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BALLISTIC EQUATIONS FOR ARTILLERY SHELLS

by

E. V. Wilms

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DEPARTMENT OF THEORETICAL AND APPLIED MECHANICS
UNIVERSITY OF ILLINOIS

Technical Report No. 620
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Artillery Carriage Dynamics
Project Supervisor: M. Stippes

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A Research Project of the
Department of Theoretical and Applied Mechanics
University of Illinois

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This report is concerned with ballistic equations for artillery shells.

The program is under the technical supervision of Rock Island Arsenal (RIA), Rock Island, Illinois, and the administrative supervision of Chicago Ordnance District.

Respectfully submitted,
University of Illinois

M Stippes

M. Stippes, Project Supervisor
Department of Theoretical and
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I. INTRODUCTION

The purpose of this report is to derive a set of equations describing the motion of artillery shells for convenient solution by computer. The effect of mass unbalance is taken into account. The equations are set up in an inertial coordinate system and the effect of the rotation of the earth is included.

The form of the ballistic equations for rockets as formulated in University of Illinois TAM Report No. 166 is preserved whenever possible, and the same notation is used.

The method of treating the rotation of the earth is that used in University of Illinois TAM Report No. 204.

II. GENERAL CONSIDERATIONS

1. Equations of Motion

The artillery shell is a rigid body, and the somewhat complex theorems of linear and angular momentum which must be used in rocket theory, take the following elementary form:

The vector sum of all the exterior forces acting on a system of particles is equal to the time rate of change of momentum of the system.

The vector sum of the moments of the exterior forces acting on a system of particles, taken about the centre of mass of the system, is equal to the time rate of change of the angular momentum of the system. In equation form:

$$\sum_{K} \bar{F}_k = \frac{d\,\bar{\mathcal{M}}}{d\,t} = \bar{G} + \bar{F} \tag{1}$$

$$\sum_{K} \bar{r}_k \times \bar{F}_k = \frac{d\,\bar{\mathcal{L}}}{d\,t} = \bar{N} \tag{2}$$

where $\sum_{K} \bar{F}_k$ and $\sum_{K} \bar{r}_k \times \bar{F}_k$ represent respectively the sum of the exterior forces acting on the shell and the sum of the moments of these forces taken about the mass centre; $\bar{\mathcal{M}}$, and $\bar{\mathcal{L}}$ are respectively the momentum of the shell, and the angular momentum of the shell about the mass centre. $\bar{G}$ is the resultant aerodynamic force, $\bar{F}$ the force due to gravity, and $\bar{N}$ the moment due to the aerodynamic forces.

2. Coordinate Systems and Basic Assumptions

In establishing the equations of motion of an artillery shell, we must apply Newton's laws of motion in an inertial coordinate system. For purposes of this discussion, we will consider an inertial system to be one which consists of the normal, tangent, and binormal to the path of the earth's centre. The particular system to be used will be rectangular X_o, Y_o, Z_o coordinates chosen in such a way that:

(a) The origin will coincide with the mass centre of the shell at launch.

(b) The direction of aim of the shell will be in the plane $Y_o = 0$.

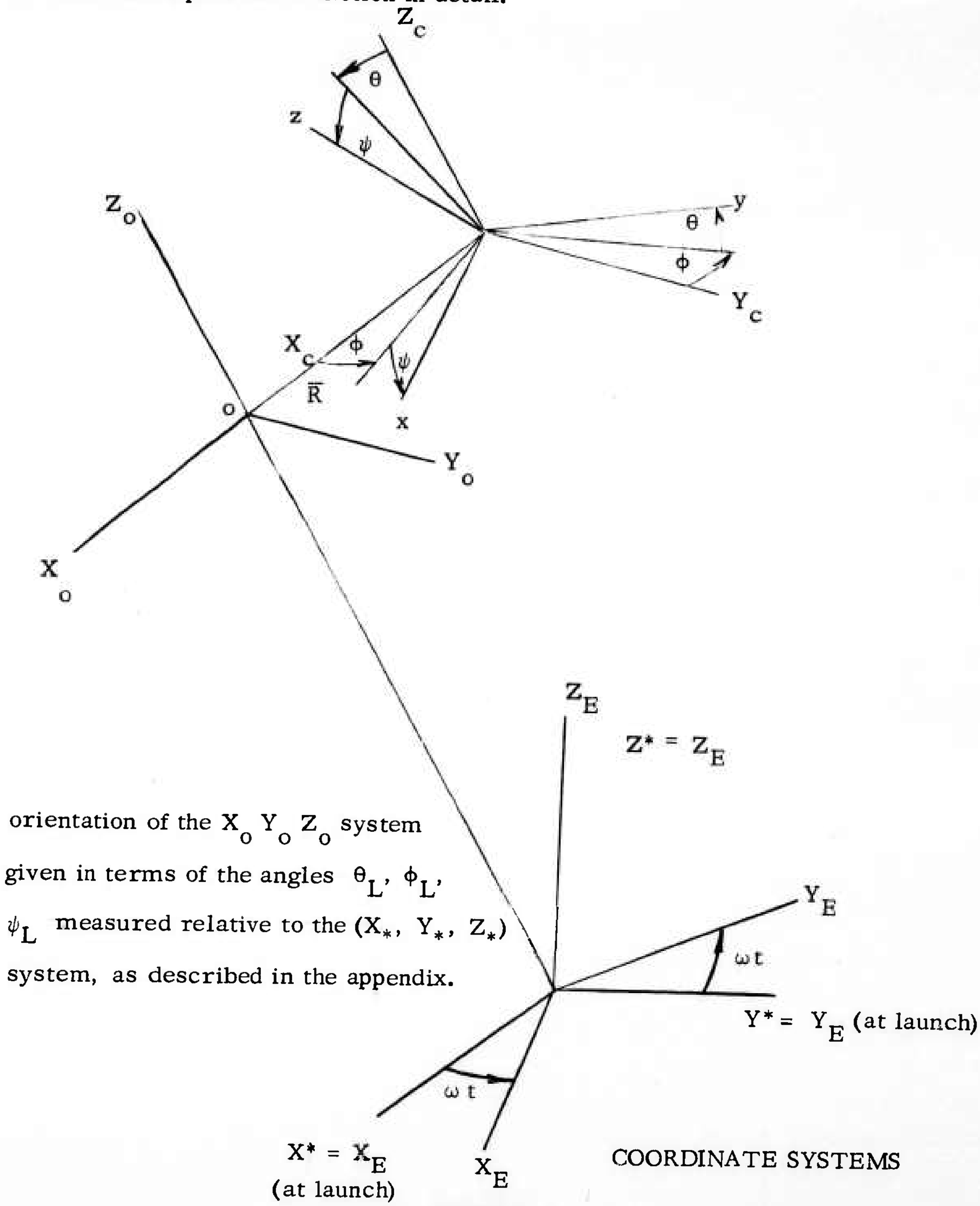
(c) The Z_o axis will pass through the "centre" of the earth.

In order to obtain the location of the shell in a system fixed with respect to the earth, we have to apply a transformation between two coordinate systems. These transformations are derived in the appendix. Let us briefly describe the nature of the two coordinate systems. The coordinates (X_E, Y_E, Z_E) (measured with respect to the earth), are defined in Eqs. (A-17) to be functions of (X_o, Y_o, Z_o) (the inertial coordinates), and also of $(\theta_L, \phi_L, \rho_L; \psi_L)$, the coordinates and direction of the shell at launch measured in a system fixed in the earth.

It may further be convenient to express the (X_E, Y_E, Z_E) coordinates in terms of the longitude, latitude, and altitude. In Eqs. (A-19) $(\rho_E, \phi_E, \theta_E)$ are expressed as functions of (X_E, Y_E, Z_E). Here ϕ_E and θ_E are related to the latitude and longitude as described in section iii of the appendix.

The wind velocities which are used in computing the aerodynamic forces must be expressed in terms of the (X_o, Y_o, Z_o) system. However, they will be given in terms of the earth coordinates $(\theta_E, \phi_E, \rho_E)$, and are specified by a direction ψ_E (azimuth) plus vertical and horizontal components P_w and Q_w.

Let us now return to the inertial $X_o \, Y_o \, Z_o$ coordinate system and talk about the equations of motion in detail.



COORDINATE SYSTEMS

FIGURE 1

Let $\bar{R}$ be the vector from the origin of the $X_o Y_o Z_o$ system to the mass centre of the shell, and let $\bar{R}$ have components (X_o, Y_o, Z_o) in $(X_o Y_o Z_o)$. Then the velocity of the mass centre is $\bar{v} = \frac{d\bar{R}}{dt}$ and has components $\frac{dX_o}{dt}$, $\frac{dY_o}{dt}$, $\frac{dZ_o}{dt}$ in $(X_o Y_o Z_o)$. At the mass centre c, we construct $(X_c Y_c Z_c)$ which moves with c but remains always parallel to $X_o Y_o Z_o$. This is illustrated in Fig. 1.

We will use the subscripts 1, 2, 3 for the components of vectors in $(X_o Y_o Z_o)$. Then, since the mass m of the shell is fixed, Eq. (1) may be written

$$m \frac{d\dot{X}_o}{dt} = G_1 + F_1$$

$$m \frac{d\dot{Y}_o}{dt} = G_2 + F_2 \tag{3}$$

$$m \frac{d\dot{Z}_o}{dt} = G_3 + F_3$$

where

$$\frac{dX_o}{dt} = \dot{X}_o$$

$$\frac{dY_o}{dt} = \dot{Y}_o \tag{4}$$

$$\frac{dZ_o}{dt} = \dot{Z}_o$$

Now, principal axes and moments of inertia for the shell may be computed. The c-xyz coordinate system may be constructed with the y-axis along the principal longitudinal axis of inertia (plai), and the x and z axes along the other two (transverse) principal axes as shown in Fig. 1. The orientation of c-xyz is defined by the three rotations ϕ , θ , ψ taken in the indicated order. The angular velocity vector of c-xyz will be denoted by $\bar{\omega}$. Let I_x, I_y, I_z denote the principal moments of inertia along the x, y, and z axes. It will also be convenient to use $\hat{i}$, $\hat{j}$, $\hat{k}$ as unit vectors in the x, y, z directions respectively. We will use the subscripts x, y, z for components of vectors in c-xyz . The angular momentum of the shell (about c), can now be written as:

$$\bar{\mathcal{L}} = \hat{i}\, I_x \omega_x + \hat{j}\, I_y \omega_y + \hat{k}\, I_z \omega_z \tag{5}$$

From Fig. 1 it is apparent that the components of angular velocity are given by:

$$\omega_x = -\frac{d\phi}{dt} \cos\theta \sin\psi + \frac{d\theta}{dt} \cos\psi ,$$

$$\omega_y = \frac{d\phi}{dt} \sin\theta + \frac{d\psi}{dt} ,$$

$$\omega_z = \frac{d\phi}{dt} \cos\theta \cos\psi + \frac{d\theta}{dt} \sin\psi ,$$

which can be inverted to yield:

$$\frac{d\phi}{dt} = \sec\theta \left[\omega_z \cos\psi - \omega_x \sin\psi \right] ,$$

$$\frac{d\theta}{dt} = \omega_x \cos\psi + \omega_z \sin\psi , \tag{6}$$

$$\frac{d\psi}{dt} = \omega_y - \tan\theta \left[\omega_z \cos\psi - \omega_x \sin\psi \right]$$

(We assume that θ is always within the bounds $|\theta| < \pi/2$).

