RACTICAL AND THEORETICAL BASES
FOR SPECIFYING
A
TRANSPORTATION VIBRATION TEST

to

kaging Branch, Weapons Installation Division
Bureau of Naval Weapons
Washington 25, D. C.

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Project RR 1175-36

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by

Reed Research, Inc.
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ABSTRACT

Each service's material probably will be carried by rail, truck, ship or airplane while being distributed from factory to the ultimate consumer. Despite similarity of total vibration environment, at least seven different vibration tests are prescribed by the government. The two reasons for this situation are: (1) there is no basis for condensing long term transport conditions into a short term laboratory test; and (2), valid description of the shipping vibration environment does not exist.

Emphasis is placed on developing a theoretical basis for a laboratory test since, if for no other reason, theory will define the numerical values of the shipping environment which should be collected. Because a shipment may occupy a long time period during which specific conditions may occur almost at random, use of modern statistical theory (in this case the theory of random vibration) is indicated. Equations are derived showing content mean square response in terms of suspension natural frequency and damping ratio and the environmental characteristics of acceleration spectral density and bandwidths.

Linear fatigue accumulation concepts are then used as a basis for realistically shortening test times. The literature not being clear concerning whether standard fatigue data (S-N data) should be fitted to semi-log or to log-log curves, two expressions are derived which equate sinusoidal test response amplitude and time with field time and the field response amplitude previously derived. The equations show that it is sufficient, when evaluating the suspension, to vibrate at a single frequency, preferably the lowest resonant.

To avoid recommending impractical tests, the capacities of existing testing equipment are reviewed. Using the results of this review as a guide, an equation relating one sinusoidal test with another is written using the principle of equal input work. This expression has the same form as the derived theoretical expression based on plotting S-N curves on log-log paper. Hence, it is not essential to insure absolute compliance with a stipulated test amplitude provided test time is varied accordingly.

By use of the foregoing theoretical and practical approaches an interim test procedure is recommended for standard use until such time as enough field vibration data can be analyzed to provide a better basis for a standard test. Further recommendations include:

a. Every effort be made to salvage as much random data from existing records as possible.

b. Future records be analyzed for contribution to a proper statistical description of the environment.

c. Experiments be conducted to verify the presumption that cumulative fatigue hypotheses apply to package cushioning devices and materials.

INTRODUCTION

All progressive package designers agree that there is no way in which the service performance of a new design can be predicted in the laboratory without including some simulation of the vibrations encountered in shipment. This statement is as true of commercial packages as it is of military packages. In fact it is standard practice to precede handling tests with a vibration test if only to measure the performance of the container in a more realistic configuration. When so used, the test can be considered as a part of the conditioning process.

Vibration isolation in missile shipping containers calls for maximum skill on the part of the container designer particularly when one considers that a suspension system also must perform as a shock isolator. Designing to provide for satisfactory isolation of large, infrequently occurring, transients and smaller frequently occurring "steady state" phenomena demands a compromise between design goals which are essentially incompatible except in special cases.

While a growing number of thoughtful writers* believe that primary emphasis should be placed on designing for vibration isolation and letting shock isolation be the secondary goal, the historical emphasis in military specialty container work has been placed on the shock isolation aspects. Nevertheless, considerable attention has been given to the vibration control problem and all services have used vibration tests to evaluate shipping containers. The picture is complex, frequently confusing and, in many cases, contradictory. This report is, therefore, devoted to an analysis of the problem and development in detail of the reasons underlying the recommendations to be made. By careful exposition of the reasoning behind a somewhat arbitrary recommendation, we aim at promoting intelligent discussion and decision.

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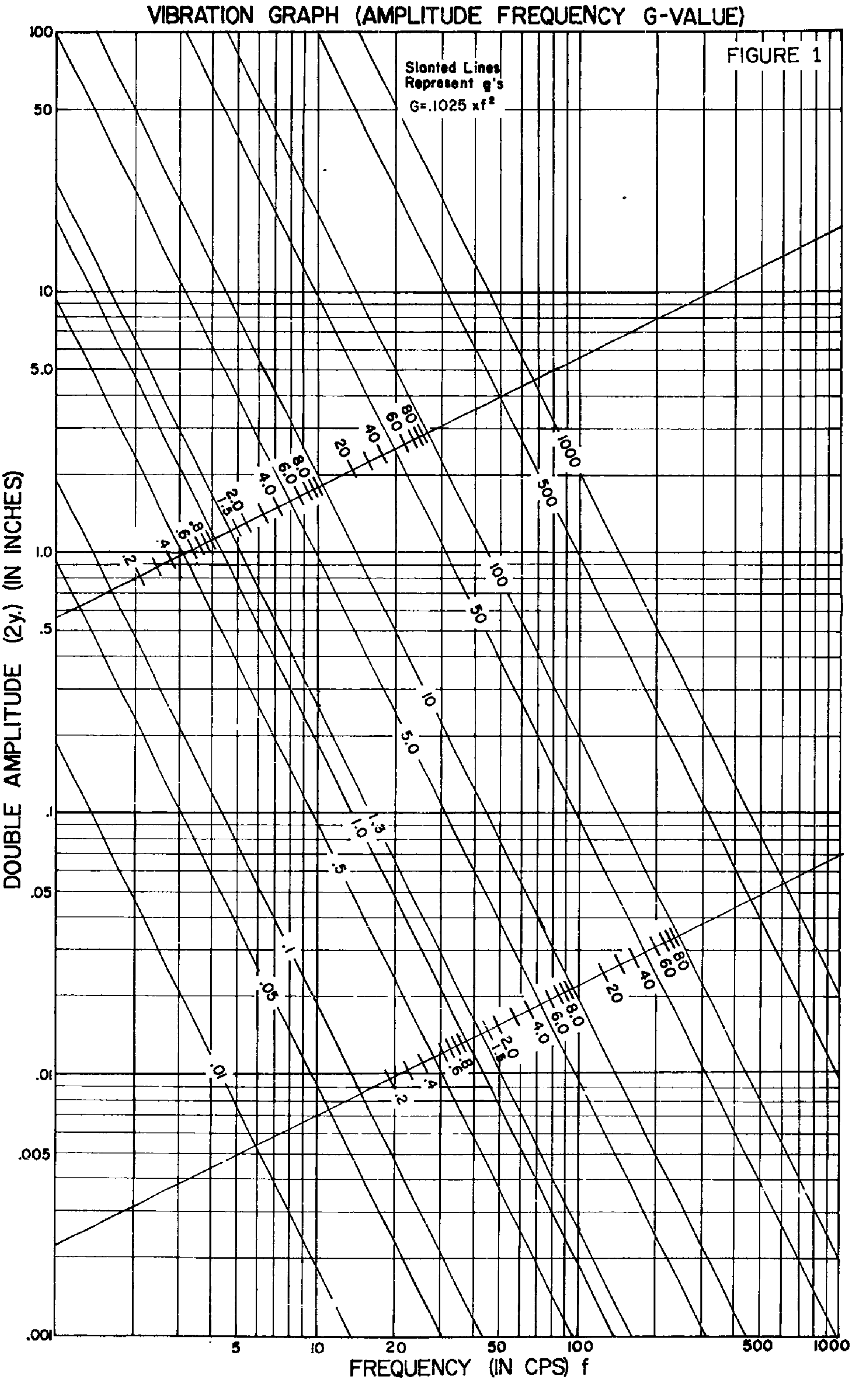
*See, for example, references (1) and (2). Throughout this report numbers in parentheses refer to literature cited at the end.


VIBRATION GRAPH (AMPLITUDE FREQUENCY G-VALUE)
FIGURE 1
Slanted Lines Represent g's
G=.1025 xf²
DOUBLE AMPLITUDE (2y.) (IN INCHES)
FREQUENCY (IN CPS) f
100
50
10
5.0
1.0
.5
.1
.05
.01
.005
.001
5
10
20
50
100
500
1000

All vibration tests consist of suitable combinations of frequency, amplitude and time. In a sinusoidal test, amplitude is controlled by stipulating linear excursion or by expressing the number of multiples of the acceleration of gravity obtained by relating acceleration, frequency and linear amplitude with the familiar approximation:

$$G = .1025\, x f^2 \qquad [1]$$

where G = number of multiples of acceleration of gravity (386 in/sec. /sec.)

x = amplitude (distance from rest position to peak,) inches

f = frequency, cycles per second

Figure 1 has been prepared for ready reference in comparing these sometimes confusing relations.

As of this writing, shipping container tests required by the government are all of the sinusoidal type. Study discloses, however, that there are almost as many tests as agencies capable of imposing the test. Figure 2 compares the seven more common tests. This figure does not include those previously required by various Bureau of Ordnance field activities prior to issuance of Specification MIL-W-21927 since such a catalogue has appeared elsewhere (3). Nor does the figure include those tests which are primarily designed to test equipment (e. g. MIL-STD-167, MIL-T-18404, MIL-E-5400, MIL-T-945, MIL-STD-353). The procedures of MIL-E-5272 and MIL-E-4970 (also intended to be equipment tests) have, however, been included. They sneak in as shipping container tests in a manner often unsuspected by those responsible for package design and development.

We are aware that, in the ballistic missile program, the phrase "transit case" in MIL-E-4970 paragraph 4. 6. 8 has sometimes been interpreted by Space Technology Laboratories as meaning shipping container even though such is not sound packaging terminollogy and in spite of the fact that paragraph 4. 6b specifically eliminates "equipment prepared for logistic or supply shipment and delivery."

The reader should also note that Air Force Specification MIL-P-9024A vibration test information is given in Figure 2. The latest version (MIL-P-9024B) has abandoned any pretense of a vibration test but relies on a shipping test with a "suitable" vibration test as an alternate. Single sample shipment testing is time consuming and, because of the small sample, of exceedingly dubious validity.

Now any rational test could be the basis for a standard test. To determine standardization potential, one must answer two questions:

(a) Is standardization necessary or desirable?

(b) Is standardization possible?

The answer to (a) is emphatically yes. With minor exceptions all missile containers are being tested for exposure to essentially the same environments, i. e. vibrations, encountered when shipping via truck, rail, ship or air. There is, therefore, no defensible excuse for knowingly continuing a situation which seems to have produced as many test procedures as technicians. In connection with another project, Reed Research received 11 replies concerning the desirability of vibration test standardization. Of these 11, 9 urged test standardization, 1 had minor reservations and 1 indicated no experience.

The answer to question (b) must be given in two parts

(1) Standardization now is possible on an arbitrary basis if certain postulates be accepted.

(2) It appears possible to devise a standard form of test with quantitative confidence that it will predict service performance provided certain further investigations are undertaken.

The remainder of this report is concerned with simultaneous development of both of these themes with the hope that the test suggested for use now can be simply modified to become the "ideal" test suggested by theory.

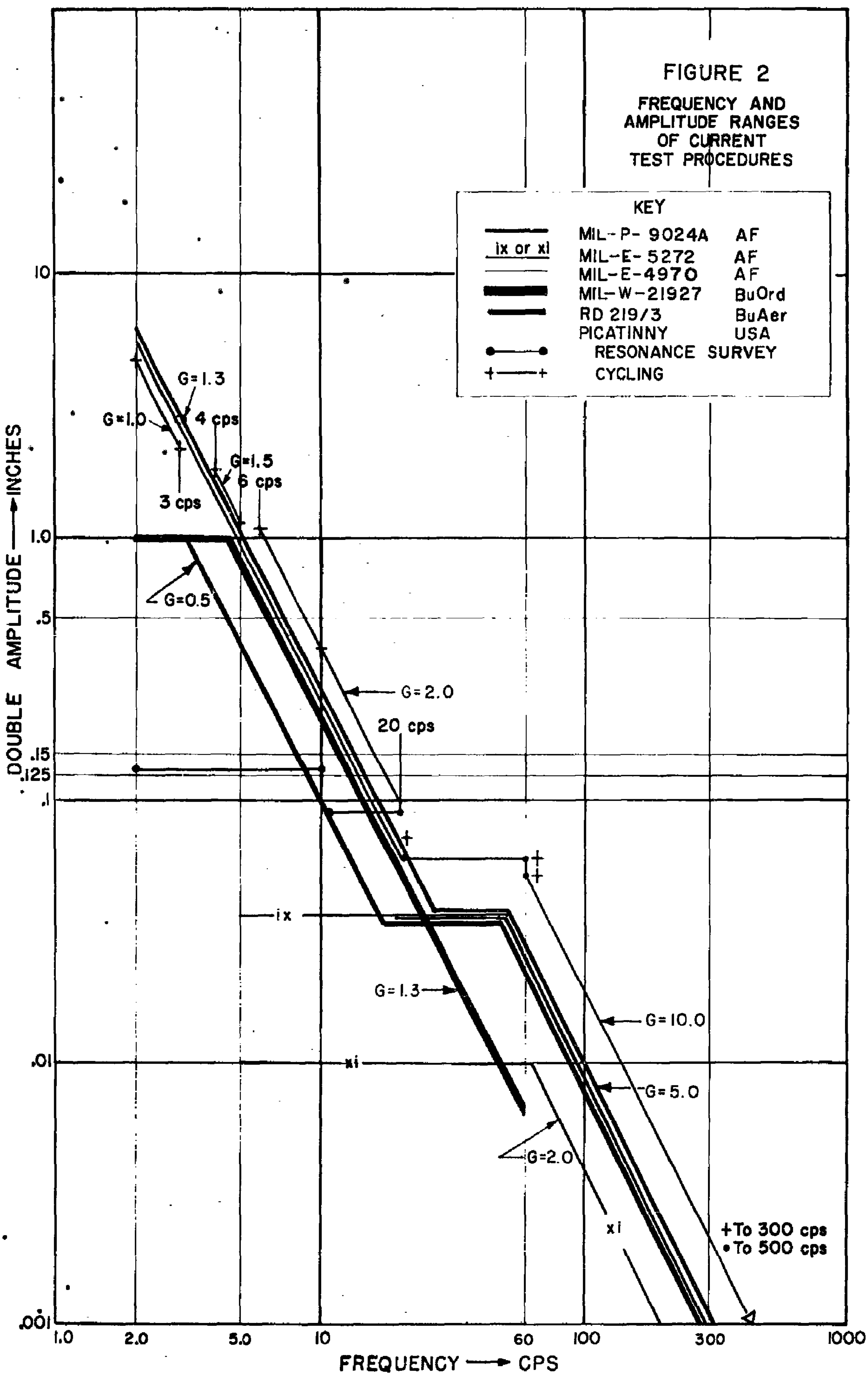


FIGURE 2
FREQUENCY AND AMPLITUDE RANGES OF CURRENT TEST PROCEDURES
KEY
MIL-P-9024A AF
ix or xi MIL-E-5272 AF
MIL-E-4970 AF
MIL-W-21927 BuOrd
RD 219/3 BuAer
PICATINNY USA
RESONANCE SURVEY
CYCLING
G=1.3
G=1.0
4 cps
G=1.5
6 cps
3 cps
G=0.5
G=2.0
20 cps
ix
G=1.3
G=10.0
xi
G=5.0
G=2.0
xi
+To 300 cps
•To 500 cps
DOUBLE AMPLITUDE → INCHES
10
1.0
.5
.15
.125
.1
.01
.001
1.0
2.0
5.0
10
60
100
300
1000
FREQUENCY → CPS

THE VIBRATION ENVIRONMENT

Such data as are available indicate that the environment includes the amplitudes and frequencies shown in Figure 3. This figure is based on the summaries contained in RD 219/3 (4) with the significant exception that the values given by Guins and Kell (5) for AAR 1915 trucks have been omitted. Reason for this omission is the very small number of such cars to be found in interchange service today and the reasonable presumption that any traffic manager can stipulate not receiving such cars for missile shipments even in dire emergency.

Some measures of statistical occurrence have been attempted as exemplified by Figure 4 taken from Ott (6). Ott's curves are cumulative. That is, a point on the curve at any given acceleration has an ordinate representing the number of peaks of that intensity or greater which may be expected during the hypothetical journey.

The existing data are subject to certain devastating criticisms:

a. Information concerning position of pickups and loading conditions of carriers is signally lacking in practically all reports.

b. Instrumentation descriptions are sketchy and statements concerning methods of data reduction leave much to be desired.

c. The data on probable number of occurrences, such as summarized by Ott, contain no data concerning the frequencies associated with the "shocks".

d. The data are presented as samples reduced, for the most part, as though the inputs were complex waves and made up of discrete frequencies and amplitudes.

The last mentioned "defect" is the one that, apparently, places the entire subject of shipping container vibration testing, and the field data upon which a test is to be based, in jeopardy. Ever since the challenging paper by Morrow and Muchmore (7) the majority of vibration workers have been insisting that the shipping environment is random and that attempting to predict

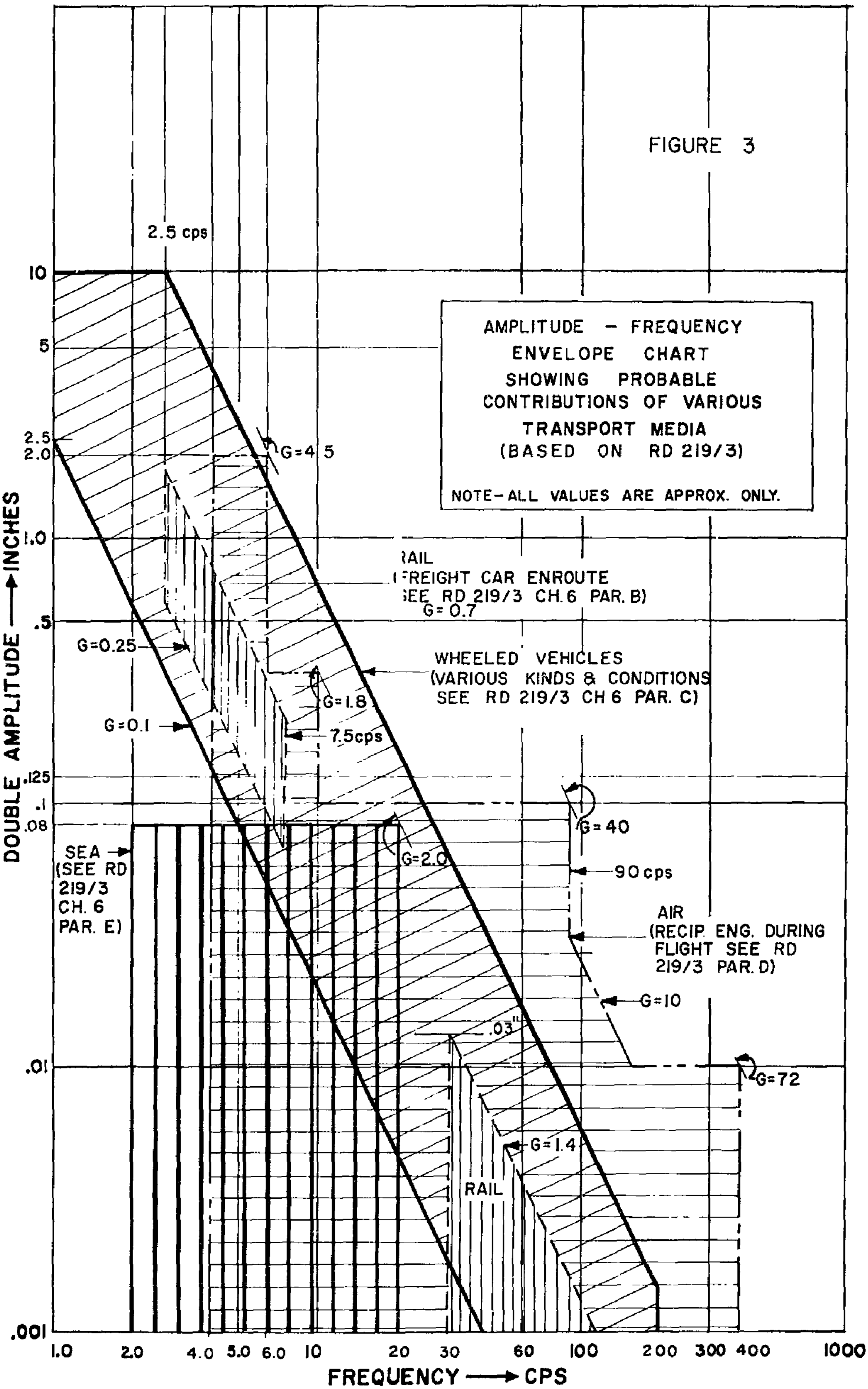


FIGURE 3
AMPLITUDE - FREQUENCY
ENVELOPE CHART
SHOWING PROBABLE
CONTRIBUTIONS OF VARIOUS
TRANSPORT MEDIA
(BASED ON RD 219/3)
NOTE-ALL VALUES ARE APPROX. ONLY.
2.5 cps
G=4.5
RAIL
FREIGHT CAR ENROUTE
SEE RD 219/3 CH.6 PAR. B)
G=0.7
G=0.25
WHEELED VEHICLES
(VARIOUS KINDS & CONDITIONS
SEE RD 219/3 CH 6 PAR. C)
G=1.8
7.5cps
G=0.1
G=40
G=2.0
90 cps
SEA
(SEE RD
219/3
CH. 6
PAR. E)
AIR
(RECIP. ENG. DURING
FLIGHT SEE RD
219/3 PAR. D)
G=10
.03"
G=72
G=1.4
RAIL
DOUBLE AMPLITUDE → INCHES
10
5
2.5
2.0
1.0
.5
.125
.1
.08
.01
.001
1.0
2.0
4.0
5.0
6.0
10
20
30
60
100
200
300
400
1000
FREQUENCY → CPS

behavior on the basis of classical vibration theory simply does not correlate with experience.

