathematical model does not explicitly reflect any particular rule for deciding which, among several customers at a service center, is to be served "next." The model and its analysis are, in fact, consistent with all rules which do not depend (directly or indirectly) upon the future routing or service time requirements of any customer. Perhaps the most commonly used rule in this category, and the one whose operation is most easily visualized, is the first-come, first-serve rule.

Results. The basic result of this paper is the equilibrium joint probability distribution of the queue lengths at the several centers in the jobshop-like queueing system just outlined. This result is stated as a theorem in Section 4, following the detailed formulation of the basic model and preliminary analysis in Sections 2 and 3. In Section 5, the theorem of Section 4 is generalized to cover cases where the immediate injection of a

new customer is triggered when the total number of customers falls below a specified limit, or where a service is deleted if a queue grows beyond a specified maximum length. Section 5 includes remarks about the theorem and points to some of its implications.

Section 6 is devoted to special cases. Systems with constant customer-arrival rate are considered; and it turns out that the main theorem of Section 5 reduces in this case to a statement that the equilibrium joint probability distribution of queue lengths is identical with what would be obtained by pretending that each individual service center is a separate queueing system independent of the others. The constant arrival-rate case is further specialized to the instance where each center is an "ordinary" multi-channel service system.³ Next, to clarify the effect of varying arrival rates, attention is directed to a special case of the general model, in which all service rates are the same. Finally, systems are considered in which the total number of customers present is held fixed.

Significance. The generalized customer-arrival process of the present paper allows for the representation of systems whose "potential customers" are generated by a stable Poisson process, but where the probability that a potential customer will actually enter the system for service depends upon how many customers are already there. In addition to the representation of "ordinary" multiserver centers, the generalized service-completion process also permits the treatment of systems at whose centers the number of servers actually functioning depends almost arbitrarily upon the queue length there, as well as plausible rough approximations to such phenomena as "speedups" and "overtime work." The further generalizations (Section 5) of the customer-

3. Which was analyzed previously in the author's paper, "Networks of Waiting Lines," Operations Research, vol. 5, no. 4 (Aug., 1957).

arrival and service-completion processes make possible crude approximations to such phenomena as the activation of "standby jobs" when a shopload falls off, and "subcontracting" when a service center becomes overloaded.

2. Notations and Restrictions

The first business of this section is to specify and provide notations for the parameters of the generic jobshop-like queueing system. As this is done, these parameters are interpreted and restrictions are stated. The model is mathematically formalized in Section 3.

States of the system. The number of service centers in the system will be denoted by N , a positive integer; and the centers will be referred to as Center 1, Center 2, etc. The states of the system are N -dimensional vectors with non-negative integer components, the n -th component representing the queue length at Center n . Such vectors will be called state vectors. If $\bar{k} = (k_1, k_2, \dots, k_N)$ is a state vector, then $S(\bar{k}) = k_1 + k_2 + \dots + k_N$, the total number of customers in the system when its state is \bar{k} .

Customer-arrival and service-completion processes.⁴ The customer-arrival process is specified by a set of parameters,

$$L = \{ \lambda(k) \mid k \in [0, \infty) \} .$$

The interpretation is that if the system is in state \bar{k} at time t , then the probability that a customer will arrive between then and time $t+h$ is $h \lambda(S(\bar{k})) + o(h)$. Similarly, the service-completion processes are specified by a set of parameters,

$$M = \{ \mu(n, k) \mid n \in [0, \infty) \} ;$$

with the interpretation that if k_n customers are present at Center n

4. If a and b are integers with $a \leq b$, then $[a, \infty)$ denotes the set of integers not smaller than a , and $[a, b]$ denotes the set of integers between a and b inclusive. The expression $o(h)$ denotes unspecified quantities such that as h tends to the limit zero, so does $o(h)/h$.

at time t , then the probability that the service of one of these customers will be completed by time $t+h$ is $h \mu(n, k_n) + o(h)$. The probability of two or more events (arrivals or completions) in a time interval of length h is $o(h)$.

From their interpretations, the $\lambda(K)$ and the $\mu(n, k)$ must be non-negative, and also each $\mu(n, 0) = 0$. However, it is convenient, and seems to result in no loss of interesting generality, to make somewhat stronger assumptions, as follows:

ASSUMPTION (2.1). Either the $\lambda(K)$ are all positive; or for some non-negative K_0 , $\lambda(K) > 0$ if $K \leq K_0$ but $\lambda(K) = 0$ if $K > K_0$.