Differentiating Eq. (5), and recalling that $\frac{d\hat{i}}{dt} = \bar{\omega} \times \hat{i}$, etc. we obtain

$$\frac{d\bar{\mathcal{L}}}{dt} = \hat{i}\left[\frac{d}{dt}(I_x\,\omega_x) + (I_z - I_y)\,\omega_z\,\omega_y\right]$$
$$+ \hat{j}\left[\frac{d}{dt}(I_y\,\omega_y) + (I_x - I_z)\,\omega_x\,\omega_z\right]$$
$$+ \hat{k}\left[\frac{d}{dt}(I_z\,\omega_z) + (I_y - I_x)\,\omega_y\,\omega_x\right]$$

We now assume that the shell is rotationally symmetric so that we can set $I_x = I_z = I$, $I_y = I'$. Then Eq. (2) can be written in the scalar form:

$$I\frac{d\,\omega_x}{dt} = -(I - I')\,\omega_y\,\omega_z + N_x$$
$$I'\frac{d\,\omega_y}{dt} = N_y \qquad (7)$$
$$I\frac{d\,\omega_z}{dt} = (I - I')\,\omega_y\,\omega_x + N_z$$

When we consider the aerodynamic forces and moments it will be convenient to have the velocities relative to the undisturbed air in the neighborhood of the shell rather than the fixed $X_o\,Y_o\,Z_o$ coordinate system. Therefore in the neighborhood of the shell the undisturbed air flow would have had a linear velocity $\bar{W}$. The velocity of the shell relative to air can then be defined by:

$$\bar{V} = \bar{v} - \bar{W}$$

so that

$$\dot{X}_c = V_1 + W_1$$
$$\dot{Y}_o = V_2 + W_2 \qquad (8)$$
$$\dot{Z}_o = V_3 + W_3$$

where W_1, W_2, W_3 are known functions of time and position. We can now substitute Eq. (8) into Eq. (3) and (4) to obtain

$$\begin{aligned} \frac{d X_o}{d t} &= V_1 + W_1 \\ \frac{d Y_o}{d t} &= V_2 + W_2 \\ \frac{d Z_o}{d t} &= V_3 + W_3 \end{aligned} \tag{9}$$

and

$$\begin{aligned} m \frac{d V_1}{d t} &= - m \dot{W}_1 + F_1 + G_1 \\ m \frac{d V_2}{d t} &= - m \dot{W}_2 + F_2 + G_2 \\ m \frac{d V_3}{d t} &= - m \dot{W}_3 + F_3 + G_3 \end{aligned} \tag{10}$$

where we have set $\dot{W}_1 = \frac{d W_1}{d t}$, etc.

The wind velocities will be specified in a coordinate system fixed in the earth. As mentioned previously, they are specified by a direction ψ_E (azimuth), plus vertical and horizontal components P_W and Q_W. Let us use the transformation of a vector from earth to fixed coordinates as indicated in the appendix (A-23, A-24).

Then, the wind velocities become:

$$W_1 = P_W \left(A_E (A'_L \cos \Omega t + D'_L \sin \Omega t) + B_E (-A'_L \sin \Omega t + D'_L \cos \Omega t) + C_E G'_L \right)$$
$$+ Q_W \left(G_E (A'_L \cos \Omega t + D'_L \sin \Omega t) + H_E (-A'_L \sin \Omega t + D'_L \cos \Omega t) + J_E G'_L \right)$$

$$W_2 = P_W \left(A_E (B'_L \cos \Omega t + E'_L \sin \Omega t) + B_E (-B'_L \sin \Omega t + E'_L \cos \Omega t) + C_E A'_L \right)$$
$$+ Q_W \left(G_E (B'_L \cos \Omega t + E'_L \sin \Omega t) + H_E (-B'_L \sin \Omega t + E'_L \cos \Omega t) + J_E H'_L \right) \qquad (11)$$

$$W_3 = P_W \left(A_E (C'_L \cos \Omega t + F'_L \sin \Omega t) + B_E (-C'_L \sin \Omega t + F'_L \cos \Omega t) + C_E J'_L \right)$$
$$+ Q_W \left(G_E (C'_L \cos \Omega t + F'_L \sin \Omega t) + H_E (-C'_L \sin \Omega t + F'_L \cos \Omega t) + J_E J'_L \right)$$

The quantities $A_E \ldots J_E$, $A'_L \ldots J'_L$ arise due to the transformation as discussed in the appendix, and are defined there.

Eqs. (6), (7), (9) and (10) constitute a system of twelve first order differential equations in the twelve dependent variables X_o, Y_o, Z_o, V_1, V_2, V_3, ϕ, θ, ψ, ω_x, ω_y, ω_z, and the independent variable t (time). It remains now to express the other terms in this set of equations in terms of these twelve generalized coordinates and the time.

We will need expressions for the direction cosines of the $X_o Y_o Z_o$ and c-xyz axes with respect to each other. From Fig. 1 to within first order terms (in ϕ) we have:

$$
\begin{aligned}
a_{x1} &= \cos\psi - \phi \sin\theta \sin\psi \, , \\
a_{x2} &= \phi \cos\psi + \sin\theta \sin\psi \, , \\
a_{x3} &= -\cos\theta \sin\psi \, , \\[1ex]
a_{y1} &= -\phi \cos\theta \, , \\
a_{y2} &= \cos\theta \, , \\
a_{y3} &= \sin\theta \, , \\[1ex]
a_{z1} &= \sin\psi + \phi \sin\theta \cos\psi \, , \\
a_{z2} &= \phi \sin\psi - \sin\theta \cos\psi \, , \\
a_{z3} &= \cos\theta \cos\psi \, .
\end{aligned}
\tag{12}
$$

Then if $\bar{u}$ is a vector we have

$$u_x = a_{x1} u_1 + a_{x2} u_2 + a_{x3} u_3 \, , \quad \text{etc.}$$

$$u_1 = a_{x1} u_x + a_{y1} u_y + a_{z1} u_z \, , \quad \text{etc.}$$

III. THE FORCE SYSTEM

1. Gravitational Force

We will consider the gravitational force to be given by :

$$\bar{F} = - m \, g_o \left(\frac{\rho_o}{\rho_E}\right)^2 \hat{g}$$

where $\hat{g}$ is a unit vector along the line joining the centre of mass of the shell, and the centre of the earth, ρ_E is the distance from the shell to the earth's centre, and ρ_o is the radius of the earth. The components of $\hat{g}$ must be expressed in the $X_o\, Y_o\, Z_o$ system. To this end, we utilize the transformation of a vector derived in the appendix, (Eqs. A-23, A-24), with $P = 0$, $Q = 1$.

Then the components of the gravitational force are:

$$\begin{aligned}
F_1 &= - m \, g_o \left(\frac{\rho_o}{\rho_E}\right)^2 \left(G_E (A'_L \cos \Omega t + D'_L \sin \Omega t) \right. \\
&\quad \left. + H_E (-A'_L \sin \Omega t + D'_L \cos \Omega t) + J_E G'_L \right) \\
F_2 &= - m \, g_o \left(\frac{\rho_o}{\rho_E}\right)^2 \left(G_E (B'_L \cos \Omega t + E'_L \sin \Omega t) \right. \\
&\quad \left. + H_E (-B'_L \sin \Omega t + E'_L \cos \Omega t) + J_E H'_L \right) \\
F_3 &= - m \, g_o \left(\frac{\rho_o}{\rho_E}\right)^2 \left(G_E (C'_L \cos \Omega t + F'_L \sin \Omega t) \right. \\
&\quad \left. + H_E (C'_L \sin \Omega t + F'_L \cos \Omega t) + J_E J'_L \right)
\end{aligned} \tag{13}$$

2. Aerodynamic Forces and Moments

The aerodynamic forces and moments are classified in three categories:

a) Those independent of angular velocity (Drag Force, Lift Force or Force due to cross velocity, Restoring Moment or Moment due to cross velocity),

b) Those depending on transverse angular velocity of the shell (Force due to cross spin, Damping Moment or Moment due to cross spin),

c) Those depending on the axial spin of the shell (Roll Moment or Moment due to spin, Magnus force and moment due to cross velocity and cross spin).

Although these forces and moments interact with each other, they are traditionally treated separately.

It is also customary to describe the aerodynamic forces and moments by means of dimensionless aerodynamic coefficients. For this purpose we introduce the average mass density of the air surrounding the shell and denote this density by ρ ; we also need a length characteristic of the shell and therefore introduce the diameter d of the shell. Also we use the linear and angular velocities of the shell where appropriate. Since we wish to base our computation on motion of the shell in undisturbed air we must account for winds by using $\bar{V}$ as the velocity of the mass centre of the shell with respect to the air.

