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ASD TECHNICAL REPORT 61-623

**ANALYTICAL FORMULATION OF DAMPED STRESS-STRAIN
RELATIONS BASED ON EXPERIMENTAL DATA WITH
APPLICATIONS TO VIBRATING STRUCTURES**

T. J. MENDEL
C. C. FU

UNIVERSITY OF MINNESOTA

NOVEMBER 1961



AERONAUTICAL SYSTEMS DIVISION

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UNIVERSITY OF MINNESOTA, Minneapolis, Minnesota. ANALYTICAL FORMULATIONS OF DAMPED STRESS-STRAIN RELATIONS BASED ON EXPERIMENTAL DATA WITH APPLICATIONS TO VIBRATING STRUCTURES, by T. J. Mentel and C. C. Fu, ^{NOVEMBER} 1961. 31p. incl. figs. (Project 7351; Task 73521) (ASD TR 61-7351; Contract No. AF 33(616)-6828) Unclassified report	UNCLASSIFIED	UNIVERSITY OF MINNESOTA, Minneapolis, Minnesota. ANALYTICAL FORMULATIONS OF DAMPED STRESS-STRAIN RELATIONS BASED ON EXPERIMENTAL DATA WITH APPLICATIONS TO VIBRATING STRUCTURES, by T. J. Mentel and C. C. Fu, ^{NOVEMBER} 1961. 31p. incl. figs. (Project 7351; Task 73521) (ASD TR 61-7351; Contract No. AF 33(616)-6828) Unclassified report	UNCLASSIFIED
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NOVEMBER 1961

DIRECTORATE OF MATERIALS AND PROCESSES
CONTRACT No. AF 33(616)-6828
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AERONAUTICAL SYSTEMS DIVISION
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE
WRIGHT-PATTERSON AIR FORCE BASE, OHIO

FOREWORD

This report was prepared by the Department of Aeronautics and Engineering Mechanics of the University of Minnesota under USAF Contract No. AF 33(616)-6828. This contract was initiated under Project No. 7351, "Metallic Materials," Task No. 73521, "Behavior of Metals." The work was administered under the direction of the Metals and Ceramics Laboratory, Directorate of Materials and Processes, Deputy for Technology, Aeronautical Systems Division, with Mr. D. M. Forney, Jr. acting as project engineer.

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ABSTRACT

A technique is presented for the constructions of stress-strain relations based on experimental, cyclic, damping data. The extension of this technique to the biaxial stress case is then shown followed by an example application involving flexural vibrations of a cantilever.

PUBLICATION REVIEW

This report has been reviewed and is approved.

FOR THE COMMANDER:

A handwritten signature in dark ink, appearing to read "W. J. Trapp", with a stylized, flowing script.

W. J. TRAPP
Chief, Strength and Dynamics Branch
Metals and Ceramics Laboratory
Directorate of Materials and Processes

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I. INTRODUCTION

The purpose of this paper is to develop and demonstrate a simple mathematical formulation of one and two dimensional, damped stress-strain relations based on a special reduction of experimental data. The motivation for this work, which might be considered as an intermediate approach, stems from the marked complexity and large number (at least a dozen) of separate physical phenomena which have been discovered as being integrant to material damping.(1)* The implication is that a rigorous approach, which perforce must reflect such involvement, could only be speculative with the present limited experimental data. In spite of the physical complexity of material damping, however, certain systematizations of the experimental data have been achieved, and it is for such a development that the present paper constructs a mathematical theory.

The simplest mathematical formulation for energy dissipation which can be used to describe structural damping is that which corresponds to the linear dashpot. Its plot on stress-strain coordinates, for sinusoidal straining action, produces smooth contoured hysteresis loops as shown in Figure 1a. If this formulation is modified by dividing the damping coefficient by the frequency (implying restriction of attention to discrete frequencies), then this removes the linear frequency dependence of the energy dissipation per cycle. This is usually termed "hysteretic damping," and is one of the most prominent characteristics of material damping. Thus for the uniaxial case, the stress-strain relation for hysteretic damping becomes

$$\sigma = E \epsilon + \frac{C}{\omega} \dot{\epsilon} \quad (1-1)$$

where σ and ϵ are respectively, the stress and strain, E is Young's modulus, C is the damping coefficient, ω is the frequency of the sinusoidal straining action and $\dot{\epsilon}$ is the strain rate.

Although it turns out that the foregoing formulation, which produces elliptical hysteresis loops, can produce excellent results in most vibration analysis problems, the experimentally obtained loops for metals have sharp corners at both ends, and

*Number in parentheses refer to a bibliography on page 26.

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furthermore, are practically linear in the central, or low stress-strain region. This suggests that the next stage of complexity in the analytical development might be an attempt at a direct, geometric description of the hysteresis loops incorporating the aforementioned characteristics. This has been done in various ways. A formulation, due to Davidenkov (2), which brings in the characteristic sharp corners as shown in Figure 1b, and which also incorporates certain dependence upon strain amplitude, is represented by the following equations:

$$\begin{aligned}\bar{\sigma} &= E \left\{ \epsilon - \frac{\mu}{m} \left[(\epsilon_0 + \epsilon)^m - 2^{m-1} \epsilon_0^m \right] \right\} \\ \bar{\sigma} &= E \left\{ \epsilon + \frac{\mu}{m} \left[(\epsilon_0 - \epsilon)^m - 2^{m-1} \epsilon_0^m \right] \right\} \quad (1-2)\end{aligned}$$

The two equations, one each for the upper and lower curves of the hysteresis loop, are now a necessary complication. The constants μ and m are left to be evaluated experimentally for the material under consideration, and ϵ_0 is the strain amplitude. This particular description of material damping has been applied to a number of problems by Pisarenko (3). The associated mathematical problem, of course, is nonlinear, but his results show that a first approximation only, derived by the use of asymptotic expansions, gives a fairly accurate result.

Another example of the foregoing technique of geometric description of hysteresis loops is that developed by Rang (4). In this case a cubic polynomial is used to describe each of the two sides of the loop, which produces sharp corners, with the constants being determined by assigning values to certain characteristic tangents of the loop and the locus of the corner points. Both of the foregoing formulations allow close approximation of experimentally observed physical phenomena, but at the expense of making subsequent mathematical analysis quite difficult. For the Rang case, an example solution is given only for a discrete, one degree of freedom, spring-mass model. Furthermore, both of the foregoing analyses are developed strictly for the uniaxial stress case and no attempt appears to have been made at incorporating them, in any systematic way, with the biaxial stress case or with other phenomena such as hereditary effects, etc. All descriptions, however, can be modified to include various types of dependence on strain amplitude.

Thus, at present, there is not only the penalty of mathematical complication in employing relatively precise descriptions of the hysteresis loop, but the added disadvantage that no suitable collection of data is available for a wide variety of materials.

In fact, the only general technique which has been shown up to now, for systematizing such data, uses the area of the hysteresis loop, plus certain fatigue properties, with no reference to loop shape. In other words, it does not contain information with respect to instantaneous rates of energy dissipation, which would therefore have to be added in some way if the more precise descriptions were to be applied. Besides this problem of expanding the collected data, there occurs the still additional problem of interpreting equivalent damping conditions under uniaxial and biaxial stress conditions. The precise phenomena involved in this connection are not fully understood at the present time. The only relevant theory for associating data obtained using uniaxial stress to the biaxial stress case is that due to Robertson and Yorgiadis (5), who define equivalent states (of material damping) as those which produce the same distortion energy, and that due to Mentel (6), who suggests that in such cases as plate vibration, some fractional part of the dilatational strain energy ought to be included in the foregoing equivalence. The latter formulation can be expressed analytically

$$\sigma_n = \sqrt{3 \tau_m^2 + \Lambda \sigma_d^2} \quad (1-3)$$

where σ_n is the maximum (normal) stress in the uniaxial case and τ_m and σ_d are the maximum shear and the "mean normal" or dilatational components of stress in the equivalent biaxial case. The development of this equation is worth special notice. Consider the two states of stress as shown in Figure 2. If we observe that the principal stresses in the biaxial stress state can be designated by

$$\sigma_1 = \sigma_d + \tau_m$$

$$\sigma_2 = \sigma_d - \tau_m$$

then the expressions for the distortional and dilatational strain energies take on the following form:

for the uniaxial stress state,

$$U_{\text{distortion}} = \frac{1 + \nu}{3E} \sigma_n^2$$

$$U_{\text{dilatation}} = \frac{1-2\nu}{6E} \sigma^2$$

for the biaxial stress state,

$$U_{\text{distortion}} = \frac{1+\nu}{3E} (3\tau_m^2 + \sigma_d^2)$$

$$U_{\text{dilatation}} = \frac{1-2\nu}{3E} 2\sigma_d^2$$

Thus, if we choose to ignore the mean normal stress σ_d , say by setting $\sigma_d = 0$, and then equate the distortional strain energies in the two cases, we obtain