ASSUMPTION (2.2). The $\mu(n, k)$ are all positive, except that each $\mu(n, 0) = 0$.

(These assumptions may seem to exclude the possibly interesting case where a service center is "shut down" if the queue length there falls to some specified lower limit. This case can, however, easily be brought into this paper's analytical framework, by redefining the n -th component of the state vector as the number of customers at Center n in excess of the lower limit there, and appropriately translating the arguments of $\lambda(K)$ and $\mu(n, k)$.)

Generation of routings. The routing-generation process is specified by a set of parameters,

$$R = \{r(m, n) \mid m \in [0, N] ; n \in [1, N+1]\} .$$

The interpretation is that, for m and $n \in [1, N]$: (i) the probability is $r(0, n)$ that Center n will be first on a routing; (ii) the probability is $r(m, N+1)$ that if Center m is i -th on a routing, then this i -th element is the last one; (iii) the probability is $r(m, n)$ that if Center m is i -th on a routing, then there is an $(i+1)$ -st element and it is Center n ; and,

for completeness, (iv) the probability is $r(0, N+1)$ that a routing will be empty. It is easily seen that the probability of routing (n_1, n_2, \dots, n_1) is the product,

$$r(0, n_1) r(n_1, n_2) \dots r(n_{i-1}, n_i) r(n_i, N+1) .$$

Two assumptions are made:

ASSUMPTION (2.3). For each $m \in [0, N]$, the set $\{ r(m, n) \mid n \in [1, N+1] \}$ is a probability distribution.

ASSUMPTION (2.4). The set of equations,

$$(2.5) \quad e(n) = r(0, n) + \sum_{m=1}^N e(m) r(m, n) , \quad n \in [1, N] ,$$

has a unique solution, $\{ e(n) \mid n \in [1, N] \}$; and the $e(n)$ are all non-negative.

Assumption (2.3) is necessary in view of the interpretation of the $r(m, n)$. In the presence of (2.3), Assumption (2.4) can be shown to be equivalent to the requirement that a routing terminate with probability one. It is intuitively obvious (and can easily be proved) that $e(n)$ is the expected value of the number of appearances of Center n on a routing.

3. Mathematical Model

The informal interpretations of Section 2 are now given mathematical substance by stating the conditional probabilities that the generic jobshop-like queueing system will be in various states $\bar{j} = (j_1, j_2, \dots, j_N)$ at time $t+h$, given that it is in state $\bar{k} = (k_1, k_2, \dots, k_N)$ at time t . These probabilities are as follows:

$$1 - h \lambda(S(\bar{k})) \sum_{n=1}^N r(0, n) - h \sum_{n=1}^N \mu(n, k_n) (1 - r(n, n)) + o(h) , \quad \text{if } \bar{j} = \bar{k} ;$$

$$h \lambda(S(\bar{k})) r(0, x) + o(h) , \quad \text{if } \bar{j} = \bar{k} \text{ except } j_x = k_x + 1 ;$$

$$h \mu(x, k_x) r(x, N+1) + o(h) , \quad \text{if } \bar{j} = \bar{k} \text{ except } j_x = k_x - 1 ;$$

$$h \mu(x, k_x) r(x, y) + o(h) , \quad \text{if } \bar{j} = \bar{k} \text{ except } j_x = k_x - 1 , j_y = k_y + 1 ;$$

$$o(h^{s-1}) , \quad \text{for } \bar{j} \text{ with } |S(\bar{j}) - S(\bar{k})| = s > 1 ;$$

where x in the second and third expressions runs through $[1, N]$; and x and y in the fourth expression run through $[1, N]$, omitting pairs with $x = y$.

The first four of the above expressions reflect, respectively, the possibility of no "events" occurring in the interval between times t and $t+h$ (except perhaps the completion of a service for a customer who then requires another service at the same center), a customer arrival, the completion of the last service on a customer's routing, and the completion of one service for a customer who then requires another service at a different center. The fifth expression reflects the need for at least $|S(\bar{j}) - S(\bar{k})|$ events to intervene between the appearances of states \bar{j} and \bar{k} .

The process defined by the preceding expressions, with the notations and subject to the assumptions of Section 2, will be referred to as Jobshop-like queueing system (N, L, M, R) , or more briefly as System (N, L, M, R) .