Drag Force and Lift Force. The aerodynamic force which acts on the shell when there is no angular velocity ($\bar{\omega} = 0$) is resolved into two components; the drag force $\bar{D}$ which is taken parallel to $\bar{V}$, and the lift force $\bar{L}$ perpendicular to $\bar{V}$. We define the drag coefficient K_D and the lift coefficient K_N by

$$D = K_D \, \rho \, d^2 \, V^2 \; ,$$

$$L = K_N \, \rho \, d^2 \, V^2 \; .$$

From physical considerations we can argue that there must be at least three principal directions such that if $\bar{V}$ is parallel to one of them the lift force is zero. The principal direction most nearly parallel to the plai of the shell is called the aerodynamic axis and the angle between the aerodynamic axis and $\bar{V}$ is called the angle of attack (denoted by a). Let $\hat{e}$ be a unit vector along the aerodynamic axis. We assume that a is a first order quantity and use the angles ξ_x and ξ_z to orient $\hat{e}$ in the c-xyz coordinate system as shown in Fig. 2. Then to first order terms

$$e_x = -\xi_z, \qquad e_y = 1, \qquad e_z = \xi_x.$$

and

$$\begin{aligned} e_1 &= -\phi \cos\theta - \xi_z \cos\psi + \xi_x \sin\psi, \\ e_2 &= \cos\theta - \sin\theta\,(\xi_x \cos\psi + \xi_z \sin\psi), \\ e_3 &= \sin\theta + \cos\theta\,(\xi_x \cos\psi + \xi_z \sin\psi). \end{aligned} \tag{14}$$

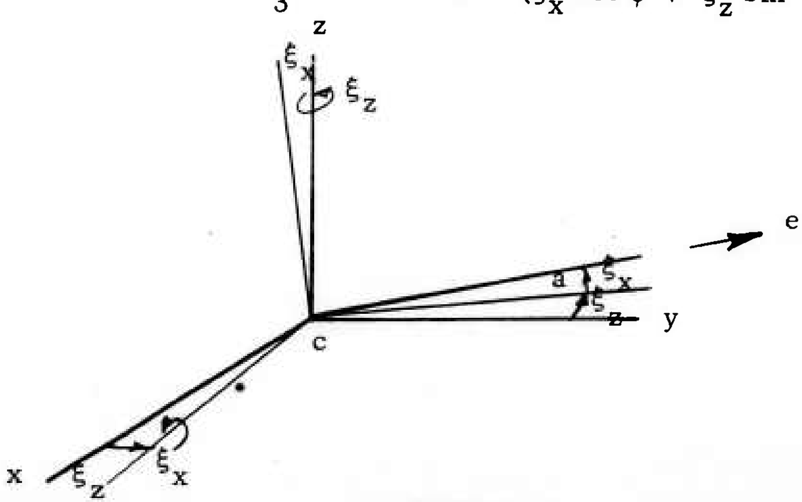


FIGURE 2

AERODYNAMIC AXIS

Also

$$a^2 = \xi_x^2 + \xi_z^2 \quad .$$

We assume that the lift force vector lies in the plane defined by $\hat{e}$ and $\bar{V}$, and since we have specified $\bar{L}$ perpendicular to $\bar{V}$ we could write

$$\bar{L} = K_L \rho d^2 \, \bar{V} \times (\hat{e} \times \bar{V})$$

$$= K_L \rho d^2 \left[V^2 \hat{e} - \bar{V} \cdot \hat{e} \; \bar{V} \right]$$

where

$$K_N = K_L \sin a = K_L \, a$$

to first order terms. We could also write $\bar{V} \cdot \hat{e} = V \cos a = V$ to first order terms so that

$$\begin{aligned} L_1 &= K_L \rho d^2 V \left[V(-\phi \cos\theta + \xi_x \sin\psi - \xi_z \cos\psi) - V_1 \right] , \\ L_2 &= K_L \rho d^2 V \left[(V\cos\theta - V_2) - V(\xi_x \cos\psi + \xi_z \sin\psi) \sin\theta \right] \\ L_3 &= K_L \rho d^2 V \left[(V\sin\theta - V_3) + V(\xi_x \cos\psi + \xi_z \sin\psi) \cos\theta \right] \end{aligned} \qquad (15)$$

The drag force is written as

$$\bar{D} = -K_D \rho d^2 V \bar{V} ,$$

so that

$$\begin{aligned} D_1 &= -K_D \rho d^2 V V_1 , \\ D_2 &= -K_D \rho d^2 V V_2 , \\ D_3 &= -K_D \rho d^2 V V_3 . \end{aligned} \qquad (16)$$

For small angles of attack both K_L and K_D can be taken as independent of α since both must be (approximately) even functions of α and therefore the dependence will be second order. The other significant variables, chief among which is the velocity, are assumed to enter only by way of the Mach number related to V and we therefore allow K_L and K_D to be functions of Mach number. If we further assume that both the sonic velocity and the mass density ρ depend only on altitude, the drag and lift forces can be determined from the generalized coordinates and the time.

Restoring Moment. The aerodynamic moment which acts on the shell when there is no angular velocity has a component perpendicular to the aerodynamic axis. Since this moment is due to the same forces as the Lift and Drag force, it follows that the restoring moment must be perpendicular to $\bar{V}$ also. We use $\bar{P}$ for the restoring moment and write

$$P = K_P \, \rho \, d^3 \, V^2 \quad ,$$

where K_P is the restoring moment coefficient, positive if $\bar{P} \cdot \hat{e} \times \bar{V} \geqq 0$. It is usually more convenient to define a centre of lift by the vector $\bar{\ell}_P$ from the mass centre,

$$\bar{\ell}_P = -\ell_P \, \hat{e}$$

such that

$$\bar{P} = \bar{\ell}_P \times \bar{L} \ ,$$

so that ℓ_P has the same sign as K_P. It follows then that

$$\bar{P} = K_L \, \rho \, d^2 \, \bar{\ell}_P \times \left[\bar{V} \times (\hat{e} \times \bar{V}) \right]$$

$$= K_L \, \rho \, d^2 \, \ell_P (\hat{e} \cdot \bar{V})(\hat{e} \times \bar{V}) \ ,$$

and to first order terms

$$P_x = K_L \rho d^2 \ell_P V \left[(V_1 + V_2 \dot{\phi}) \sin\psi - (V_2 \sin\theta - V_3 \cos\theta) \cos\psi - \xi_x (V_2 \cos\theta + V_3 \sin\theta) \right] ,$$

$$P_y = 0 , \tag{17}$$

$$P_z = -K_L \rho d^2 \ell_P V \left[(V_1 + V_2 \dot{\phi}) \cos\psi + (V_2 \sin\theta - V_3 \cos\theta) \sin\psi + \xi_z (V_2 \cos\theta + V_3 \sin\theta) \right] .$$

It is presumed that ℓ_P is a known function of time and the Mach number associated with V. Dependence on all other parameters, including angle of attack, is ignored. Generally, the location of the centre of lift is taken as a function of Mach number only, while the location of the mass centre is a function of time only.

Force and Moment Due to Cross Spin. If the shell has an angular velocity $\bar{\omega}$, the resultant aerodynamic force and moment will, in general, be different from the case when angular velocity is absent. For if $\bar{\rho}$ is a vector from the mass centre to some point on the surface of the shell, the velocity of this point will be $\bar{V} + \bar{\omega} \times \bar{\rho}$ with respect to the air, and among other things, the aerodynamic force exerted on a small part of the surface depends on the velocity of that surface with respect to the nearby undisturbed air. The component of the angular velocity which is perpendicular to the aerodynamic axis, $\bar{\omega}_s$, is called the cross spin. Clearly

$$\bar{\omega}_s = e \times (\bar{\omega} \times \hat{e}) = \bar{\omega} - (\hat{e} \cdot \bar{\omega})\, \hat{e} .$$

The additional aerodynamic force due to cross spin is denoted by $\bar{S}$ and is assumed to be perpendicular to both $\hat{e}$ and $\bar{\omega}_s$;

$$\bar{S} = K_s \rho d^3 V \hat{e} \times \bar{\omega}_s$$

$$= K_s \rho d^3 V \hat{e} \times \bar{\omega}$$

where K_s is the cross spin force coefficient. To within first order terms,

$$S_1 = - K_s \rho d^3 V \left[(\omega_x + \xi_z \omega_y) \sin \psi - (\omega_z - \xi_x \omega_y) \cos \psi \right] ,$$

$$S_2 = K_s \rho d^3 V \left[(\omega_x + \xi_z \omega_y) \cos \psi + (\omega_z - \xi_x \omega_y) \sin \psi \right] \sin \theta \quad (18)$$

$$S_3 = -K_s \rho d^3 V \left[(\omega_x + \xi_z \omega_y) \cos \psi + (\omega_z - \xi_x \omega_y) \sin \psi \right] \cos \theta .$$

A similar argument can be advanced for the existence of a moment due to cross spin, $\bar{H}$, which must be parallel to the cross spin. We therefore write

$$\bar{H} = -K_H \rho d^4 V \bar{\omega}_s$$

$$= -K_H \rho d^4 V \hat{e} \times (\bar{\omega} \times \hat{e})$$

so that, to first order terms,

$$H_x = -K_H \rho d^4 V \left[\omega_x + \xi_z \omega_y \right] ,$$

$$H_y = 0 , \quad (19)$$

$$H_z = -K_H \rho d^4 V \left[\omega_z - \xi_x \omega_y \right] ,$$

where K_H , the cross spin moment coefficient is also called the damping moment coefficient. This name follows from the fact that for $K_H > 0$, the moment $\bar{H}$ is proportional to the cross spin and tends to damp it out.

The coefficients K_s and K_H are usually taken to equal zero since their effect is generally small compared to the effects of other causes. In any event, we presume that K_s and K_H are at most known functions of the Mach number associated with V . Sometimes it is convenient to define a cross spin centre of pressure by the vector $\bar{\ell}_s$ from the mass centre, $\bar{\ell}_s = -\ell_s \hat{e}$ where $K_H = K_s \ell_s$. Note that ℓ_s can now be a function of Mach number and time. Also, if the cross spin centre of pressure is behind the mass centre ($\ell_s > 0$) the damping moment does indeed oppose the cross spin.

Roll Moment. To this point we have accounted for the aerodynamic force and moment due to the linear and angular velocity of the shell in still air, except for the component of moment along the aerodynamic axis due to linear velocity, and the force and moment due to the component of angular velocity along the aerodynamic axis. Aerodynamically, we assume that the shell is a surface of revolution about the aerodynamic axis. Due to skin friction, the component of angular velocity along the aerodynamic axis causes a friction moment about this axis. If $\bar{R}$ denotes this moment, then we can write

$$\bar{R} = -K_F \, \rho \, d^4 V (\bar{\omega} \cdot \hat{e}) \hat{e} ,$$

or to first order terms,

$$\begin{aligned} R_x &= K_F \, \rho \, d^4 V \omega_y \xi_z , \\ R_y &= -K_F \, \rho \, d^4 V \omega_y , \\ R_z &= -K_F \, \rho \, d^4 V \omega_y \xi_x . \end{aligned} \qquad (20)$$

We suppose that K_F , the roll friction coefficient, is at most a function of the Mach numbers associated with V and $\frac{d}{2} \omega$.