$$\frac{1+\nu}{3E} \sigma_n^2 = \frac{1+\nu}{E} \tau_m^2$$

or

$$\sigma_n = \sqrt{3\tau_m^2}$$

(1-4)

which is the Robertson-Yorgiadis criterion. On the other hand, if we retain all quantities and equate the total strain energies in the two cases, we find the equivalent expression to be

$$\sigma_n = \sqrt{2(1+\nu)\tau_m^2 + 2(1-\nu)\sigma_d^2} \quad (1-5)$$

If we now set $\nu = 1/3$, then (1-5) becomes

$$\sigma_n = \sqrt{\frac{8}{3}\tau_m^2 + \frac{4}{3}\sigma_d^2} \quad (1-6)$$

Alternatively, if regard the damping phenomenon as being associated primarily with a "plastic type" straining process wherein $\nu = \frac{1}{2}$, we obtain

$$\sigma_n = \sqrt{3 \tau_m^2 + \sigma_d^2} \quad (1-7)$$

The coefficient of τ_m^2 , as given by (1-6) and (1-7), is thus observed to be at most only moderately altered from its value in (1-4). Hence, in order to allow for a variable damping influence between distortional and dilatational straining, the form given by (1-3) is suggested as a simple, one parameter description for empirical reduction of experimental data. Although the uniaxial to biaxial damping equivalence given by (1-3) carries the implication that the shear stress criterion, as expressed by (1-4), gives the lower bound for material damping, it should be observed that no argument is given to exclude negative values of Λ which means, for example, that a nullifying type of interaction could be described. Note that the first stress invariant I_1 , for the biaxial case, is given by

$$I_1 = \sum \sigma_{ij} = 2 \sigma_d$$

so that (1-3) can be rewritten

$$\sigma_n = \sqrt{3 \tau_m^2 + \frac{1}{4} \Lambda I_1^2} \quad (1-8)$$

This reduction can be carried one step further by introducing the second stress invariant

$$I_2 = \sigma_1 \sigma_2 = \sigma_d^2 - \tau_m^2$$

which gives

$$\tau_m^2 = \frac{1}{4} I_1^2 - I_2$$

so that (1-8) can be further modified to

$$\sigma_n = \sqrt{\frac{1}{4} (3 + \Lambda) I_1^2 - 3 I_2} \quad (1-9)$$

It is observed, however, that I_2 introduces no new functional forms beyond those contained in (1-8), so that a simple, one parameter description, based on (1-8), is

$$\sigma_n = \sqrt{3 \tau_m^2 + \Lambda^* I_1^2} \quad (1-10)$$

which is the expression used in this paper. A two parameter description, based on (1-9), would be

$$\sigma_n = \sqrt{\Lambda_1 I_1^2 + \Lambda_2 I_2} \quad (1-11)$$

This latter formulation might be the most promising for the tri-axial stress case but this problem is left for future studies.

In view of the foregoing unclear situation concerning material damping under biaxial stress, it was concluded that a useful interim step for theoretical response work, would be to construct a mathematical theory which was counterpart only to the degree of systematization thus far shown for experimental data. This implies neglect of loop shape, so that the simplest mathematical shape, viz., the ellipse, might as well be taken. The chief effect of this simplification, whose primary geometric characteristic is the elimination of the sharp corners at the loop tips, might be expected to be significant only at the highest modes and frequencies of vibration. No such effect, however, is demonstrated since the analysis is carried out only for vibrations in the neighborhood of the fundamental resonance. In this case, when the loops correspond to structural metals at stress amplitudes below yield, the area only and not the loop shape appears to be significant [3]. Further discussion of this question is included in subsequent sections of this paper.

The one parameter equivalence relation between uniaxial and biaxial stress states, as expressed by (1-10) and the experimental uniaxial damping data are then used as a basis for generating the stress strain relations for the biaxial case. These relations permit the formulation of the vibration response problem for plates exhibiting material damping. Example solutions for the case of a circular plate are given in [7].

II. ANALYTICAL FORMULATION OF MATERIAL DAMPING

2.1 General Problem

It was observed in the foregoing section, that at least a dozen individual phenomena (e.g., magnetostriction, intercrystal thermal currents, interstitial solute atoms, etc.) have been identified as being significant in the production of material damping. Any widely applicable reduction of damping data can therefore be expected to be either extremely elaborate, which singles out the aforementioned phenomena and their associated parameters, or extremely simple, which suppresses them. At the present time, only the latter approach has led to significant success based on a broad assimilation of data. This development has been due to Lazan who has shown, empirically, that the fatigue strength of material can be used as an important parameter in the reduction of experimental damping data (see Figure 2.10 of Ref. (8)). This parameter is the ratio of the cyclic stress S_0 (in mathematical notation, the stress amplitude σ_0) to the fatigue strength S_E , corresponding to 2×10^7 cycles, both quantities corresponding to uniaxial, sinusoidal, constant amplitude and fixed frequency conditions. The experimental results show, that for the given selection of structural materials, the energy dissipation is essentially independent of both the frequency of cycling and the stress history when the stress amplitude is kept below a certain critical value. This observation confirms the common notion of hysteretic damping, which serves adequately in many simplified analysis of vibration response.

The aforementioned critical value of stress amplitude is called the cyclic stress sensitivity limit S_L . Both the frequency of cycling and stress history develop marked influence for cyclic stressing above this stress limit. An empirical determination of this critical value, due to Lazan (1), is 80 percent of the fatigue strength corresponding to 2×10^7 cycles. Fortunately, from the analytical point of view, this value is relatively high compared to stress levels nominally encountered in structures subjected to long term dynamic loading.

The conclusion derived from the foregoing sections is that at stress amplitudes below the cyclic stress sensitivity limit, the cyclic material damping is adequately expressed merely as a function of the stress amplitude S_0 . Such a description has been found by Lazan (8) to be

$$D_0 = J S_0^\alpha \quad (2.1-1)$$

where J and α are adjusted for the given material, and D_0 is the cyclic energy dissipation per unit volume. The range of values taken by the parameters J and α is relatively small (no changes of order of magnitude) for a large selection of structural materials. The total cyclic energy dissipation D_m , in a given volume of material is then given by

$$D_m = \int_{\text{Volume}} D_0 dV \quad (2.1-2)$$

For reasons which will be apparent later, the alternative expression

$$D_0 = \sum_n J_n S_0^n \quad (2.1-3)$$

where n is restricted to being an integer, will be introduced as a replacement of (2.1-1).

2.2 Uniaxial Stress

We have already introduced the stress strain relations of Davidenkov as equations (1-2) in section 1. If these equations are integrated over a complete cycle in order to obtain the net loop area, and hence the cyclic energy dissipation D_0 , we find that

$$D_0 = \int_{-\epsilon_0}^{+\epsilon_0} \frac{\sigma}{\epsilon} d\epsilon + \int_{+\epsilon_0}^{-\epsilon_0} \frac{\sigma}{\epsilon} d\epsilon = \frac{2^{m+1} (m-1) \mu E}{m(m+1)} \epsilon_0^{m+1}. \quad (2.2-1)$$

We now note that

$$S_0 \equiv \sigma_0 = E \epsilon_0 \left\{ 1 - \frac{\mu}{m} (2 \epsilon_0)^{m-1} \right\},$$

so that if we assume small strains (and small nonlinearity) in the sense that

$$\frac{\mu}{m} (2 \epsilon_0)^{m-1} \ll 1$$

we can rewrite equation (2.2-1) in the form

$$D_0 = \frac{2^{m+1} (m-1) \mu}{m(m+1) E^m} S_0^{m+1}. \quad (2.2-2)$$

Equation (2.2-2) is now in the same form as the empirical expression (2.1-1) and we obtain an identity if we choose

$$m = \alpha - 1$$

$$\text{and } \mu = \frac{\alpha(\alpha-1)E^{\alpha-1}}{2^{\alpha}(\alpha-2)}J \quad (2.2-3)$$

The fact of being able to demonstrate the foregoing equivalence lends additional validity both to the experimentally derived equation (2.1-1) and the Davidenkov equations (1-2). However, the usefulness of the Davidenkov equations is limited by the complication of having to keep track of the proper branch of the hysteresis loop for different material points in addition to having mathematical nonlinearity. Furthermore, the value of α in (2.1-1) lies between 2 and 3 for most structural metals with the value being nearer 2 for most steels (8) ($\alpha = 2$ for 1020 steel, $\alpha = 2.3$ for Sandvik steel Q-T). This corresponds to $m = 1$ in (1-2), for which value these equations collapse to

$$\sigma = E(1 - \mu)\epsilon$$

which describes simply the undamped elastic case. Thus, for integral values of α , equations (1-2) are useful only for the extreme case of $\alpha = 3$ if material damping is to be included.