Time-dependent state probabilities. Let $P(\bar{k}, t)$ be the probability that System (N, L, M, R) is in state \bar{k} at time t ; and let $P(\bar{k}, t' | \bar{j}, t)$ be the conditional probability of state \bar{k} at time t' , given state \bar{j} at time t . From elementary probability theory,

$$P(\bar{k}, t+h) = \sum P(\bar{j}, t) P(\bar{k}, t+h | \bar{j}, t) ,$$

the sum extending over all state vectors \bar{j} ; for each state vector \bar{k} . If the conditional probabilities given previously are substituted in this equation; and then $P(\bar{k}, t)$ is subtracted from both sides; and then both sides are divided by h ; and then, finally, the limit is taken as h tends to zero; the final result is the following differential-difference equation:

$$(3.1) \quad \frac{dP(\bar{k}, t)}{dt} = -[\lambda(S(\bar{k})) + \sum_n \mu(n, k_n) (1 - r(n, n))] P(\bar{k}, t) \\ + \sum_n \lambda(S(\bar{k}) - i) r(0, n) P(\bar{h}(n), t) \\ + \sum_n \mu(n, k_n + 1) r(n, N+1) P(\bar{i}(n), t) \\ + \sum_{mn} \mu(n, k_n + 1) r(n, m) P(\bar{j}(m, n), t) ;$$

where the sums extend over $[1, N]$, except that pairs with $m = n$ are omitted from the double sum, and terms are omitted for which the vector argument of P has a negative component; $\bar{h}(n) = \bar{k}$, except that its n -th component is $k_n - 1$; $\bar{i}(n) = \bar{k}$, except that its n -th component is $k_n + 1$; and $\bar{j}(m, n) = \bar{k}$, except that its m -th component is $k_m - 1$ and its n -th component is $k_n + 1$.

4. Equilibrium for Jobshop-Like Queueing System (N, L, M, R)

By definition, an equilibrium state probability distribution for System (N, L, M, R) is a probability distribution, $\{p(\bar{k})\}$, over state vectors \bar{k} , such that $P(\bar{k}, t) \equiv p(\bar{k})$ specifies a (constant) solution to equations (3.1). If such a distribution exists, it is unique, and for each state vector \bar{k} ,

$$\lim_{t \rightarrow \infty} P(\bar{k}, t) = p(\bar{k}),$$

the limits being independent of the "initial" state of the system.⁵ Thus, the equilibrium state probability distribution provides a rather complete description of the long-run-average behavior of the state of the system. The theorem of this section gives a condition under which such a distribution exists, and specifies the distribution.

5. That the $P(\bar{k}, t)$ approach limits follows from Theorem 2.9, p. 102, A.T. Barucha-Reid, Elements of the Theory of Markov Processes and their Applications (McGraw-Hill, 1960). It can be argued, using Assumptions (2.1) and (2.2) that every non-transient state of System (N, L, M, R) communicates with every other, and hence that if a limiting probability distribution exists, it is unique. The assertions then follow from the observation that the equilibrium state probability distribution is one possible "initial" distribution over state vectors.

The following notations will be used in stating the theorem, and subsequently:⁶

$$(4.1) \quad W(K) = \prod_{i=0}^{K-1} \lambda(i) \quad , \text{ for } K = 0, 1, 2, \dots ;$$

$$(4.2) \quad w(\bar{k}) = \prod_{n=1}^N \prod_{i=1}^k [e(n)/\mu(n,i)] \quad , \text{ for } \bar{k} = (k_1, k_2, \dots, k_N)$$

a state vector;

$$(4.3) \quad T(K) = \sum w(\bar{k}) \quad , \text{ summed over state vectors } \bar{k} \text{ with } S(\bar{k}) = K \quad ,$$

for $K = 0, 1, 2, \dots ;$

$$(4.4) \quad \pi = \left\{ \sum_{K=0}^{\infty} W(K) T(K) \right\}^{-1} \quad , \text{ if the sum converges,}$$

$= 0 \quad , \text{ otherwise.}$

It is easy to verify that the sum in (4.4) converges to a positive number or diverges to plus infinity.