Magnus Force and Moment. We have finally to consider the Magnus force and moment. These are significant only when there is appreciable angular velocity along the aerodynamic axis. Then there will be a force perpendicular to both $\bar{V}$ and $\hat{e}$ (if they are not parallel) which can be represented by

$$\bar{F}_M = K_M \rho d^3 \omega_e \times \bar{V}$$

where $\bar{\omega}_e = (\bar{\omega} \cdot \hat{e}) \hat{e}$ is the component of angular velocity along the aerodynamic axis, and K_M is the Magnus force coefficient due to cross velocity, assumed to be at most a function of the Mach number associated with $\frac{d}{2} \omega_e$ and V. To first order terms

$$\bar{F}_M = K_M \rho d^3 \omega_y \hat{e} \times \bar{V}$$

so that

$$\begin{aligned} F_{M1} &= -K_M \rho d^3 \omega_y \left[V(\xi_x \cos \psi + \xi_z \sin \psi) + V_2 \sin \theta - V_3 \cos \theta \right] , \\ F_{M2} &= K_M \rho d^3 \omega_y \left[V_1 \sin \theta + V_3 (\phi \cos \theta - \xi_x \sin \psi + \xi_z \cos \psi) \right] \qquad (21) \\ F_{M3} &= -K_M \rho d^3 \omega_y \left[V_1 \cos \theta + V_2 (\phi \cos \theta - \xi_x \sin \psi + \xi_z \cos \psi) \right] . \end{aligned}$$

Now because the cross spin adds a cross velocity to points away from the mass centre (in the amount $\bar{\omega}_e \times \bar{\rho}$ where $\bar{\rho}$ is the position vector from the mass centre) we might expect the cross velocity to affect the Magnus force. We therefore suppose K_M to be measured at zero cross spin and consider a Magnus force due to cross spin, $\bar{F}_M'$ with coefficient K_M'. We therefore write

$$\bar{F}'_M = K'_M \rho d^4 \omega_y \hat{e} \times (\hat{e} \times \bar{\omega}_s)$$

$$= -K'_M \rho d^4 \omega_y \bar{\omega}_s \tag{21}$$

$$= -K'_M \rho d^4 \omega_y \left[\hat{i}(\omega_x + \omega_y \xi_z) + \hat{k}(\omega_z - \omega_y \xi_x)\right]$$

so that

$$F'_{M1} = -K'_M \rho d^4 \omega_y \left[(\omega_x + \omega_y \xi_z) \cos\psi + (\omega_z - \omega_y \xi_x) \sin\psi\right]$$

$$F'_{M2} = -K'_M \rho d^4 \omega_y \left[(\omega_x + \omega_y \xi_z) \sin\psi - (\omega_z - \omega_y \xi_x) \cos\psi\right] \sin\theta, \tag{22}$$

$$F'_{M3} = K'_M \rho d^4 \omega_y \left[(\omega_x + \omega_y \xi_z) \sin\psi - (\omega_z - \omega_y \xi_x) \cos\psi\right] \cos\theta .$$

We should also expect a Magnus moment due to cross velocity and one due to cross spin. We define a Magnus cross velocity centre of pressure by the vector $\bar{\ell}_M = -\ell_M \hat{e}$ from the mass centre and allow ℓ_M to be a function of the Mach number associated with V, $\frac{d}{2}\omega_e$ and possibly the time. Then we write

$$\bar{M}_M = \bar{\ell}_M \times \bar{F}_M$$

$$= -K_M \rho d^3 \ell_M \omega_y \hat{e} \times (\hat{e} \times \bar{V})$$

$$= K_M \rho d^3 \ell_M \omega_y \left[\bar{V} - (\hat{e} \cdot \bar{V})\hat{e}\right] ,$$

so that

$$M_{Mx} = K_M \rho d^3 \ell_M \omega_y \left[V\xi_z + (V_1 + V_2\phi) \cos\psi + (V_2 \sin\theta - V_3 \cos\theta) \sin\psi\right]$$

$$M_{My} = 0 , \tag{23}$$

$$M_{Mz} = K_M \rho d^3 \ell_M \omega_y \left[-V\xi_x + (V_1 + V_2\phi) \sin\psi - (V_2 \sin\theta - V_3 \cos\theta) \cos\psi\right] .$$

Similarly, we define a Magnus cross spin centre of pressure by $\bar{\ell}'_M = -\ell'_M \hat{e}$ and write

$$\bar{M}'_M = \bar{\ell}' \times \bar{F}'_M$$

$$= K'_M \rho\, d^4 \ell'_M \;\omega_y \hat{e} \times \overline{\omega}_s$$

$$= K'_M \rho\, d^4 \ell'_M \;\omega_y \left[\hat{i}(\omega_z - \omega_y \xi_x) - \hat{k}(\omega_x + \omega_y \xi_z)\right]$$

so that

$$M'_{Mx} = K'_M \rho\, d^4 \ell'_M \omega_y (\omega_z - \omega_y \xi_x) \;,$$

$$M'_{My} = 0 \;, \qquad (24)$$

$$M'_{Mz} = -K'_M \rho\, d^4 \ell'_M \omega_y (\omega_x + \omega_y \xi_z) \;.$$

IV. SUMMARY

The equations of motion along with the necessary notation are collected in this chapter for easy reference. The independent variable is time denoted by t. The equations of motion are in the form of a system of twelve first order differential equations in twelve coordinates:

$X_o\ Y_o\ Z_o$	components of the displacement vector of the mass centre of the shell from the launching site (measured in inertial coordinates)
$V_1\ V_2\ V_3$	inertial components of the velocity vector of the mass centre relative to the air
$\phi,\ \theta,\ \psi$	angles orientating the shell
$\omega_x,\ \omega_y,\ \omega_z$	components of the angular velocity vector of the shell

The twelve equations of motion are

$$\frac{d X_o}{d t} = V_1 + W_1$$

$$\frac{d Y_o}{d t} = V_2 + W_2 \tag{9}$$

$$\frac{d Z_o}{d t} = V_3 + W_3$$

$$m \frac{d V_1}{d t} = -m \dot{W}_1 + F_1 + G_1$$

$$m \frac{d V_2}{d t} = -m \dot{W}_2 + F_2 + G_2 \tag{10}$$

$$m \frac{d V_3}{d t} = -m \dot{W}_3 + F_3 + G_3$$

$$\frac{d\phi}{dt} = \sec\theta \left[\omega_z \cos\psi - \omega_x \sin\psi \right]$$

$$\frac{d\theta}{dt} = \omega_x \cos\psi + \omega_z \sin\psi \tag{6}$$

$$\frac{d\psi}{dt} = \omega_y - \tan\theta \left[\omega_z \cos\psi - \omega_x \sin\psi \right]$$

$$I \frac{d\omega_x}{dt} = (I' - I)\,\omega_y \omega_z + N_x$$

$$I' \frac{d\omega_y}{dt} = N_y \tag{7}$$

$$I \frac{d\omega_z}{dt} = (I - I')\,\omega_y \omega_x + N_z$$

We now define the individual terms in the above equations:

$$\begin{aligned} W_1 = {} & P_W (A_E (A'_L \cos\Omega t + D'_L \sin\Omega t) + B_E (-A'_L \sin\Omega t + D'_L \cos\Omega t) \\ & \qquad + C_E G'_L) \\ & + Q_W (G_E (A'_L \cos\Omega t + D'_L \sin\Omega t) + H_E (-A'_L \sin\Omega t + D'_L \cos\Omega t) \\ & \qquad + J_E G'_L) \\ W_2 = {} & P_W (A_E (B'_L \cos\Omega t + E'_L \sin\Omega t) + B_E (-B'_L \sin\Omega t + E'_L \cos\Omega t) \\ & \qquad + C_E A'_L) \\ & + Q_W (G_E (B'_L \cos\Omega t + E'_L \sin\Omega t) + H_E (-B'_L \sin\Omega t + E'_L \cos\Omega t) \\ & \qquad + J_E H'_L) \end{aligned} \tag{11}$$

$$\frac{d\phi}{dt} = \sec\theta \left[\omega_z \cos\psi - \omega_x \sin\psi \right]$$

$$\frac{d\theta}{dt} = \omega_x \cos\psi + \omega_z \sin\psi \qquad (6)$$

$$\frac{d\psi}{dt} = \omega_y - \tan\theta \left[\omega_z \cos\psi - \omega_x \sin\psi \right]$$

$$I \frac{d\omega_x}{dt} = (I' - I)\, \omega_y \omega_z + N_x$$

$$I' \frac{d\omega_y}{dt} = N_y \qquad (7)$$

$$I \frac{d\omega_z}{dt} = (I - I')\, \omega_y \omega_x + N_z$$

We now define the individual terms in the above equations:

$$W_1 = P_W (A_E (A'_L \cos\Omega t + D'_L \sin\Omega t) + B_E (-A'_L \sin\Omega t + D'_L \cos\Omega t) + C_E G'_L)$$
$$+ Q_W (G_E (A'_L \cos\Omega t + D'_L \sin\Omega t) + H_E (-A'_L \sin\Omega t + D'_L \cos\Omega t) + J_E G'_L)$$

$$W_2 = P_W (A_E (B'_L \cos\Omega t + E'_L \sin\Omega t) + B_E (-B'_L \sin\Omega t + E'_L \cos\Omega t) + C_E A'_L) \qquad (11)$$
$$+ Q_W (G_E (B'_L \cos\Omega t + E'_L \sin\Omega t) + H_E (-B'_L \sin\Omega t + E'_L \cos\Omega t) + J_E H'_L)$$

$$W_3 = P_W (A_E (C'_L \cos \Omega t + F'_L \sin \Omega t) + B_E (-C'_L \sin \Omega t + F'_L \cos \Omega t) + C_E J'_L)$$

$$+ Q_W (G_E (C'_L \cos \Omega t + F'_L \sin \Omega t) + H_E (-C'_L \sin \Omega t + F'_L \cos \Omega t) + J_E J'_L) \tag{11}$$

$$F_1 = -m\, g_o \left(\frac{\rho_o}{\rho_E} \right)^2 \left(G_E (A'_L \cos \Omega t + D'_L \sin \Omega t) + H_E (-A'_L \sin \Omega t + D'_L \cos \Omega t) + J_E G'_L \right)$$

$$F_2 = -m\, g_o \left(\frac{\rho_o}{\rho_E} \right)^2 \left(G_E (B'_L \cos \Omega t + E'_L \sin \Omega t) + H_E (-B'_L \sin \Omega t + E'_L \cos \Omega t) + J_E H'_L \right) \tag{13}$$

$$F_3 = -m\, g_o \left(\frac{\rho_o}{\rho_E} \right)^2 \left(G_E (C'_L \cos \Omega t + F'_L \sin \Omega t) + H_E (-C'_L \sin \Omega t + F'_L \cos \Omega t) + J_E J'_L \right)$$

$$A_E = \cos \theta_E \cos \phi_E \cos \psi_E + \sin \theta_E \sin \psi_E$$

$$B_E = \sin \theta_E \cos \phi_E \cos \psi_E - \cos \theta_E \sin \psi_E$$

$$C_E = -\sin \phi_E \cos \psi_E \tag{A-25}$$

$$G_E = \cos \theta_E \sin \phi_E$$

$$H_E = \sin \theta_E \sin \phi_E$$

$$J_E = \cos \phi_E$$

$$\rho_E^{\ 2} = X_E^{\ 2} + Y_E^{\ 2} + Z_E^{\ 2}$$

$$\phi_E = \cos^{-1} Z_E / \sqrt{X_E^{\ 2} + Y_E^{\ 2} + Z_E^{\ 2}} \qquad \text{(A-19)}$$

$$\theta_E = \sin^{-1} Y_E / \sqrt{X_E^{\ 2} + Y_E^{\ 2}}$$

$$X_E = X_o (A_L \cos \Omega t + B_L \sin \Omega t) + Y_o (D_L \cos \Omega t + E_L \sin \Omega t)$$