Now classical theory, in effect, tells us that if we can identify all the components of a vibration record in terms of certain relatively simple functions (sinusoidal, step, etc.) we could predict behavior. In the case of transportation vibrations, however, there are so many variables affecting the vibration that it seems to many thoughtful people that it is impossible to predict precisely what will happen in the next instant of time*. Probability theory can be used, however, to predict what might reasonably be expected to happen over a given period of time on the assumption that any of the possibilities could happen next. In effect, we treat the problem much as we would treat the problem of experimental error. Hence the phrase "random vibration".

Mathematicians call the models derived to handle such problems stochastic processes**. The stochastic process is used in much the same way as a simple sinusoidal description is used. It must be remembered that both are simplifications of reality but the stochastic is considered a better simplification. In fact, Morrow (11) states "For missiles the most common and most damaging tactical vibration is random".

As implied, the concept of random vibration involves applications of mathematical statistics to mechanical problems. This is not altogether a new concept. Einstein's (12) 1905 theory of Brownian motion (which may be considered a special case of random vibration) was extended to strings and certain beams in 1931 (13). In 1933 Wiener (14) introduced such concepts as power spectral density (to be defined later) in his generalized harmonic analysis giving a powerful analytical tool. The big impetus towards current use of random theory was given by S. O. Rice in his classic papers on random noise in electric circuits (15), (16). A complete summary of available knowledge is beyond the scope of this work. Of particular interest, however, are the analyses of random

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*For much of what follows concerning history and definitions we are indebted to the clear exposition by Crandall (8, 9) and by James Nichols and Phillips (10).

**Stochastic could, if one wished, be considered a ten-dollar word implying chance.

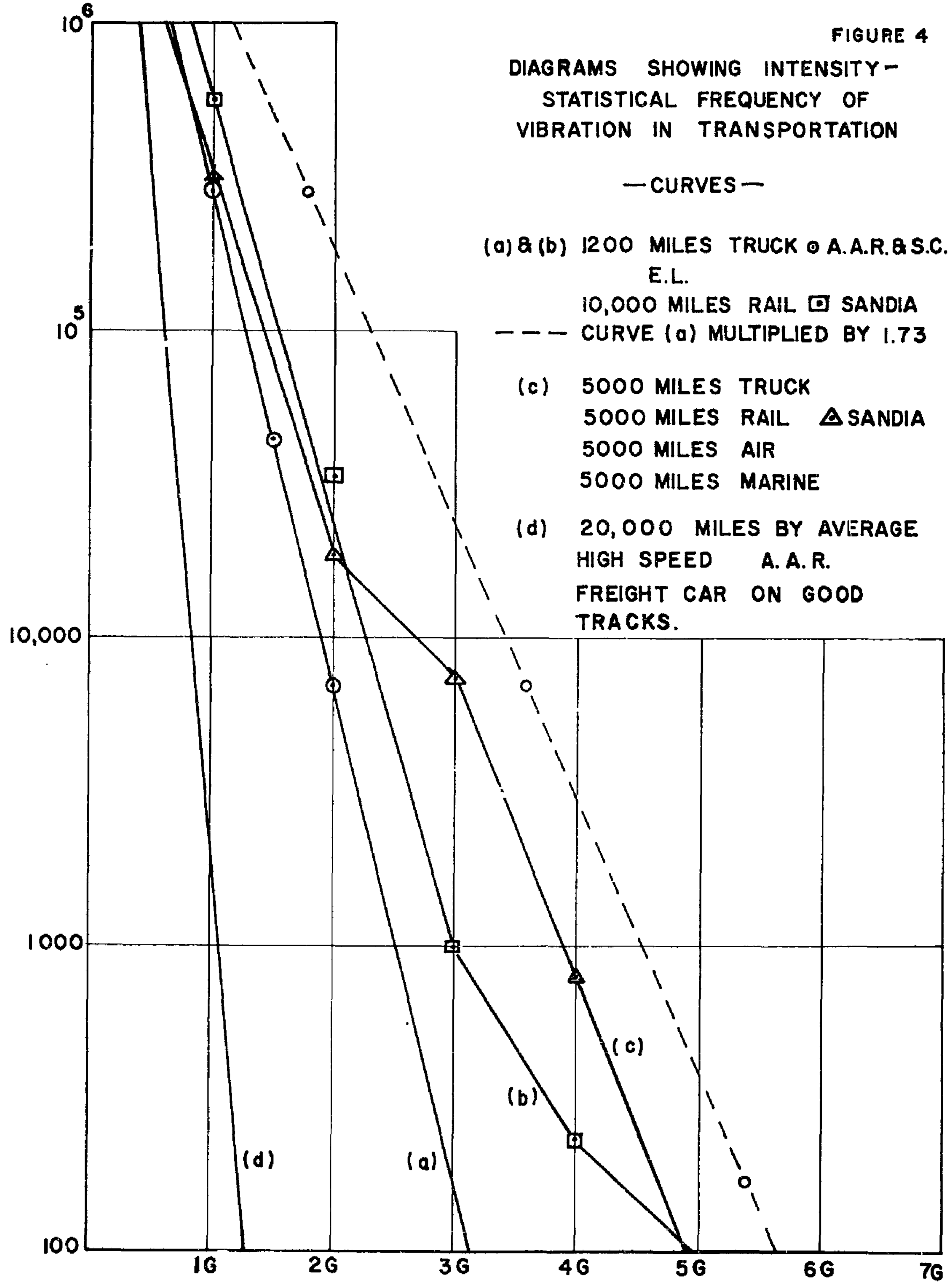
FIGURE 4
DIAGRAMS SHOWING INTENSITY-STATISTICAL FREQUENCY OF VIBRATION IN TRANSPORTATION
—CURVES—
(a) & (b) 1200 MILES TRUCK ⊙ A.A.R. & S.C. E.L.
10,000 MILES RAIL ⊡ SANDIA
— — — CURVE (a) MULTIPLIED BY 1.73
(c) 5000 MILES TRUCK
5000 MILES RAIL △ SANDIA
5000 MILES AIR
5000 MILES MARINE
(d) 20,000 MILES BY AVERAGE HIGH SPEED A.A.R. FREIGHT CAR ON GOOD TRACKS.
10^6
10^5
10,000
1000
100
1G
2G
3G
4G
5G
6G
7G
(a)
(b)
(c)
(d)

vibrations in beams (13), (17), (18), (19), in plates (17), (20) and in shells (21).

The statistical approach to vibration analysis requires the introduction of certain concepts which may be new to those grounded in classical vibration concepts. For clarity, we reintroduce these concepts, basing our exposition largely on references (9) and (10).

Let us consider any motion, $x(t)$, in which x denotes a displacement and t indicates that x is a function of time. This motion may be the motion of the bed of a truck or railroad car or airplane or other support for a package and represents the forcing function in the dynamical problem under study. We will designate by $a(t) = \frac{d^2x}{dt^2}$ the absolute acceleration of this motion.

Three statistical measures are of particular interest with respect to this motion. The first is the mean square defined by

$$\overline{x}^2 = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x^2(t)\,dt \qquad [2]$$

which, as the name implies, is a very general measure of intensity. Sometimes the square root of the mean square, referred to as the root mean square (and also as the standard deviation), is used and this is $\sigma = \sqrt{\overline{x}^2}$.

These average intensity measures give no indication of the oscillatory nature of the motion. This is pictured by the power spectral density, $S_x(\omega)$ which, as the name implies, shows the power content of the vibratory components as a function of their circular frequencies, ω .

The power spectral density has the property that its integral is the mean square of the motion. That is

$$\overline{x}^2 = \frac{1}{2\pi} \int_{0}^{\infty} S_x(\omega)\,d\omega \qquad [3]$$

This simply means that the average power in the motion is the sum of the contributions of each infinitesimal component at each frequency. It is considered characteristic of random motions that power spectral density is a bounded function. In the case of transportation vibration it is also considered reasonable that the function will contain peaks in the neighborhood of the resonant frequencies of the vehicle.

The power spectral density is also defined mathematically in terms of the Fourier Transform. Thus, *if we consider a function

$X_T(t) = X(t)$ when $-T<t<T$ with T a particular value of time and $X_T(t) = 0$ otherwise

and write $A_T(\omega) = \int_{-\infty}^{\infty} X_T(t)\, e^{-i\omega\tau} dt = \int_{-T}^{T} X_T(t) e^{-i\omega\tau} dt$

then $\lim A_T(\omega)$ is the Fourier Transform of $X(t)$. If $X(t)$ is considered to extend indefinitely on the time scale, then this limit does not exist. However, the limit $\lim_{T\to\infty} \frac{1}{T}|A_T(\omega)|^2$ does exist, if $X(t)$ is a random motion, and this limit defines the power spectral density. That is

$$S_x(\omega) = \lim_{T\to\infty} \frac{1}{T} \left| A_T(\omega) \right|^2$$

Such power spectral density functions, when plotted against the frequency, often show marked peaks at certain frequencies. A useful description of these peaks is given by the peak value, the frequency at the peak and the band width. The latter is conventionally measured by the frequency difference between the points on each side where the spectral density has fallen to one half the peak value.

The third important statistical measure of random vibration is the probability density function. The underlined phrase is drawn, at least in part, from the language of mechanical engineering. Because of mathematical similarity, it is convenient to consider that the probability of a value falling within a certain bounded region is equal to the mass of that region. It is usual to assume that the probability density of a random motion is normal or Gaussian. That is to say, the probability of encountering a particular value of $X(t)$ is as shown in Figure 5.

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*Alternatively, consider the autocorrelation function

$$R_x(\tau) \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)x(t+\tau)\, dt$$

Then the power spectral density is

$$S_x(\omega) = 2\int_{-\infty}^{\infty} R_x(\tau) e^{-i\omega\tau} dt$$

That is, the power spectral density is twice the Fourier Transform of the autocorrelation function.

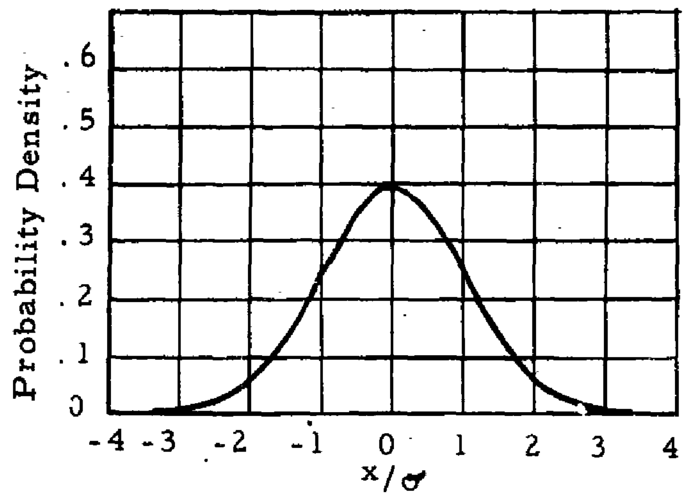

FIGURE 5. NORMAL OR GAUSSIAN PROBABILITY DENSITY

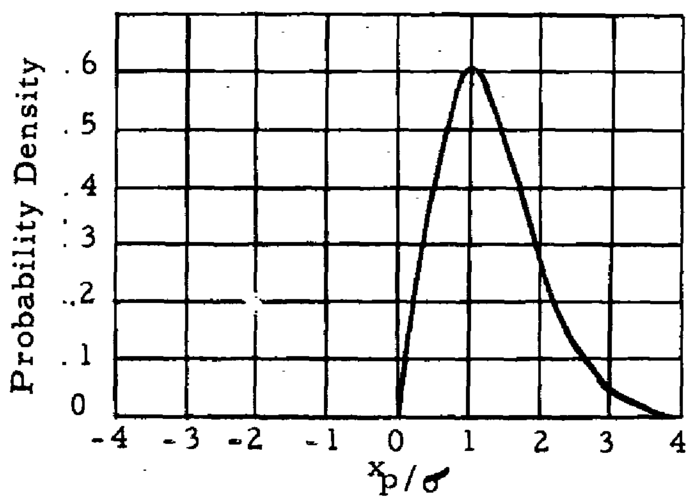

FIGURE 6. RAYLEIGH PROBABILITY DENSITY

When the response of a resonant system to a random excitation is considered, an envelope of peak values can be drawn. If the excitation is Gaussian, and the Q of the system is high*, then the probability density function for the envelope is a Rayleigh density as shown in Figure 6.

The equation of this curve is: $$p(x_p) = \frac{x_p}{\sigma^2} e^{\frac{-x_p^2}{2\sigma^2}} \quad (6)$$

where:

$p(x_p)$ is the probability density of x_p; i.e., $p(x_p)\Delta x_p$ is the probability that x_p lies between x_p and $x_p + \Delta x_p$

x_p is a peak value of x or an ordinate of the envelope curve.

σ is the rms value of x.

It is clear from the foregoing that an adequate description of the environment useful in a statistical approach requires data concerning the envelope of all possible acceleration spectra together with a measure of the bandwidths of the peaks therein and an approximate measure of possible duration. It is also clear that currently available information concerning vibrations encountered in shipment cannot be considered a statistical description of the environment. About all that can be said about these data is that the recorded frequencies do occur and that the reported amplitudes occur a sufficient number of times to be noticeable. Statistical measures are, however, still insufficient for completely logical test design. In effect, therefore, no matter what test is suggested at present writing, it must be accepted that no quantitative confidence figure can be derived concerning faithful prediction of service performance.

The impossibility of deriving a quantitative confidence level should not deter the government from selecting a "standard" test

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*That is, to say, the system is lightly damped. By definition

$$Q = \frac{1}{2\zeta} = \frac{C_c}{2C_D} = \frac{\sqrt{Km}}{C_D} \quad \text{where } \zeta = \text{ damping ratio,}$$

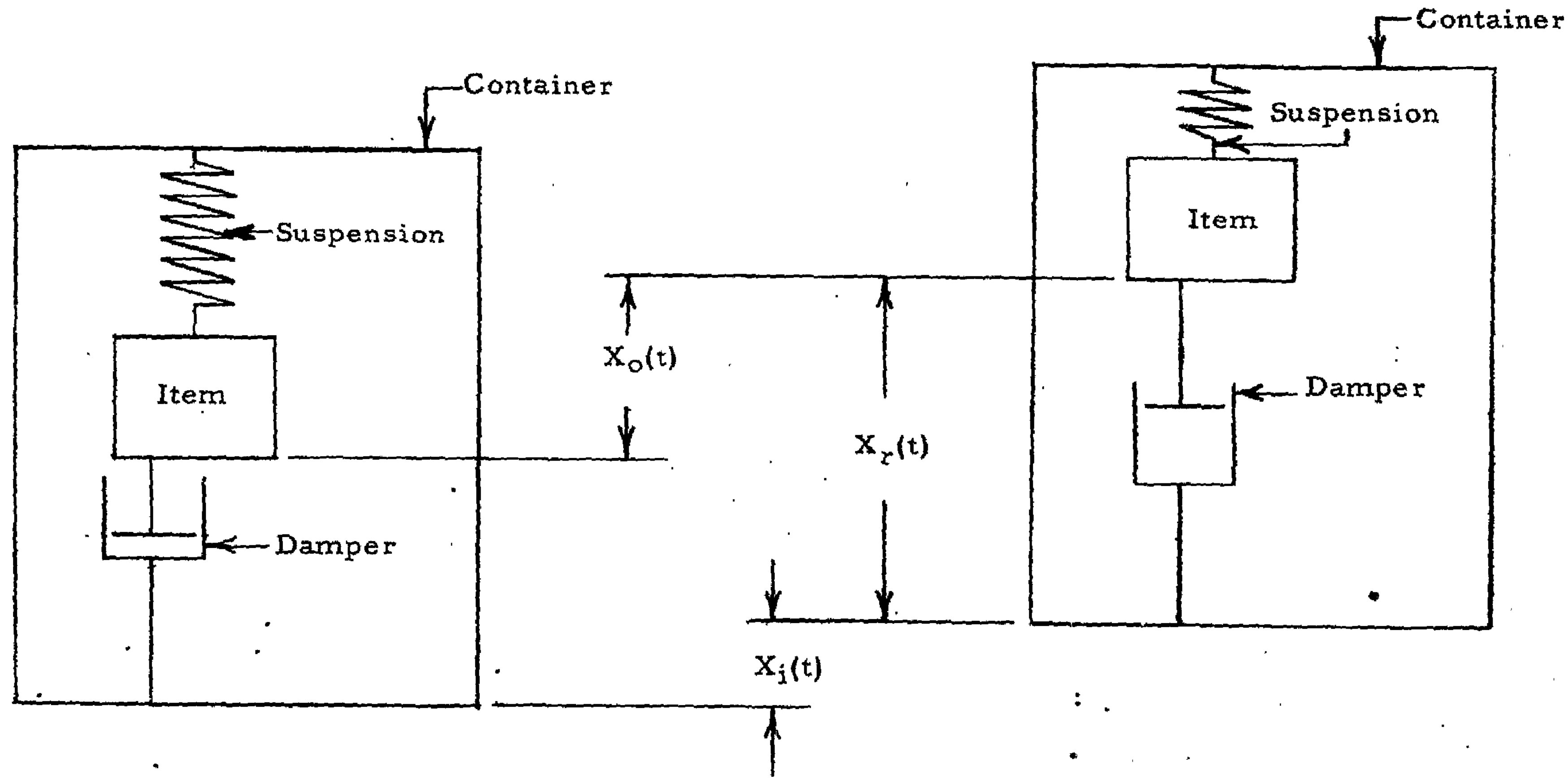
C_D = damping coefficient, lb. sec. /in, C_c = critical damping coefficient, K = spring rate of suspension and m mass.

based on qualitative confidence levels derived from experience provided it is recognized that an arbitrary "standard" has been established and that work should continue towards achieving a better standard. The major difficulty with selecting a standard now is that, objectively, each method now in use has, with certain exceptions, equal standing with every other method, including the possible standard to be suggested here.

Our problem is to establish a theoretical basis for a transportation vibration test assuming that a valid statistical description of the environment exists and then examining what practical compromises are necessary to permit performance of any test now. Even to go this far we must proceed from two assumptions which we will, hereafter, regard as axiomatic:

a. Shipping container vibration tests evaluate only shipping containers. In other words, one is testing performance of the isolator system, not performance of the missile. "Trick" frequencies and amplitudes are a problem for the missile designer. If they exist in the range to be considered, supplementary requirements may be written but <u>not</u> by packaging personnel.

b. Any test is a simulation not a duplication of the environment. In a laboratory test, the time scale is reduced and, most frequently, severity is increased to compensate.



MODEL OF A PACKAGED ITEM

Figure 7

RESPONSE OF A PACKAGED ITEM TO RANDOM MOTION

With the foregoing assumptions in hand, we proceed to an analysis of the response of a package with a linear damped spring system such as shown in Figure 7. For simplicity we will restrict ourselves to the linear case.* If the system be linear then, by use of the concept of normal modes, it is possible to treat the motional response of the item with respect to the container as a single degree of freedom system.

The input motion "coordinate" is $x_i(t)$ with corresponding acceleration $a_i(t) = \frac{d^2 x_i}{dt^2}$. The absolute response coordinate is $X_o(t)$ and the relative motion $X_r(t)$. The absolute response acceleration is $a_o(t) = \frac{d^2 x_o}{dt^2}$. It must be recognized that the absolute response may require a vector pair to describe its translational and rotational components. The response coordinate $x_o(t)$ is simply a magnitude which characterizes the amplitude of the response in the normal mode under consideration. The same thing is true of the relative motion. So far as the input is concerned, it can generally be assumed to consist of a rectilinear motion, although in principle it could just as well be a rotation or a combined motion.

