THE APPLICATION OF STATISTICS TO
THE FLIGHT VEHICLE VIBRATION PROBLEM

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DIVISION OF THOMPSON RAMO WOOLDRIDGE INC.
CANOGA PARK, CALIFORNIA

DECEMBER 1961

AERONAUTICAL SYSTEMS DIVISION

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NOTICES

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AERONAUTICAL SYSTEMS DIVISION
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE
WRIGHT-PATTERSON AIR FORCE BASE, OHIO

FOREWORD

This report was prepared by Julius S. Bendat, Loren D. Enochson, G. Harold Klein and Allan G. Piersol of Ramo-Wooldridge, a division of Thompson Ramo Wooldridge Inc., Canoga Park, California. It contains results obtained from July 1960 to June 1961 on Air Force Contract No. AF33(616)-7434, "The Application of Statistics to the Flight Vehicle Vibration Problem." The work was accomplished under Project No. 1370, "Dynamic Problems in Flight Vehicles," Task No. 14094, "Methods of vibration Prediction, Control and Measurement." This report was administered under O. R. Rogers, Chief, Vehicle-Kinetics Section, Dynamics Branch, Flight Dynamics Laboratory, Aeronautical Systems Division, with Robert F. Wilkus initially in charge of the project. He was succeeded by Otto F. Maurer.

Professor William T. Thomson, Engineering Department University of California at Los Angeles, participated in physical studies for the investigation and consulted on other phases of the work.

ABSTRACT

This report presents a critical analysis of the application of statistics to flight vehicle vibration problems. The general problem is discussed from many aspects, and results are applicable to many other physical areas besides vibration. Detailed analytical engineering procedures are proposed for determining statistical properties of a single vibration record, and for establishing the over-all vibration environment from a collection of vibration records. Jet aircraft and several categories of missiles are broken down into definite operating phases as regards their vibration environment. Simple statistical techniques are developed for reducing the amount of data that needs to be gathered for later processing. This technique can provide a low probability of missing an unexpected event, and a high probability of covering the range of expected events. Straight-forward statistical tests are developed for testing fundamental assumptions of randomness, stationarity, and normality. Mathematical and physical distinctions are explained between different information obtained by measuring numerous important statistical parameters, such as an instantaneous amplitude probability density function, or a mean square acceleration power spectral density function, or an autocorrelation function. An engineering discussion is given of related instrumentation equipment available today in many laboratories, with emphasis on their statistical accuracy in measuring desired information. An experimental laboratory and flight test program is outlined for verifying these measurement accuracies and other theoretical statistical results contained in the report, such as material on repeated experiments and random sampling techniques. Important physical applications are explored indicating how statistical information can be helpful in predicting the response of continuous structures to random excitation, and for evaluating loading effects on a structure. The report concludes with recommendations for future work.

PUBLICATION REVIEW

This report has been reviewed and is approved.

FOR THE COMMANDER

WILLIAM C. NIELSEN
Colonel, USAF
Chief, Flight Dynamics Laboratory

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GLOSSARY OF SYMBOLS

$b(x:N, p)$	Binomial probability density function
B	Realizable bandwidth in cycles per second
c	Physical definition: Damping coefficient
c	Statistical definition: Number of defects
c_{cr}	Critical damping coefficient, $2\sqrt{km}$
C	Electrical capacitance
$C(f)$	Co-Spectral density function
d_2	Sample range to sample standard deviation conversion factor
e	Length of unexpected event
E	Physical definition: Young's modulus
E	Statistical definition: Expected value
f	Physical definition: Frequency in cycles per second
f	Statistical definition: Number of observations in a class interval
F	Variable with F distribution
F_{max}	Ratio of largest to smallest variance in a set of several variances
F_i	Expected number of observations in class interval i
g	Acceleration due to gravity, 386 in/sec^2
$G(f), G(\omega), S(f), S(\omega)$	Power spectral density functions
$h(t)$	Weighting function
$h_n(z)$	Probability density function for extreme values
$H(\omega), H(f)$	Complex frequency response function
$H_n(z)$	Cumulative probability distribution function for extreme values
i, j	$\sqrt{-1}$
1	Moment of inertia
j, i	$\sqrt{-1}$
k	Physical definition: Spring rate (spring constant)