The approach to the development of damped stress-strain relations proposed in this paper, is to use strain rate as the device to introduce elliptical hysteresis loops (in the manner of complex damping [9, 10]). The advantage of this approach is that it allows direct and simple usage of the full hysteresis loop. The stress-strain relation can thus be written in the form

$$\sigma = E(\epsilon + C_1\epsilon^{\beta} + C_2\dot{\epsilon}) \quad (2.2-4)$$

where C_1 and β account for elastic nonlinearity and C_2 accounts for the damping. The available experimental data, however, deals primarily with structural metals, where, in the range of stress levels at which (2.1-1) is applicable, the elastic component is strictly linear. Hence, in place of (2.2-4), we are able to use the simpler relation

$$\sigma = E(\epsilon + C\dot{\epsilon}) \quad (2.2-5)$$

where the damping coefficient C is expected to be a function of the strain amplitude, frequency and the elastic constants.

Integrating (2.2-5) for the cyclic damping, we obtain

$$D_0 = E C \int_{\text{Cycle}} \dot{\epsilon} d\epsilon. \quad (2.2-6)$$

If the straining action is now assumed to have the sinusoidal variation

$$\epsilon = \epsilon_0 \cos \omega_s t \quad (\epsilon_0 \geq 0)$$

the stress strain relation gives

$$\sigma = E \epsilon_0 (\cos \omega_s t - C \omega_s \sin \omega_s t),$$

so that

$$S_0 = E \epsilon_0 (1 + C^2 \omega_s^2)^{\frac{1}{2}} \quad (2.2-7)$$

the energy dissipation, as expressed by (2.2-6), then becomes

$$D_0 = E C \int_0^{\frac{2\pi}{\omega_s}} \dot{\epsilon}^2 dt = \pi E C \omega_s \epsilon_0^2 \quad (2.2-8)$$

substitution of (2.2-7) into (2.2-1) gives

$$D_0 = J E^\alpha \epsilon_0^\alpha (1 + C^2 \omega_s^2)^{\frac{1}{2}} \quad (2.2-9)$$

which can be compared with (2.2-8). Requiring the equivalence of (2.2-8) with (2.2-9) thus gives

$$\pi C \omega_s = J E^{\alpha-1} \epsilon_0^{\alpha-2} (1 + C^2 \omega_s^2)^{\frac{\alpha}{2}}. \quad (2.2-10)$$

Equation (2.2-10) identifies the functional dependence required of the damping parameter C . This expression can be simplified if we observe that, for most structural applications, we can expect $C^2 \omega_s^2 \ll 1$. The resulting expression for C is

$$C = \frac{J}{\pi \omega_s} E^{\alpha-1} \epsilon_0^{\alpha-2} \quad (2.2-11)$$

Recalling that for structural metals α usually has some non-integral value between 2 and 3, the alternative expression (2.1-3) can be employed to retain the simplicity of being able to stay with integers. In particular, we can take

$$D_0 = J_2 S_0^2 + J_4 S_0^4 \quad (2.2-12)$$

which provides an excellent approximation of (2.1-1) for any α in the aforementioned range. The reason for stepping up to the fourth power in (2.2-12) is to obviate any necessity of having to keep track of the sign of S_0 in subsequent applications. The corresponding expression for the damping coefficient C is

$$C = \frac{E}{\pi \omega_s} (J_2 + E^2 J_4 \epsilon_0^2) \quad (2.2-13)$$

where again the condition $C^2 \omega_s^2 \ll 1$ has been applied. Note that, for the case of $\alpha = 2$ in the prior formulation (2.2-11), we obtain

$$C = \frac{E}{\pi \omega_s} J_2 \quad (2.2-14)$$

which is the first term of (2.2-13).

We thus conclude, that for the one dimensional case, a modified stress strain relation based on an elliptical hysteresis loop, can be readily constructed such that the resulting energy dissipation is the same as that given by the empirical relation (2.1-1), or its substitute, (2.2-12). This construction is considerably simplified by the restriction $C^2 \omega_s^2 \ll 1$, which, in physical terms, is equivalent to saying that the stress amplitude is determined substantially by the elastic component (of strain) only.

2.3 Biaxial Stress

The method of extending the stress-strain relation (2.2-5) to the biaxial case is not obvious. The reasons behind this were noted in the introduction. The most general uniaxial to biaxial stress state equivalence presently anticipated, as expressed by

(1-3), is therefore chosen for analysis. Using (1-3) and (2.1-1), the cyclic damping for biaxial stress becomes

$$D_0 = J (3 \tau^2 + \Lambda^* I_1^2)^{\frac{\alpha}{2}} \quad (2.3-1)$$

while using (1-3) and (2.2-11), we obtain

$$D_0 = \sum_n J_n (3 \tau^2 + \Lambda^* I_1^2)^{\frac{n}{2}} \quad (2.3-2)$$

The criterion that the stress amplitudes remain sufficiently small so that no effects of frequency or stress history might enter is now interpreted as

$$\tau + \sigma_d \leq S_L \quad (2.3-3)$$

Either of the formulations (2.3-1) or (2.3-2) enable us to generalize the stress-strain relation (2.2-5) to the biaxial stress case. Suppose we have a plane stress problem. Consider the following construction for the stress-strain relations written in principal coordinates

$$\begin{aligned} \sigma_x &= \frac{E}{1-\nu^2} (\epsilon_x + \nu \epsilon_y) + E^* (\dot{\epsilon}_x + \nu^* \dot{\epsilon}_y), \\ \sigma_y &= \frac{E}{1-\nu^2} (\epsilon_y + \nu \epsilon_x) + E^* (\dot{\epsilon}_y + \nu^* \dot{\epsilon}_x). \end{aligned} \quad (2.3-4)$$

The motivation behind the construction is clear: the first terms of (2.3-4) are the usual relations of elasticity theory, and the second terms are modeled on the first, with constants E^* and ν^* providing the analogous extension of constant C in (2.2-5). Again, by introducing a harmonic straining action described by

$$\begin{aligned} \epsilon_x &= \epsilon_{0x} \cos \omega_s t, \\ \epsilon_y &= \epsilon_{0y} \cos \omega_s t, \end{aligned} \quad (2.3-5)$$

we can integrate (2.3-4) for the cyclic energy dissipation. The result is

$$\begin{aligned} D_0 &= E^* \int_0^{\frac{2\pi}{\omega_s}} (\dot{\epsilon}_x^2 + 2\nu^* \dot{\epsilon}_x \dot{\epsilon}_y + \dot{\epsilon}_y^2) dt \\ &= \pi E^* \omega_s (\epsilon_{ox}^2 + 2\nu^* \epsilon_{ox} \epsilon_{oy} + \epsilon_{oy}^2) \quad (2.3-6) \end{aligned}$$

which expresses the energy dissipation in terms of the amplitudes of the principal strains.

We proceed next to construct the alternative expression for D_0 by solving for the equivalent normal (or uniaxial) stress and substituting this into the empirically based expression (2.3-1). Thus the stress-strain relations may be written

$$\sigma_x = \frac{E}{1-\nu^2} (\epsilon_{ox} + \nu \epsilon_{oy}) \cos \omega_s t + E^* \omega_s (\epsilon_{ox} + \nu^* \epsilon_{oy}) \sin \omega_s t,$$

$$\sigma_y = \frac{E}{1-\nu^2} (\epsilon_{oy} + \nu \epsilon_{ox}) \cos \omega_s t + E^* \omega_s (\epsilon_{oy} + \nu^* \epsilon_{ox}) \sin \omega_s t.$$

By subtracting and adding the two foregoing equations, we obtain

$$\sigma_x - \sigma_y = \frac{E}{1+\nu} (\epsilon_{ox} - \epsilon_{oy}) \cos \omega_s t + E^* \omega_s (1-\nu^*) (\epsilon_{ox} - \epsilon_{oy}) \sin \omega_s t,$$

$$\sigma_x + \sigma_y = \frac{E}{1-\nu} (\epsilon_{ox} + \epsilon_{oy}) \cos \omega_s t + E^* \omega_s (1+\nu^*) (\epsilon_{ox} + \epsilon_{oy}) \sin \omega_s t,$$

from which we can identify the maximum shear stress and dilational components to be

$$\tau^2 = \frac{1}{4} \left\{ \left(\frac{E}{1+\nu} \right)^2 + (E^* \omega_s)^2 (1-\nu^*)^2 \right\} (\epsilon_{ox} - \epsilon_{oy})^2,$$

$$I_1^2 = \left\{ \left(\frac{E}{1-\nu} \right)^2 + (E^* \omega_s)^2 (1+\nu^*)^2 \right\} (\epsilon_{ox} + \epsilon_{oy})^2.$$

Substituting these results into (2.3-1) finally gives

$$D_0 = J \left\{ \frac{3}{4} \left[\left(\frac{E}{1+\nu} \right)^2 + (E^* \omega_s)^2 (1-\nu^*)^2 \right] (\epsilon_{0x} - \epsilon_{0y})^2 \right. \\ \left. + \Lambda^* \left[\left(\frac{E}{1-\nu} \right)^2 + (E^* \omega_s)^2 (1+\nu^*)^2 \right] (\epsilon_{0x} + \epsilon_{0y})^2 \right\}^{\frac{\alpha}{2}}. \quad (2.3-7)$$

The prior expression, given by (2.3-6) is now compared with (2.3-7), from which we obtain the condition that

$$J \left\{ \frac{3}{4} \left[\left(\frac{E}{1+\nu} \right)^2 + (E^* \omega_s)^2 (1-\nu^*)^2 \right] (\epsilon_{0x} - \epsilon_{0y})^2 \right. \\ \left. + \Lambda^* \left[\left(\frac{E}{1-\nu} \right)^2 + (E^* \omega_s)^2 (1+\nu^*)^2 \right] (\epsilon_{0x} + \epsilon_{0y})^2 \right\} = \\ = \pi E^* \omega_s (\epsilon_{0x}^2 + 2 \nu^* \epsilon_{0x} \epsilon_{0y} + \epsilon_{0y}^2). \quad (2.3-8)$$

A difficulty which now arises is that the foregoing expression (2.3-8) is the only condition which emerges from this development, whereas there are now two, as yet unspecified damping parameters E^* and ν^* , in addition to the special parameter

Λ^* . Equation (2.3-8) therefore merely provides a constraint between the three quantities E^* , ν^* and Λ^* . In spite of this indeterminacy, several useful, specific formulations are possible provided we accept certain restrictive assumptions.