$$+ (Z_o + \rho)(G_L \cos \Omega t + H_L \sin \Omega t)$$

$$Y_E = X_o (-A_L \sin \Omega t + B_L \cos \Omega t) + Y_o (-D_L \sin \Omega t + E_L \cos \Omega t)$$

$$+ (Z_o + \rho)(-G_L \sin \Omega t + H_L \cos \Omega t) \qquad \text{(A-17)}$$

$$Z_E = C_L X_o + F_L Y_o + J_L (Z_o + \rho_E)$$

$$A_L = \cos \theta_L \cos \phi_L \cos \psi_L + \sin \theta_L \sin \psi_L$$

$$B_L = \sin \theta_L \cos \phi_L \cos \psi_L - \cos \theta_L \sin \psi_L$$

$$C_L = - \sin \phi_L \cos \psi_L$$

$$D_L = \cos \theta_L \cos \phi_L \sin \psi_L - \sin \theta_L \cos \psi_L$$

$$E_L = \sin \theta_L \cos \phi_L \sin \psi_L + \cos \theta_L \cos \psi_L \qquad \text{(A-18)}$$

$$F_L = - \sin \phi_L \sin \psi_L$$

$$G_L = \cos \theta_L \sin \phi_L$$

$$H_L = \sin \theta_L \sin \phi_L$$

$$J_L = \cos \phi_L$$

$$A'_L = \cos\psi_L \cos\phi_L \cos\theta_L + \sin\psi_L \sin\theta_L$$

$$B'_L = \sin\psi_L \cos\phi_L \cos\theta_L - \cos\psi_L \sin\theta_L$$

$$C'_L = \sin\phi_L \cos\theta_L$$

$$D'_L = \cos\psi_L \cos\phi_L \sin\theta_L - \sin\psi_L \cos\theta_L$$

$$E'_L = \sin\psi_L \cos\phi_L \sin\theta_L + \cos\psi_L \cos\theta_L \qquad \text{(A-13)}$$

$$F'_L = \sin\phi_L \sin\theta_L$$

$$G'_L = -\cos\psi_L \sin\phi_L$$

$$H'_L = -\sin\psi_L \sin\phi_L$$

$$J'_L = \cos\phi_L$$

$$G_1 = D_1 + L_1 + S_1 + F_{M1} + F'_{M1} \quad \text{etc.}$$

$$D_1 = -K_D \rho d^2 V V_1 ,$$

$$D_2 = -K_D \rho d^2 V V_2 , \qquad (16)$$

$$D_3 = -K_D \rho d^2 V V_3 .$$

$$L_1 = K_L \rho d^2 V (V(-\phi \cos\theta + \xi_x \sin\psi - \xi_z \cos\psi) - V_1)$$

$$L_2 = K_L \rho d^2 V ((V \cos\theta - V_2) - V(\xi_x \cos\psi + \xi_z \sin\psi) \sin\theta) \qquad (15)$$

$$L_3 = K_L \rho d^2 V ((V \sin\theta - V_3) + V(\xi_x \cos\psi + \xi_z \sin\psi) \cos\theta) .$$

$$S_1 = -K_s \rho d^3 V \left((\omega_x + \xi_z \omega_y) \sin \psi - (\omega_z - \xi_x \omega_y) \cos \psi \right)$$

$$S_2 = K_s \rho d^3 V \left((\omega_x + \xi_z \omega_y) \cos \psi + (\omega_z - \xi_x \omega_y) \sin \psi \right) \sin \theta$$

$$S_3 = -K_s \rho d^3 V \left((\omega_x + \xi_z \omega_y) \cos \psi + (\omega_z - \xi_x \omega_y) \sin \psi \right) \cos \theta \quad (18)$$

$$F_{M1} = -K_M \rho d^3 \omega_y \left(V (\xi_x \cos \psi + \xi_z \sin \psi) + V_2 \sin \theta - V_3 \cos \theta \right) ,$$

$$F_{M2} = K_M \rho d^3 \omega_y \left(V_1 \sin \theta + V_3 (\phi \cos \theta - \xi_x \sin \psi + \xi_z \cos \psi)\right) \quad (21)$$

$$F_{M3} = -K_M \rho d^3 \omega_y \left(V_1 \cos \theta + V_2 (\phi \cos \theta - \xi_x \sin \psi + \xi_z \cos \psi)\right) .$$

$$F'_{M1} = -K'_M \rho d^4 \omega_y \left((\omega_x + \omega_y \xi_z) \cos \psi + (\omega_z - \omega_y \xi_x) \sin \psi \right) ,$$

$$F'_{M2} = -K'_M \rho d^4 \omega_y \left((\omega_x + \omega_y \xi_z) \sin \psi - (\omega_z - \omega_y \xi_x) \cos \psi \right) \sin \theta \quad (22)$$

$$F'_{M3} = K'_M \rho d^4 \omega_y \left((\omega_x + \omega_y \xi_z) \sin \psi - (\omega_z - \omega_y \xi_x) \cos \psi \right) \cos \theta$$

$$N_x = P_x + H_x + R_x + M_{Mx} + M'_{Mx} , \quad \text{etc.}$$

$$P_x = K_L \rho d^2 \ell_P V \left((V_1 + V_2 \phi) \sin \psi - (V_2 \sin \theta - V_3 \cos \theta) \cos \psi - \xi_x (V_2 \cos \theta + V_3 \sin \theta) \right)$$

$$P_y = 0 . \quad (17)$$

$$P_z = -K_L \rho d^2 \ell_P V \left((V_1 + V_2 \phi) \cos \psi + (V_2 \sin \theta - V_3 \cos \theta) \sin \psi + \xi_z (V_2 \cos \theta + V_3 \sin \theta) \right) .$$

$$H_x = -K_H \rho d^4 V (\omega_x + \xi_z \omega_y)$$

$$H_y = 0 , \quad (19)$$

$$H_z = -K_H \rho d^4 V (\omega_z - \xi_x \omega_y) ,$$

$$R_x = K_F \rho d^4 V \omega_y \xi_z$$

$$R_y = -K_F \rho d^4 V \omega_y \qquad (20)$$

$$R_z = -K_F \rho d^4 V \omega_y \xi_x$$

$$M_{Mx} = K_M \rho d^3 \ell_M \omega_y (V \xi_z + (V_1 + V_2 \phi) \cos \psi + (V_2 \sin \theta - V_3 \cos \theta) \sin \psi)$$

$$M_{My} = 0 \qquad (23)$$

$$M_{Mz} = K_M \rho d^3 \ell_M \omega_y (-V \xi_x + (V_1 + V_2 \phi) \sin \psi - (V_2 \sin \theta - V_3 \cos \theta) \cos \psi)$$

$$M'_{Mx} = +K_M \rho d^4 \ell'_M \omega_y (\omega_z - \omega_y \xi_x) ,$$

$$M'_{My} = 0 , \qquad (24)$$

$$M'_{Mz} = -K'_M \rho d^4 \ell'_M \omega_y (\omega_x + \omega_y \xi_z) .$$

Notation

$$V^2 = V_1^2 + V_2^2 + V_3^2$$

W_1, W_2, W_3 - components of the wind velocity in the neighborhood of the shell, relative to the inertial reference system

m	mass of the shell (constant)
I	principal transverse moment of inertia of the shell (constant)
I'	principal longitudinal moment of inertia of the shell (constant)
ξ_x, ξ_z	aerodynamic malalignment angles (constant)
ρ	density of the air (function of position), i. e. θ_E, ρ_E, ϕ_E
d	shell diameter
D_1, D_2, D_3	aerodynamic drag force
L_1, L_2, L_3	aerodynamic lift force
S_1, S_2, S_3	aerodynamic cross spin force
F_{M1}, F_{M2}, F_{M3}	Magnus force due to cross velocity
F'_{M1}, F'_{M2}, F'_{M3}	Magnus force due to cross spin
P_x, P_y, P_z	aerodynamic restoring moment (due to cross velocity)
H_x, H_y, H_z	aerodynamic damping moment (due to cross spin)
R_x, R_y, R_z	aerodynamic roll moment
M_{Mx}, M_{My}, M_{Mz}	Magnus moment due to cross velocity
M'_{Mx}, M'_{My}, M'_{Mz}	Magnus moment due to cross spin
K_D	aerodynamic drag coefficient (function of Mach number)
K_L	aerodynamic lift coefficient (function of Mach number)
K_S	aerodynamic cross spin force coefficient (function of Mach number)
K_M	Magnus force coefficient due to cross velocity (function of Mach number)

$K_{H^{\circ}}$	aerodynamic damping moment coefficient (function of Mach number)
K_F	aerodynamic roll friction coefficient (function of Mach number)
ℓ_P	distance between mass centre and centre of lift (function of Mach number)
ℓ_M'	distance between mass centre and Magnus cross spin centre of pressure (function of Mach number)
P_w, Q_w	horizontal and vertical components of wind velocity
ψ_E	direction of wind velocity as given in coordinates fixed in the earth
θ_E, ϕ_E, ρ_E	coordinates of a point measured in earth coordinates
θ_L, ϕ_L, ρ_L	coordinates of the shell at time of launch (fixed with respect to the earth)
Ω	angular velocity of the earth $= 2\pi$ radians/day

K_H	aerodynamic damping moment coefficient (function of Mach number)
K_F	aerodynamic roll friction coefficient (function of Mach number)
ℓ_P	distance between mass centre and centre of lift (function of Mach number)
ℓ'_M	distance between mass centre and Magnus cross spin centre of pressure (function of Mach number)
P_w, Q_w	horizontal and vertical components of wind velocity
ψ_E	direction of wind velocity as given in coordinates fixed in the earth
θ_E, ϕ_E, ρ_E	coordinates of a point measured in earth coordinates
θ_L, ϕ_L, ρ_L	coordinates of the shell at time of launch (fixed with respect to the earth)
Ω	angular velocity of the earth $= 2\pi$ radians/day

V. APPENDIX

COORDINATE SYSTEMS

i We will first derive some general transformations, and then apply them to specific applications. Let us consider 4 such transformations.

ii General Transformations

ii-1 Rotation of Coordinate Axes

Let us successively make rotations about the X, Y, and Z axes respectively; through the angles θ, ϕ, and ψ. Then we will obtain the following relations between unit vectors:

$$
\begin{aligned}
\hat{i}' &= \hat{i}\,A + \hat{j}\,B + \hat{k}\,C & \qquad \hat{i} &= \hat{i}'\,A' + \hat{j}'\,B' + \hat{k}'\,C' \\
\hat{j}' &= \hat{i}\,D + \hat{j}\,E + \hat{k}\,F & \qquad \hat{j} &= \hat{i}'\,D' + \hat{j}'\,E' + \hat{k}'\,F' \\
\hat{k}' &= \hat{i}\,G + \hat{j}\,H + \hat{k}\,J & \qquad \hat{k} &= \hat{i}'\,G' + \hat{j}'\,H' + \hat{k}'\,J'
\end{aligned}
\qquad \text{A-1}
$$

Here $\hat{i}$, $\hat{j}$, $\hat{k}$, and $\hat{i}'$, $\hat{j}'$, $\hat{k}'$, are unit vectors along the X, Y, Z, and X', Y', Z' axes respectively.