The proof of the following theorem (after the preceding development) is a straightforward matter of verifying that (4.6) does define a probability distribution ("obvious"), and that equations (3.1) are satisfied by this distribution (by direct substitution and a moderately tedious, but routine, algebraic reduction):

THEOREM (4.5). If $\pi > 0$, then a unique equilibrium state probability distribution exists for Jobshop-like queueing system (N,L,M,R) , and is given by

$$(4.6) \quad p(\bar{k}) = \pi w(\bar{k}) W(S(\bar{k})) \quad ,$$

for \bar{k} a state vector.

An example. The above theorem is extended in Section 5, and the statement of remarks and derivative conclusions will be deferred until this is done. As a preliminary illustration of the theorem, consider a

6. Empty products are assigned the conventional value, +1 .

two-service-center jobshop-like queueing system for which the parameters are as follows:⁷

$$(4.7) \quad \lambda(K) = a/(K+1)^x, \text{ where } a > 0 \text{ and } x \geq 0;$$

$$(4.8) \quad \mu(1,k) = b k^y, \text{ where } b > 0 \text{ and } y \geq 0;$$

$$(4.9) \quad \mu(2,k) = c k^z, \text{ where } c > 0 \text{ and } z \geq 0;$$

$$(4.10) \quad r(0,1) = r(1,2) = r(2,3) = 1; \text{ the other } r(m,n) = 0.$$

The specification of R implies that each customer requires exactly two services, the first at Center 1 and the second at Center 2; and it follows that $e(1) = e(2) = 1$.

After substituting in (4.1) through (4.4), it is not difficult to verify that $\pi > 0$ unless both (i) $x = 0$ and (ii) either $y = 0$ and $a/b \geq 1$ or $z = 0$ and $a/c \geq 1$. If $\pi > 0$, then Theorem (4.5) provides the joint equilibrium probability distribution of queue lengths at the two centers, as follows:

$$(4.11) \quad p(i,j) = \frac{\pi(a/b)^i (a/c)^j}{((i+j)!)^x (i!)^y (j!)^z}.$$

The numerical value of π can be obtained for specific values of the parameters by using the fact that the $p(i,j)$ must sum to one.

Variations upon this example will be used again in subsequent sections.

5. Triggered Arrivals and Service Deletions

In this section, the concept of System (N,L,M,R) is generalized to include cases where the immediate injection of a new customer is

7. The particular example used here was suggested by R. W. Conway and W. L. Maxwell, "A Queueing Model with State Dependent Service Rates," Journal of Industrial Engineering, vol. XII, no. 2 (1962); which treats a queueing system equivalent to a single one of the example's two interacting service centers.

triggered whenever the total number of customers falls below a specified limit, or where a service is deleted if a queue grows beyond a specified maximum length. The generalized model, to be referred to subsequently as Jobshop-like queueing system $(N,L,M,R)^*$, or as System $(N,L,M,R)^*$, is identical with System (N,L,M,R) , except for the following:

(i) There is some non-negative integer, K^* ; such that the initial state \bar{k} satisfies $S(\bar{k}) \geq K^*$; and if the last service on some customer's routing is completed when the system is in a state \bar{k} with $S(\bar{k}) = K^*$, then the arrival of a new customer occurs automatically at that instant.

(ii) For each Center n , there is a k_n^* , which may be a positive integer or $+\infty$; such that the initial state, $\bar{k} = (k_1, k_2, \dots, k_N)$, satisfies $k_n \leq k_n^*$ for $n \in [1, N]$; and if a customer arrives at center n when the queue length there is k_n^* , then the imminent service required by that customer at Center n is deleted from his routing, and he proceeds from there immediately, as if he had been served.³

It is implicit in the above statements that $K^* \leq k_1^* + k_2^* + \dots + k_N^*$. If $K^* > 0$, then the $\lambda(K)$ for $K < K^*$ are obviously meaningless; and if $k_n^* < +\infty$, then the $\mu(n, k)$ for $k > k_n^*$ are obviously meaningless.