Let us consider the special case where $\alpha = 2$. Equation (2.3-8) is then satisfied if we choose

$$\pi E^* \omega_s = J \left\{ \Lambda^* \left[\left(\frac{E}{1-\nu} \right)^2 + (E^* \omega_s)^2 (1+\nu^*)^2 \right] + \frac{3}{4} \left[\left(\frac{E}{1+\nu} \right)^2 + (E^* \omega_s)^2 (1-\nu^*)^2 \right] \right\}, \\ \pi E^* \omega_s \nu^* = J \left\{ \Lambda^* \left[\left(\frac{E}{1-\nu} \right)^2 + (E^* \omega_s)^2 (1+\nu^*)^2 \right] - \frac{3}{4} \left[\left(\frac{E}{1+\nu} \right)^2 + (E^* \omega_s)^2 (1-\nu^*)^2 \right] \right\},$$

If we now apply the restriction that

$$\frac{E^* \omega_s (1+\nu)(1-\nu^*)}{E} \ll 1 \quad (2.3-9)$$

and

$$\frac{E^* \omega_s (1-\nu)(1+\nu^*)}{E} \ll 1$$

the foregoing equations simplify to the following:

$$E^* = \frac{J E^2}{\pi \omega} \left\{ \frac{\Lambda^*}{(1-\nu)^2} + \frac{3}{4(1+\nu)^2} \right\}, \quad (2.3-10)$$

$$\nu^* = \frac{4 \Lambda^* (1+\nu)^2 - 3(1-\nu)^2}{4 \Lambda^* (1+\nu)^2 + 3(1-\nu)^2}.$$

It is tempting, at this stage, to investigate the condition $\nu = \nu^*$, which is strongly suggested by the way in which the original equations (2.3-4) were formulated. This condition gives

$$E^* = \frac{3 J E^2}{2 \pi \omega (1+\nu)(1-\nu^2)} \quad (2.3-11)$$

$$\Lambda^* = \frac{3(1-\nu)}{4(1+\nu)},$$

where the expression for Λ^* is particularly interesting if we recall equations (1-7) and (1-10). This indicated value of Λ^* is 1/4 for the limiting case of Poisson's ratio of 1/2, which checks with the aforementioned equations, in particular, with the conditions set down for (1-7). On the other hand, for Poisson's ratio equal to 1/4, the indicated value of Λ^* is 3/8, which, although an interesting result, must be treated as merely the consequence of an expedient postulate.

An interesting result may also be obtained for the more general case where $2 < \alpha \leq 3$, which includes the entire range of experimental values applicable to structural metals at low stress levels. In this case, in order to avoid dealing with fractional values of α , we turn to the expression given by (2.2-12). This two-term approximation gives

$$D_0 = J_2 Q + J_4 Q^2 \quad (2.3-12)$$

where

$$Q = \left\{ \frac{3}{4} \left[\left(\frac{E}{1+\nu} \right)^2 + (E^* \omega_s)^2 (1-\nu^*)^2 \right] (\epsilon_{0x} - \epsilon_{0y})^2 \right. \\ \left. + \Lambda^* \left[\left(\frac{E}{1-\nu} \right)^2 + (E^* \omega_s)^2 (1+\nu^*)^2 \right] (\epsilon_{0x} + \epsilon_{0y})^2 \right\}$$

Equating (2.3-6) with (2.3-12), and applying the restrictions given by (2.3-9), we obtain

$$\begin{aligned} & \left\{ \frac{3}{4} \left(\frac{E}{1+\nu} \right)^2 (\epsilon_{0x} - \epsilon_{0y})^2 + \Lambda^* \left(\frac{E}{1-\nu} \right)^2 (\epsilon_{0x} + \epsilon_{0y})^2 \right\} \\ & \left\{ J_2 + J_4 \left[\frac{3}{4} \left(\frac{E}{1+\nu} \right)^2 (\epsilon_{0x} - \epsilon_{0y})^2 + \Lambda^* \left(\frac{E}{1-\nu} \right)^2 (\epsilon_{0x} + \epsilon_{0y})^2 \right] \right\} \\ & = \pi E^* \omega_s (\epsilon_{0x}^2 + 2\nu^* \epsilon_{0x} \epsilon_{0y} + \epsilon_{0y}^2) \end{aligned}$$

(2.3-13)

Equation (2.3-13) is satisfied, if, in particular, we choose

$$\begin{aligned} E^* &= \frac{E}{\pi \omega_s} \left\{ J_2^* + J_4^* (\epsilon_{0x}^2 + 2\nu^* \epsilon_{0x} \epsilon_{0y} + \epsilon_{0y}^2) \right\}, \\ \nu^* &= \frac{4\Lambda^*(1+\nu)^2 - 3(1-\nu)^2}{4\Lambda^*(1+\nu)^2 + 3(1-\nu)^2} \end{aligned} \quad (2.3-14)$$

where

$$\begin{aligned} J_2^* &= E J_2 \left\{ \frac{\Lambda^*}{(1-\nu)^2} + \frac{3}{4(1+\nu)^2} \right\} \\ J_4^* &= E J_4 \left\{ \frac{J_2^*}{J_2} \right\}. \end{aligned}$$

Once again, if we look at the special case where $\nu = \nu^*$, we find

$$\begin{aligned} E^* &= \frac{3E^2}{2\omega_s(1-\nu)(1+\nu)^2} \left\{ J_2 + \frac{3J_4E^2}{2(1-\nu)(1+\nu)^2} (\epsilon_{0x}^2 + 2\nu\epsilon_{0x}\epsilon_{0y} + \epsilon_{0y}^2) \right\} \\ \Lambda^* &= \frac{3(1-\nu)}{4(1+\nu)}. \end{aligned}$$

It should be noted that the strain amplitudes ϵ_{ox} and ϵ_{oy} are not necessarily positive quantities, although the sign of one determines the sign of the other on account of (2.3-5). However, (2.3-6), which gives the dissipation, is quadratic in the strains, so that the sign is unimportant in this case.

III. VIBRATING BEAM EXAMPLE

3.1 General Considerations

We consider the specific problem of a cantilever beam of constant cross section which is excited in its fundamental flexural mode by a forced periodic rotation of its support. This problem has already been studied by Pisarenko who used Davidenkov's equations (1-2) to describe the beam material. An additional study of this same problem, but using the stress-strain relations proposed herein, is thus provided with an interesting comparison. For example, it can be argued that the stress-strain relations developed in the foregoing section have such complication that one might as well drop any pretense of approximate formulations and work from a much more general development. It is one of the objectives of this section to throw more light on this question.

Pisarenko's analysis, which appears in Chapter Two of [2], uses asymptotic expansions to generate successively higher ordered approximations of the solution to the steady state problem. He finds, however, that only the first order approximation is necessary to produce acceptable accuracy. It is assumed that the same situation holds true when the new stress-strain relations are used in place of Davidenkov's. A comparison of results is then possible using first order approximations only.

3.2 Equations of Motion

The basic geometry and coordinate system for describing the vibrating, deformed beam is shown in Figure 3, with the following notation:

x = horizontal coordinate axis coinciding with the undeformed, unrotated centerline of the beam

$\bar{y}$ = total deflection of the centerline from the x -axis due to rotation and deformation

θ = small angle of rotation of the clamped end of the beam

$y = \bar{y} - \theta x$, the deflection of the centerline due to deformation from the rigidly rotated position.

This description is based on the concept of the elastic line whose properties are derived for a rectangular cross section of width b and height h as shown in Figure 4.