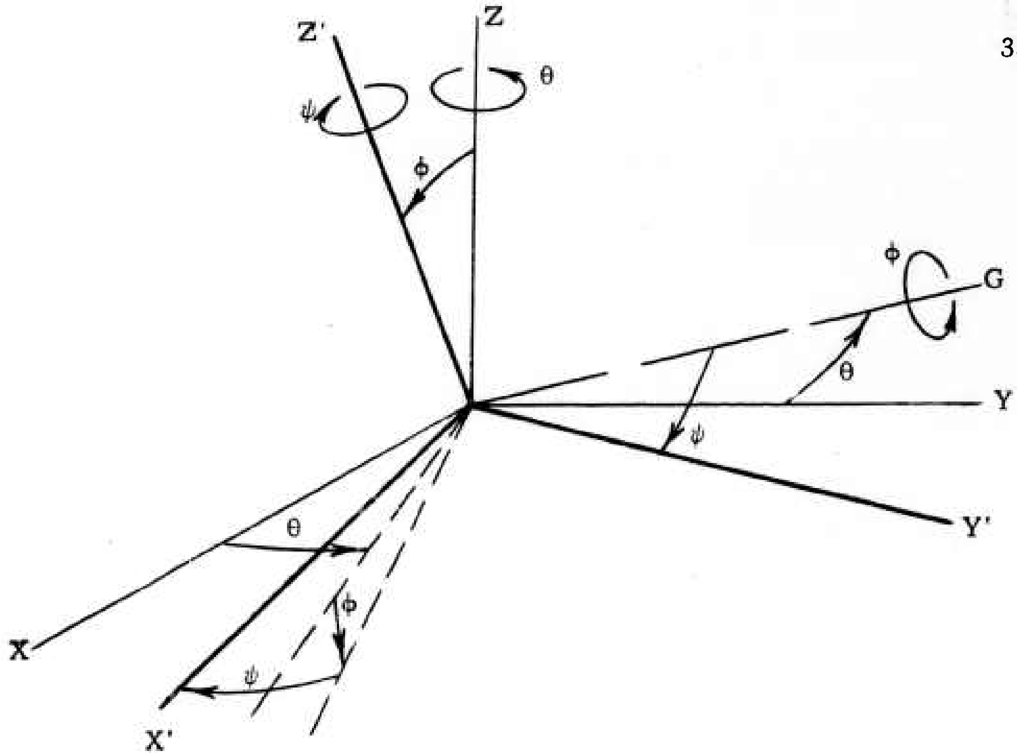


Figure A-1

Also:

$A = \cos\theta \cos\phi \cos\psi + \sin\theta \sin\psi$ $\quad$ $A' = \cos\psi \cos\phi \cos\theta + \sin\psi \sin\theta$

$B = \sin\theta \cos\phi \cos\psi - \cos\theta \sin\psi$ $\quad$ $B' = \sin\psi \cos\phi \cos\theta - \cos\psi \sin\theta$

$C = -\sin\phi \cos\psi$ $\quad$ $C' = \sin\phi \cos\theta$

$D = \cos\theta \cos\phi \sin\psi - \sin\theta \cos\psi$ $\quad$ $D' = \cos\psi \cos\phi \sin\theta - \sin\psi \cos\theta$

$E = \sin\theta \cos\phi \sin\psi + \cos\theta \cos\psi$ $\quad$ $E' = \sin\psi \cos\phi \sin\theta + \cos\psi \cos\theta$ A-2

$F = -\sin\phi \sin\psi$ $\quad$ $F' = \sin\phi \sin\theta$

$G = \cos\theta \sin\phi$ $\quad$ $G' = -\cos\psi \sin\phi$

$H = \sin\theta \sin\phi$ $\quad$ $H' = -\sin\psi \sin\phi$

$J = \cos\phi$ $\quad$ $J' = \cos\phi$

Then any point may be located by the vector:

$$\overline{r} = \hat{i}\, X + \hat{j}\, Y + \hat{k}\, Z = \hat{i}'\, X' + \hat{j}'\, Y' + \hat{k}'\, Z'$$

where:

$$X = AX' + DY' + GZ' \qquad X' = A'X + D'Y + G'Z$$
$$Y = BX' + EY' + HZ' \qquad Y' = B'X + E'Y + H'Z \qquad \text{A-3}$$
$$Z = CX' + FY' + JZ' \qquad Z' = C'X + F'Y + J'Z$$

ii-2 Translation of Coordinate Axes:

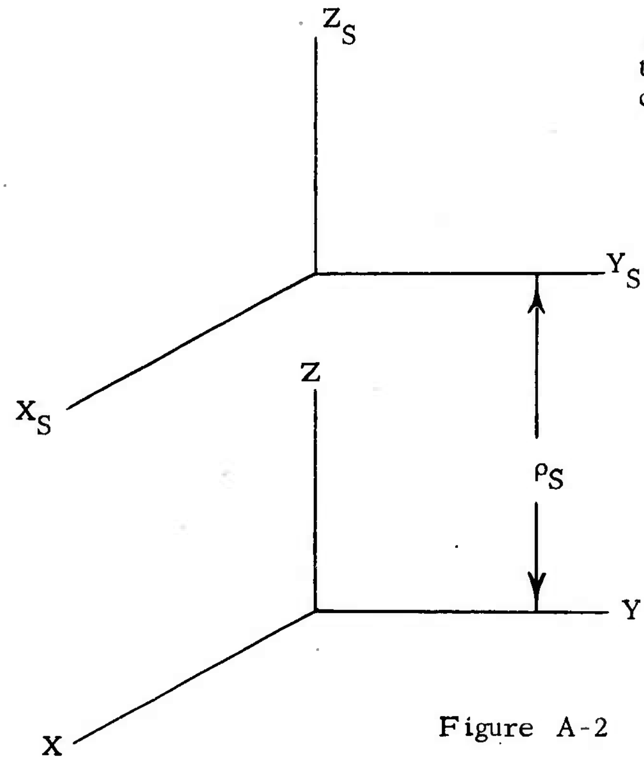


Figure A-2

It will frequently be convenient to consider the translation of coordinates given by:

$$X_S = X$$
$$Y_S = Y \qquad \text{A-4}$$
$$Z_S = Z - \rho_S$$

ii-3 Rotation of the Earth

Consider X Y Z to be a rectangular system of coordinates fixed in the earth, with the Z axis along the axis of rotation of the earth. Let X_*, Y_*, Z_* be an inertial system such that Z_* will coincide with Z . The X, Y, Z system then rotates with a constant angular velocity $\hat{k}\ \Omega$ with respect to the $X_*\ Y_*\ Z_*$ system.

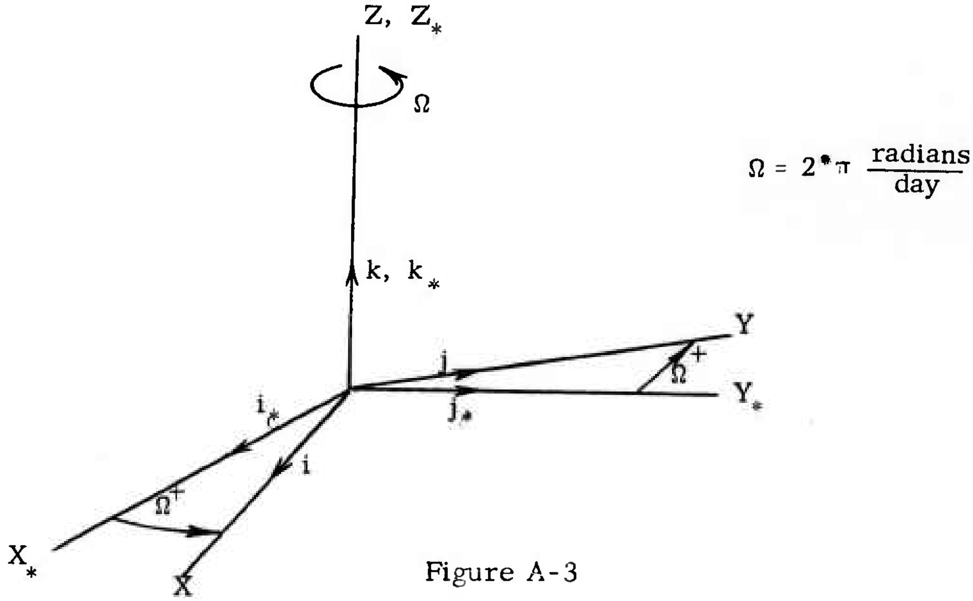


Figure A-3

From Figure 3:

$$\hat{i} = \hat{i}_* \cos \Omega t + \hat{j}_* \sin \Omega t; \qquad \hat{i}_* = \hat{i} \cos \Omega t - \hat{j} \sin \Omega t$$

$$\hat{j} = -\hat{i}_* \sin \Omega t + \hat{j}_* \cos \Omega t \qquad \hat{j}_* = \hat{i} \sin \Omega t + \hat{j} \cos \Omega t \qquad \text{A-5}$$

$$\hat{k} = \hat{k}_* \qquad \hat{k}_* = \hat{k}$$

Then, any vector may be represented by:

$$\bar{r} = \hat{i} X + \hat{j} Y + \hat{k} Z = \hat{i}_* X_* + \hat{j}_* Y_* + \hat{k}_* Z_*$$

where:

$$X = X_* \cos \Omega t + Y_* \sin \Omega t; \qquad X_* = X \cos \Omega t - Y \sin \Omega t$$

$$Y = -X_* \sin \Omega t + Y_* \cos \Omega t; \qquad Y_* = X \sin \Omega t + Y \cos \Omega t \qquad \text{A-6}$$

ii-4 Spherical Coordinates:

It will be convenient to use spherical coordinates in addition to the rectangular coordinates.