If x_i is a complex exponential function of time, $x_i = X_i e^{i\omega t}$ then the output is $x_o = X_o e^{i\omega t}$ and the relative motion is $x_r = X_r e^{i\omega t}$

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*For delicate items the mounting most often consists of shear mounts which are substantially linear in the range under consideration. This restriction does not, therefore, result in an academic enterprise.

where, in general, X_i, X_o and X_R, are complex functions of the circular frequency, ω. The ratios (22)

$$\frac{x_o}{x_i} = \frac{X_o}{X_i} = T_o = \frac{1 + i2\zeta\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + i2\zeta\frac{\omega}{\omega_n}} \tag{7}$$

and

$$\frac{x_r}{x_i} = \frac{X_r}{X_i} = H\left(\frac{\omega}{\omega_n}\right)^2 = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + i2\zeta\frac{\omega}{\omega_n}} \tag{8}$$

in which H is the complex frequency response or magnification factor, permit us to write

$$X_o = T_o X_i \tag{9}$$

and

$$X_R = H\left(\frac{\omega}{\omega_n}\right)^2 X_i \tag{10}$$

where ω_n is the natural frequency of the normal mode under consideration

and ζ is the ratio of the damping to the critical damping.

When the input acceleration, a_i, is given the response acceleration is a_o and

$$\frac{a_o}{a_i} = \frac{x_o}{x_i} = T_o \tag{11}$$

The relative motion, x_r, is given by

$$\frac{x_r}{a_i} = -\frac{1}{\omega^2}\frac{x_r}{x_i} = \frac{1}{\omega_n^2} H \tag{12}$$

From these relations between excitation and the response thereto it is possible to obtain relations between the statistical measures of each. Thus, if $S_{ix}(\omega)$ denote the power spectral

density of the input motion and $S_{ox}(\omega)$ and $S_{rx}(\omega)$ the spectral

densities of the absolute and relative responses, respectively, then (23)

$$S_{ox}(\omega) = |T_o|^2 S_{ix}(\omega) = \frac{1 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right] + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2} \text{Six (w)} \tag{13}$$

$$S_{rx}(\omega) = \left| H\left(\frac{\omega}{\omega_n}\right)^2 \right|^2 S_{ix}(\omega) = \frac{\left(\frac{\omega}{\omega_n}\right)^4}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2} S_{ix}(\omega) \quad (14)$$

As the input is more apt to be given in terms of acceleration and as the absolute acceleration response is of more interest than the absolute response motion, we denote the input acceleration spectral density by $S_{ia}(\omega)$ and the acceleration spectral density of the absolute response by $S_{oa}(\omega)$.

Then

$$S_{oa}(\omega) = |T_o|^2 S_{ia}(\omega) \quad (15)$$

$$S_{rx}(\omega) = \left| \frac{H}{-\omega_n^2} \right|^2 S_{ia}(\omega) \quad (16)$$

From these the mean square absolute acceleration response can be determined as:

$$\overline{a_o}^2 = \frac{1}{2\pi}\int_0^\infty |T_o|^2 S_{ia}(\omega)\,d\omega \quad (17)$$

and the mean square relative motion response as:

$$\overline{x_r}^2 = \frac{1}{2\pi}\int_0^\infty \left| \frac{H}{\omega_n^2} \right|^2 S_{ia}(\omega)\,d\omega \quad (18)$$

If the input acceleration spectral density is flat in the neighborhood of the natural frequency, then, approximately,

$$\overline{a_o}^2 \approx \frac{\omega_n}{8\zeta}\left(4\zeta^2 + 1\right) S_{ia}(\omega_n) \quad (19)$$

and

$$\overline{x_r}^2 \approx \frac{1}{8\zeta\omega_n^3} S_{ia}(\omega_n) \quad (20)$$

Where $S_{ia}(\omega_n)$ denotes the value of the input acceleration density when the input frequency is equal to the natural frequency of the suspended mass. If the input acceleration spectral density has a peak at $\omega = \omega_n$, as is to be expected, $S_{ia}(\omega)$ is a variable during the integration of equations 17 and 18. The following function is considered descriptive of the variation of $S_{ia}(\omega)$:

$$S_{ia}(\omega) = \frac{B_i^2\left(\frac{\omega}{\omega_n}\right)^2}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + B_i^2\left(\frac{\omega}{\omega_n}\right)^2} S_{ia}(\omega_n) \quad (21)$$

This function requires for its specification only its peak value, $S_{ia}(\omega_n)$, the corresponding frequency, ω_n, and its bandwidth, $\omega_n B_i$, which is the frequency difference between the points on each side of the peak at which the ordinate of the curve is half the peak value. Putting equation [21] into equation [17] and integrating (see Appendix I) gives the means square of the output acceleration in terms of the natural frequency of the suspension system and its damping ratio, as well as the input acceleration density at $\omega = \omega_n$ and its bandwidth:

$$\overline{a_o^2} = \frac{\omega_n (1 + 4\zeta^2)}{8\zeta \left(1 + \frac{2\zeta}{B_i}\right)} S_{ia}(\omega_n) \tag{22}$$

Similar substitution in equation [18] produces the following expression of the mean square relative motion response:

$$\overline{X_r^2} = \frac{1}{8\,\omega_n^3\,\zeta \left(1 + \frac{2\zeta}{B_i}\right)} S_{ia}(\omega_n) \tag{23}$$

SINUSOIDAL STEADY STATE
versus
RANDOM VIBRATION TESTING

Equipment for producing random motion in the laboratory is very expensive and not readily available to all who might be involved in the packaging of complex items. On the other hand sinusoidal type testing machines and their associated devices and controls are considerably less expensive. If they could be used a meaningful test would be within the reach of a greater number of operators and testing costs (hence the cost of packaging) could be greatly reduced. We now address ourselves to this problem.

If a sinusoidal test motion with amplitude X_{iT} is applied at the natural frequency of the package, a uniform response amplitude will result. The mean square value of this response, $\overline{X}^2_{nT}$,

is given by $$\overline{X}^2_{nT} = \frac{1}{2} X^2_{nT} = \frac{1}{8\zeta^2} X^2_{iT} \qquad (24)$$

For one form of equivalence, mean square responses in the test and in transportation could be made equal. Combining [24] and [23]

$$X^2_{iT} = \frac{\zeta}{\omega_n^3 \left(1 + \frac{2\zeta}{B_i}\right)} S_{ia}(\omega_n) \qquad (25)$$

The test acceleration amplitude A_{iT} for such equivalent

motion will, since $A_{iT} = \omega_n^2 X_{iT}$, be given by

$$A^2_{iT} = \frac{\zeta \omega_n}{1 + \frac{2\zeta}{B_i}} S_{ia}(\omega_n) \qquad (26)$$

Such a test would involve power dissipation and heat generation within the package substantially equal to that in service. It would fail to simulate service conditions, however, with respect to the distribution of amplitudes. For example, Figure 8 shows that, if the distribution of amplitudes in service is a Rayleigh distribution, the sinusoidal test amplitude (which is $\sqrt{2}$ times the rms value) is exceeded 40% of the time in service. A service amplitude twice the sinusoidal test amplitude is exceeded about 1% of the time and thrice the test amplitude 0.01% of the time.

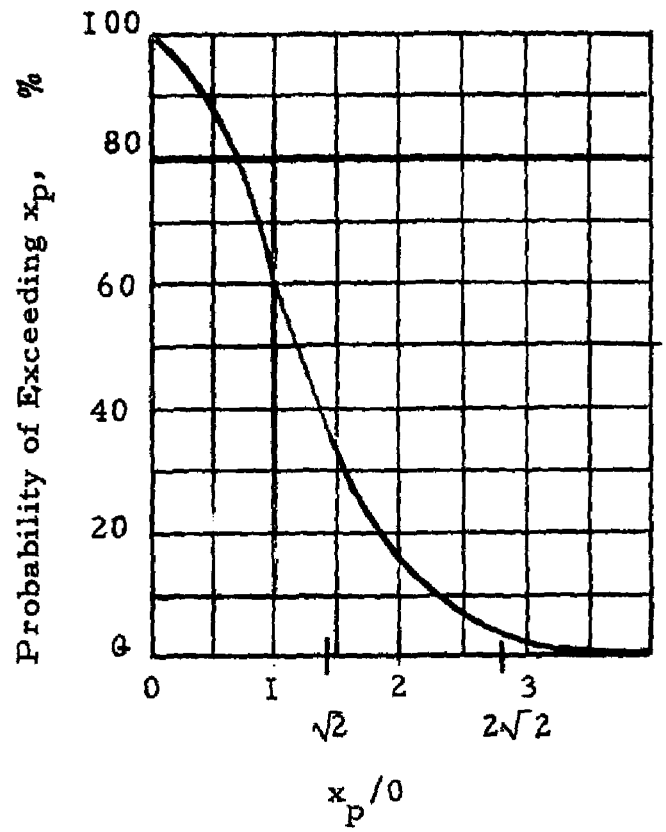


FIGURE 8. PROBABILITY OF EXCEEDING ANY VALUE OF PEAK DISPLACEMENT, IF THE PEAKS FOLLOW THE RAYLEIGH PROBABILITY DENSITY

The possibility exists, therefore, and particularly where heat is not the cause of failure, that service amplitudes in excess of rms value of the test amplitudes would occur a sufficient number of times to cause a fatigue failure of the suspension.

It should also be noted that equations [25] and [26] have no test duration factors and hence imply that the test time must approximate the transit time. Thus, such a method of stipulating the test parameters fails on two counts. In the next section, therefore, we explore the nature of fatigue failures with a view to determining what an equivalent test time could be with increased X_T or A_T above the values indicated by equations [25] and [26].

FATIGUE AND SHORTENED TEST TIMES

Shortening test time by increasing severity is a procedure which has been followed intuitively in almost all disciplines for years. Modern philosophical justification for so doing in a vibration test is based on cumulative fatigue damage concepts developed by Palmgren (24), Miner (25), Corten and Dolan (26) and others*. These hypotheses, based on metal fatigue, postulate that a stress at some level reduces the life of the test object by a stipulated fraction of its fatigue life at that stress. Repeating the shocks produces cumulative damage eventually resulting in fatigue failure. Miner's original hypothesis showed linear damage accumulation characterized by the relation

$$\sum_i \frac{n_i}{N_i} = C \qquad (27)$$

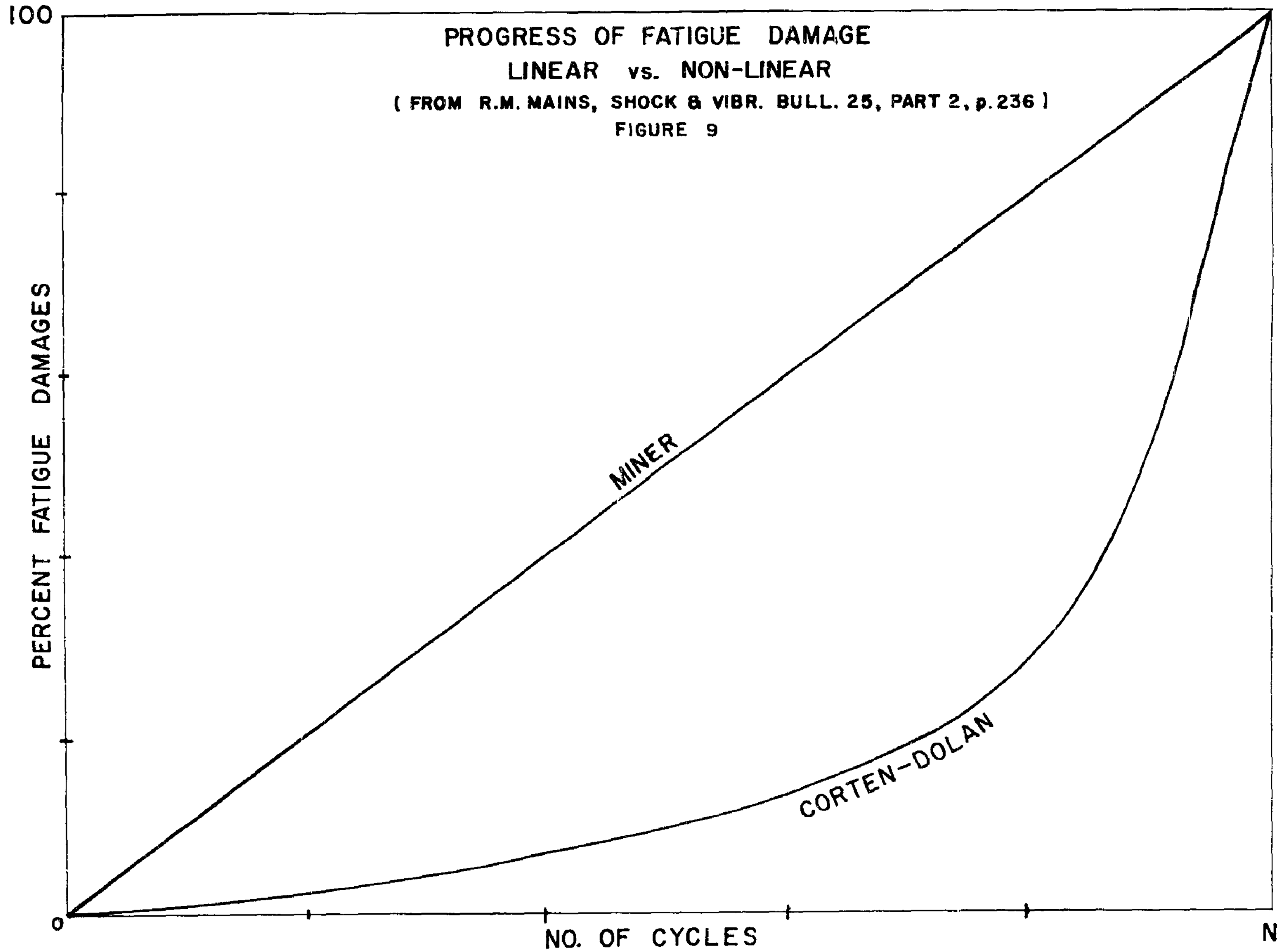
in which n_i represents the actual number of cycles at a given stress level, N_i the fatigue life at that same stress level and C is a constant. Miner proposed that $C = 1.0$ but limited experimental results show a range for C from 0.3 to about 3.0, depending upon whether the high-stress cycles come early or late in the test. Corten and Dolan's non-linear accumulation proposal correlates with experimental data on cold-drawn wire and is characterized by the equation

$$N_g = \frac{N_1}{\alpha_1 + \alpha_2\left(\frac{S_2}{S_1}\right)^d + \alpha_3\left(\frac{S_3}{S_1}\right)^d + \dots \alpha_i\left(\frac{S_i}{S_1}\right)^d} \qquad (28)$$

where N_g represents the total number of cycles to failure; N_1, the fatigue life at the maximum stress; $\alpha_1, \alpha_2, \dots \alpha_i$ are the percentages of cycles at stresses $S_1, S_2, \dots S_i$; and d is an experimentally determined exponent which, for a specific cold-drawn wire, was found to be 6.57. Figure 9 compares the progress of damage at a given stress level under the two theories.

Miles (28) advances the argument that presently available fatigue data on metals scarcely warrant non-linearity complications

*McClintock (27) gives a good resumé of many of the theoretical investigations.


PROGRESS OF FATIGUE DAMAGE
LINEAR vs. NON-LINEAR
(FROM R.M. MAINS, SHOCK & VIBR. BULL. 25, PART 2, p.236)
FIGURE 9
100
PERCENT FATIGUE DAMAGES
MINER
CORTEN-DOLAN
0
NO. OF CYCLES
N

in predicting fatigue life of a single sample. He further demonstrates that using non-linear approaches has no appreciable effect when the stress amplitudes are continually distributed over a wide range. Mains (29), using Miner's approach with $C = 0.3$ and Corten-Dolan's with $d = 6.57$ encountered the "interesting coincidence" that life prediction in one case was substantially the same. Since the transportation input vibration is wide ranged and of long duration Miles' argument may well be applicable to the transport condition. The test condition is assumed at a constant stress amplitude. We feel free, therefore, to use Miner's hypothesis as a point of departure.

Miner's hypothesis can be restated as

$$E = \sum_{S} \frac{n(S)}{N(S)} \tag{29}$$

where $S =$ the stress level, which must be greater than the endurance limit, S_e, i.e. the stress at which fatigue life is infinite.

In an extended duration random motion increasingly finer stress gradations occur and we are led to

$$E = \int \frac{dn}{N(S)} \tag{30}$$

in which n is now considered the number of stress cycles which do not exceed S.

Since n is a function of time and stress is proportional to relative motion we may write the following expression for life expenditure as a function of time

$$E(t) = \int \frac{dn}{N_x} \tag{31}$$

in which N_x is the number of cycles to failure at an amplitude X_p. It remains to express n and N_x as functions of X_p in order to carry out the integration.

We have shown that the relative motion response, X_n, is reasonably narrow banded due to the peaked character of the excitation spectral density curve. Thus the number of peaks occurring per second is approximated by $\frac{\omega_n}{2\pi}$. If we further assume that the suspension system has a high Q, i.e. $Q \geq 10$, then the number of peaks is $\frac{\omega_n}{2\pi}$ per second.* The number of such occurrences for which the peak value lies between X_p and $X_p + dX_p$ is (30):

$$dn = \frac{\omega_n t}{2\pi} \frac{X_p}{\sigma_n^2} e^{-\frac{X_p^2}{2\sigma_n^2}} dX_p \qquad (32)$$

in which σ_n = RMS value of the response motion ($\sigma_n^2 = \overline{X_n^2}$)

X_p = peak value of X_n

The form of the relation between maximum stress in a fatigue test and number of cycles to failure is expressible as the semi-logarithmic function

$$S = \alpha - \beta \log N^2 \qquad S > S_e \qquad (33)$$

which in terms of relative motion in the package can be written as

$$\left.\begin{aligned} X_p &= A - B \log N_x \\ \text{or} \quad \frac{1}{N_x} &= e^{-\frac{A}{B}} e^{\frac{X_p}{B}} \end{aligned}\right\} X_p > X_e \qquad (34)$$

where A and B are empirical constants (B corresponding to the slope of the S-N curve) and X_e is the amplitude corresponding to S_e.

Substituting [32] and [34] into [31] and performing the indicated integration (see Appendix I) results in

$$E(t) = \frac{\omega_n t}{2\pi} e^{\frac{\sigma_n^2}{2B^2} - \frac{A}{B}} \left[\frac{\sigma_n}{B} \sqrt{\frac{\pi}{2}} \left(1 - \mathrm{erf}(K)\right) + e^{-K^2} \right] \qquad (35)$$

---

*Since packages usually have a $Q < 5$ this appears a sweeping assumption but it is not necessarily crippling to what follows. Failure can be intellectually divided into heat failure and fatigue. If heat is the cause the temperature will build up rather rapidly until some equilibrium is reached. This will or will not cause failure. If the system has a very low undamped natural frequency, auxiliary dampers will be used. These are expected to fail first and then we do have a high Q package.

in which

σ_r = RMS value of relative motion response to the field environment defined by equation [23] .

$K = \frac{X_e}{\sqrt{2}\,\sigma_r}$

erf (K) = the probability integral $\frac{2}{\sqrt{\pi}} \int_0^K e^{-x^2} dx$, values of which can be obtained from tables, (31)

Miles (28) has suggested that "it seems reasonable to ignore the possibility of an endurance limit when................stress amplitudes are distributed over a wide range". If this be true in fact, and it is easy to visualize many distributed cushioning materials in which an endurance limit would be very poorly defined or even non-existent, then equation [35] simplifies to

$$E(t) = \frac{\omega_n t}{2\pi} e^{\left(\frac{\sigma_r^2}{2B^2} - \frac{A}{B}\right)} \left(1 + \frac{\sigma_r}{B}\sqrt{\frac{\pi}{2}}\right) \tag{36}$$

The equation analogous to equation [35] for a sinusoidal vibration test is

$$E_T(t) = \frac{\omega_n t_T}{2\pi} e^{\left(\frac{X_{rT}}{B} - \frac{A}{B}\right)} \tag{37}$$

where t_T = test time and X_{rT} = test response amplitude.

To account for possible variation from Miner's failure at unity hypothesis we set

$$E_T(t) = C\,E(t) \tag{38}$$

where C = the actual value assigned to Miner's constant.