The appropriate stress-strain relation, as developed in Section 2, would be based on equations (2.2-4), (2.2-12) and (2.2-13) which give

$$\sigma = E \left\{ \epsilon + \frac{E}{\pi \omega_s} (J_2 + E^2 J_4 \epsilon_0^2) \dot{\epsilon} \right\} \quad (3.2-1)$$

However, it is instructive to base the stress-strain relation more directly on the Davidenkov description. Thus if we determine the constants in (2.2-12) by using (2.2-3) where we arbitrarily choose $m = 2$, the corresponding stress-strain relation becomes

$$\bar{\sigma} = E \left\{ \epsilon + \frac{4\mu}{3\pi\omega_s} \epsilon_0 \dot{\epsilon} \right\}. \quad (3.2-2)$$

Using the expression for infinitesimal strains (Figure 3)

$$\epsilon = z \frac{\partial^2 y}{\partial x^2},$$

(3.2-1) and (3.2-2) become respectively

$$\sigma = E z \left\{ \frac{\partial^2 y}{\partial x^2} + \frac{E}{\pi \omega_s} (J_2 + E^2 J_4 \epsilon_0^2) \frac{\partial^3 y}{\partial t \partial x^2} \right\}, \quad (3.2-3)$$

and

$$\bar{\sigma} = E z \left\{ \frac{\partial^2 y}{\partial x^2} + \frac{4\mu}{3\pi\omega_s} \epsilon_0 \frac{\partial^3 y}{\partial t \partial x^2} \right\}. \quad (3.2-4)$$

The bending moments, M and $\bar{M}$ for the two cases, at any cross section are then given by

$$M = EI \left\{ 1 + \frac{E J_2}{\pi \omega_s} \frac{\partial}{\partial t} \right\} \frac{\partial^2 y}{\partial x^2} + \frac{2 b J_4 E^4}{\pi \omega_s} \int_0^{\frac{h}{2}} \epsilon_0^2 \frac{\partial^3 y}{\partial t \partial x^2} z^2 dz, \quad (3.2-5)$$

$$\bar{M} = EI \frac{\partial^2 y}{\partial x^2} + \frac{8 \mu b}{\pi \omega_s} \int_0^{\frac{h}{2}} \epsilon_0 \frac{\partial^3 y}{\partial t \partial x^2} z^2 dz. \quad (3.2-6)$$

On neglecting the effects of shear deformation and rotatory inertia, the equations of motion are

$$\bar{y} = y + \theta x ,$$

$$\rho \frac{\partial^2 \bar{y}}{\partial t^2} + \frac{\partial^2 M}{\partial x^2} = 0 . \quad (3.2-7)$$

Substituting (3.2-5) and (3.2-6) into (3.2-7) gives respectively

$$\begin{aligned} \rho \left\{ \frac{\partial^2 y}{\partial t^2} + x \frac{\partial^2 \theta}{\partial x^2} \right\} + EI \left\{ 1 + \frac{E J_2}{\pi \omega_s} \frac{\partial}{\partial t} \right\} \frac{\partial^4 y}{\partial x^4} + \\ + \frac{2 b J_4 E^4}{\pi \omega_s} \frac{\partial^2}{\partial x^2} \int_0^{\frac{h}{2}} \epsilon_0^2 \frac{\partial^3 y}{\partial t \partial x^2} z^2 dz = 0 , \end{aligned} \quad (3.2-8)$$

$$\begin{aligned} \rho \left\{ \frac{\partial^2 y}{\partial t^2} + x \frac{\partial^2 \theta}{\partial x^2} \right\} + EI \frac{\partial^4 y}{\partial x^4} + \\ + \frac{8 \mu b}{\pi \omega} \frac{\partial^2}{\partial x^2} \int_0^{\frac{h}{2}} \epsilon_0 \frac{\partial^3 y}{\partial t \partial x^2} z^2 dz = 0 , \end{aligned} \quad (3.2-9)$$

as the general equations of motion for the elastic line. The corresponding boundary conditions are

$$\begin{aligned} y = \frac{\partial y}{\partial x} = 0 \quad \text{at } x = 0 , \\ \frac{\partial^2 y}{\partial x^2} = \frac{\partial^3 y}{\partial x^3} = 0 \quad \text{at } x = l . \end{aligned} \quad (3.2-10)$$

The analyses of (3.2-8) and (3.2-9) are essentially parallel to each other: the former (3.2-8) is the one suggested for general application, while the latter (3.2-9) is for the specific purpose of making a comparison with Pisarenko's solution. The subsequent analysis is therefore restricted to (3.2-9) only.

We reformulate (3.2-9) and (3.2-10) by introducing the dimensionless quantities $x = l \zeta$, $z = h \eta$, and $y = \lambda l u$, where λ is an arbitrary, small parameter. Furthermore, on assuming that the damping effect and the excitation, characterized in this case by the change in the angle θ , are both small, we denote

$$\theta = \lambda^2 \theta_0 \cos \omega t ,$$

$$\epsilon_0(x, y) = \lambda \epsilon_0^*(\zeta, u) .$$

The equation of motion then becomes

$$EI \frac{\partial^4 u}{\partial \zeta^4} + \lambda \frac{8\mu b h^3}{\pi \omega_s} \frac{\partial^2}{\partial \zeta^2} \int_0^{\frac{1}{2}} \epsilon_0^* \frac{\partial^3 u}{\partial \tau \partial \zeta^2} \eta^2 d\eta +$$

$$+ \rho l^4 \frac{\partial^2 u}{\partial \tau^2} = \lambda l^4 \theta_0 \zeta \cos \omega \tau ,$$

(3.2-11)

with the boundary conditions

$$u = \frac{\partial u}{\partial \zeta} = 0 \quad \text{at } \zeta = 0 ,$$

$$\frac{\partial^2 u}{\partial \zeta^2} = \frac{\partial^3 u}{\partial \zeta^3} = 0 \quad \text{at } \zeta = 1 .$$

(3.2-12)

3.3 Method of Solution and Zeroth Approximation

The method of solution to be applied to the foregoing equations is similar to that proposed by Pisarenko. We assume that the excitation frequency ω , is in the neighborhood of one of the linear natural frequencies; say the fundamental natural frequency ω_0 , and introduce the dimensionless time

$$\tau = \omega \tau + \psi ,$$

where ψ is an arbitrary constant. Using the new variable τ , (3.2-11) can be rewritten

$$\left(\frac{\omega}{\omega_0}\right)^2 \frac{\partial^2 u}{\partial \tau^2} + \frac{EI}{\rho l^4 \omega_0^2} \frac{\partial^4 u}{\partial \zeta^4} + \lambda \gamma \left(\frac{\omega}{\omega_s}\right)^2 \frac{\partial^2}{\partial \zeta^2} \int_0^{\frac{1}{2}} \epsilon_0^* \frac{\partial^3 u}{\partial \tau \partial \zeta^2} \eta^2 d\eta$$

$$= \lambda \theta_0 \left(\frac{\omega}{\omega_0}\right)^2 \zeta \cos(\tau - \psi) , \quad (3.3-1)$$

where $\gamma = \frac{8\mu b h^3}{\pi \rho l^4} .$

We now assume a solution in the form of the expansions

$$u = u_0 + \lambda u_1 + \dots$$

$$\omega^2 = \omega_0^2 (1 + \lambda \omega_{01} + \dots)$$

$$\theta = \theta_0 (1 + \lambda \theta_{01} + \dots) ,$$

(3.3-2)

where

$$u_0 = \delta \phi(\zeta) \cos \tau. \quad (3.3-3)$$

On substituting (3.3-2) into (3.3-1), we obtain the following set of linear differential equations:

$$\frac{\partial^2 u_0}{\partial \tau^2} + \frac{EI}{\rho l^4 \omega_0^2} \frac{\partial^4 u_0}{\partial \zeta^4} = 0, \quad (3.3-4)$$

$$\frac{\partial^2 u_1}{\partial \tau^2} + \frac{EI}{\rho l^4 \omega_0^2} \frac{\partial^4 u_1}{\partial \zeta^4} = \theta_0 \zeta \cos(\tau - \psi) - \omega_{01} \frac{\partial^2 u_0}{\partial \tau^2} - \gamma \frac{\partial^2}{\partial \zeta^2} \int_0^{\frac{1}{2}} \epsilon_0^*(u_0) \frac{\partial^3 u_0}{\partial \tau \partial \zeta^2} \eta^2 d\eta, \quad (3.3-5)$$

etc., where $\omega = \omega_s$. The corresponding boundary conditions are

$$\begin{aligned} u_i &= \frac{\partial u_i}{\partial \zeta} = 0 & \text{at } \zeta = 0, \\ \frac{\partial^2 u_j}{\partial \zeta^2} &= \frac{\partial^3 u_j}{\partial \zeta^3} = 0 & \text{at } \zeta = 1. \end{aligned} \quad (3.3-6)$$