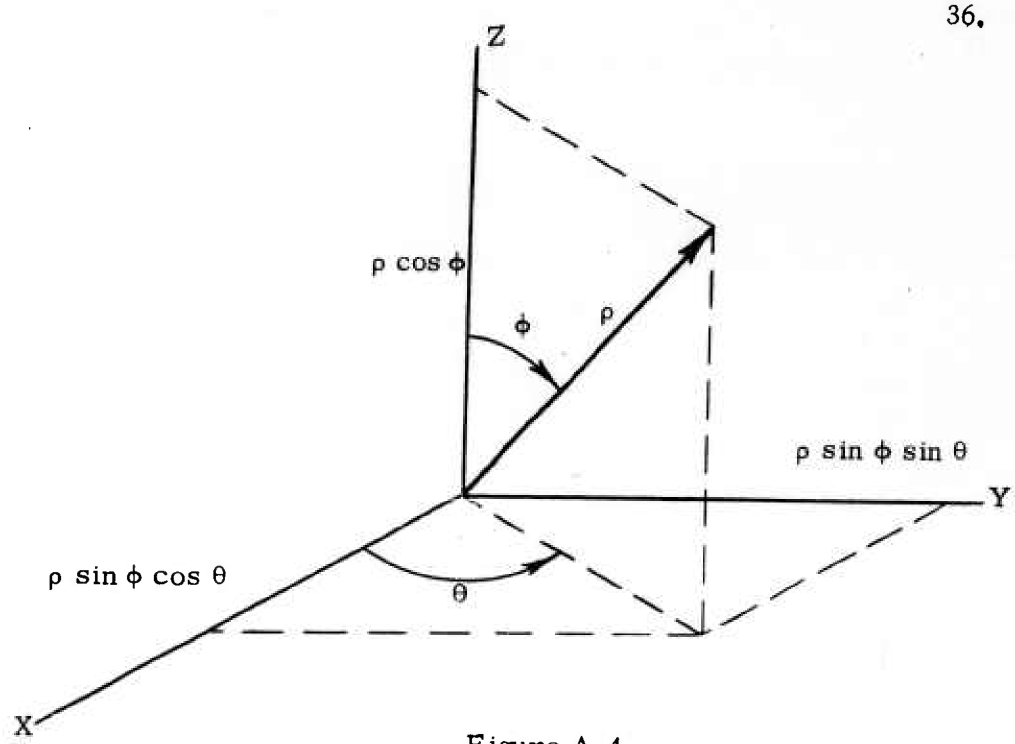


Figure A-4

We have:

$$X = \rho \sin\phi \cos\theta \qquad \rho^2 = X^2 + Y^2 + Z^2$$

$$Y = \rho \sin\phi \sin\theta \qquad \phi = \cos^{-1} Z / \sqrt{X^2 + Y^2 + Z^2} \qquad \text{A-7}$$

$$Z = \rho \cos\phi \qquad \theta = \sin^{-1} Y / \sqrt{X^2 + Y^2}$$

iii Location of Points; and Direction of Vectors in Space

Pertinent quantities, such as air densities, wind velocities, and gravity forces are available in a reference system fixed with respect to the earth. Any point in this system may be located by specifying two angles θ_E, ϕ_E, and a length ρ_E. These quantities are the longitude, latitude, and distance from the center of the earth respectively.

We may also determine a direction at the point $(\theta_E, \phi_E, \rho_E)$.

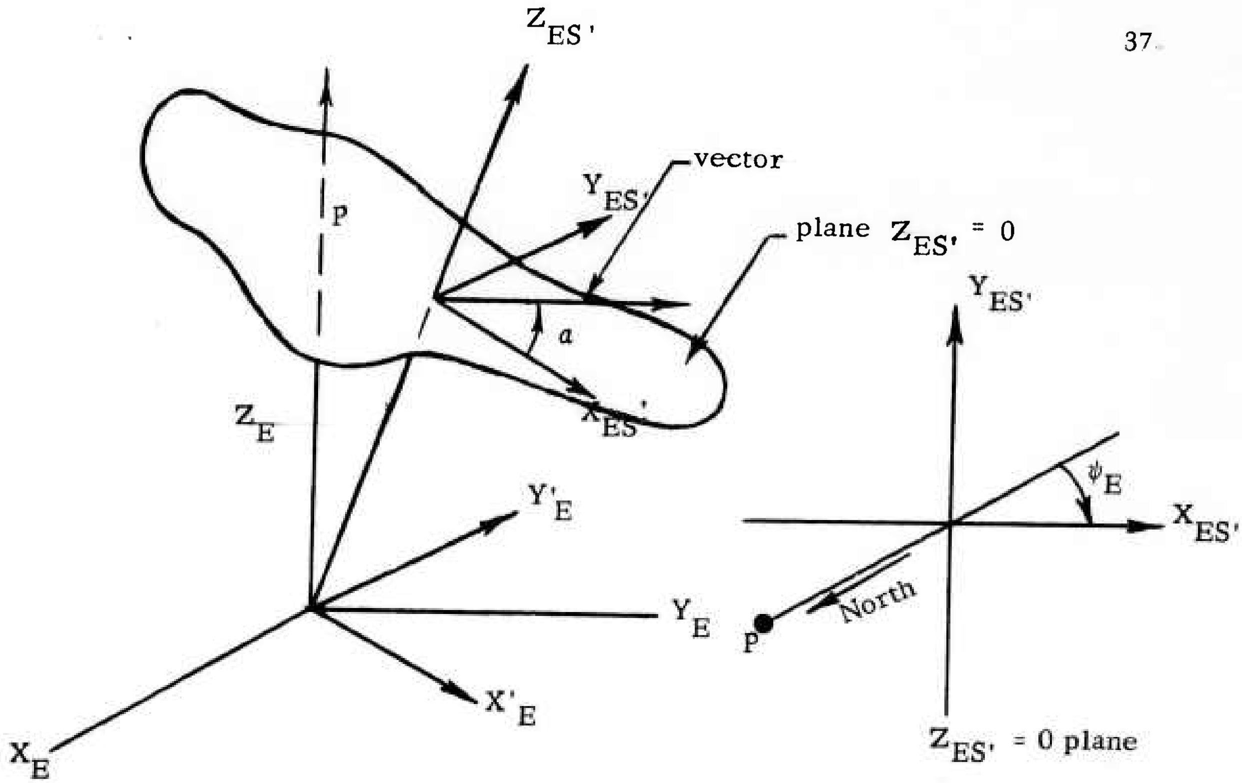


Figure A-5

Consider an additional rotation ψ_E as performed in section II-1. Then if ψ_E is chosen so that the direction (or vector) lies in the plane $Y'_{ES} = 0$, the vector may be represented by 2 components; one along the X'_{ES} axis, and one along the Z'_{ES} axis. The angle $\psi_E + 180^o$ is referred to as the azimuth of the vector; and the angle a between the vector and the X'_{ES} axis is called its elevation.

The angle θ_E varies from 0 to 360^o. As θ_E goes through the variations: $0 \to 90^o \to 180^o \to 270^o \to 360^o$ longitude goes through the variations: $0 \to 90^o E \to 180^o E \to 90^o W \to 0^o$.

The angle ϕ_E varies from 0 to 180^o. As ϕ_E goes through the variations: $0 \to 45^o \to 90^o \to 135^o \to 180^o$ latitude goes through the variations: $90^o N \to 45^o N \to 0^o \to 45^o S \to 90^o S$.

The angle ψ_E varies from 0^o to 360^o. As ψ_E goes through the variation: $0 \rightarrow 90^o \rightarrow 180^o \rightarrow 270^o \rightarrow 360^o$, azimuth goes through $180^o \rightarrow 270^o \rightarrow 360^o \rightarrow 90^o \rightarrow 180^o$, and the more familiar directions go through $S \rightarrow W \rightarrow N \rightarrow E \rightarrow S$.

iv <u>Transformation of the Coordinates of a Point From a System Fixed in the Earth to the X'_{oS}, Y'_{oS}, Z'_{oS} (or $X_o Y_o Z_o$) System</u>

The transformation of coordinates from the system fixed in the earth to the inertial system may be carried out using 4 successive transformations. (We will assume that the location of a point is specified by ϕ, θ, ρ).

1. From the (ρ_E, ϕ_E, θ_E) system to the X_E, Y_E, Z_E system (both fixed in the earth), using ii-4.

$$\begin{aligned} X_E &= \rho_E \sin \phi_E \cos \theta_E \\ Y_E &= \rho_E \sin \phi_E \sin \theta_E \\ Z_E &= \rho_E \cos \phi_E \end{aligned} \qquad \text{A-8}$$

2. From the $X_E Y_E Z_E$ system fixed in the earth to the $X_* Y_* Z_*$ inertial system using ii-3

$$\begin{aligned} X_* &= X_E \cos \Omega t - Y_E \sin \Omega t \\ Y_* &= X_E \sin \Omega t + Y_E \cos \Omega t \\ Z_* &= Z_E \end{aligned} \qquad \text{A-9}$$

3. From the $X_* Y_* Z_*$ inertial system to the $X'_o Y'_o Z'_o$ inertial system using ii-1

$$\begin{aligned} X'_o &= A'_L X_* + D'_L Y_* + G'_L Z_* \\ Y'_o &= B'_L X_* + E'_L Y_* + H'_L Z_* \\ Z'_o &= C'_L X_* + F'_L Y_* + J'_L Z_* \end{aligned} \qquad \text{A-10}$$

Here the angles ϕ_L , θ_L , ψ_L appearing in A'_L - - - J'_L are the angles giving initial position and orientation of the missile at launch.

4. From the X'_o, Y'_o, Z'_o inertial system to the X'_{oS} , Y'_{oS}, Z'_{oS} (or X_o, Y_o, Z_o) inertial system using ii-2.

$$
\begin{aligned}
X_o &= X'_{oS} = X'_o \\
Y_o &= Y'_{oS} = Y'_o \\
Z_o &= Z'_{oS} = Z'_o - \rho_E
\end{aligned}
\qquad \text{A-11}
$$

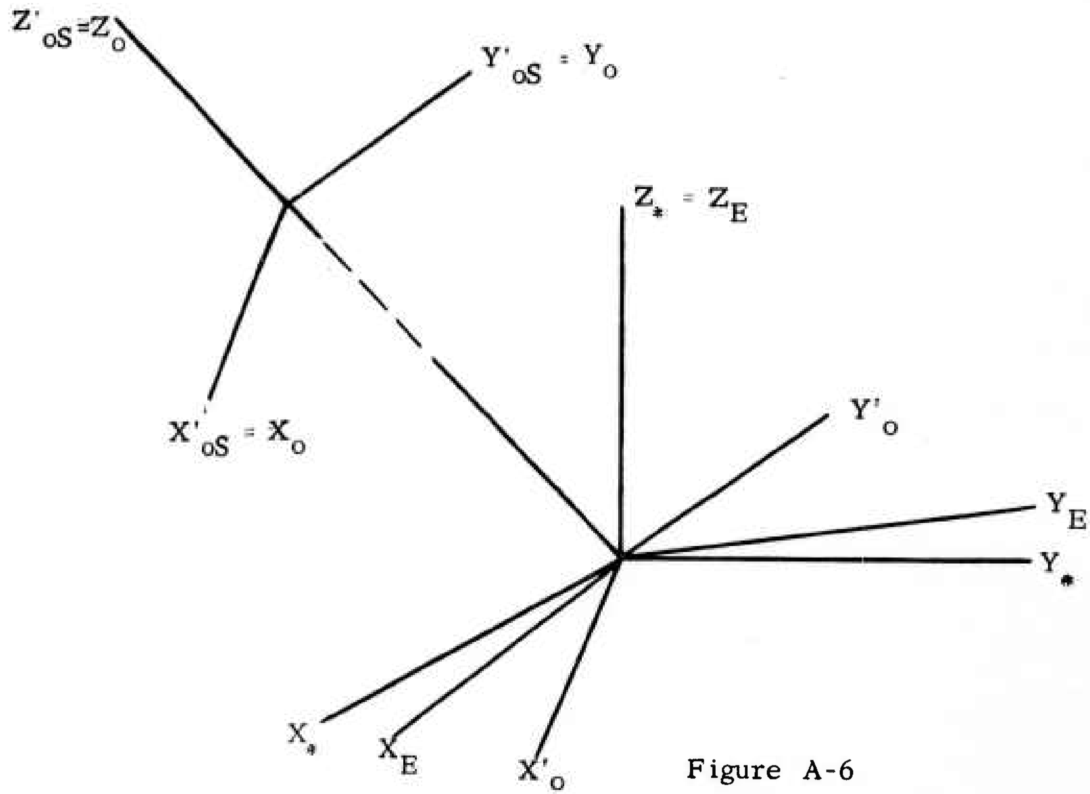


Figure A-6

Then the transformation becomes:

$$x/\rho_E = \sin\phi_E\,(A'_L(\cos\theta_E\cos\Omega t - \sin\theta_E\sin\Omega t) + D'_L(\cos\theta_E\sin\Omega t + \sin\theta_E\cos\Omega t)) + G'_L\cos\phi_E$$

$$y/\rho_E = \sin\phi_E\,(B'_L(\cos\theta_E\cos\Omega t - \sin\theta_E\sin\Omega t) + E'_L(\cos\theta_E\sin\Omega t + \sin\theta_E\cos\Omega t)) + H'_L\cos\phi_E \qquad \text{A-12.}$$

$$z/\rho_E = \sin\phi_E(C'_L(\cos\theta_E\cos\Omega t - \sin\theta_E\sin\Omega t) + F'_L(\cos\theta_E\sin\Omega t + \sin\theta_E\cos\Omega t)) + J'_L\cos\phi_E - 1$$

where:

$$\begin{aligned}
A'_L &= \cos\psi_L\cos\phi_L\cos\theta_L + \sin\psi_L\sin\theta_L \\
B'_L &= \sin\psi_L\cos\phi_L\cos\theta_L - \cos\psi_L\sin\theta_L \\
C'_L &= \sin\phi_L\cos\theta_L \\
D'_L &= \cos\psi_L\cos\phi_L\sin\theta_L - \sin\psi_L\cos\theta_L \qquad \text{A-13} \\
E'_L &= \sin\psi_L\cos\phi_L\sin\theta_L + \cos\psi_L\cos\theta_L \\
F'_L &= \sin\phi_L\sin\theta_L \\
G'_L &= -\cos\psi_L\sin\phi_L \\
H'_L &= -\sin\psi_L\sin\phi_L \\
J'_L &= \cos\phi_L
\end{aligned}$$

Similarly, by reversing the order of the transformations, we may express the coordinates ρ_E, θ_E, ϕ_E in terms of $X_o Y_o$ and Z_o.

5. Applying ii-2

$$\begin{aligned}
X'_o &= X'_{oS} = X_o \\
Y'_o &= Y'_{oS} = Y_o \qquad \text{A-14} \\
Z'_o &= Z'_{oS} + \rho_E = Z_o + \rho_E
\end{aligned}$$

6. Applying ii-1

$$X_* = A_L X'_o + D_L Y'_o + G_L Z'_o$$
$$Y_* = B_L X'_o + E_L Y'_o + H_L Z'_o \qquad \text{A-15}$$
$$Z_* = C_L X'_o + F_L Y'_o + J_L Z'_o$$

7. Applying ii-3

$$X_E = X_* \cos \Omega t + Y_* \sin \Omega t$$
$$Y_E = -X_* \sin \Omega t + Y_* \cos \Omega t \qquad \text{A-16}$$
$$Z_E = Z_*$$

Or, we may write:

$$X_E = X_o(A_L \cos \Omega t + B_L \sin \Omega t) + Y_o(D_L \cos \Omega t + E_L \sin \Omega t)$$
$$+ (Z_o + \rho)(G_L \cos \Omega t + H_L \sin \Omega t)$$
$$Y_E = X_o(-A_L \sin \Omega t + B_L \cos \Omega t) + Y_o(-D_L \sin \Omega t + E_L \cos \Omega t)$$
$$+ (Z_o + \rho)(-G_L \sin \Omega t + H_L \cos \Omega t) \qquad \text{A-17}$$
$$Z_E = C_L X_o + F_L Y_o + J_L (Z_o + \rho_E).$$

where:

$$A_L = \cos \theta_L \cos \phi_L \cos \psi_L + \sin \theta_L \sin \psi_L$$
$$B_L = \sin \theta_L \cos \phi_L \cos \psi_L - \cos \theta_L \sin \psi_L$$
$$C_L = -\sin \phi_L \cos \psi_L$$
$$D_L = \cos \theta_L \cos \phi_L \sin \psi_L - \sin \theta_L \cos -_L \qquad \text{A-18}$$
$$E_L = \sin \theta_L \cos \phi_L \sin \psi_L + \cos \theta_L \cos \psi_L$$
$$F_L = -\sin \phi_L \sin \psi_L$$
$$G_L = \cos \theta_L \sin \phi_L$$
$$H_L = \sin \theta_L \sin \phi_L$$
$$J_L = \cos \phi_L$$

8. And applying ii-4:

$$\rho_E^2 = X_E^{\ 2} + Y_E^{\ 2} + Z_E^{\ 2}$$

$$\phi = \cos^{-1} Z_E / \sqrt{X_E^{\ 2} + Y_E^{\ 2} + Z_E^{\ 2}} \qquad \text{A-19}$$

$$\theta = \sin^{-1} Y_E / \sqrt{X_E^{\ 2} + Y_E^{\ 2}}$$

v Transformation of a Vector From the System Fixed in the Earth to the Inertial System (x, y, z)

From Fig. 5, we see that a vector may be expressed as:

$$\bar{r} = P\, \hat{i}'_{ES} + Q\, \hat{k}'_{ES}$$

We wish to express this vector in terms of its components in the $\hat{i}'_{oS}$, $\hat{j}'_{oS}$, $\hat{k}'_{oS}$ (or $\hat{i}_0\ \hat{j}_0\ \hat{k}_0$) directions.

This may be done by 3 successive transformations:

1. From the $\hat{i}'_{ES}$, $\hat{j}'_{ES}$, $\hat{k}'_{ES}$ (or $\hat{i}'_E$, $\hat{j}'_E$, $\hat{k}'_E$) to the $\hat{i}_E$, $\hat{j}_E$, $\hat{k}_E$ components. Applying ii-1

$$\hat{i}'_E = \hat{i}_E A_E + \hat{j}_E B_E + \hat{k}_E C_E$$

$$\hat{k}'_E = \hat{i}_E G_E + \hat{j}_E H_E + \hat{k}_E J_E \qquad \text{A-20}$$

2. From the $\hat{i}_E$, $\hat{j}_E$, $\hat{k}_E$ to the $\hat{i}_*$ $\hat{j}_*$ $\hat{k}_*$ components. Applying ii-3:

$$\hat{i}_E = \hat{i}_* \cos \Omega t + \hat{j}_* \sin \Omega t$$

$$\hat{j}_E = -\hat{i}_* \sin \Omega t + \hat{j}_* \cos \Omega t \qquad \text{A-21}$$

$$\hat{k}_E = \hat{k}_*$$

3. From the $\hat{i}_*, \hat{j}_*, \hat{k}_*$ to the $\hat{i}'_o, \hat{j}'_o, \hat{k}'_o$ components. Applying ii-1:

$$\begin{aligned} \hat{i}_* &= \hat{i}'_o A'_L + \hat{j}'_o B'_L + \hat{k}'_o C'_L \\ \hat{j}_* &= \hat{i}'_o D'_L + \hat{j}'_o E'_L + \hat{k}'_o F'_L \\ \hat{k}_* &= \hat{i}'_o G'_L + \hat{j}'_o H'_L + \hat{k}'_o J'_L \end{aligned} \qquad \text{A-22}$$

But

$$\hat{i}'_{oS} = \hat{i}'_o = \hat{i}_o; \quad \hat{j}'_{oS} = \hat{j}'_o = \hat{j}_o; \quad \hat{k}'_{oS} = \hat{k}'_o = \hat{k}_o$$

Then the vector $\bar{v}$ may be written:

$$\begin{aligned} \bar{v} &= P\,\hat{i}'_E + Q\,\hat{k}'_E \\ &= R\,\hat{i}'_{oS} + S\,\hat{j}'_{oS} + T\,\hat{k}'_{oS} \\ &= R\,\hat{i}_o + S\,\hat{j}_o + T\,\hat{k}_o \end{aligned} \qquad \text{A-23}$$

where:

$$\begin{aligned} R = {} & P(A_E(A'_L \cos \Omega t + D'_L \sin \Omega t) + B_E(-A'_L \sin \Omega t + D'_L \cos \Omega t) \\ & + C_E G'_L) \\ & + Q(G_E(A'_L \cos \Omega t + D'_L \sin \Omega t) + H_E(-A'_L \sin \Omega t + D'_L \cos \Omega t) \\ & + J_E G'_L) \end{aligned}$$

$$\begin{aligned} S = {} & P(A_E(B'_L \cos \Omega t + E'_L \sin \Omega t) + B_E(-B'_L \sin \Omega t + E'_L \cos \Omega t) \\ & + C_E A'_L) \\ & + Q(G_E(B'_L \cos \Omega t + E'_L \sin \Omega t) + H_E(-B'_L \sin \Omega t + E'_L \cos \Omega t) \\ & + J_E H'_L) \end{aligned} \qquad \text{A-24}$$

$$T = P(A_E(C'_L \cos \Omega t + F'_L \sin \Omega t) + B_E(-C'_L \sin \Omega t + F'_L \cos \Omega t) + C_E J'_L)$$

$$+ Q(G_E(C'_L \cos \Omega t + F'_L \sin \Omega t) + H_E(-C'_L \sin \Omega t + F'_L \cos \Omega t) + J_E J'_L)$$

and:

$$\begin{aligned}
A_E &= \cos \theta_E \cos \phi_E \cos \psi_E + \sin \theta_E \sin \psi_E \\
B_E &= \sin \theta_E \cos \phi_E \cos \psi_E - \cos \theta_E \sin \psi_E \\
C_E &= -\sin \phi_E \cos \psi_E \\
G_E &= \cos \theta_E \sin \phi_E \\
H_E &= \sin \theta_E \sin \phi_E \\
J_E &= \cos \phi_E
\end{aligned} \qquad \text{A-25}$$

A'_L - - - J'_L have been defined in (13).

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