k	Statistical definition: Number of flights; or number of successes in Binomial distribution
K	Physical definition. Time constant of RC circuit, $K = RC$
K	Statistical definition: Tolerance factor
$\bar{L}$,	Mean time between samples
LCL	Lower control limit
m	Physical definition: Mass
m	Statistical definition: Sample mean value
M	Physical definition: Concentrated mass
M	Statistical definition: Size of population
$\mathfrak{M}$	Expected number of maxima per unit time
n	Number of degrees of freedom
N	Sample size
N_α	Number of crossings per unit time at level α
N_0	Number of zero crossings per unit time
p	Percent value; or fraction defective; or probability of success for Binomial distribution
$p(k; \lambda)$	Poisson probability density function
$p(x, \theta)$	General probability density function of random variable x with parameter θ
$p(x)$	Probability density function
P	Physical definition: Period, $(1/f)$
P	Statistical definition: Proportion
P_a	Probability of acceptance
$P(x)$	(Cumulative) probability distribution function
q	Probability of failure for Binomial distribution
Q	Mechanical Q, $(1/2\zeta)$
$Q(f)$	Quad-Spectral density function

r	Number of runs in "run test"
r_{ij}	Sample correlation coefficient
R	Physical definition: Electrical resistance
R	Statistical definition: Sample range
$R_{1.23\ldots k}$	Sample multiple correlation coefficient
$R(t_1, t_2), R(\tau), R(\tau, T,$	Correlation functions
s^2	Sample variance (s = sample standard deviation)
S	Physical definition: Stress level
S	Statistical definition: Total population size
S. R.	Sweep rate or scan rate
$S(f), S(\omega), G(f), G(\omega)$	Power spectral density functions
t	Physical definition: Time
t	Statistical definition: Variable with Student's "t" distribution
T	Physical definition: Time interval
T	Statistical definition: Sample length (record length)
UCL	Upper control limit
Var(x)	Variance of x (second moment about the mean)
x	Any variable
$x(t), y(t), z(t)$	Amplitude as a function of time
$\overline{x(t)}$	Mean value of $x(t)$
$\overline{x^2(t)}$	Mean square value of $x(t)$
z	Standardized normal variate (zero mean, unit variance)
$Z(\omega)$	Mechanical impedance
α	Level of significance (i.e., probability of Type I Error); or arbitrary level crossing of a random process
$(1 - \alpha)$	Confidence coefficient

β	Probability of Type II error
$(1 - \beta)$	Power of test
$\gamma^2(f)$	Coherence function at frequency f
Γ	Correlation coefficient
$\Gamma(n)$	Gamma function
δ	Dirac delta function
ϵ^2	Mean square percentage error, (i.e., normalized mean square error)
ϵ	Standard error, $+\sqrt{\epsilon^2}$
ζ	Damping ratio, (c/c_{cr})
η	Multiple correlation coefficient
θ	Phase angle
λ	Number of standard deviations; or mean of Poisson distribution
$\lambda(f)$	Spectral bandwidth
μ	Population mean value
ξ	Generalized variable
ϕ	Covariance function
σ^2	Population variance
σ	Population standard deviation, $+\sqrt{\sigma^2}$
τ	Time difference
φ	Mechanical mode shape
χ^2	Variable with Chi-square distribution
ω	Circular frequency, $2\pi f$
ω_n	Natural circular frequency
$\wedge$	Estimate of
$\langle \ \rangle$	Ensemble average
$\begin{pmatrix} a \\ b \end{pmatrix}$	Number of combinations, $\dfrac{a!}{b!\,(a-b)!}$

I. INTRODUCTION

1.1 OBJECTIVES OF CONTRACT

The objectives of the contract were to:

o Determine vibration response at a single point on a structure of a flight vehicle during entire operational life history to assist in future fatigue and reliability investigations.

o Determine vibration environment for equipment mounted on structures

o Determine vibration environment for human comfort problem in future space flights.

These objectives resulted from the need to:

* Improve design of jet aircraft and missiles to reduce structural failures due to vibration.

* Establish better laboratory testing specifications for structures and equipment.

* Develop over-all scientific statistical procedures as opposed to limited special-purpose techniques.

o Bridge gaps between known theoretical ideas and practical methods.

Principal activities of the contract were divided into three phases as follows

Phase I. Preliminary Study of Application of Statistics to Flight Vehicle Vibration Problems.

Phase II. Development of Techniques for Estimating Desired Statistical Characteristics of Vibratory Time Histories.

Phase III. Outline of Experimental Program for Evaluation of Estimation Procedures by (a) Laboratory Testing, and (b) Flight Testing.

The work was designed toward the following main goals:

- Provide material to aid in understanding the proper application of various proposed statistical techniques through consideration and verification of basic assumptions which must be satisfied.

- Emphasis on practical statistical procedures for estimating vibration responses at a single point on a structure under steady operating conditions.

- Preparation of an experimental program cognizant of equipment instrumentation problems and statistical sources of error.

    In addition, there were many other important goals such as:

- Over-all physical description of vibration environment for different categories of jet aircraft and missiles.