Taking [35], [37] and [38] and solving for time produces

$$\frac{t_T}{Ct} = e^{\frac{\sigma_r^2 - 2BX_{rT}}{2B^2}} \left[\frac{\sigma_r}{B}\sqrt{\frac{\pi}{2}}\left(1 - \operatorname{erf}(K)\right) + e^{-K^2}\right] \tag{39}$$

or, substituting with [36] in lieu of [35]:

$$\frac{t_T}{Ct} = e^{\frac{\sigma_r^2 - 2BX_{rT}}{2B^2}} \left(1 + \frac{\sigma_r}{B}\sqrt{\frac{\pi}{2}}\right) \tag{40}$$

Miles (28) has also suggested that it is equally convenient to express fatigue life as a log-log relation; $\log S = \log \alpha - \beta \log N_S$. If we follow this through (see Appendix I) we arrive at the following complete statement of the ratio of test time to service life

$$\frac{t_T}{C_t} = X_{\eta T}^{-\frac{1}{B}} \left(\sqrt{2}\, \sigma_n\right)^{1/B} \Gamma\left(1+\frac{1}{2B}\right)\left[1 - \frac{\gamma\left(1+\frac{1}{2B}, K\right)}{\Gamma\left(1+\frac{1}{2B}\right)}\right] \tag{41}$$

where $\Gamma\left(1+\frac{1}{2B}\right)$ is the complete Gamma function which can be evaluated from tables or curves (32).

$\gamma\left(1+\frac{1}{2B}, K\right)$ is the incomplete Gamma function. Curves for some values of the ratio $\frac{\gamma}{\Gamma}$ are given in Jahnke-Emde (32) while extensive tables are given by Pearson (33).

If K be sufficiently small to permit ignoring its contribution [41] becomes

$$\frac{t_T}{C_t} = X_{\eta T}^{-\frac{1}{B}} \left(\sqrt{2}\, \sigma n\right)^{1/B} \cdot \Gamma\left(1+\frac{1}{2B}\right) \tag{42}$$

It is interesting to compare the effects of varying test amplitudes on test times. For a specific package design, the extensive modifiers will remain constants. Thus, the semilog equations can be rearranged to produce

$$\frac{t'_T}{t''_T} = e^{\frac{X''_{RI} - X'_{RI}}{B}} = e^{\frac{H}{B}\left(X''_{iT} - X'_{iT}\right)} \tag{43}$$

while the log-log S-N curve equations lead to

$$\frac{t'_T}{t''_T} = \left(\frac{X''_{RT}}{X'_{RT}}\right)^{1/B} = \left(\frac{X''_{iT}}{X'_{iT}}\right)^{\frac{1}{B}} \tag{44}$$

It is appropriate to review, at this point, what the theoretical approach does and does not tell us. The equations developed herein indicate that:

a. For a specific package a quantitative relationship between sinusoidal test time and service life in a random environment can be computed if:

1. Suspension systems follow the cumulative damage hypotheses and provided, further, that:

(i) The slope of and at least one point on the S, N curve for various materials can be found experimentally.

(ii) The level of the endurance limit, if it exists, can also be found.

(iii) Suitable values to use for Miner's constant can be found.

2. The response characteristics of the particular package can be determined. Note that σ_R is a function of system damping and input characteristics.

3. Adequate descriptions of the environment are available. These descriptions must include acceleration spectral density, bandwidth and time. Unless we have these, we cannot compute σ_R nor can we compute K.

Since we do not now have the critical information shown above, it would appear, at first glance, that further progress towards standardization is impractical. This is true, however, only if we insist that we must have, now, a quantitative service life prediction. There are certain other teachings implicit in equations [39] through [44] These are:

1. A sinusoidal test can be an adequate simulation of a random environment.

2. The time of test can be varied provided there be a concomitant variation in test response amplitude. The minimum such amplitude should be such as to produce a stress greater than the endurance limit. The maximum is the one time failure stress. In practice, other considerations such as available clearance should probably govern.

3. Since test severity is directly related to response amplitude, it is only necessary, to evaluate life of the suspension, to vibrate where response amplitude is the greatest, i.e. at the lowest resonant frequency.

These three major factors permit logical simplification of the confusion engendered by the conflicts inherent in Figure 2. However, it must be recognized that actual standardization is only possible if certain arbitrary postulates be accepted. In addition to the two postulates stated at the end of the section on environment we will proceed on the assumptions that the following are self-evident:

a. A test written as a specification requirement today must be capable of performance on equipment available today. If not capable of performance it dies from non-use. Besides, how could it be a quasi-legal basis for acceptance or rejection under contract?

b. Since it is clear that some arbitrary decisions must be made, it is possible for all concerned to be arbitrary in the same way.

With these thoughts in mind, we now proceed to a discussion of the arbitrary decisions to be made, lacing our argument with the lessons of the theoretical discussion just concluded and various practical matters inherent in specialty containers.

SELECTION OF FREQUENCY RANGE TO BE COVERED

The theoretical discussion has shown that we need only test at the lowest resonant frequency. We now examine which particular frequency limits can be placed on the specification requirements. There appears to be no question but that the tests should include the very low frequency range of 2-10 cycles per second. From the evidence available the peak amplitudes of the random environment can be expected in this range. Further, container shock isolation systems for very delicate components quite often exhibit natural frequencies below 10 cps. Such mounting systems are quite "soft", frequently require supplementary dampers, and resonance can produce large magnifications and considerable heat built-up which, in itself, can damage the materials used in mounts or cushions.

Now RD 219/3 shows that higher frequencies, up to 7,000 cps, are within the realm of possibility. The sinusoidal amplitude, however, even at 5g (an amplitude not claimed for the 7,000 cps frequency) is extremely small, i.e., 0.00000102 inches, which is hardly worth considering to be contributory to any response motion of the item in its shock isolation system. At 100 cycles per second, amplitude is only .005 inches for 5g accelerations. At 60 cycles per second, 1.3g produces an amplitude of only .003 inches. It is highly probable that such amplitudes would be damped out by almost any simple structure such as a single layer of corrugated fibreboard. In a missile container, of course, there is considerably more structure than a single layer of fibreboard between the source of the excitation and the suspended mass. We have never personally favored higher frequency vibration testing of shipping containers.* Because of possible prejudice, therefore, an industry survey went into the question of what is learned when testing containers over 60 cps. Of eleven replies received 4

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*Some will recall an early version of MIL-P-7936 in its uncoordinated form which was prepared in a considerably greater state of ignorance by one of the authors some six years ago. Here testing over 55 cps was not required except for rigidly mounted components in all-metal containers.

flatly stated that they had never learned anything, 2 stated they had no experience, 2 didn't answer this question. One pled lack of experience but doubted necessity. One said, "not a great deal although some secondary resonances were found and a single case of poor attenuation was noted." The last responder stated that about all he had proved is that "you end up testing the resonant frequency of the fixtures that are supposed to transmit the vibration from the table or head to the box".

The last remark, delivered in half joking language, truly represents a basic facet of the problem. With very few exceptions (the giant machine developed by Norair Division of Northrop or the special devices, only one of which is currently known to be in existence, developed by Royal Jet) there are no machines which will shake the whole container at these frequencies if the container be much larger than a case of canned goods. Some "cobbled rig" is almost always necessary to transmit force from the relatively small geometry of the shaker unit's moving element to the very much larger container. The higher frequency low amplitude vibrations simply get lost in this structure.

Another way of looking at the utility of testing above 60 cps is to consider G-factor vs. height of drop which would produce a natural frequency at 60 cps. Inasmuch as, for a linear system,

$$f_n = 3.13 \sqrt{\frac{K}{W}} \tag{45}$$

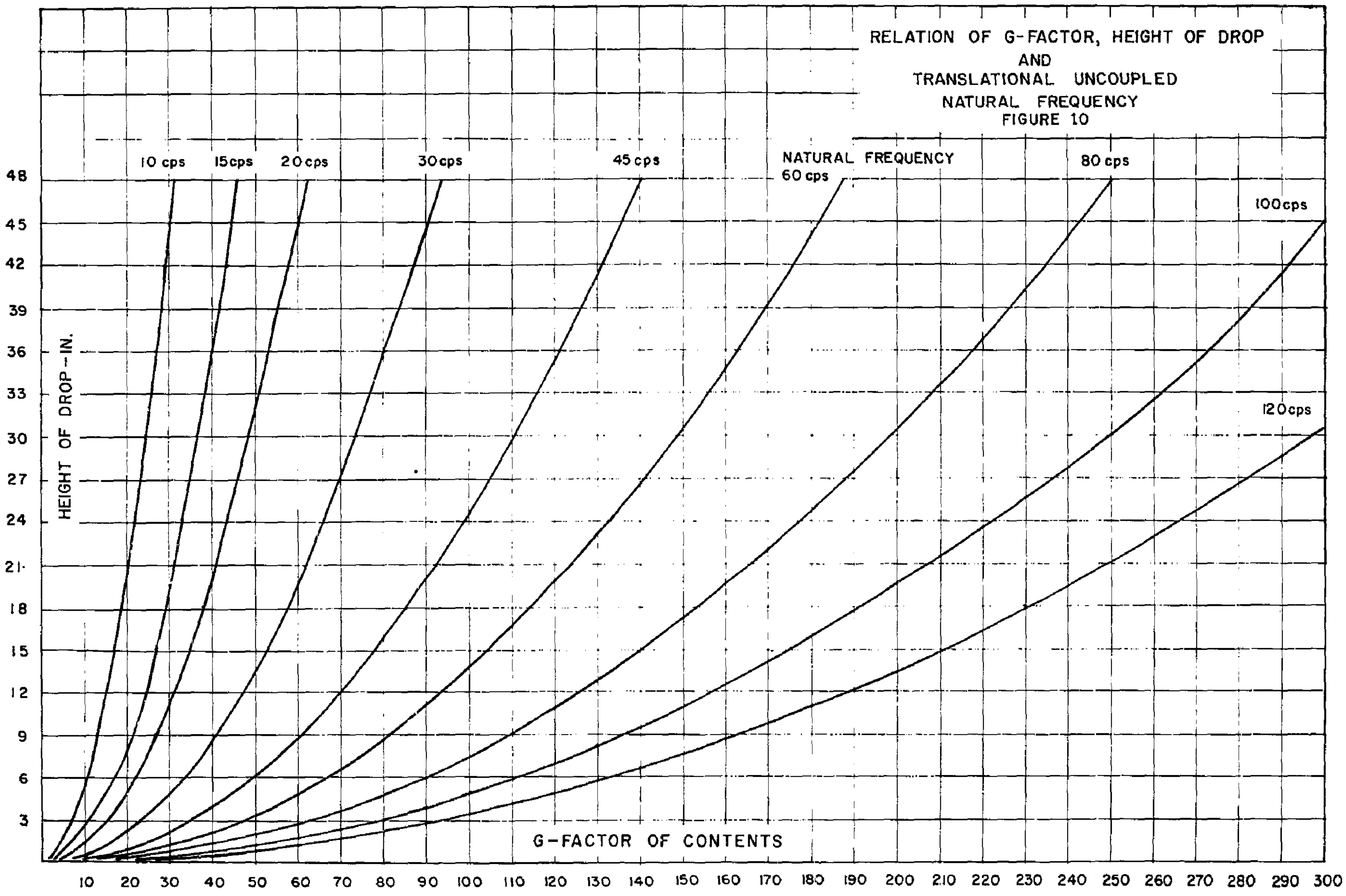
and

$$K = \frac{W (G+1)^2}{2h} \tag{46}$$

therefore

$$G = .453 f_n \sqrt{h} - 1 \tag{47}$$

Figure 10 is a plot of this last equation for various values of f_n. This graph shows that, to achieve a natural frequency of 60 cps at a reasonable drop height, the contents must be very rugged indeed. It is difficult, except in unusual circumstances, to envision stress levels on the order of 1/50th to 1/160th of a conservatively estimated breaking stress resulting in fatigue damage to the isolator system.



RELATION OF G-FACTOR, HEIGHT OF DROP
AND
TRANSLATIONAL UNCOUPLED
NATURAL FREQUENCY
FIGURE 10

In other words, so long as design emphasis is on shock isolation, high frequency testing does not appear to evaluate any significant life property of the suspension.

In the light of all the above, we consider that the standard shipping container vibration test should cover only the frequency range 2-60 cycles per second.

CYCLING vs. RESONANCE TESTING

The next question concerns whether the test should be cycling, resonance or combinations thereof. Practically all tests used require a search through the stipulated frequency range to find the points of resonance or maximum magnification. After finding resonances, the container is vibrated at these points at a stipulated amplitude for a stipulated period of time. The major exception is Picatinny Arsenal which follows the complex schedule* shown in Table I.

One could adopt the dangerous premise that a standard should not accept the Table I schedule on the simple grounds that the other agencies don't use it. Such a procedure is, of course, scientific nonsense and our reasons for favoring a simpler approach should be spelled out. We favor a procedure which involves searching out the fundamental "resonant frequency" and vibrating at that frequency for fairly long time because:

a. We are evaluating the fatigue resistance of the vibration isolation system.

b. Maximum system response is obtained at the fundamental resonant frequency. Hence, stress levels are at the maximum which can be obtained with motion generating equipment being used.

c. The cumulative damage hypotheses indicate that the maximized stresses will, for the shortest test time, produce the most damage.

d. Reversing c, if a system passes the test, confidence in service reliability of the system is obviously maximized.

e. While it is true that one can select the wrong stress or time parameter, or both, further studies of the environment can improve the validity of these parameters. Our theoretical approach supports this thesis.

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*The schedule reproduced here covers only ambient conditions. Picatinny also requires vibration testing with different time constants at temperature extremes.

f. All other things being equal, the simplest approach is to be preferred. This paper indicates that a simple approach is feasible.

TABLE I

Picatinny Arsenal Vibration Test Syllabus

Resonance & Transmissibility		Cycling		Resonance (2)	
C. P. S.	D. A.	C. P. S.	D. A.	C. P. S.	D. A.
2-10-2	.13	2-3-2	1g	2-10-2	.13
11-20-11	.09	4-5-4	1.5g	11-20-11	.09
21-60-21	.06	6-20-6	2g	21-60-21	.06
61-300-61	± 10g	21-60-21	.06	61-300-61	± 10g
301-500-301 (1)	± 10g	61-300-61	± 10g	301-500-301 (1)	± 10g
		301-500-301 (1)	± 10g		

Notes:

(1) If no predominant resonances noted between 60 and 300 cps, do not test above 300 cps.

(2) Between 2 and 20 cps vibrate intermittently at resonance for 30 mins. based on dwell time at resonance of 5 min. intervals, total vibration time 40 minutes. Above 20 cps vibrate 30 minutes steady steady state.

TEST AMPLITUDES

Having disposed of the questions of frequency range and type of test, the difficult problems of controlling amplitude and time to simulate the effects of the environment now follow. For convenience, amplitude is discussed first although time is closely related. Because of this interrelation we will discuss amplitude as a target optimum and discuss possible variations in amplitude and time later.

Our theoretical study indicates that there is no input amplitude function (such as displacement-frequency or, acceleration-frequency) which would result in equal life expenditure for every suspension system. Thus, the specification writer's ideal of a simple test applicable to all suspensions is a will-o'-the-wisp. On the other hand, any reduction in the variety of tests currently prescribed is an obvious desideratum.

Equations [43] and [44] show that input amplitude can be varied depending upon the response characteristics of the item on its suspension. These response characteristics (σ_n), are, in turn, functions of the acceleration spectral density, $S_{ia}(\omega_n)$. In the absence of statistical measures of $S_{ia}(\omega_n)$ we do not have a logical basis in theory for stipulating test time or response or input amplitude. While it should be possible eventually to obtain this information, and the results of such knowledge acquisition may be totally different amplitudes from those now specified, it is illogical to force vibration testing equipment development along a road, the direction and length of which cannot begin to be estimated at this time. That is to say, we have no right to ask for more capacity than is available until we know we need it.

In furtherance of this thought, therefore, we investigated the capacity of various commercially available vibration testers. Figures 11 through 16 show the capacity of various machines to move the several weights shown through various amplitudes.* One point that stands out above all others is that there is only one shaker which can cover all the amplitudes and frequencies shown

*The drawings are not cumulative but, rather, represent the maximum capacity of the existing known machines. Obiviously, large machines can all be used for testing small packages.

in Figure 2 to be currently required between 2 - 60 cps.* When amplitudes are restricted to those producing an acceleration of 1.3g on 5000 lbs. gross weight, the Wyle W-1000 type can perform at all frequencies.** Maximum acceleration output, at 5000 lbs. gross for the MB HC-50 hydraulic types is 1g.

Obtaining more than 1 inch double amplitude below 5 cycles per second is a very expensive endeavor. As it now stands, the only location where this could be done today is at Norair Division, Northrop Corp., Hawthorne, California*** which would add considerably to the cost of testing a container designed and fabricated east of the Rockies. Let us, therefore, look a little more closely at the requirements of those specifications which ostensibly call for double amplitudes in excess of 1 inch. Such a procedure discloses the following:

a. MIL-P-9024A and MIL-E-4970A show the very high double amplitudes below 5 cps identified in Figure 2. Both MIL-P-9024A and MIL-E-4970 permit significant deviation as exemplified by the following from paragraph 4.6.8.1 of MIL-E-4970A: "...... at a suitable sweep frequency rate and with a vibratory double amplitude or a peak acceleration input of <u>sufficient magnitude</u> to establish significant resonant frequencies.".

b. Picatinny Arsenal's procedure states "When the natural frequency lies in the 2 - 20 cps range input when cycling through resonance shall <u>be reduced</u> to those values listed in the resonance test for 2 cps below to 2 cps above the resonant frequency". Hence,

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* This is the Northrop electro-hydraulic system of which the Wyle company is a licensee for the shaker components. Northrop has estimated that development costs for their system were around $700,000.00 and that cost of duplication of their facility would be about $300,000.00.

** An informal cost estimate from Wyle for a system of this nature is $48,000 and 6 months delivery.

*** Northrop has expressed willingness to make time available on their machine to others and has stated that, once accepted, outside tasks will have equal priority standing with in-house work.

for resonant testing in the low range, the double amplitude is really 0.13 inches.

c. Until quite recently, there were no commercially available equipments which could shake a container through an approximately 8 inch stroke. Hence, most previous tests have been run at 1-inch double amplitude obtainable from the standard package testers covered by ASTM D999-48T.

In view of the foregoing, practically no vibration tests have been run at a double amplitude in excess of 1.0 inches and yet the vast number of tests which have been run at this amplitude gives an indirect measure of confidence that can be relied upon in a very large number of cases.* We have machines available to us which will vibrate at this amplitude between 2.5 and 5 cps, the L.A.B. and Gates package testers**. With minor changes in pulleys, the machines can be modified to operate between 2 and 4.5 cps. As will be brought out later, 4.5 cps (1g at 1 inch double amplitude) is considered high enough. Aside from the fact that most tests have been run at this amplitude, this simple change makes it possible to substitute a machine costing around $5,000 max. for one costing almost 10 times as much. This, then, represents our recommendation for the 2 - 4.5 cps range.