On substituting (3.3-3) into (3.3-4) and considering the boundary conditions (3.3-6), a set of eigenfunctions

$$\begin{aligned} \phi_j = \frac{1}{2 \sin k_j \sinh k_j} \{ & (\cos k_j + \cosh k_j)(\cosh k_j \zeta - \cos k_j \zeta) + \\ & + (\sin k_j - \sinh k_j)(\sinh k_j \zeta - \sin k_j \zeta) \} \end{aligned} \quad (3.3-7)$$

is generated; and the corresponding eigenvalues k_j are determined by

$$\cos k_j \cosh k_j + 1 = 0 \quad (3.3-8)$$

where $k_j^4 = \frac{\rho l^4 \omega_j^2}{EI}$

Hence, the zeroth approximation is

$$\omega_0^2 = \frac{k_0^4 EI}{\rho l^4} \quad (3.3-9)$$

$$\phi(\zeta) = \phi_0(\zeta) \quad (3.3-10)$$

3.4 First Order Approximation

Recalling that

$$\epsilon = \frac{h}{l} \eta \frac{\partial^2 u}{\partial \zeta^2}$$

we have

$$\begin{aligned} \epsilon(u_0) &= \frac{\lambda h \delta}{l} \eta \phi_0'' \cos \tau \\ \epsilon_0^*(u_0) &= \frac{h \delta}{l} \eta \phi_0'' \quad \text{for } \eta > 0 \end{aligned} \quad (3.4-1)$$

If (3.3-10) and (3.4-1) is now placed in (3.3-5), we obtain

$$\begin{aligned} \frac{\partial^2 u_1}{\partial \tau^2} + \frac{1}{k_0^4} \frac{\partial^4 u_1}{\partial \zeta^4} &= \theta_0 \zeta \cos(\tau - \psi) + \\ &+ \omega_{01} \delta \phi_0 \cos \tau + \frac{\gamma h \delta^2}{32 l} (k_0^4 \phi_0 \phi_0'' + \phi_0'''^2) \sin \tau \end{aligned} \quad (3.4-2)$$

which is the differential equation for the first approximation. The corresponding response equation is obtained by multiplying (3.4-2) by $\phi_0 \cos \tau$ and $\phi_0 \sin \tau$ separately, and integrating each result over the length of the beam for one cycle. This eliminates the time dependence and, of course, the energies must balance over the cycle. We thus obtain

$$\begin{aligned} \int_0^l \{ \theta_0 \zeta \cos \psi + \omega_{01} \delta \phi_0 \} \phi_0 d\zeta &= 0 \\ \int_0^l \{ \theta_0 \zeta \sin \psi + \frac{\gamma h \delta^2}{32 l} (k_0^4 \phi_0 \phi_0'' + \phi_0'''^2) \} \phi_0 d\zeta &= 0 \end{aligned} \quad (3.4-3)$$

Carrying out the integrations and substituting for ω_{01} from (3.3-2), finally leads to the response equation for the first order approximation.

$$\left(\frac{\omega}{\omega_0} \right)^2 = 1 \pm \frac{\lambda}{\delta} \left\{ \left[\frac{4 \theta_0 (\cos k_0 + \cosh k_0)}{k_0^2 \sin k_0 \sinh k_0} \right]^2 - \left[\frac{\gamma h \Gamma}{8 l} \right]^2 \delta^4 \right\}^{\frac{1}{2}} \quad (3.4-4)$$

where

$$\tau = \int_0^1 \{ k_0^4 \phi_0 \phi_0'' + \phi_0'''^2 \} \phi_0 d\zeta.$$

The first order approximation of the solution for the displacement is now readily constructed by using the method of Fourier expansion. We thus assume the solution

$$u_1(\tau, \zeta) = \psi_1(\zeta) \cos \tau + \psi_1^*(\zeta) \sin \tau \quad (3.4-5)$$

which, on substitution into (3.4-2), gives

$$\begin{aligned} \psi_1'''' - k_0^4 \psi_1 &= k_0^4 \{ \theta_0 \zeta \cos \psi + \delta \omega_0 \phi_0 \} \\ \psi_1^{*''''} - k_0^4 \psi_1^* &= k_0^4 \left\{ \theta_0 \zeta \sin \psi + \frac{\gamma h \delta^2}{32 l} (k_0^4 \phi_0 \phi_0'' + \phi_0'''^2) \right\} \end{aligned} \quad (3.4-6)$$

On expanding

$$\psi_1 = \sum_{j=1}^{\infty} a_j \phi_j \quad \psi_1^* = \sum_{j=1}^{\infty} a_j^* \phi_j \quad (3.4-7)$$

we find

$$\begin{aligned} a_j &= \frac{4 k_0^4 \theta_0 \cos \psi}{k_j^4 - k_0^4} \int_0^1 \phi_j \zeta d\zeta, \\ a_j^* &= \frac{4 k_0^4}{k_j^4 - k_0^4} \int_0^1 \left\{ \theta_0 \zeta \sin \psi + \frac{\gamma h \delta^2}{32 l} (k_0^4 \phi_0 \phi_0'' + \phi_0'''^2) \right\} \phi_j d\zeta. \end{aligned}$$

3.5 Numerical Result and Comparison

The result obtained in the foregoing section is now applied to the specific example studied by Pisarenko. The beam is made of steel with the following given data: $m = 2.7$, $\mu = 18.6$, $b = 1.12$ in., $h = 0.59$ in., $l = 15.95$ in., $E = 2.96 \times 10^7$ psi., and $\lambda^2 \theta_0 = 10^{-4}$.

The plot of the response curve given by (3.4-5) is shown in Figure 5; also shown in the same figure are the corresponding solu-

tions up to the first and second order approximations by Pisarenko. Obviously his second approximation blows up at the peak point of the response curve where $\psi = \pi/2$. This singularity can also be noticed by studying equations 12.4 and 12.5 of [3], which indicate that 12.4 cannot be satisfied at $\psi = \pi/2$. However, this singularity can be removed if we modify Pisarenko's method by expanding the amplitude of the excitation instead of the phase shift. This modification leaves the zeroth and the first approximations unchanged. Therefore, although the second approximation (by Pisarenko) blows up at one point it does not imply the invalidity of the first approximation. Also, the result far away from the singularity, which falls closely on that given by (3.4-4) might still be used.

Another reason for the difference in amplitude and shape of the response curves between the two cases is the strain hardening effect of the Davidenkov stress-strain relation. This is easily seen by plotting the locus of the tips of the hysteresis loops as shown in Figure 6. The cyclic stress sensitivity limits (S_1) for some typical structural metals are also indicated in the figure, beyond which values both formulations are invalid because of stress history effects. All of these points are noted to be below the corresponding yield limits, which justifies the linear formulation of the elastic component in (2.2-5); otherwise, as stated in Section 2, the typical cubic or some other term would have to be included.

If the difference in the response (Figure 5) between the two cases is mainly due to the method of solution and magnitude of non-linear elastic components, then the use of the elliptical hysteresis loop should yield a satisfactory result. At least, the result should be as good as that given by Pisarenko's solution. This conjecture is enhanced by plotting the two corresponding hysteresis loops, given by (1-2) and (3.2-2) respectively, as shown in Figure 7. The strain amplitude is taken at a value slightly below the cyclic stress sensitivity limit of high strength steel. Nevertheless the geometric appearance of the two loops appears to be practically the same, except for the strain hardening effect.

IV. CONCLUSIONS

The analytical technique of describing material damping by elliptical hysteresis loops has been applied to experimental damping data to produce damped, stress-strain relations. The objective of this approach is to produce an easily tractable mathematical description of the stress-strain relation which at the same time adheres closely to experimentally observed behavior. An extension of this development to the biaxial stress case is also demonstrated, although

certain special assumptions concerning the role of distortional and dilatational strain energies are necessary in order to achieve this. An example application to the flexural vibration of a cantilever beam is shown to illustrate the aforementioned properties.

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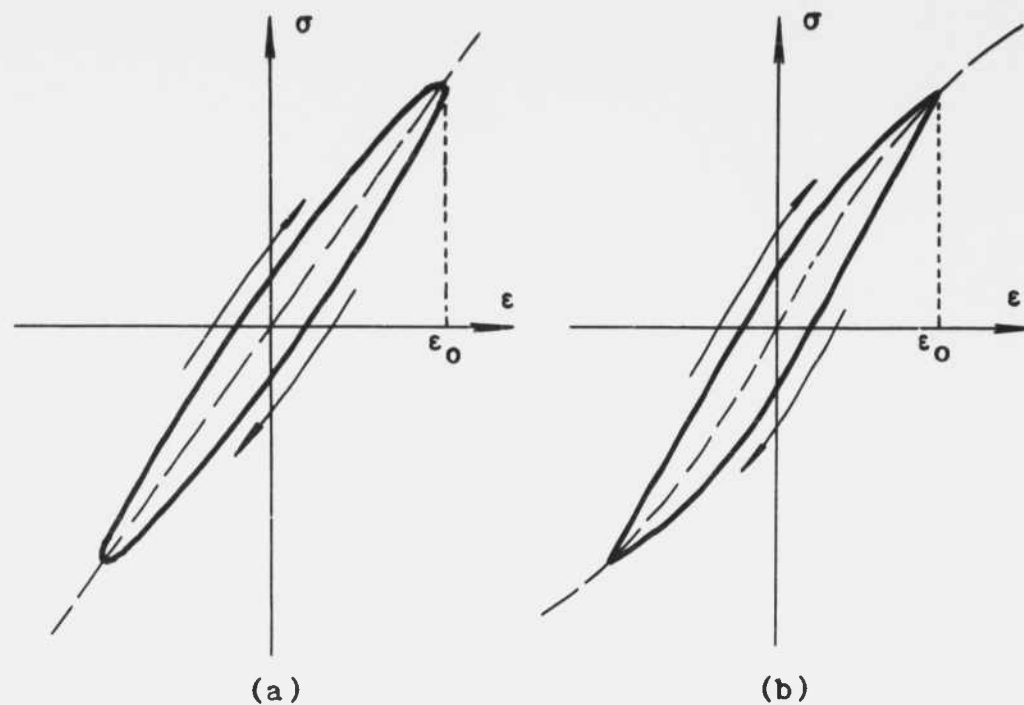