System (N,L,M,R) is the special case of System $(N,L,M,R)^*$ with $K^* = 0$ and each $k_n^* = +\infty$. To formalize System $(N,L,M,R)^*$ would involve rewriting the conditional probabilities (beginning of Section 3) for cases where $S(\bar{k}) = K^*$ and/or one or more $k_n = k_n^*$, and rewriting equations (3.1) correspondingly. This will not be done here, because the modified equations are of insufficient interest to justify the development of (necessarily

3. Or, more generally, some customer is emitted from the center; with the restriction that the choice of the customer to be emitted must not depend upon the future routing or service requirements of any customer.

rather formidable) appropriate notations. But it is necessary to give generalizations of equations (4.1) through (4.4) to prepare for the generalization of Theorem (4.5):

$$(5.1) \quad W^*(K) = 0, \text{ for } K < K^*, \\ = \prod_{i=K^*}^{K-1} \lambda(i), \text{ for } K \geq K^* ;$$

$$(5.2) \quad w^*(\bar{k}) = \prod_{n=1}^N \prod_{i=1}^{k_n} [e(n)/\mu(n,i)], \text{ if } k_n \leq k_n^* \text{ for } n \in [1,N], \\ = 0, \text{ otherwise;}$$

$$(5.3) \quad T^*(K) = \sum w^*(\bar{k}), \text{ summed over } \bar{k} \text{ with } S(\bar{k}) = K;$$

$$(5.4) \quad \pi^* = \left\{ \sum_{K=0}^{\infty} W^*(K) T^*(K) \right\}^{-1}, \text{ if the sum converges,} \\ = 0, \text{ otherwise}$$

If $K^* = 0$ and each of the $k_n^* = +\infty$, then equations (5.1) through (5.4) specialize to equations (4.1) through (4.4), upon removal of the asterisks.

The following theorem, which can be proved by substitution in the modified version of equations (3.1), differs from Theorem (4.5) only in the presence of the asterisks:

THEOREM (5.5). If $\pi^* > 0$, then a unique equilibrium state probability distribution exists for Jobshop-like queueing system $(N,L,M,R)^*$, and is given by

$$(5.6) \quad p(\bar{k}) = \pi^* w^*(\bar{k}) W^*(S(\bar{k})),$$

for \bar{k} a state vector.

Remarks. It is natural to conjecture that if $\pi^* = 0$, then no equilibrium state probability distribution exists for System $(N,L,M,R)^*$, but I have been unable to prove this except in special cases. If either (i) there is some (finite) K_0 such that $\lambda(K) = 0$ for $K > K_0$, or

(ii) all of the k_n^* are finite; then only finitely many terms in the sum of (5.4) can be positive, so it is assured that $\pi^* > 0$, and the theorem applies.

The "discovery" of Theorem (4.5) resulted from making a sequence of "guesses" concerning more and more general jobshop-like queueing systems, and proving successively more general versions of the theorem. The way in which this took place may be suggested by the discussion of certain special cases in Section 6.

Some corollary results. Suppose that System $(N, L, M, R)^*$ satisfies the condition, $\pi^* > 0$, so that Theorem (5.5) applies. Write $p(S=K)$ for the equilibrium probability that the total number of customers in the system is K ; i.e., that the state \bar{k} satisfies $S(\bar{k}) = K$. If $p(S = K) > 0$, and \bar{k} is a state vector, write $p(\bar{k}|K)$ for the equilibrium conditional probability of state \bar{k} , given that the total number of customers in the system is K ; i.e., $p(\bar{k}|K) = p(\bar{k})/p(S=K)$ if $S(\bar{k}) = K$, but otherwise $p(\bar{k}|K) = 0$. It follows directly from the theorem that:

$$(5.7) \quad p(S = K) = \pi^* T^*(K) W^*(K) ;$$

$$(5.8) \quad p(\bar{k}|S(\bar{k})) = w^*(\bar{k})/T^*(S(\bar{k})), \text{ if } p(S = S(\bar{k})) > 0 .$$

The probabilities (5.7) are of interest in themselves, because there are instances where the total number of customers in the system is of primary concern. They also lead to the overall mean arrival rate of customers under equilibrium, $E(\lambda)$:

$$(5.9) \quad E(\lambda) = \sum_{K=0}^{\infty} \lambda(K) p(S = K) \\ = \pi^* \sum_{K=0}^{\infty} W^*(K+1) T^*(K) .$$

Equation (5.9) is potentially of special interest, because it allows the determination of "loss rates" when the variation in the $\lambda(K)$ is due to the failure of "potential customers" to enter the system for service when too many customers are already present.

The main interest of the conditional probabilities (5.8) perhaps stems from the observation that they do not depend upon the customer-arrival process -- except for its role in determining whether $\pi^* > 0$, and possibly in determining the values of K for which $p(S = K) > 0$. In a sense, the effect of particular customer-arrival-process parameters is concentrated upon the determination of the $p(S = K)$.