Over 4.5 cps one must choose between the 1.5g (stipulated by Picatinny up to 7 cps) the 1.3g of MIL-P-21927 (NOrd), MIL-P-9024A and MIL-E-4970 and some of the lesser units. Here we take the view that the maximum practical number of machines should be usable. Looking briefly at the 5000 lb. capacity curves

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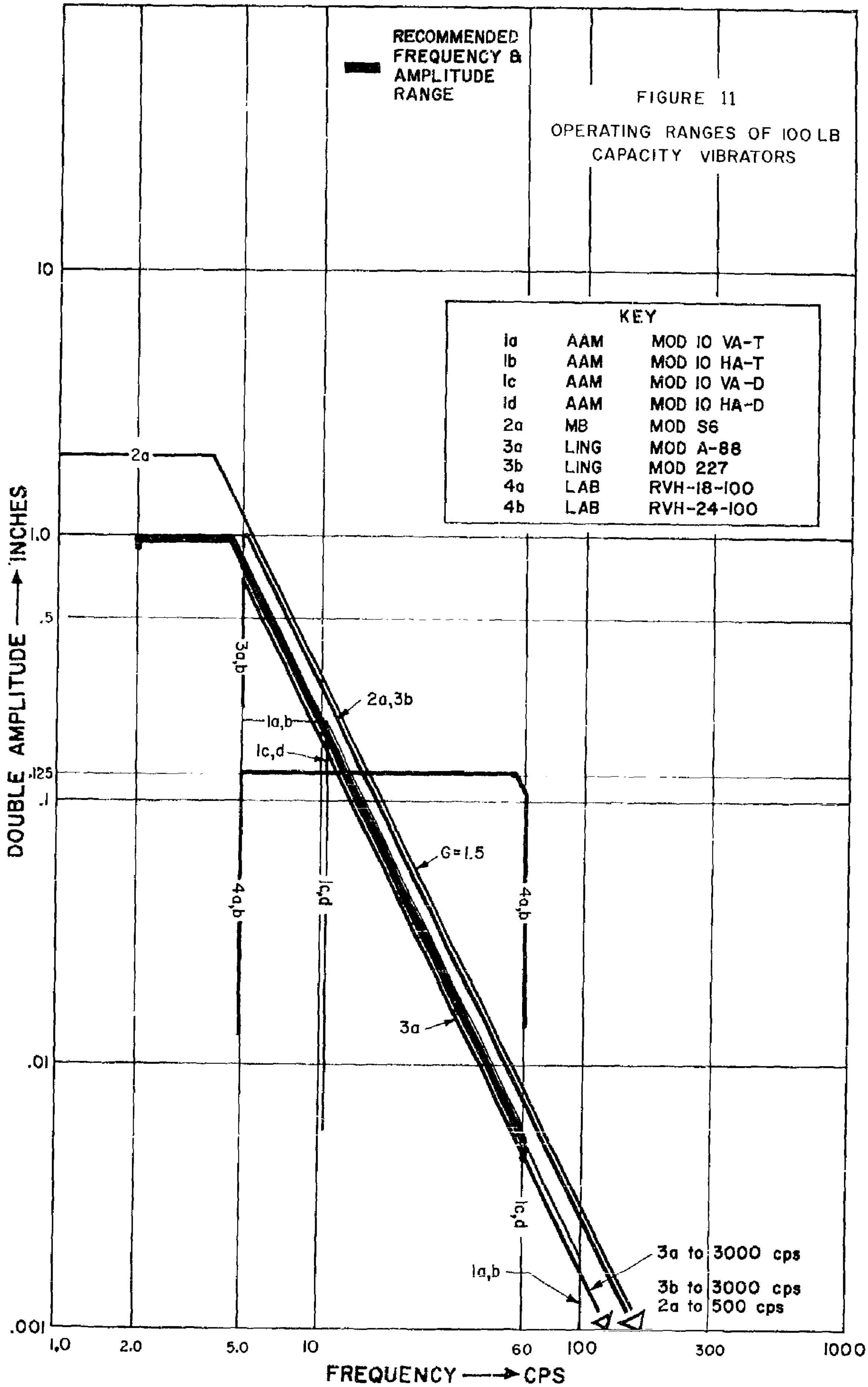
* There is no way one can avoid the implications of the National Safe Transit Committee program. One of the authors has personally conducted such vibration tests many times. Among other things that have been predicted successfully and cures found in the laboratory are vibration damage to TV sets and electronic organs, fatigue failures of an overloaded shock mount system, fretting corrosion damage to empty aluminum foil dinner trays, improved design of hot water heaters, causes of shipment damage to large electron tubes, etc., etc. The basic NSTC procedure is not as bad as some people think. When properly used and interpreted, it can be and is a powerful tool.

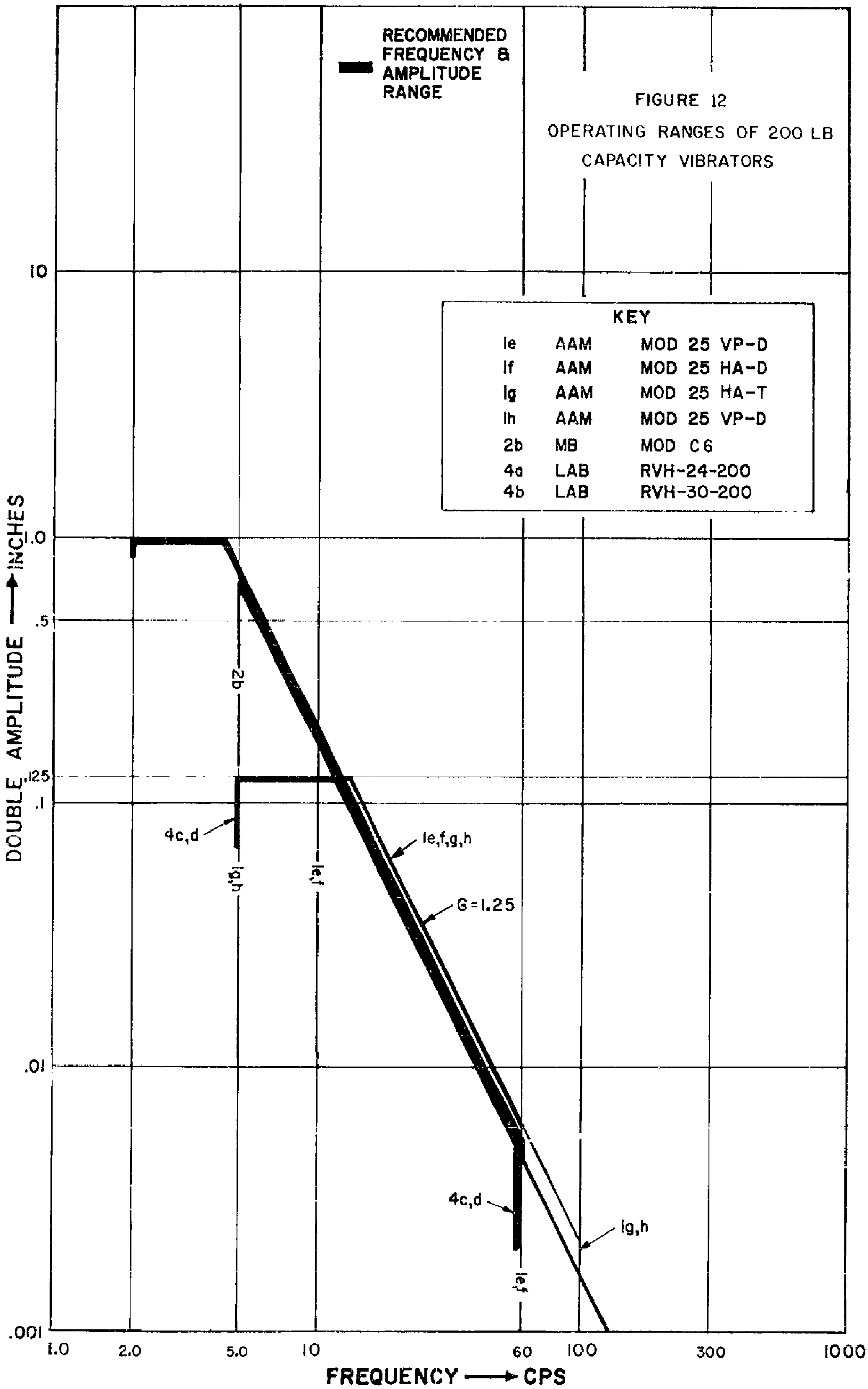
** The Gates testers are not identified as such in the various graphs. In general, what L.A.B. can do, Gates can do and at a competitive price.

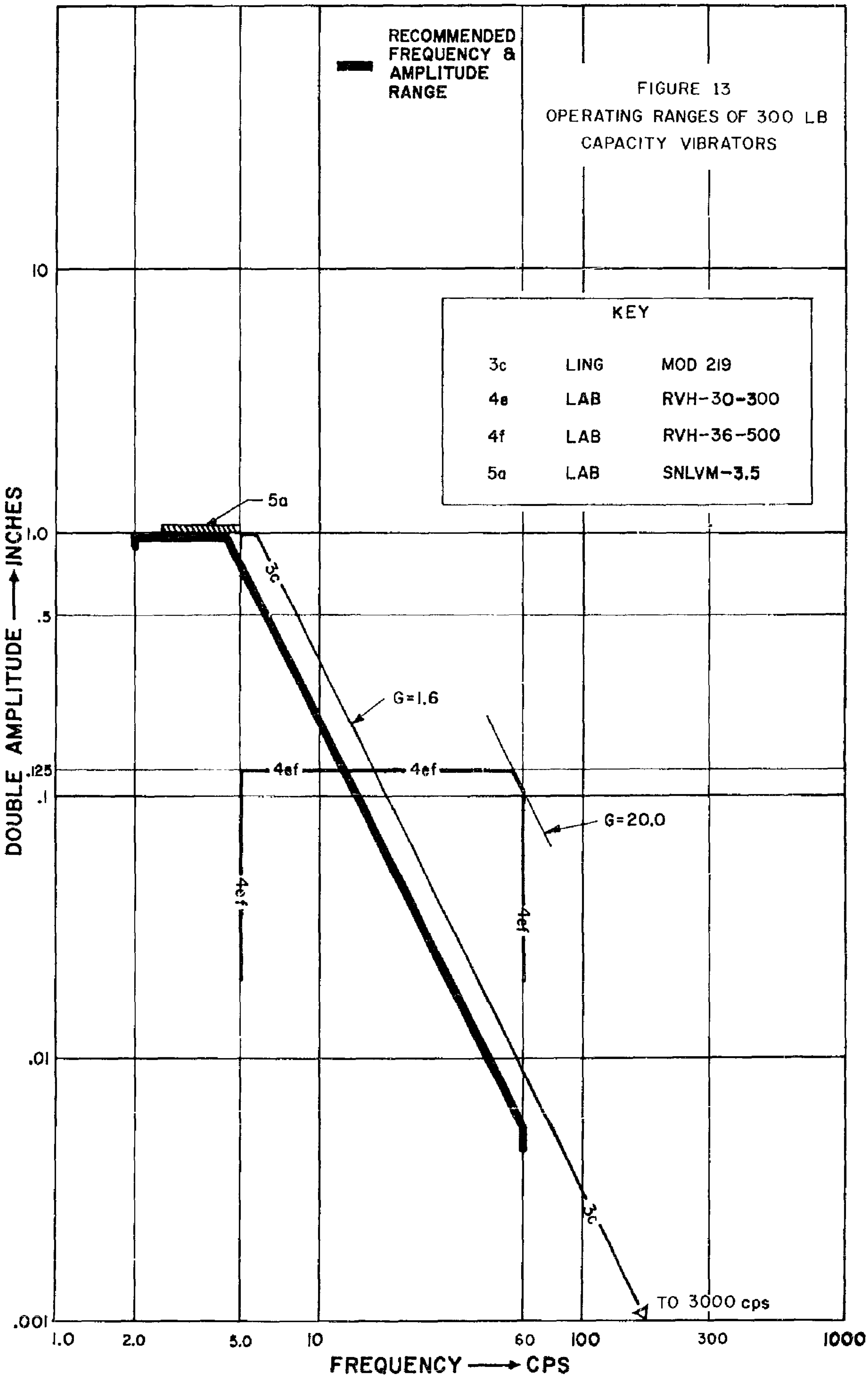
(Figure 11) we find an awkward gap between 4.5 cps and 5 cps at the 1.3g level (assuming that the LAB machine has been modified to get down to 2 cps) where only one machine in existence can produce the necessary frequency and amplitudes. This same gap exists all the way down to 100 pound capacity machines.* Over 5 cps, at 5000 lbs. MB C70 and Ling 246 have the theoretical capacity. If we drop the input to lg, however, we bring into the picture the MB HC50 family of hydraulic shakers plus Royal Jet's special and, above 5 cps, the whole family of MB electromechanical shakers. In other words, stipulating lg above 4.5 or 5 cycles per second results in maximum machine availability for performing the necessary tests. Of particular importance, the MB line of machines is made usable. Regardless of any merits or demerits of these particular devices, the MB concern has sold more of their machines than any other company, partly because they were among the very first in the field. In the absence of a statistically sound description of the environment, we can see no compelling reason for eliminating the majority of available devices from consideration just because of an arbitrary interpretation of that environment. We therefore, recommend an equally arbitrary decision setting the amplitude at lg between 4.5 and 60 cps. The complete frequency amplitude spectrum recommended as a possible standard is shown in Figure 17.

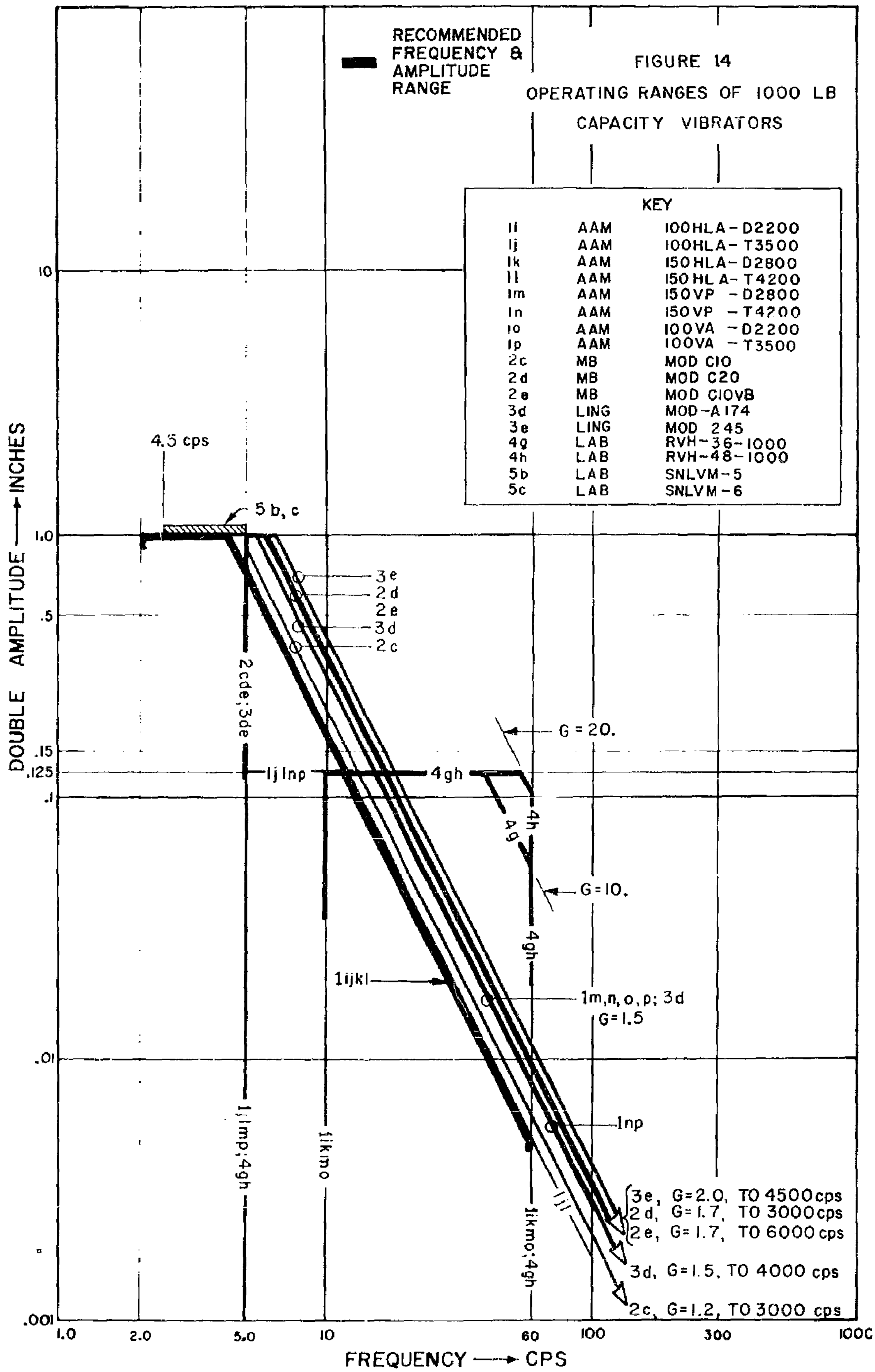
* This gap is not always present. Thus, it is possible to modify a "package tester" so that it will cover both 2 and 5 cps. Further at 5000 lbs. the Ling and MB machines might be run down that extra 1/2 cycle. If the load is considerably less than maximum capacity, you do have a chance of covering the gap with a machine of considerably greater capacity than would normally be rated for the job.



RECOMMENDED FREQUENCY & AMPLITUDE RANGE
FIGURE 11
OPERATING RANGES OF 100 LB CAPACITY VIBRATORS
KEY
1a AAM MOD 10 VA-T
1b AAM MOD 10 HA-T
1c AAM MOD 10 VA-D
1d AAM MOD 10 HA-D
2a MB MOD S6
3a LING MOD A-88
3b LING MOD 227
4a LAB RVH-18-100
4b LAB RVH-24-100
DOUBLE AMPLITUDE ——→ INCHES
FREQUENCY ——→ CPS
2a
3a,b
1a,b
1c,d
2a,3b
G=1.5
4a,b
1c,d
4a,b
3a
1c,d
1a,b
3a to 3000 cps
3b to 3000 cps
2a to 500 cps


RECOMMENDED FREQUENCY & AMPLITUDE RANGE
FIGURE 12
OPERATING RANGES OF 200 LB CAPACITY VIBRATORS
KEY
Ie AAM MOD 25 VP-D
If AAM MOD 25 HA-D
Ig AAM MOD 25 HA-T
Ih AAM MOD 25 VP-D
2b MB MOD C6
4a LAB RVH-24-200
4b LAB RVH-30-200
10
1.0
.5
.125
.1
.01
.001
2b
4c,d
Ig,h
Ie,f
Ie,f,g,h
G=1.25
4c,d
Ig,h
Ie,f
DOUBLE AMPLITUDE → INCHES
1.0
2.0
5.0
10
60
100
300
1000
FREQUENCY → CPS


RECOMMENDED FREQUENCY & AMPLITUDE RANGE
FIGURE 13
OPERATING RANGES OF 300 LB CAPACITY VIBRATORS
KEY
3c LING MOD 219
4e LAB RVH-30-300
4f LAB RVH-36-500
5a LAB SNLVM-3.5
5a
3c
G=1.6
4ef
G=20.0
TO 3000 cps
10
1.0
.5
.125
.1
.01
.001
DOUBLE AMPLITUDE ——→ INCHES
1.0
2.0
5.0
10
60
100
300
1000
FREQUENCY ——→ CPS


RECOMMENDED FREQUENCY & AMPLITUDE RANGE
FIGURE 14
OPERATING RANGES OF 1000 LB CAPACITY VIBRATORS
KEY
1i AAM 100HLA-D2200
1j AAM 100HLA-T3500
1k AAM 150HLA-D2800
1l AAM 150HLA-T4200
1m AAM 150VP -D2800
1n AAM 150VP -T4200
1o AAM 100VA -D2200
1p AAM 100VA -T3500
2c MB MOD C10
2d MB MOD C20
2e MB MOD C10VB
3d LING MOD-A174
3e LING MOD 245
4g LAB RVH-36-1000
4h LAB RVH-48-1000
5b LAB SNLVM-5
5c LAB SNLVM-6
4.5 cps
5 b, c
3e
2d
2e
3d
2c
2cde;3de
G = 20.
1j1np
4gh
4h
4g
G=10.
4gh
1ijkl
1m,n,o,p; 3d G=1.5
1np
1jlmp;4gh
1ikmo
1ikmo;4gh
3e, G=2.0, TO 4500cps
2d, G=1.7, TO 3000cps
2e, G=1.7, TO 6000 cps
3d, G=1.5, TO 4000 cps
2c, G=1.2, TO 3000 cps
DOUBLE AMPLITUDE ——→ INCHES
FREQUENCY ——→ CPS