FIG. 1. HYSTERESIS LOOP SHAPES

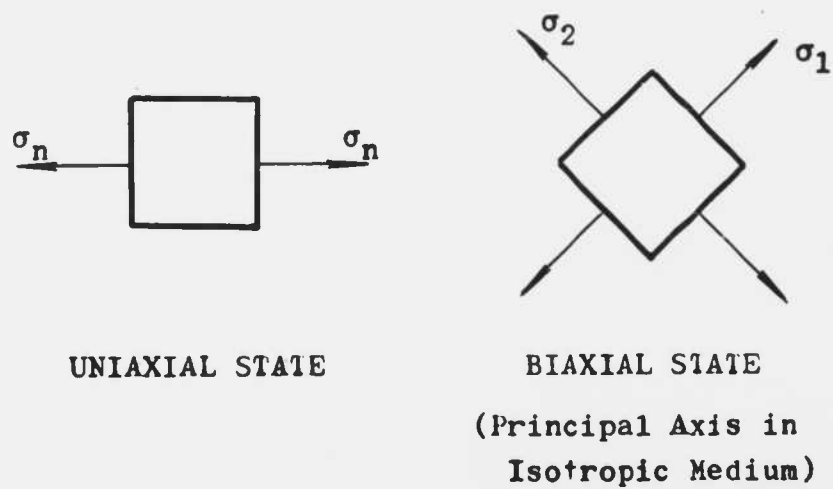


FIG. 2. COMPARISON OF STRESS STATES

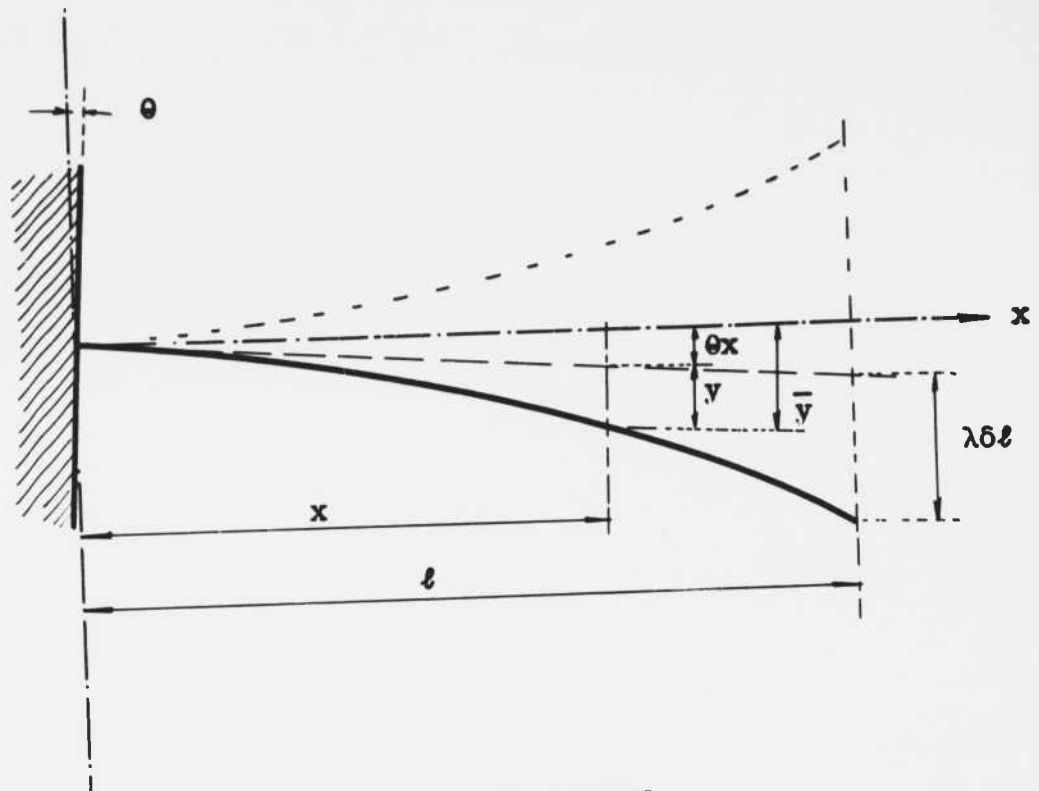


FIG. 3. BEAM MODEL

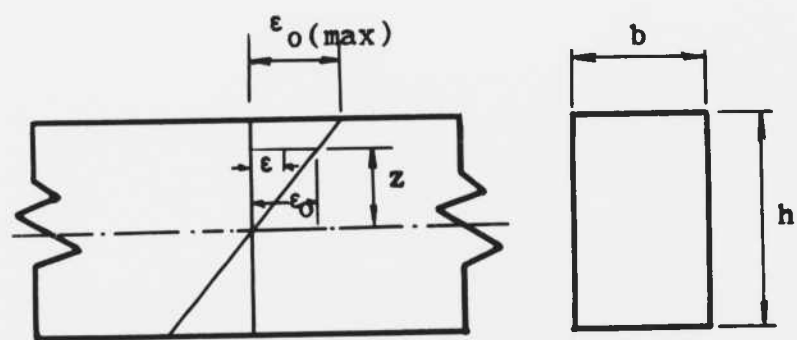


FIG. 4. BEAM CROSS SECTIONS

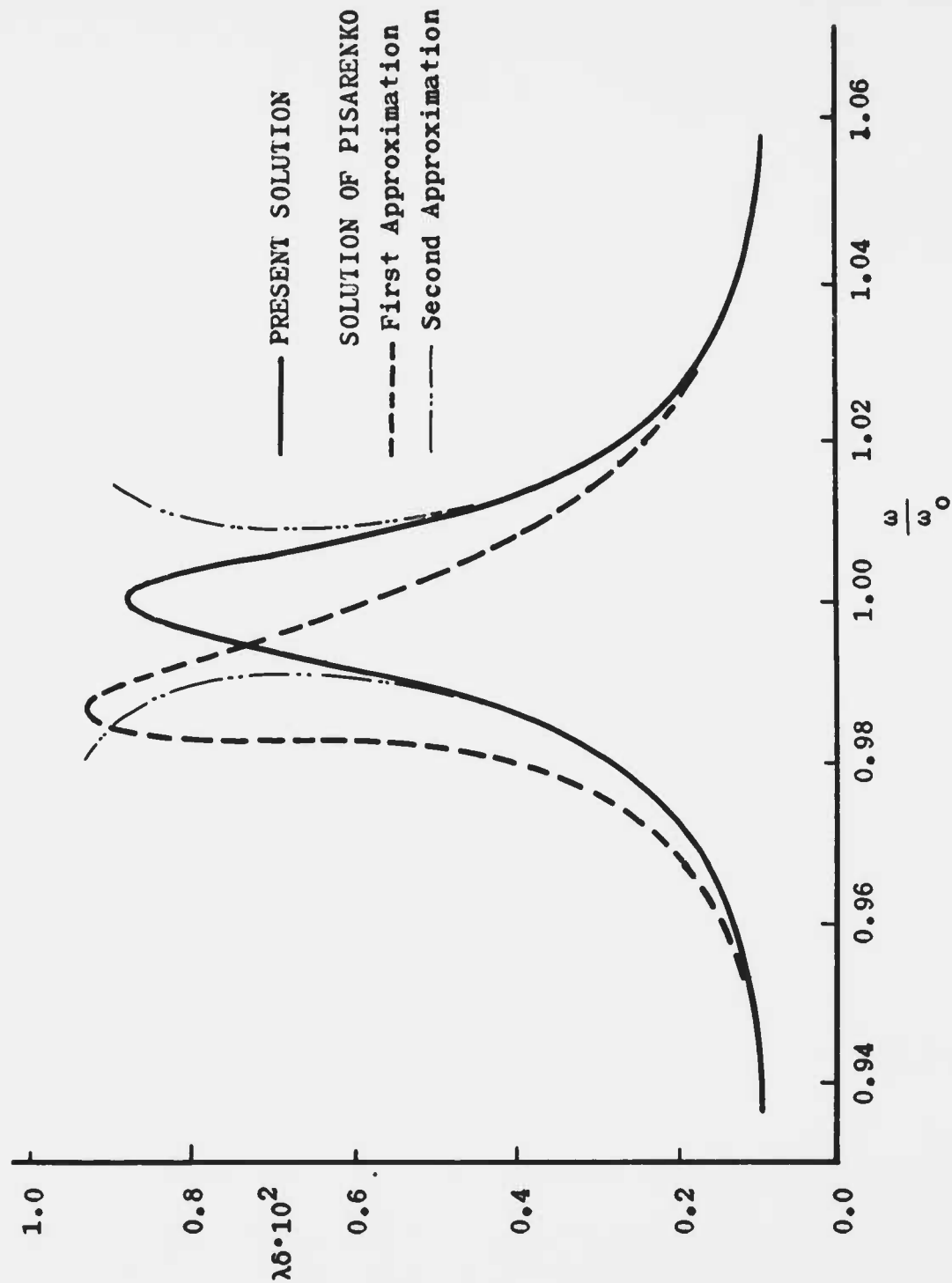