Note also that the parameters of the routing-generation process influence the equilibrium state probability distribution only by way of the $e(n)$. It was remarked previously that $e(n)$ is the expected number of appearances of Center n on a routing.

Example. Consider a system identical with the example given at the end of Section 4, except that for some K^* , k_1^* , and k_2^* : (i) if the total number of customers in the system falls below K^* , then a new customer is instantly "created"; (ii) if a $(k_1^* + 1)$ -st customer joins the queue at Center 1, he immediately moves on to Center 2, without being served at Center 1; and (iii) if a $(k_2^* + 1)$ -st customer joins the queue at Center 2, he immediately leaves the system without being served there. Equation (4.11) is still valid for the $p(i,j)$ such that $i + j \geq K^*$, $i \leq k_1^*$, and $j \leq k_2^*$; but all other $p(i,j) = 0$. There is a change, of course, in the numerical value of π (which might be replaced by π^* for consistency with the notation of the present section).

6. Special Cases

The first part of this section is restricted to the special case of System $(N,L,M,R)^*$ in which $\lambda(K) \equiv \lambda(0)$, to be referred to as the constant-arrival-rate case of System $(N,L,M,R)^*$. For this case, it is convenient to define the following notations, for $n \in [1,N]$:

$$(6.1) \quad w_n(k) = \prod_{i=1}^k [\lambda(0) e(n)/\mu(n,i)] , \text{ for } k \leq k_n^* , \\ = 0 , \text{ for } k > k_n^* ;$$

$$(6.2) \quad p_n(k) = w_n(k) / \sum_{i=0}^{\infty} w_n(i) , \text{ if the sum converges,} \\ = 0 , \text{ otherwise, for } k = 0, 1, 2, \dots .$$

The following theorem is the direct specialization of Theorem (5.5) to the constant-arrival-rate case:

THEOREM (6.3). If, in the constant-arrival-rate case of System $(N,L,M,R)^*$, $p_n(0) > 0$ for $n \in [1,N]$; then a unique equilibrium state probability distribution exists for the system and is given by the product,

$$(6.4) \quad p(k_1, k_2, \dots, k_N) = p_1(k_1) p_2(k_2) \dots p_N(k_N) ,$$

for (k_1, k_2, \dots, k_N) a state vector.

Interpretation. When Theorem (6.3) applies, it is immediate from (6.2) that the $p_n(k)$ for fixed n form a probability distribution over $k = 0, 1, 2, \dots$. This distribution is, in fact, the equilibrium distribution of queue length for the one-service-center queuing system where customer arrivals form a Poisson process with mean rate $\lambda(0) e(n)$, and whose service-completion process is identical to that of Center n (which is well known, and also follows from the one-service-center case of the theorem).

But $e(n)$ is the mean number of appearances of Center n on a routing in the generic system to which Theorem (6.3) refers, whence $\lambda(0) e(n)$ is in fact the mean rate of customer arrivals at Center n in this system. Thus, Theorem (6.3) can be interpreted as stating that for the constant-arrival-rate case of System $(N,L,M,R)^*$, the equilibrium probability distributions of queue lengths at the individual centers are independent; and also each of these distributions is identical with that for a one-service-center queueing system "similar" to the center concerned.

Further specializations. To represent an "ordinary" $M(n)$ -channel Center n , with exponentially distributed holding times whose mean is $1/\mu_n$, set $\mu(n,k) = \mu_n \min(k, M(n))$. If all centers are thus specialized in Theorem (6.3), the result is the theorem of the paper cited in footnote 3. In this instance, the sum of the $w_n(i)$ can easily be expressed in closed form. Under this specialization, the one-service-center case of Theorem (6.3) gives the well-known equilibrium distribution of queue length for an "ordinary" multi-channel service center with Poisson customer-arrival process and exponentially distributed holding times.

Example. Suppose that in the example at the end of Section 4, $x = 0$; and, to assure the existence of equilibrium, assume it is false that either $y = 0$ and $a/b \geq 1$ or $z = 0$ and $a/c \geq 1$. Then

$$p(i,j) = p_1(i) p_2(j) ;$$

where $p_1(i) = p_1(0) (a/b)^i / (i!)^x$ and $p_2(j) = p_2(0) (a/c)^j / (j!)^y$; the $p_n(0)$ being determined by the conditions that $\sum_i p_n(i) = 1$, for $n = 1, 2$.