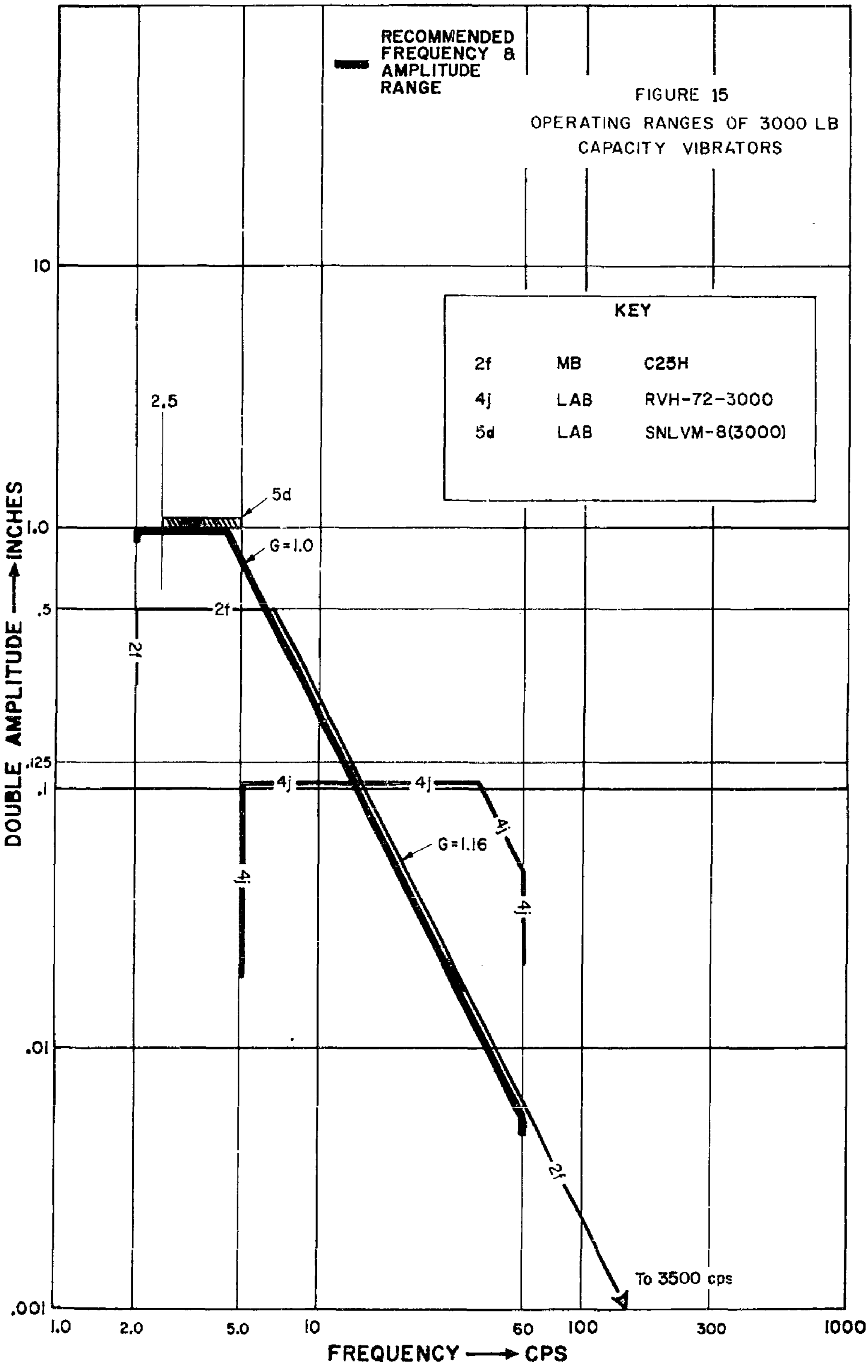
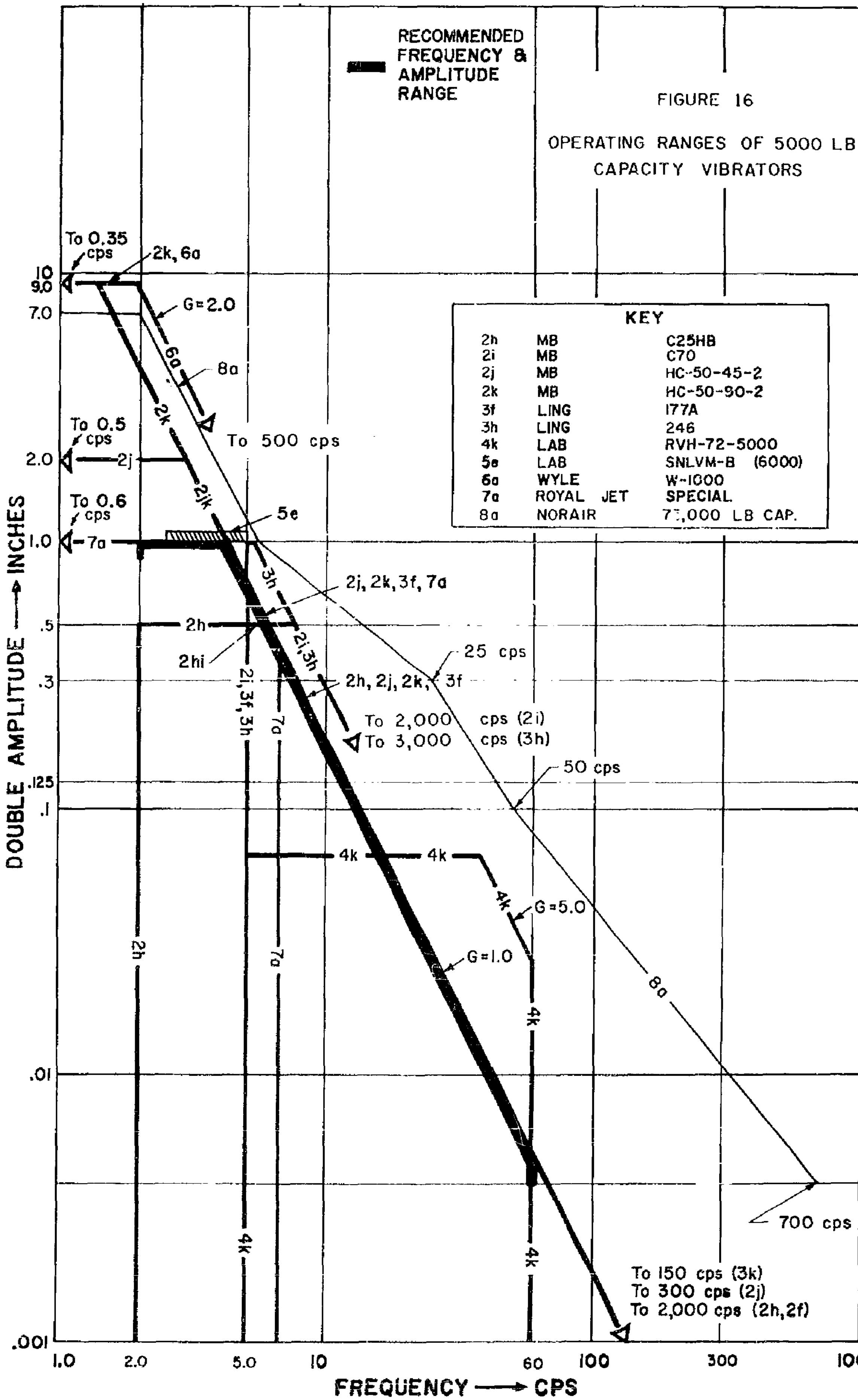


FIGURE 15
OPERATING RANGES OF 3000 LB CAPACITY VIBRATORS



RECOMMENDED FREQUENCY & AMPLITUDE RANGE
FIGURE 16
OPERATING RANGES OF 5000 LB CAPACITY VIBRATORS
KEY
2h MB C25HB
2i MB C70
2j MB HC-50-45-2
2k MB HC-50-90-2
3f LING 177A
3h LING 246
4k LAB RVH-72-5000
5e LAB SNLVM-B (6000)
6a WYLE W-1000
7a ROYAL JET SPECIAL
8a NORAIR 75,000 LB CAP.
To 0.35 cps
2k, 6a
G=2.0
6a
8a
2k
To 500 cps
To 0.5 cps
2j
2jk
To 0.6 cps
5e
7a
3h
2j, 2k, 3f, 7a
2h
2hi
2i,3h
25 cps
2h, 2j, 2k, 3f
2i, 3f, 3h
7a
To 2,000 cps (2i)
To 3,000 cps (3h)
50 cps
4k
4k
4k
G=5.0
G=1.0
2h
7a
8a
4k
700 cps
4k
4k
To 150 cps (3k)
To 300 cps (2j)
To 2,000 cps (2h, 2f)
DOUBLE AMPLITUDE → INCHES
10
9.0
7.0
2.0
1.0
.5
.3
.125
.1
.01
.001
FREQUENCY → CPS
1.0
2.0
5.0
10
60
100
300
1000

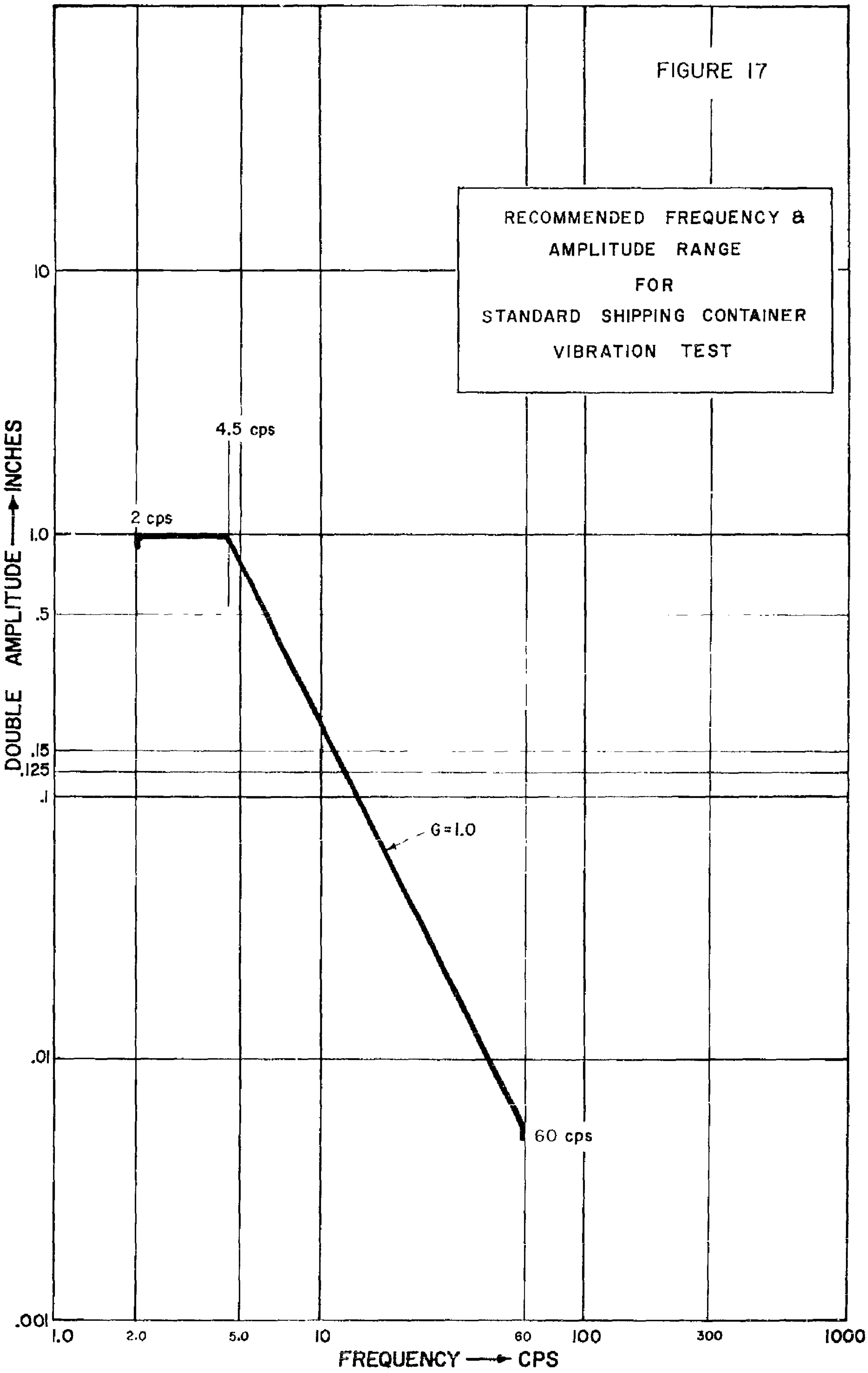


FIGURE 17
RECOMMENDED FREQUENCY & AMPLITUDE RANGE FOR STANDARD SHIPPING CONTAINER VIBRATION TEST
DOUBLE AMPLITUDE ——► INCHES
FREQUENCY ——► CPS
10
1.0
.5
.15
.125
.1
.01
.001
1.0
2.0
5.0
10
60
100
300
1000
4.5 cps
2 cps
G = 1.0
60 cps

TEST TIMES

Before discussing possible equivalent variances in amplitudes to further broaden the range, let us first examine the time parameter. Our theoretical studies disclose no basis, with currently available data, for selecting an optimum test time for the amplitudes selected. We proceed, therefore, to reasoning by analogy.

Most of the tests require 1/2 hour at resonance but at the 1.3 g intensity level. In the range of frequencies covered by the National Safe Transit Committee procedures, and with which the author has had the most experience, it is considered that a sound system should be capable of withstanding 1 hour at "resonance". Commercial tests are based on this time scale at a 1" double amplitude and it hardly seems reasonable to require less in guided missile containers. The authors own personal experience is that reliable systems will last almost indefinitely at these amplitudes and frequencies whereas many which are marginal require more than 30 minutes to fail*.

Inasmuch as we have recommended a 1g intensity in lieu of the more commonly specified 1.3g at frequencies in excess of 4.5 cycles per second some increase in the time at resonance over the 30 minutes usually stipulated for this range appears warranted. For conformity with the low frequency test requirement, 1 hour is recommended**.

---

* A specific case of a hi-fi unit is recalled in which the chassis was supported on soft rubber mounts. An alarming number of instances of transformer mounting lug failure had been reported. In the laboratory after about 40 minutes at 3.5 cycles 1" double amplitude, fatigue failure of the lugs occurred. The transformer then "took off" and "cleaned out" the inside of the hi-fi set. Appropriate corrections were then devised and no failures occurred in 1 hr. No further failures in shipment were then encountered. While this was actually a case of faulty unit design, the principle disclosed is valid for the package.

** Since we are also recommending vibrating only at the fundamental resonance, this increase in time at this point will still result in a net decrease in overall test time.

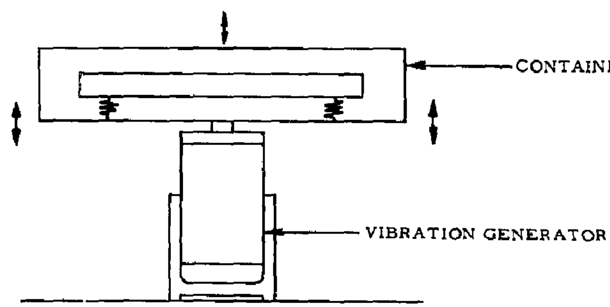
Heretofore, we have been discussing testing as though the entire container could be moved in conformity with the input motions stipulated. There are, however, many containers that simply will not fit on existing equipment without substantial, and quite often mechanically significant, modifications in the tester. Figure 18 summarizes some of the problems associated with testing when machines of adequate size and force capacity are not available. If a practical test procedure is to be written, then the practical test procedure must recognize the practical difficulties of compliance. At the same time, it is felt that a practical test should take into account the possibility of further increasing severity and reducing test time. As pointed out earlier, current specifications permit reduction in amplitude but do not make any attempt to modify the time parameter. Equations [43] and [44] , on the other hand, indicate that change in test time with change in amplitude is a logical necessity. In the absence, however, of appropriate data concerning proper values for the slope of the S-N curve (B), direct use of these equations is not possible at this time.

If we are to propose a standard at this time which takes into account the practical difficulties of complying with a stipulated amplitude, we must again be arbitrary while still remaining consistent with our theoretical study. In any case, a specification must be consistent with itself even though it may not be consistent with reality. One way in which time might reasonably be varied is to equate the total input work. Since work for a given sine wave cycle is conveniently measured by the mean square amplitude, we find that total work is the same when the ratio of test times is inversely proportional to the mean square amplitudes, or

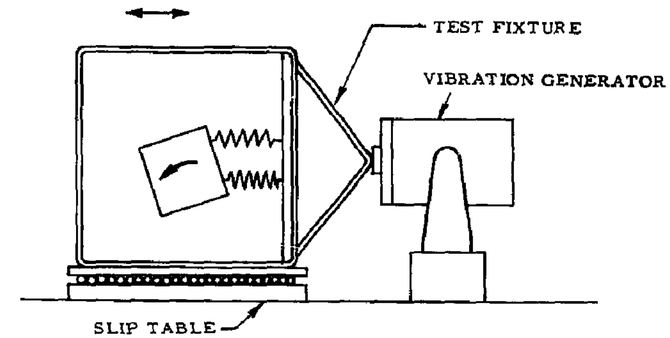
$$\frac{t'_T}{t''_T} = \left(\frac{X''_{iT}}{X'_{iT}}\right)^2 \qquad (48)$$

This expression is the same as the similar expression for the log-log S-N curve, simply through substitution of $B = 1/2$ in equation [44] . Thus, our criterion of maintaining some consistency with the theoretical approach is satisfied.

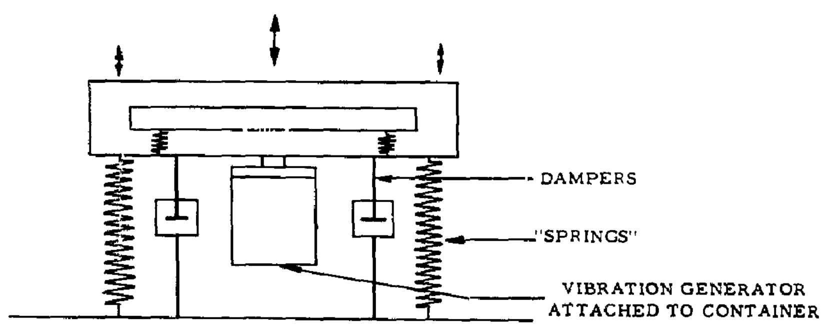
By the cumulative damage hypotheses the relation given above can only be valid in the range X_{iT} = at least large enough to just exceed the infinite number of stresses line on the S-N curve to X_{iT} the maximum one time stress the system is capable of



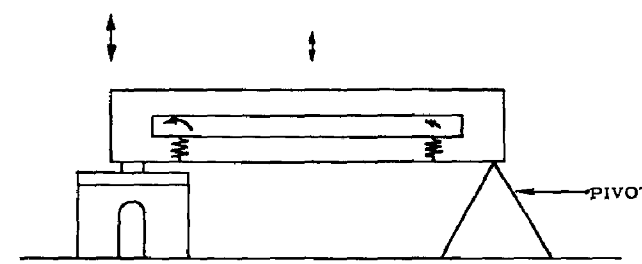
A. BAD OVERHANG. END MOTION GREATER THAN CENTER. VERY SEVERE ON CONTAINER. IF CONTAINER MOTION AT ITEM MOUNT LOCATIONS SAME AS STIPULATED, COULD BE ROUGHLY EQUIVALENT TO SPECIFICATION.



B. CONTAINER TOO HEAVY, TEST ON SIDE. CONTAINER MOTION CAN BE HELD TO REQUIRED VALUES IF FIXTURE IS PROPERLY DESIGNED (SOMETIMES EXPENSIVE). MAJOR DISADVANTAGE IS PRODUCTION OF UNDESIRED ROTATIONAL MODE IN MOUNT SYSTEM.



C. CONTAINER TOO HEAVY OR TOO LONG, RESONANT BEAM TEST. CONTAINER IS USED AS BEAM OR SOMETIMES BEAM IS UNDERNEATH. RESONANT FREQUENCY MUST BE FOUND FIRST AND THEN SPRINGS AND DAMPERS MUST BE DESIGNED TO PRODUCE REQUIRED MOTION. MOTION AT GENERATOR IS GREATER THAN AT ENDS BECAUSE OF BEAM DEFLECTIONS.



D. MODIFIED RESONANT BEAM. DOUBLES CAPACITY OF ANY MACHINE. MOTION AT CENTER ONLY 1/2 MOTION AT MOVING END. THE MOUNT INPUT MOTIONS ARE PROPORTIONAL TO SPACING. AS A MINIMUM, SWAP ENDS HALFWAY THROUGH TESTS. TO BE ROUGHLY EQUIVALENT, MOUNT LOCATION MOTION SHOULD BE EQUAL TO SPECIFIED VALUES. NOTE: SOME ROTATIONAL COUPLING INEVITABLE.

FIGURE 18

SOME VARIATIONS ON BRUTE FORCE VIBRATION TESTING withstanding. Since these values are not known these limits are somewhat theoretical. We take refuge in analysis of Figure 18 which tells us, in effect, that it is not essential that the input amplitude be less than 1/2 the stipulated value*. For a high limit we can take refuge in the opinion expressed by one industry responder that 10 minutes at resonance with a high amplitude input should normally be sufficient. A plot of the relation given above in terms of t''_T = 60 minutes and within the limits $X''/X' \geqq .5$ and $t''_T \geqq 10$ is shown in Figure 19 which is recommended for adoption.