FIG. 5. COMPARISON OF RESPONSE CURVES

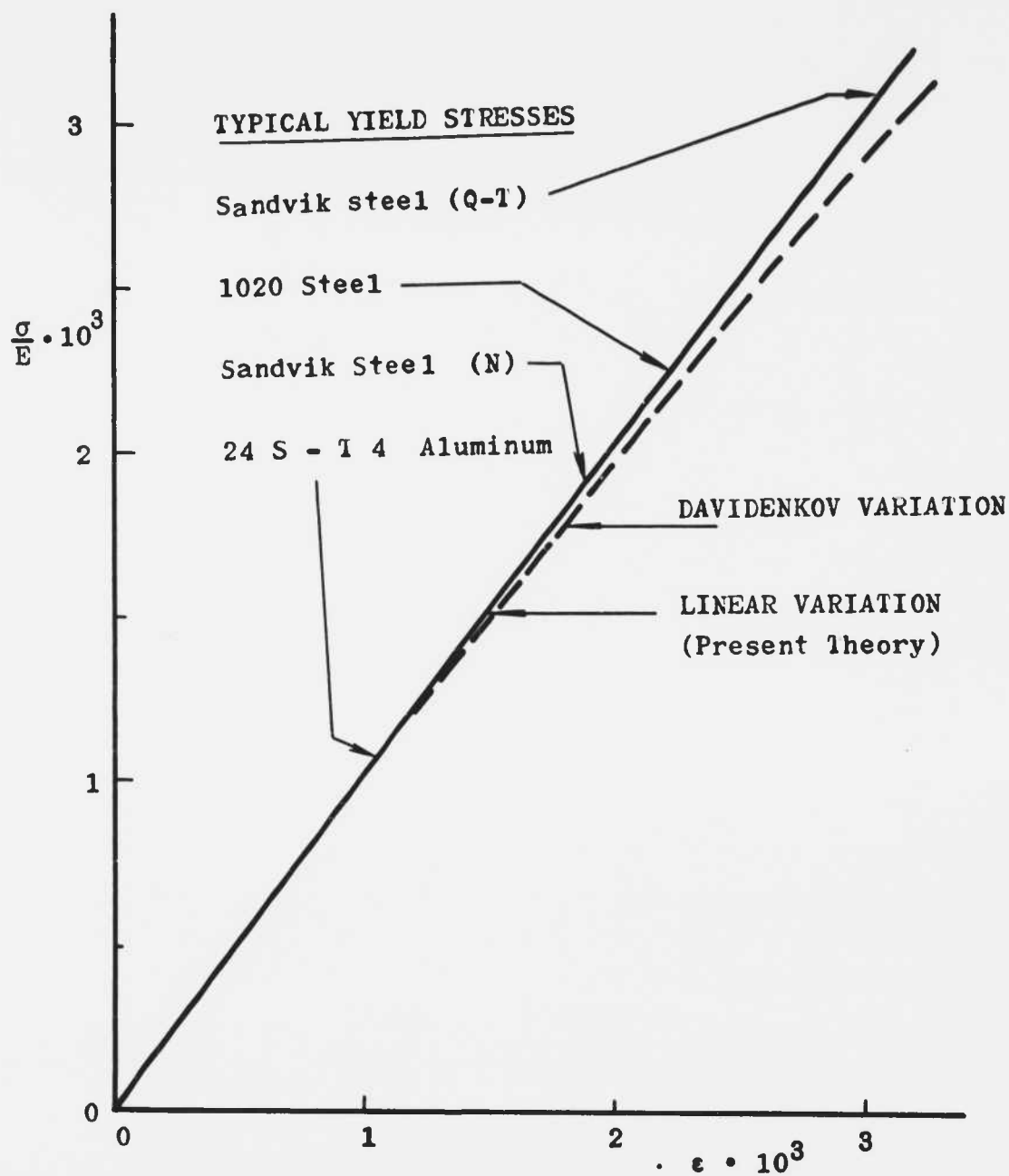


FIG. 6. COMPARISON OF LOCUS CURVES

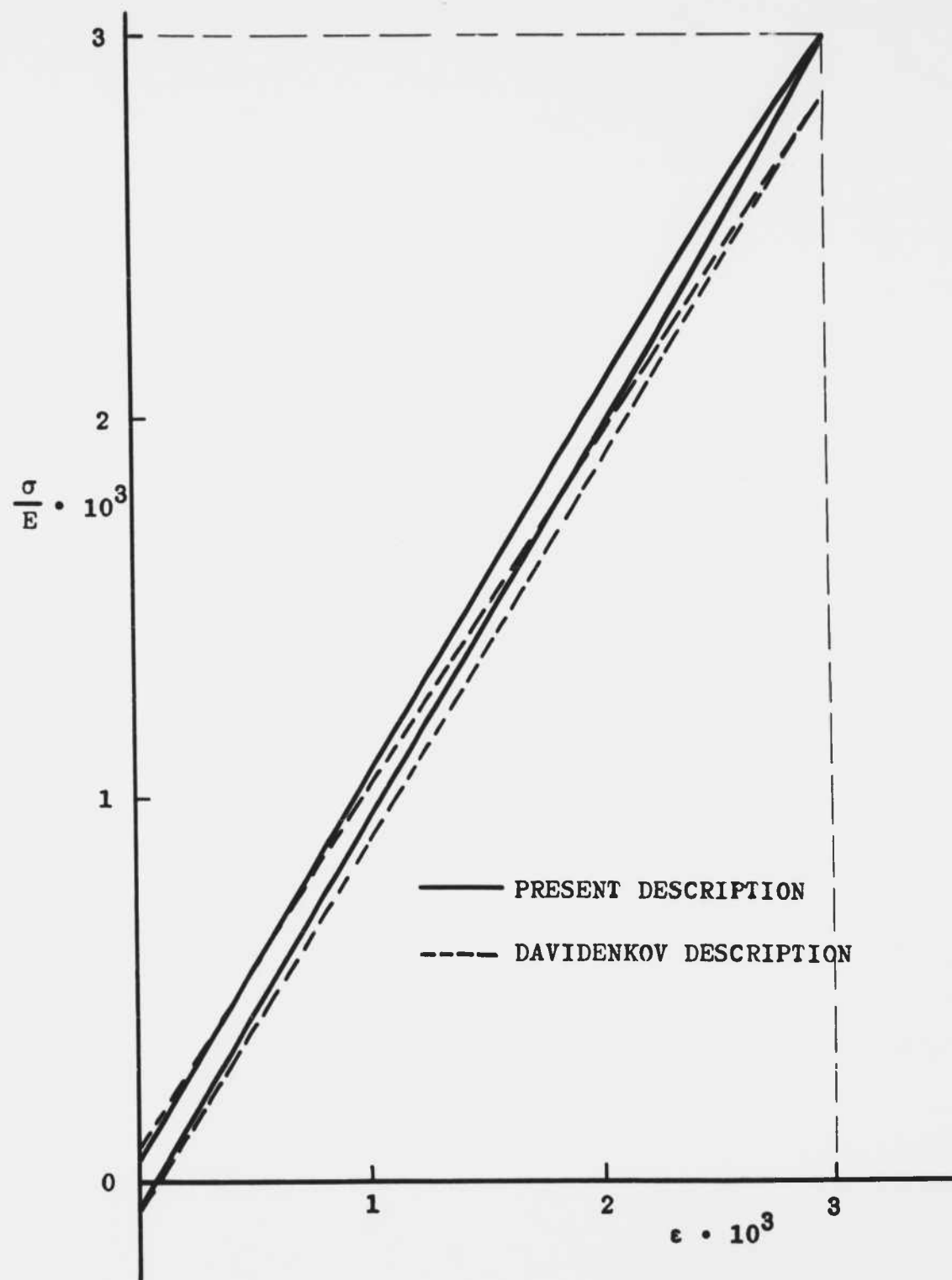


FIG. 7. COMPARISON OF HYSTERESIS LOOP SHAPES
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