- Greater physical and mathematical insight into the response of structures to random excitation.

- Extended application of this material to other physical applications involving statistical analysis of random phenomena.

- Clearer appreciation of limitations of these statistical techniques when dealing with phenomena failing to satisfy basic assumptions.

1.2 HISTORY OF CONTRACT

This contract resulted from a Wright Air Development Division (WADD) Request for Proposal P.R. No. 96129, dated 30 December 1959, and subsequent technical discussions between Dr. O. R. Rogers, WADD, Mr. R. F. Wilkus, WADD, and Dr. J. S. Bendat, Ramo-Wooldridge (R-W), on 11 March 1960. Work commenced on 1 July 1960.

Progress reports were submitted monthly to WADD beginning 10 August 1960, and interim technical reports were submitted when completed on various portions of the investigations. A preliminary draft of this final report was submitted for a technical and format review on 4 May 1961. Briefings between WADD and R-W personnel on the subject contract took place at WADD on 8 September 1960, at R-W on 21 December 1960, and at WADD on 4 May 1961.

1.3 PERSONNEL OF CONTRACT

Project manager for the contract was Dr. J. S. Bendat. He was responsible for coordinating the entire effort, and for assigning and reviewing various work undertaken by other Ramo-Wooldridge personnel. Mathematical portions of this report were of special concern to him, as well as proper emphasis of statistical and engineering material.

Three members of the technical staff from Ramo-Wooldridge contributed greatly to the project: Mr. L. D. Enochson, Mr. G. H. Klein, and Mr. A. G. Piersol. Mr. Enochson was responsible for much of the basic statistical procedures and tests contained in this report, and assisted in design of the experimental program. Mr. Klein was responsible for describing the physical vibration aspects of the various categories of jet aircraft and missiles under consideration; also, he worked on the random sampling technique, instrumentation problems, and flight test program. Mr. Piersol was responsible for investigating structural fatigue problems, non-linear effects, instrumentation problems, and the experimental laboratory program. A fourth member from Ramo-Wooldridge, Mr. R. S. Boyle, worked briefly on initial statistical questions.

Professor William T. Thomson, Engineering Department, University of California at Los Angeles, was engaged as a consultant at the start of the contract. He participated actively both in group discussions and in personal studies incorporated in this final report dealing with the response of structures to random excitation and the effects of loading.

A small subcontract was let to Norair, a division of Northrop Corporation, Hawthorne, California for assistance in formulating engineering aspects of flight vehicles and reviewing certain other Norair material. This work was done principally by Mr. William Roberts and Mr. Robert White of Norair. In addition, Mr. Roy Mustain of Nortronics, another division of Northrop Corporation, was helpful in establishment of the flight test program discussed in this report.

Conferences were held also with many engineers from other companies in the Los Angeles area to survey their vibration programs and needs. These individuals provided valuable information on the current state of vibration

analysis and, although they will be nameless here, their advice is grate-
fully acknowledged.

The contents and recommended procedures of this report, of course,
are the responsibility only of the four authors and do not represent the
practices or views of any other individuals or companies.

2. SUMMARY OF MAIN RESULTS

2.1 MATHEMATICAL ANALYSIS AND STATISTICAL PROCEDURES

An over-all scientific analysis is presented of the vibration environment to be expected in various flight vehicles wherein:

* Proper emphasis is given to different types of statistical information concerning the vibrations.

* Statistical techniques are explained in straight-forward language so as to be readily comprehensible to individuals concerned with analysis problems of vibration phenomena.

* Simple statistical tests are developed for verifying basic assumptions instead of accepting them without proof.

* Analytical procedures are proposed for analyzing the pertinent statistical properties of a single vibration time history record.

* Analytical procedures are proposed for establishing the over-all vibration environment given the statistical properties of each of a collection of vibration time history records.

Mathematical material in the report explains from a broad viewpoint:

* Elementary ideas of probability theory.
* Fundamental concepts of random processes.
* General matters of statistical estimation problems.

To mention but a few topics of interest, the presentation includes discussions on:

* Probability density and distribution functions.
* Power spectral density functions.
* Correlation functions.

* Threshold crossings.
* Extreme value properties.
* Peak value distribution of wide-band and narrow-band noise.
* Measurement of linear system frequency response function.

Various statistical concepts, tables and curves are presented including:

* Normal (Gaussian) distribution.
* Chi-square distribution.
* "t" distribution and F distribution.
* Statistical results from repeated experiments.
* Quality control procedures.
* Operating characteristic curves.
* Analysis of variance techniques.
* Multiple regression techniques.

2.2 THEORETICAL INVESTIGATIONS

A number of significant theoretical (mathematical and physical) results have been obtained during the course of the investigation. These include:

* Preliminary study of a random sampling technique to reduce the amount of data to be gathered.