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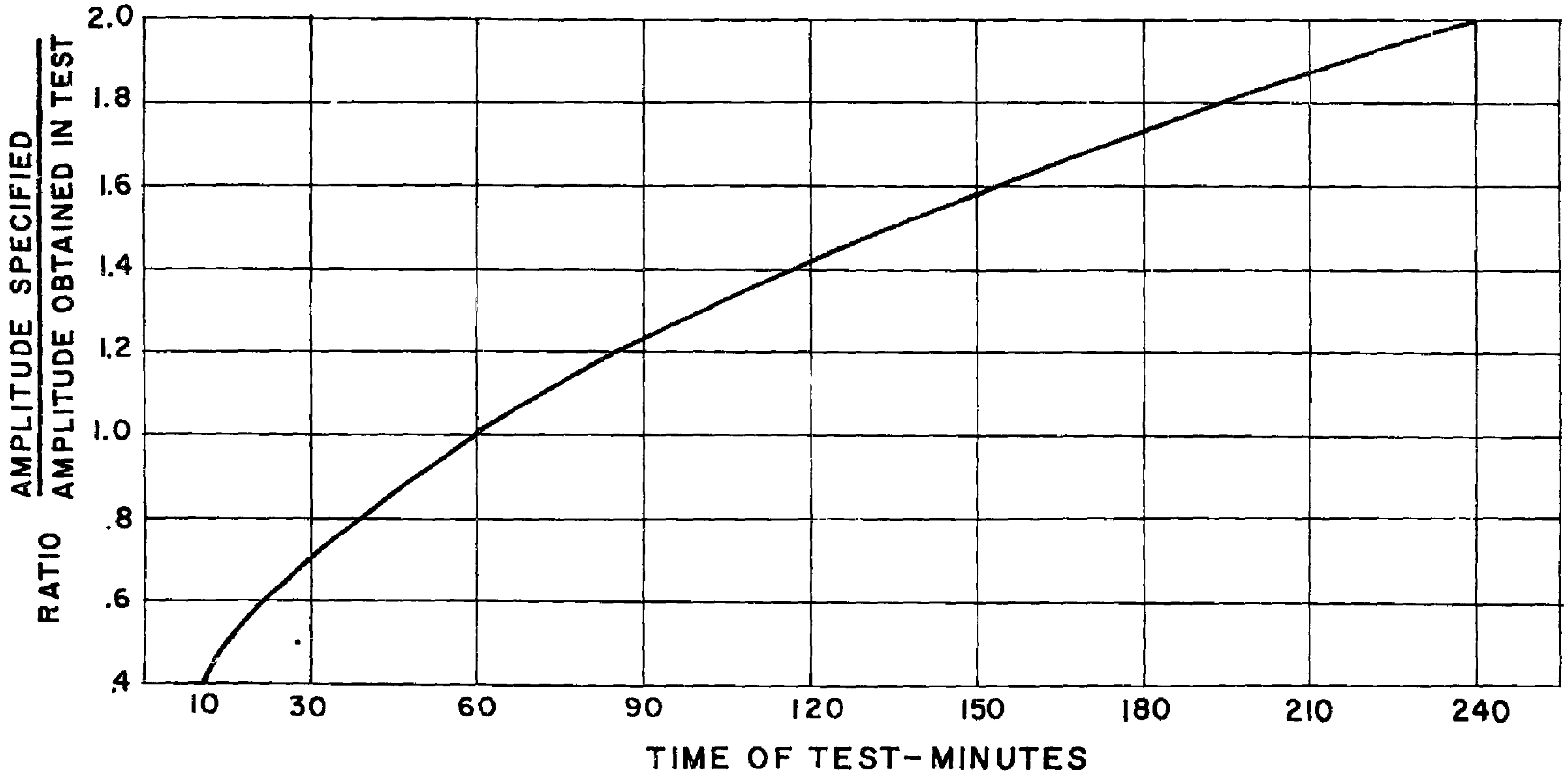
* In fairness it is not proven that this is a valid lower limit. The author has been "burned" by a system vibrated at approximately 3 cps with one end on an LAB package tester as in scheme D of Figure 18. With this low natural frequency the shear mount system had auxiliary friction dampers such that transmissibility was less than 3. After considerable cut and try, including use of teflon inserts in the damper members, it was possible to get the system to pass the contract requirements of 1/2 hour input at each end. When the system was placed in service, however, the dampers heated up, seized and then parted company. Many instances of broken mounts were found. Eventually, the system had to be completely redesigned to a different natural frequency before a serviceable container was achieved. In the questionnaire results many of the responders regarded this particular situation as a freak. But what good is a specification if it doesn't at least screen out the freaks? All we know for sure is that 1 hour at half the stipulated amplitude wasn't enough and that a four hour test might have caught it. Or, four hour testing certainly should have given us a better chance to find this element of unreliability than the 1 hour test which failed to prove it.

FIGURE 19

SUGGESTED TIME OF TEST AT AMPLITUDES DIFFERING FROM THOSE GIVEN IN FIGURE 17



DIRECTION OF APPLICATION

Practically all of the test procedures analyzed require that input vibrations be applied to the three orthogonal axes of the container on the assumption, apparently, that the unit has equal chance of being stowed in the carrier on any face. The minute, however, that the container has so much as a "This Side Up" marked on it the chances are no longer equal that any specific container will be stored on any face*. The moment a container has a definite base defined in some manner, such as by skids, etc., carriage on any other face becomes a difficult achievement.

From the standpoint of the designer, the possibility of vibration on any face requires careful design analysis, frequently imposing complexities in the design which might otherwise be unnecessary. Almost all knowledgeable designers of large containers, therefore, seek deviation from the requirement in one fashion or another and, in most cases, this deviation is granted, tacitly, if not explicitly. Since the deviation appears to have been granted more often than not, correct specification practice dictates placing all designers on equal footing. Stipulation that vibration be applied only through the base on skidded containers appears to be an obvious requirement.

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*In spite of our usual cynicism in these matters, some people do read and heed. Incidentally, a carrier can be held liable for damaged goods if it can be shown he ignored such a warning.

FASTENING TO THE VIBRATION TABLE

There is considerable variation in stipulating the method of fastening the container to the vibration sou ce. MIL-W-21927 (NOrd) prescribes tying down as it would be in the rail car while other specifications tend towards use of the phrase "rigidly attached". While ideal infinite rigidity is impossible, vibration test technicians much prefer the prettier results obtained from a "rigid" attachment. This occasionally produces the odd result that the tie downs are more massive than the container! In many cases, the tie downs are considerably more expensive than what would be used in practice. This element of unreality is further emphasized by a quick glance at most carloading practices for heavy containers. In most blocking arrangements, the restraints are primarily fore and aft with vertical restraints consisting mostly of far from rigid steel strapping. For smaller containers the vertical restraint usually consists only of the weight of superimposed containers plus friction of adjacent containers. Possibility always exists that some significant portion of the journey could well be without any restraint.

A completely unrestrained container, on the other hand, could "take off". Aside from the fact that falling off the vibration rig could be disastrous, the element of personnel safety enters.

Now the NSTC test procedure does not actually call for testing at the point at which maximum resonance occurs. The procedure stipulates that one increases the frequency until such time as the resultant causes the outer container to lift off the table approximately 1/32 inch. This may or may not be the frequency producing maximum transmissibility but it is not far removed from it. From the number of tests conducted, this slight extra input at each cycle seems to improve the confidence in the test results. In any case, the slight additional motion, while defying generalized analysis, can be considered as a possible allowable increased motion which does not "ruin" the test even though it certainly will confuse the record. We, therefore, make the suggestion that this amount of motion be allowed. It should be noted in passing that holding acceleration to 1 g in lieu of 1.3 g, greatly reduces the strength requirements for hold downs and, therefore, contributes to a lesser cost of testing.

PROPOSED LANGUAGE FOR A STANDARD TEST

With the above reasoning in mind, it is possible to write a test procedure requirement which could be adopted as standard and which approaches our self imposed criteria. This proposed language follows.

Basic Test

The container, loaded with its contents or with a dummy of equal weight and equivalent principal moments of inertia, shall be vibrated in such a fashion as to discover the frequency producing the maximum transmissibility within the frequency range of 2 to 60 cycles per second. The container shall then be suitably fastened (maximum permissible motion freedom approximately 1/32nd of an inch) to the table of a vibration testing device producing an essentially sine wave motion and vibrated for one hour at the frequency producing maximum transmissibility and at the double amplitude shown in Figure 17.

The test shall be applied to each of the three principal orthogonal axes of the shipping container except that where containers are equipped with a definite base, defined by skids or equivalent, vibration inputs shall be applied only through such base. This test shall preferably be applied just before any required shock tests.

Permissible Alternates

a. When a device such as one meeting ASTM specification D996 is used for frequencies less than 5 cycles per second, the horizontal motion inherent in such machines is acceptable. Wherever possible, however, machines shall be adjusted to produce "vertical-linear" motion.

b. When testing devices are not available to produce the required double amplitude over the entire container, alternate methods will be acceptable provided the motion at the locations of mounts (or the centroids of each cushion on either side of the item center of gravity) is the same as that specified. A test procedure which places one end of the container on a pivot while the other end is moved is also acceptable provided the position of the vibration generator is changed from one end of the container to the other as rapidly as possible half-way through the test.

c. Variations in stipulated double amplitudes are permitted provided that the time of test be adjusted in accordance with Figure 19. The minimum time at resonance shall be 10 minutes and the maximum time at resonance shall be 4 hours.

DISCUSSION

It is regretted that a search for transportation vibration data in a form useful for computing in accordance with the theories advanced herein produced negative results. Most literature data are already reduced to a form useful for the particular study in hand. If original records are still available, it might be possible to reduce the original data in the form needed here. It is considered of fundamental importance that:

a. Every effort be made to salvage as much random data from still existing records as possible.

b. Future records be analyzed for whatever purposes desired but also be analyzed for contribution to a proper statistical description of the environment.

Achieving a useful description of the environment is of fundamental importance regardless of the ultimate fate of the theory advanced herein. The theory cannot be completely discounted until such time as the environment is shown not to conform to our assumptions.

We have also, however, developed a theory of possible behavior based on the presumption that cumulative fatigue hypotheses apply to package shock and vibration isolation devices and materials. This assumption lacks any experimental verification except for steel springs. The scope of this study precluded such experimentation which, in any case, should not have been undertaken until a theory needing verification existed. Now that the theory exists it is considered a most urgent necessity that work be undertaken to verify, on a spot check basis as a minimum, whether or not shock isolation systems obey cumulative damage hypotheses.

Complete verification of all facets of our theory cannot come immediately. After studying the various test methods currently specified the only reasonable conclusion is that some form of standardization is an immediate requirement. We have, therefore, sought a logical means of cutting through the fog to some basis which, although of necessity rather arbitrary, recognizes the practical limitations of conducting any such test and is consistent with at least one of the theoretical lines developed herein. To the best of our knowledge, no other testing method proposal can make such a claim while still maintaining the virtue of compliance ability on a broad national basis. Accordingly, it is urged that the test method proposed be adopted as an interim measure. The fact that this interim may last a number of years simply buttresses any arguments against waiting for a perfected method.

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APPENDIX I

MATHEMATICAL DERIVATIONS

In the interest of maintaining continuity of reasoning some rather tedious mathematical operations have been omitted from the main body of the report. This appendix details these operations for the record.

A. <u>Derivation of Equation [22].</u>

Equation [17] reads

$$\alpha_o^{-2} = \frac{1}{2\pi}\int_0^\infty |T_o|^2 S_{ia}(\omega)\,d\omega \tag{A-1}$$

while equation [21] reads

$$S_{ia}(\omega) = \frac{B_i^2\left(\frac{\omega}{\omega_n}\right)^2}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + B_i^2\left(\frac{\omega}{\omega_n}\right)^2} S_{ia}(\omega_n) \tag{A-2}$$

By direct substitution (A-1) becomes

$$\alpha_o^{-2} = \frac{1}{2\pi}\int_0^\infty |T_o|^2 \frac{B_i^2\left(\frac{\omega}{\omega_n}\right)^2}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + B_i^2\left(\frac{\omega}{\omega_n}\right)^2} S_{ia}(\omega_n)\,d\omega \tag{A-3}$$

Equation [7] gives a value of

$$T_o = \frac{1 + i\,2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2 + i\,2\zeta\frac{\omega}{\omega_n}} \tag{A-4}$$

Squaring and substituting into (A-3) leads to the rational expression

$$\alpha_o^{-2} = \frac{1}{2\pi}\int_0^\infty \frac{\left[1+4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2\right] B_i^2\left(\frac{\omega}{\omega_n}\right)^2}{\left[\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2+4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2\right]\left[\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2+B_i^2\left(\frac{\omega}{\omega_n}\right)^2\right]} S_{ia}(\omega_n)\,d\omega \tag{A-5}$$

For simplicity let $\frac{\omega}{\omega_n} = X$, then $d\omega = \omega_n dX$ and apply the further reasonable restrictions that $\zeta < 1$, $B_i < 2$ and $2\zeta \neq B_i$. For a given set of conditions ω_n, B_i and $S_{ia}(\omega_n)$ will be constants. Hence (A-5) goes over to

$$a_o^2 = \frac{\omega_n}{2\pi} B_i^2 S_{ia}(\omega_n) \int_0^\infty \frac{(1+4\zeta^2 x^2) x^2}{[(1-x^2)^2+4\zeta^2 x^2][(1-x^2)^2+B_i^2 x^2]} dx \quad \text{(A-6)}$$

Call the integrand $f_1(X)$ and let $I_1 = \int_0^\infty f_1(x)\,dx$ and it can be seen that we have a rational function of X which can be integrated if expressed in terms of partial fractions.

The left hand member of the denominator in the integrand of equation (A-6) expands and then factors into

$$\begin{aligned}(1-x^2)^2 + 4\zeta^2 X^2 &= X^4 + 2X^2(2\zeta^2 - 1) + 1 \\ &= (X^2 + \alpha X + 1)(X^2 - \alpha X + 1)\end{aligned} \quad \text{(A-7)}$$

where $\alpha = 2\sqrt{1-\zeta^2}$

Similarly the right hand term of the same denominator expands and factors into

$$(1-x^2)^2 + B_i^2 X^2 = X^4 + X^2(B_i^2 - 2) + 1 \quad \text{(A-8)}$$

$$= (X^2 + \beta x + 1)(x^2 - \beta x + 1)$$

where $\beta = \sqrt{4 - B_i^2}$

Hence $f_1(X)$ will assume the form

$$f_1(X) = \frac{A + Kx}{X^2 + \alpha x + 1} + \frac{C + Dx}{X^2 - \alpha x + 1} + \frac{M + NX}{X^2 + \beta x + 1} + \frac{P + QX}{X^2 - \beta x + 1} \quad \text{(A-9)}$$

Expanding this function, we have

$$f_1(x) = \frac{[(A+Kx)(x^2-\alpha x+1)+(C+Dx)(x^2+\alpha x+1)][(1-x^2)^2+B_i^2x^2]}{[(1-X^2)^2+4\zeta^2X^2][(1-x^2)^2+B_i^2X^2]} + \tag{A-10}$$

$$\frac{[(M+Nx)(x^2-\beta x+1)+(P+Qx)(x^2+\beta x+1)][(1-X^2)^2+4\zeta^2X^2]}{[(1-x^2)^2+4\zeta^2X^2][(1-X^2)^2+B_i^2X^2]}$$

Now, let

$$E_1 = K+D,\ E_2 = A+C+\alpha(D-K),\ E_3 = D+K+\alpha(C-A),$$
$$E_4 = A+C,\ G_1 = N+Q,\ G_2 = M+P+B(Q-N),$$
$$G_3 = Q+N+\beta(P-M) \quad \text{and} \quad G_4 = D+M$$

then multiplying the left hand sections (in braces) of the numerators of the two fractions and substituting, we obtain

$$f_1X = \frac{[E_1X^3+E_2X^2+E_3X+E_4][(1-X^2)^2+B_i^2X^2]+[G_1X^3+G_2X^2+G_3X+G_4][(1-X^2)^2+4\zeta^2X^2]}{[(1-X^2)^2+4\zeta^2X^2][(1-X^2)^2+B_i^2X^2]} \tag{A-11}$$

Expanding and equating to the appropriate coefficients of X^n we obtain

(A-12)

$$(E_1+G_1)x^7 = 0 \tag{a}$$
$$(E_2+G_2)x^6 = 0 \tag{b}$$
$$[E_3+G_3+E_1(B_i^2-2)+G_1(4\zeta^2-2)]x^5 = 0 \tag{c}$$
$$[E_4+G_4+E_2(B_i^2-2)+G_2(4\zeta^2-2)]X^4 = 4\zeta^2 \tag{d}$$
$$[E_1+G_1+E_3(E_i^2-2)+G_3(4\zeta^2-2)]X^3 = 0 \tag{e}$$
$$[E_2+G_2+E_4(B_i^2-2)+G_4(4\zeta^2-2)]X^2 = 1 \tag{f}$$
$$(E_3+G_3)X = 0 \tag{g}$$
$$(E_4+G_4)X = 0 \tag{h}$$

A-12 (a), (b), (g) and (h) give us

$$\left.\begin{aligned} &E_1 + G_1 = 0,\ E_1 = -G_1 \\ &E_2 + G_2 = 0,\ E_2 = -G_2 \\ &E_3 + G_3 = 0,\ E_3 = -G_3 \\ &E_4 = -G_4 \end{aligned}\right\} \qquad \text{(A-13)}$$

Substituting in A-12 (c), we have

$$E_1(B_i^2 - 4\zeta^2) = 0 \qquad \text{(A-14)}$$

$$E_1 = G_1 = 0$$

A-12 (d) reduces to

$$E_2 = -G_2 = \frac{4\zeta^2}{B_i - 4\zeta^2} \qquad \text{(A-15)}$$

A-12 (e) gives

$$E_3 = G_3 = 0 \qquad \text{(A-16)}$$

and A-12 (f) gives

$$E_4 = -G_4 = \frac{1}{B_i^2 - 4\zeta^2} \qquad \text{(A-17)}$$

Dropping the intermediate substitutions we write

$$\begin{aligned} &E_1 = K + D = 0 \\ &E_2 = A + C + \alpha(D - K) = \frac{4\zeta^2}{B_i^2 - 4\zeta^2} \\ &E_4 = A + C = \frac{1}{B_i^2 + 4\zeta^2} \end{aligned} \qquad \text{(A-18)}$$

which leads directly to

$$D = -K = \frac{4\zeta^2 - 1}{2\alpha(B^2 - 4\zeta^2)} \qquad \text{(A-19)}$$

also, since $E_3 = D + K + \alpha(A - C) = 0$

$$A = C = \frac{1}{2(B_i^2 - 4\zeta^2)} \qquad \text{(A-20)}$$

Similarly, we find

$$\left.\begin{aligned} N = -Q &= \frac{4\zeta^2-1}{2\beta(\beta_i^2-4\zeta^2)} \\ P = M &= \frac{-1}{2(\beta_i^2-4\zeta^2)} \end{aligned}\right\} \qquad \text{(A-21)}$$

Using these values of the arbitrary constants equation (A-9) becomes

$$f_1(x) = \frac{1}{2(\beta_i^2-4\zeta^2)}\left[\frac{1+\frac{1-4\zeta^2}{\alpha}x}{x^2+\alpha x+1} + \frac{1+\frac{4\zeta^2-1}{\alpha}x}{x^2-\alpha x+1} - \frac{1+\frac{1-4\zeta^2}{\beta}x}{x^2+\beta x+1} - \frac{1+\frac{4\zeta^2-1}{\beta}x}{x^2-\beta x+1}\right] \qquad \text{(A-22)}$$

Let $m = 1-4\zeta^2$. Then the integrand becomes

$$I_1 = \frac{1}{2(\beta_i^2-4\zeta^2)}\int_0^\infty \frac{1+\frac{m}{\alpha}x}{x^2+\alpha x+1}dx + \int_0^\infty \frac{1-\frac{m}{\alpha}x}{x^2-\alpha x+1}dx - \int_0^\infty \frac{1+\frac{m}{\beta}x}{x^2+\beta x+1}dx - \int_0^\infty \frac{1-\frac{m}{\beta}x}{x^2-\beta x+1}dx \qquad \text{(A-23)}$$

Integrating each part using Pierce's formulas [67] and [72], we find

$$\begin{aligned} \int \frac{1+\frac{m}{\alpha}x}{x^2+\alpha x+1}dx &= \int \frac{dx}{x^2+\alpha x+1} + \frac{m}{\alpha}\int \frac{x\,dx}{x^2+\alpha x+1} \\ &= \frac{1}{\zeta}\tan^{-1}\frac{2x+\alpha}{2\zeta} + \frac{m}{\alpha}\left[\frac{1}{2}\log(x^2+\alpha x+1) - \frac{\alpha}{2}\int\frac{dx}{x^2+\alpha x+1}\right] \\ &= \frac{1}{\zeta}\tan^{-1}\frac{2x+\alpha}{2\zeta} + \frac{m}{\alpha}\left[\frac{1}{2}\log(x^2+\alpha x+1) - \frac{\alpha}{2\zeta}\tan^{-1}\frac{2x+\alpha}{2\zeta}\right] \\ &= \frac{2-m}{2\zeta}\tan^{-1}\frac{2x+\alpha}{2\zeta} + \frac{m}{2\alpha}\log(x^2+\alpha x+1) \end{aligned}$$

(A-24)

Similarly

$$\int \frac{1-\frac{m}{\alpha}x}{x^2-\alpha x+1}\,dx = \frac{2-m}{2\delta}\tan^{-1}\frac{2x-\alpha}{2\delta} - \frac{m}{2\alpha}\log\left(x^2-\alpha x+1\right) \qquad \text{(A-25)}$$

$$\int \frac{1+\frac{m}{\beta}x}{x^2+\beta x+1}\,dx = \frac{2-m}{B_i}\tan^{-1}\frac{2x+\beta}{B_i} + \frac{m}{2\beta}\log\left(x^2+\beta x+1\right) \qquad \text{(A-26)}$$

$$\int \frac{1+\frac{m}{\beta}}{x^2-\beta x+1}\,dx = \frac{2-m}{B_i}\tan^{-1}\frac{2x-\beta}{B_i} - \frac{m}{2\beta}\log\left(x^2-\beta x+1\right) \qquad \text{(A-27)}$$

Hence

$$I_1 = \int_0^\infty f_1(x)\,dx = \frac{1}{2(B_i^2-4\delta^2)}\left[\frac{m}{2\alpha}\log\frac{x^2+\alpha+1}{x^2-\alpha x+1} - \frac{m}{2\beta}\log\frac{x^2+\beta x+1}{x^2-\beta x+1} + \right.$$

$$\left.\frac{2-m}{2\delta}\left(\tan^{-1}\frac{2x+\alpha}{2\delta}+\tan^{-1}\frac{2x-\alpha}{2\delta}\right) - \frac{2-m}{B_i}\left(\tan^{-1}\frac{2x+\beta}{B_i}+\tan\frac{2x-\beta}{B_i}\right)\right]_0^\infty$$

$$= \frac{1}{2(B_i^2-4\delta^2)}\left[\pi\left(\frac{2-m}{2\delta} - \frac{2-m}{B_i}\right)\right]$$

$$= \frac{\pi}{2(B_i^2-4\delta^2)}\left(\frac{1+4\delta^2}{2\delta} - \frac{1+4\delta^2}{B_i}\right)$$

$$= \frac{\pi(1+4\delta^2)}{4B_i\delta(B_i+2\delta)} \qquad \text{(A-28)}$$

Therefore

$$\bar{a}_o^2 = \frac{\omega_n}{2\pi} B_i^2 S_{ia}(\omega_n) \cdot \frac{\pi(1+4\zeta^2)}{4 B_i \zeta (B_i + 2\zeta)}$$

$$= \frac{\omega_n (1+4\zeta)}{8\zeta\left(1+\frac{2\zeta}{B_i}\right)} S_{ia}(\omega_n) \qquad \text{(A-29)}$$

Which is the form expressed for equation [22] in the main body of the report.

