

UNCLASSIFIED

---

AD 294 118

*Reproduced  
by the*

ARMED SERVICES TECHNICAL INFORMATION AGENCY  
ARLINGTON HALL STATION  
ARLINGTON 12, VIRGINIA



---

UNCLASSIFIED

NOTICE: When government or other drawings, specifications or other data are used for any purpose other than in connection with a definitely related government procurement operation, the U. S. Government thereby incurs no responsibility, nor any obligation whatsoever; and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use or sell any patented invention that may in any way be related thereto.

294118

MEMORANDUM  
RM-3001-PR  
MARCH 1962

AD No. \_\_\_\_\_  
ASTIA FILE COPY

# POISSON SUMMATION FORMULAS FOR GROUPS-1: FINITE GROUPS

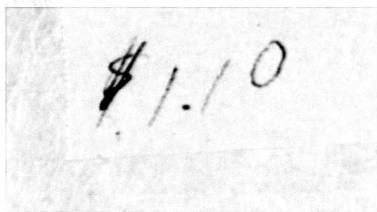
Richard Bellman



PREPARED FOR:

UNITED STATES AIR FORCE PROJECT RAND

The **RAND** Corporation  
SANTA MONICA • CALIFORNIA



W0-2

MEMORANDUM

RM-3001-PR

MARCH 1962

POISSON SUMMATION FORMULAS  
FOR GROUPS-1: FINITE GROUPS

Richard Bellman

This research is sponsored by the United States Air Force under Project RAND—Contract No. AF 49(638)-700—monitored by the Directorate of Development Planning, Deputy Chief of Staff, Research and Technology, Hq USAF. Views or conclusions contained in this Memorandum should not be interpreted as representing the official opinion or policy of the United States Air Force. Permission to quote from or reproduce portions of this Memorandum must be obtained from The RAND Corporation.

---

*The* **RAND** *Corporation*

1700 MAIN ST • SANTA MONICA • CALIFORNIA

---

PREFACE

As a part of its research program, The RAND Corporation engages in basic supporting studies in mathematics. Progress in one branch of mathematics often results from observing analogies to known results in another branch. The Poisson summation formula has long been a basic tool in mathematical analysis. The present research extends the idea of this formula to make it applicable to certain mathematical structures (groups) that are of basic importance in many fields of mathematics.

SUMMARY

↙ A summation formula for finite groups is established  
analogous in form and proof to the classical Poisson  
summation formula. ↘

CONTENTS

|                                   |     |
|-----------------------------------|-----|
| PREFACE. . . . .                  | 11  |
| SUMMARY. . . . .                  | 111 |
| Section                           |     |
| 1. INTRODUCTION . . . . .         | 1   |
| 2. FINITE-GROUP VERSION . . . . . | 2   |
| 3. PROOF. . . . .                 | 3   |

POISSON SUMMATION FORMULAS FOR GROUPS—I:  
FINITE GROUPS1. INTRODUCTION

The classical Poisson summation formula asserts that under appropriate conditions concerning the function  $f(x)$ , we have the relation

$$(1.1) \quad \sum_{-\infty < n < \infty} f(n) = \sum_{-\infty < n < \infty} \left( \int_{-\infty}^{\infty} f(x) e^{-2\pi i n x} dx \right).$$

A formal derivation which shows the group-theoretic origin of this formula is the following. Consider the function

$$(1.2) \quad F(y) = \sum_n f(n + y),$$

periodic with period 1. Expanding  $F(y)$  in a Fourier series, we have

$$(1.3) \quad F(y) = \sum_m a_m e^{2\pi i m y},$$

where the coefficients are determined in the following way:

$$\begin{aligned} (1.4) \quad a_m &= \int_0^1 F(y) e^{-2\pi i m y} dy \\ &= \sum_{-\infty < n < \infty} \int_0^1 f(n + y) e^{-2\pi i m y} dy \\ &= \int_{-\infty}^{\infty} f(y) e^{-2\pi i m y} dy. \end{aligned}$$

Setting  $y = 0$ , we have (1.1). To obtain group-theoretic versions of (1.1), we need only to interpret each of these steps in a suitable fashion.

Since the Poisson summation formula can be used to obtain such fundamental manifestations of duality as the transformation formulas for the theta functions and Gauss sums, it is to be expected that the corresponding formulas for groups could be used equally to obtain duality theorems for groups. These matters and generalizations of the results obtained here will be discussed subsequently.

## 2. FINITE-GROUP VERSION

Let  $H$  be a finite group of order  $N$ , and  $G$  a proper subgroup of order  $M$ . Let  $x_1, x_2, \dots, x_M$  be the elements of  $G$ , and let  $y$  denote an element of  $H$ . Further, let  $f(y)$  be defined for  $y \in H$ .

Let  $\{X_i(y)\}$  be a complete set of characters with the property that any function defined on  $H$  can be expanded in the form  $\sum_1 a_i X_i(y)$ . Then we have the following result.

### Theorem.

$$(2.1) \quad \frac{1}{M} \sum_1 f(x_i) = \frac{1}{N} \sum_{X_G} \left[ \sum_y f(y) X(y^{-1}) \right],$$

where  $X_G$  denotes the set of characters satisfying the relation

$$(2.2) \quad X(x_i) = 1 \quad \text{for} \quad x_i \in G.$$

3. PROOF

To establish the foregoing result, we consider the character expansion of the function

$$(3.1) \quad F(y) = \sum_1 f(x_1 y).$$

Writing

$$(3.2) \quad F(y) = \sum_j a_j X_j(y),$$

we see that the invariance,  $F(y) = F(yx)$  for  $x \in G$ , requires that

$$(3.3) \quad a_j(1 - X_j(x)) = 0.$$

Hence  $a_j = 0$  if  $X_j(x) \neq 1$  for  $x \in G$ .

The orthogonality property of the characters yields the representation for the coefficients:

$$(3.4) \quad a_j = \frac{1}{N} \sum_y F(y) X_j(y^{-1}).$$

Using the expression for  $F(y)$ , we have

$$\begin{aligned} (3.5) \quad a_j &= \frac{1}{N} \sum_y \left( \sum_1 f(x_1 y) \right) X_j(y^{-1}) \\ &= \frac{1}{N} \sum_1 \left( \sum_y f(x_1 y) X_j(y^{-1}) \right) \\ &= \frac{1}{N} \sum_1 X_j(x_1^{-1}) \left( \sum_y f(y) X_j(y^{-1}) \right) \\ &= \frac{M}{N} \sum_y f(y) X_j(y^{-1}), \end{aligned}$$

since  $x_j(x_i^{-1}) = 1$  for  $x_i$  and the admissible  $j$ -values.

Hence

$$(3.6) \quad \sum_i f(x_i y) = \frac{M}{N} \sum_j \left( \sum_{y_1} f(y_1) x_j(y_1^{-1}) \right) x_j(y).$$

Setting  $y = I$ , the identity element, we obtain the stated result.

Observe that each step has been the analogue of the corresponding step in Sec. 1.