Varying arrival rate, uniform service rates. To focus on the effect of varying arrival rates, consider the special case of System $(N,L,M,R)^*$ in which $e(n)/\mu(n,i) = 1$ for all $i > 0$. Then $w^*(\bar{k}) = 1$ for every state vector \bar{k} . It is easy to show that

$$T^*(K) = \binom{N-1+K}{N-1}, \text{ for } K = 0, 1, 2, \dots$$

It then follows that a sufficient (but not necessary) condition that $\pi^* > 0$ is that $\limsup \lambda(K) < 1$. Suppose that $\pi^* > 0$. If \bar{k} and \bar{j} are two states, with $S(\bar{k}) = K \geq J = S(\bar{j})$, where $p(S = K)$ and $p(S = J)$ are positive; then, from Theorem (5.5):

$$p(\bar{k})/p(\bar{j}) = \prod_{i=J}^{K-1} \lambda(i) ; ;$$

$$\frac{p(S = K)}{p(S = J)} = \prod_{i=J}^{K-1} \frac{(1+N)\lambda(i)}{1+i} ;$$

$$p(\bar{k}|K) = \binom{N-1+K}{N-1}^{-1} .$$

The interested reader can improve his "feel" for the effect of varying arrival rates by working out more specialized examples, including ones in which $p(S = K) > 0$ for only finitely many different values of K .

Total number of customers held fixed. Consider, finally, the special case of System $(N, L, M, R)^*$ in which $\lambda(K) = 0$ for $K \geq K^*$. In this system, customer arrivals occur when and only when other customers leave the system; so the total number present will eventually remain fixed at K^* . It follows from Theorem (5.5) that $p(\bar{k}) = w^*(\bar{k})/T^*(S(\bar{k}))$ if $S(\bar{k}) = K^*$, and otherwise $p(\bar{k}) = 0$.

In this instance, the model can also be interpreted as representing a closed system through which the same customers circulate eternally. In this interpretation, the probability that a customer served at Center m will next require service at Center n is $r(m, n) + r(m, N+1) r(0, n)$, in the notation of the present paper. If the "givens" of such a system include probabilities $s(m, n)$ that a customer served at Center m will next require service at Center n , for m and $n \in [1, N]$; these

probabilities can be appropriately converted for the application of Theorem (5.5) by setting $r(m,n) = s(m,n)$ for $m \in [1, N-1]$ and $n \in [1, N]$, $r(N, N+1) = 1$, and $r(0, n) = s(N, n)$ for $n \in [1, N]$.

As a numerical example, consider a closed system made up of one "machine operator" and two "repairmen," in which there are two "machines." The machine operator functions like a service center of the kind treated in the present paper; but the "service" he renders is to cause the machine to "fail." The time to failure is exponentially distributed with mean 10. When a machine fails it must be worked on first by one repairman, then by the other, in a specified order. Each repairman functions like a service center, with mean service rate $1/3$ when he has one machine to work on, but with mean service rate $1/2$ when he has two. Designating the machine operator as Center 1, and the repairmen as Centers 2 and 3, what has been described is an instance of System $(N, L, M, R)^*$, with: $N = 3$; $K^* = 2$ and $\lambda(k) = 0$ for $k \geq 2$; $\mu(1,1) = \mu(1,2) = 1/10$, $\mu(2,1) = \mu(3,1) = 1/3$, and $\mu(2,2) = \mu(3,2) = 1/2$; and $r(0,1) = r(1,2) = r(2,3) = r(3,4) = 1$, whence $e(1) = e(2) = e(3) = 1$. Using (5.2), (5.3), and the fact that $p(\bar{k}) = w^*(\bar{k})/T^*(2)$ for $S(\bar{k}) = 2$; it turns out that the equilibrium state probabilities are as follows: $p(0,0,2) = p(0,2,0) = 6/181$; $p(0,1,1) = 9/181$; $p(1,0,1) = p(1,1,0) = 30/181$; $p(2,0,0) = 100/181$. It can be concluded, for instance, that the long-run-average proportion of time during which the machine operator will actually have a machine to operate is $160/181$, or about 88.4 percent.

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