B. <u>Derivation of Equations [35] and [36].</u>

Equation [31] reads

$$E(t) = \int \frac{dn}{N_x} \qquad \text{(B-1)}$$

while [32] and [34] are

$$dn = \frac{\omega_n t}{2\pi} \cdot \frac{x_p}{\sigma_n^2} e^{-\frac{x_p^2}{2\sigma_n^2}} dx_p \qquad \text{(B-2)}$$

$$\frac{1}{N_x} = e^{-\frac{A}{B}} e^{\frac{x_p}{B}} \qquad \text{(B-3)}$$

Direct substitution produces

$$E(t) = \frac{\omega_n t}{2\pi} \cdot \frac{e^{-\frac{A}{B}}}{\sigma_n^2} \int_{x_e}^{\infty} x_p e^{\frac{x_p}{B} - \frac{x_p^2}{2\sigma_n^2}} dx_p \qquad \text{(B-4)}$$

Let $z^2 = \frac{x_p^2}{2\sigma_n^2}$ from which we immediately obtain

$$\left.\begin{aligned} X_p &= \sqrt{2}\,\sigma_n z \\ dX_p &= \sqrt{2}\,\sigma_n dz \\ X_p dX_p &= 2\sigma_n^2 z\, dz \end{aligned}\right\} \qquad \text{(B-5)}$$

Let us also introduce an appropriate value for the endurance limit motion X_e, such that $X_e = \sqrt{2}\,\sigma_n K$. Then [B-4] becomes

$$E(t) = \frac{\omega_n t}{\pi} e^{-\frac{A}{B}} \int_K^{\infty} z e^{\frac{\sqrt{2}\,\sigma_n}{B} z - z^2} dz \qquad \text{(B-6)}$$

Now let $a = \frac{\sqrt{2}\,\sigma_n}{B}$ and we write the integrand as

$$\int_K^{\infty} z e^{az - z^2} dz = \int_0^{\infty} z e^{az - z^2} dz - \int_0^{K} z e^{az - z^2} dz \qquad \text{(B-7)}$$

$$= I_1 - I_2$$

In evaluating I_1, complete the square

$$I_1 = \int_0^{\infty} z e^{az - z^2} dz = \int_0^{\infty} z e^{\frac{a^2}{4} - \left(z - \frac{a}{2}\right)^2} dz$$

$$= e^{\frac{a^2}{4}} \int_0^{\infty} z e^{-\left(z - \frac{a}{2}\right)^2} dz \qquad \text{(B-8)}$$

For simplicity, let $y = z - \frac{a}{2}$ so that $z = y + \frac{a}{2}$ and $dz = dy$. Then

$$I_1 = e^{\frac{a^2}{4}} \int_0^{\infty} \left(y + \frac{a}{2}\right) e^{-y^2} dy$$

$$= e^{\frac{a^2}{4}} \left[\int_0^{\infty} y e^{-y^2} dy + \frac{a}{2} \int_0^{\infty} e^{-y^2} dy \right]$$

$$= e^{\frac{a^2}{4}} \left(I_3 + I_4 \right)$$

(B-9)

$$I_3 = -\frac{1}{2} e^{-y^2} \Big]_0^{\infty} = -\frac{1}{2}(0-1) = \frac{1}{2} \qquad \text{(B-10)}$$

Pierce, 492:

$$I_4 = \frac{a}{2}\frac{\sqrt{\pi}}{2} = \frac{a}{4}\sqrt{\pi} \qquad \text{(B-11)}$$

Substituting in [B-9] we have

$$I_1 = \frac{e^{\frac{a^2}{4}}}{2}\left(1 + \frac{a\sqrt{\pi}}{2}\right) \qquad \text{(B-12)}$$

Turning to I_2 we again complete the square

$$I_2 = \int_0^K z e^{\frac{a^2}{4} - \left(z - \frac{a}{2}\right)^2} dz \qquad \text{(B-13)}$$

which, by making the same substitutions as before becomes

$$I_2 = e^{\frac{a^2}{4}}\left[\int_0^K y e^{-y^2} dy + \frac{a}{2}\int_0^K e^{-y^2} dy\right] = e^{\frac{a^2}{4}}(I_5 + I_6) \qquad \text{(B-14)}$$

Therefore

$$I_5 = -\frac{1}{2} e^{-y^2}\Big]_0^K = -\frac{1}{2}\left(e^{-K^2} - 1\right) \qquad \text{(B-15)}$$

Multiplying I_6 by $\frac{\sqrt{\pi}}{2} \cdot \frac{2}{\sqrt{\pi}}$ transposes the integrand of I_6 into the probability integral

$$I_6 = \frac{a}{4}\sqrt{\pi}\left(\frac{2}{\sqrt{\pi}}\int_0^K e^{-y^2} dy\right) = \frac{a\sqrt{\pi}}{4} \operatorname{erf}(K) \qquad \text{(B-16)}$$

Hence,

$$I_2 = e^{\frac{a^2}{4}}\left[\frac{a\sqrt{\pi}}{4}\operatorname{erf}(K) - \frac{1}{2}\left(e^{-K^2} - 1\right)\right]$$

$$= \frac{e^{\frac{a^2}{4}}}{2}\left[1 - e^{-K^2} + \frac{a\sqrt{\pi}}{2}\operatorname{erf}(K)\right] \qquad \text{(B-17)}$$

Values of $\operatorname{erf}(K)$ are tabulated in the literature, e.g. Pierce page 116.

Substituting in [B-7] we have, getting rid of the artificial constant Q,

$$\int_K^{\infty} z e^{az - z^2} dz = \frac{e^{\frac{a^2}{4}}}{2}\left(1 + \frac{a}{2}\sqrt{\pi}\right) - \frac{e^{\frac{a^2}{4}}}{2}\left[1 - e^{-K^2} + \frac{a\sqrt{\pi}}{2}\, \mathrm{erf}(K)\right]$$

$$= \frac{e^{\frac{a^2}{4}}}{2}\left[\frac{a\sqrt{\pi}}{2}\left(1 - \mathrm{erf}(K)\right) + e^{-K^2}\right] \quad \text{(B-18)}$$

$$= e^{\frac{\sigma_n^2}{2B^2}}\left[\frac{\sigma_n}{B}\sqrt{\frac{\pi}{2}}\left(1 - \mathrm{erf}(K)\right) + e^{-K^2}\right]$$

From which [B-6] can be written as

$$E(t) = \frac{\omega_n t}{\pi} e^{\frac{\sigma_n^2}{2B^2} - \frac{A}{B}}\left[\frac{\sigma_n}{B}\sqrt{\frac{\pi}{2}}\left(1 - \mathrm{erf}(K)\right) + e^{-K^2}\right] \quad \text{(B-19)}$$

which is the form of equation [35]

It can also readily be seen that when K is very small, or non-existent, [B-17] will simplify to

$$E(t) = \frac{\omega_n t}{\pi} e^{\frac{\sigma_n^2}{2B^2} - \frac{A}{B}}\left(1 + \frac{\sigma_n}{B}\sqrt{\frac{\pi}{2}}\right) \quad \text{(B-20)}$$

which is the form given for equation [36]

C. Derivation of Equations [41] and [42].

The equation for a log-log S-N relation is

$$\log x_p = \log A - B \log N_x \qquad X_p > X_e \quad \text{(C-1)}$$

From which $N_x = \left(\frac{A}{X_p}\right)^{\frac{1}{B}}$

or

$$\frac{1}{N_x} = \left(\frac{X_p}{A}\right)^{\frac{1}{B}} \quad \text{(C-2)}$$

Substituting in equation [31] we have

$$E(t) = \frac{\omega_n t}{2\pi A^{\frac{1}{B}} \sigma_n^2} \int_{K\sigma}^{\infty} x_p^{1+\frac{1}{B}} e^{-\frac{x_p^2}{\sigma_n^2}} dx_p \qquad \text{(C-3)}$$

Introducing the notation $z^2 = \frac{x_p^2}{2\sigma_n^2}$ and its corollaries (C-3) becomes

$$E(t) = \frac{\omega_n t}{2\pi A^{\frac{1}{B}} \sigma_n^2} \int_{K}^{\infty} z^{\frac{B+1}{B}} \left(\sqrt{2}\sigma_n\right)^{1+\frac{1}{B}} e^{-z^2} \sqrt{2}\, \sigma_n dz$$

$$= \frac{\omega_n t}{\pi} \left(\frac{\sqrt{2}\sigma_n}{A}\right)^{\frac{1}{B}} \int_{K}^{\infty} z^{\frac{1+B}{B}} e^{-z^2} dz \qquad \text{(C-4)}$$

Now, the complete Gamma function in z^2 is defined by the relations

$$\Gamma(n) = 2\int_0^{\infty} z^{2n-1} e^{-z^2} dz$$

and

$$\Gamma(n+1) = 2\int_0^{\infty} z^{2n} e^{-z^2} dz \qquad \text{(C-5)}$$

while the incomplete Gamma function is defined by

$$\gamma(n, K) = 2\int_0^{K} z^{2n-1} e^{-z^2} dz$$

$$\gamma(n+1, K) = 2\int_0^{K} z^{2n} e^{-z^2} dz \qquad \text{(C-6)}$$

It is evident that the integrand of equation (C-4) can be expressed in terms of Gamma functions with $n = \frac{1}{2B} + 1$.

Then, we have

$$\int_K^{\infty} z^{\frac{1+B}{B}} e^{-z^2} dz = \int_0^{\infty} z^{1+\frac{1}{B}} e^{-z^2} dz - \int_0^{K} z^{1+\frac{1}{B}} e^{-z^2} dz$$

$$= \frac{1}{2}\left[\Gamma\left(1+\frac{1}{2B}\right) - \gamma\left(1+\frac{1}{2B}, K\right)\right]$$

$$= \frac{\Gamma\left(1+\frac{1}{2B}\right)}{2}\left[1 - \frac{\gamma\left(1+\frac{1}{2B}, K\right)}{\Gamma\left(1+\frac{1}{2B}\right)}\right] \qquad \text{(C-7)}$$

from which (C-4) becomes

$$E(t) = \frac{\omega_n t}{2\pi}\left(\frac{\sqrt{2}\,\sigma_n}{A}\right)^{\frac{1}{B}} \cdot \Gamma\left(1+\frac{1}{2B}\right) \cdot \left[1 - \frac{\gamma\left(1+\frac{1}{2B}, K\right)}{\Gamma\left(1+\frac{1}{2B}\right)}\right] \qquad \text{(C-8)}$$

The life expenditure function for a sinusoidal vibration test with a log-log S-N curve is written rather simply as

$$E(t) = \frac{\omega_n t_T}{2\pi}\left(\frac{A}{X_{nT}}\right)^{\frac{1}{B}} \qquad \text{(C-9)}$$

Combining and solving for time, we have

$$\frac{t_T}{C_T} = X_{nT}^{-\frac{1}{B}}\left(\sqrt{2}\,\sigma_n\right)^{\frac{1}{B}} \cdot \Gamma\left(1+\frac{1}{2B}\right) \cdot \left[1 - \frac{\gamma\left(1+\frac{1}{2B}, K\right)}{\Gamma\left(1+\frac{1}{2B}\right)}\right] \qquad \text{(C-10)}$$

which is the form given for equation [41] .

It is directly evident that, if K be very small or non-existent, one can ignore the contribution of the integral $\int_0^K z^{1+\frac{1}{B}} e^{-z^2} dz$ and the relation of (C-10) can be simplified to

$$\frac{t_T}{C_T} = X_{nT}^{-\frac{1}{B}}\left(\sqrt{2}\,\sigma_n\right)^{\frac{1}{B}} \cdot \Gamma\left(1+\frac{1}{2B}\right) \qquad \text{(C-11)}$$

which is the form given for equation [42] .

To compute solutions, it is only necessary to note that tables and curves of $\Gamma(n)$ and related functions can be found in Jahnke-Emde who also refer to other extensive tabulations of these functions. Jahnke-Emde also give curves of the ratio $\frac{\gamma(n+1, x)}{\Gamma(n+1)}$ while Pearson has compiled extensive tables of the incomplete Gamma function. Hence our solutions are in computable form.

APPENDIX II

Glossary of Mathematical Notation

A — empirical constant in equation [34] and used in derivation of equation [22] Appendix 1A.

$A_T(\omega)$ — defined by equation [4]. The $\lim_{T \to 0} A_T(\omega)$ is the Fourier transform of $x(t)$

A_{iT} — test input acceleration amplitude corresponding to sinusodial test input motion X_{iT}.

$a(t) = \frac{d^2 x}{dt^2}$ — the acceleration of motion $x(t)$

$a_o(t)$ — absolute response acceleration corresponding to absolute response motion $X_o(t)$

$a_i(t)$ — input acceleration

$\bar{a}_o^2$ — mean square absolute acceleration response

a — used in [B-7] $a = \frac{\sqrt{2}\, \sigma_n}{B}$

B — the constant first used in equating [34], which is the motion constant equivalent to the slope of the S-N curve.

B_i — the bandwidth. $\omega_n B_i$ is the frequency difference between the points on each side of the peak at which the ordinate of the curve is half the peak value.

C_D — damping coefficient, lbs. sec./in.

C_c — critical damping coefficient.

C — Miner's constant, ratio of stress cycles occurring to stress cycles to failure (ostensibly 1.0). Also used in derivation of equation [22] Appendix 1A.

D — used in derivation of equation [22] Appendix 1A.

d — experimentally determined exponent in the Corten and Dolan non-linear fatigue constant.

E Life expenditure in Miner's hypothesis

$E(t)$ life expenditure as a function of time.

$E_T(t)$ life expenditure time function in sinusoidal test

E_1, E_2, E_3, E_4 used in derivation of equation [22] Appendix 1A.

$erf(K) = \frac{2}{\sqrt{\pi}} \int_0^K e^{-x^2} dx$ = the error function of K sometimes called the probability integral.

f frequency, cycles per second.

f_n natural frequency, cycles per second.

$f_1(x)$ used in appendix 1A.

G number of multiples of acceleration of gravity.

G_1, G_2, G_3, G_4 used in derivation of equation [22] Appendix 1A.

H complex frequency response or magnification factor defined by equation [8].

h height of drop equation [46], [47]

$I_1, I_2, I_3, I_4, I_5, I_6$ symbols to identify integrals in Appendix 1.

K used in derivation of equation [22] Appendix 1A.

k spring rate of suspension, also $K = \frac{X_e}{\sqrt{2}\,\sigma_\eta}$ in equation [B-6]

M used in derivation of equation [22] Appendix 1A.

m mass

N used in derivation of equation [22] Appendix 1A.

N_1 fatigue life at maximum stress (Corten and Dolan)

N_X number of cycles to failure at amplitude X_p.

N_i fatigue life at stress level n_i used in Miner's hypothesis characterizing the linear damage accumulation in fatigue considerations.

N_g used in Corten and Dolan s non-linear damage accumulation proposal as the total number of cycles to failure.

n_i acutal number of cycles at a given stress level in Miner's hypothesis

P used in derivation of equation [22] Appendix 1A.

$P(X_p)$ probability density of X_p defined by equation [6]

Q used in derivation of equation [22] Appendix 1A.

Q sometimes called quality factor $Q \cong \frac{1}{2\zeta}$ where

ζ is damping ratio.

$R_x(\tau)$ autocorrelation function defined in a footnote to equation [4]

S_e endurance limit, i. e., stress at which fatigue is infinite

S stress level

$S_x(\omega)$ power spectral density shows the power content of the vibratory components as a function of their circular frequency. ω.

$S_{ix}(\omega)$ power spectral density of the input motion.

$S_{ox}(\omega)$ power spectral density of absolute response.

$S_{rx}(\omega)$ power spectral density of relative response.

$S_{ia}(\omega)$ acceleration spectral density.

$S_{oa}(\omega)$ acceleration spectral density of the absolute response.

$S_{ia}(\omega_n)$	input acceleration spectral density corresponding to the natural frequency of the suspension
T	a particular value of time
T_o	the transmissibility ratio between complex functions, defined by equation [7]
t	time coordinate
t_T	test time
t'_T	test time corresponding to response amplitude X'_{nT}
t''_T	test time corresponding to response amplitude X''_{nT}, see equations [42], [44]
W	weight
X	amplitude (distance from rest position to peak) inches
$\bar{X}^2$	mean square amplitude defined by equation [2]
$\bar{X}_n^2$	mean square relative amplitude
$x(t)$	function representing motion in which x denotes displacement and t indicates x is a function of time.
X_p	peak value of X, or an ordinate of the envelope curve.
$X_i(t)$	input motion coordinate
$X_o(t)$	absolute response motion coordinate
$X_n(t)$	relative response motion coordinate
X_o, X_i, X_n	complex functions of the circular frequency ω see equation [7]
X_{it}	amplitude of a sinusoidal test motion

X_{nT} Uniform response amplitude to sinusodial test motion.

X'_{iT}, X''_{iT} test input amplitudes

X'_{nT}, X''_{nT} test relative response amplitudes, see equations [42] [44]

X_T amplitude of sinusodial test motion

$\bar{X}^2_{nT}$ mean square relative response amplitude in test.

X_e amplitude corresponding to endurance limit.

Z used in evaluating integral in Appendix I, $Z^2 = \frac{X_p^2}{2\sigma_n^2}$

α used in Appendix A-7

α used as a constant in semi-logarithmic stress function.

α_i percentages of cycles at stresses S_i (Corten and Dolan).

β used as a constant in semi-logarithmic stress function.

β used in Appendix A-8.

Γ complete Gamma function.

γ the incomplete Gamma function.

ζ ratio of damping to critical damping $\zeta = \frac{C_i}{C_c}$

σ called root mean square and also standard deviation.

σ_n r. m. s. value of response motion $\left(\sigma_n^2 = \bar{X}_n^2\right)$

ω circular frequency, radians per second

ω_n natural frequency, radians per second