* Establishment of simple quantitative statistical tests for randomness, stationarity, and normality.

* Physical applications of measurable statistical information from vibration data for predicting the response of continuous structures to random excitation, and for evaluating loading effects on a structure.

2.3 INSTRUMENTATION STUDY

* Statistical effects are analyzed fully for actual measurements of various vibration characteristics such as mean square values, power spectral density functions, amplitude probability density functions, and correlation functions.

* A survey is presented of certain available instrumentation
equipment, with emphasis on their practical physical limita-
tions and accuracy to perform desired measurements.

2.4 EXPERIMENTAL PROGRAM

An experimental program is outlined for verifying statistical proce-
dures developed in the report as appropriate to:

* Laboratory Test Program
* Flight Test Program.

  Special attention is given to matters such as:

* Tests of basic assumptions for stationarity, randomness, and
normality.

* Measurement of various vibration characteristics.
* Verification of random sampling procedures.
* Increasing over-all prediction capability of the entire vibration
environment through repeated experiments.

* Statistical design of experiments to minimize number of
experiments.

* Practical considerations in laboratory testing and flight
testing.

2.5 PHYSICAL RESULTS

Many physical topics on response of structures have been investigated
during the course of the contract. These include:

* Response of linear structures to random excitation.
* Continuous structures excited by correlated random forces.
* Modification of response due to loading.
* Vibration induced structural fatigue.
* Effect of nonlinearities on response statistics.

2.6 SECTIONS OF REPORT

Section 3 discusses the flight vehicle vibration problem, reviews certain well known results on the response of linear structures to periodic and random disturbances, and summarizes some past experimental analyses of vibration data from flight vehicles. Sections 4 and 5 provide comprehensive background theoretical material on mathematical and statistical fundamentals which are important for analyzing and evaluating vibration phenomena. Section 6 contains analytical engineering procedures for carrying out a sequence of statistical tests to establish the complete vibration environment (at an arbitrary point on a structure) on a sound statistical basis. Section 7 discusses instrumentation problems relative to measuring desired vibration characteristics, while Section 8 outlines an experimental program to verify the analytical procedures developed in Section 6. The reader may prefer to read Sections 6, 7, and 8 prior to Section 4 and 5, these earlier two sections furnishing theoretical material for special study when needed. Section 9 contains some advanced physical applications on the response of structures which indicate that, for certain problems, statistical information is available from vibration data which is not presently being explored fully, while, for other problems, more appropriate statistical information has still to be developed. The final Section 10 gives a brief review of each Section, 3 through 9, and concludes with some specific recommendations for future work.

3. PHYSICAL DISCUSSION OF FLIGHT VEHICLE
VIBRATION PROBLEMS

3.1 INTRODUCTION

For many years, engineers have been aware of various problems in flight vehicle performance due to the vibration environment. However, it was not until the advent of jet-powered aircraft, rocket aircraft, and missiles, that statistics and random process theory were found to be important for these problems. The main emphasis to date has been to use statistical parameters only for the description of the vibration environment. However, even in this area very little has been done in establishing an accurate knowledge of the errors of these parameters. Various fundamental assumptions have been accepted frequently without proof.

In addition there has been an almost complete absence of application of statistics to determine the number of records that should be taken, the length of each record, and when during a flight, the vibration should be recorded. This has often resulted in either too much data, or not enough, which in turn would produce an inaccurate or biased estimate about the vibration levels of flight vehicles.

Statisticians have known for some time that a properly selected sample can provide a very accurate picture of the entire population or process from which it was taken, and that there is also a point of diminishing returns. Namely, no matter how large the population or how big the process, there exists a sample size, which results in obtaining a maximum amount of information for a given confidence in the results. Increasing the sample size beyond this number yields only an insignificant increase in accuracy. (See Section 6.2.3, Figures 6.9 and 6.10.)

One of the purposes of this report is to place the selection of samples of vibration records in flight vehicles on a sound statistical basis. At the same time it should be realized that this selection cannot be placed entirely on an objective basis. Engineering judgment and a basic knowledge of the over-all flight vehicle vibration problem will still be required. To aid engineers in the process of determining the point where judgement leaves off and the tools of statistics can be applied, a discussion of the various factors involved are developed in an orderly fashion as follows:

3.2 FLIGHT VEHICLES AND THEIR OPERATING CHARACTERISTICS

The type of vehicle, aircraft or missile, large or small, and the number of flights that can be made, help determine the answers to the following main questions:

1. How many samples should be taken?
2. How long should each sample be?
3. When should the sample be taken?

An answer to these questions requires a knowledge of the vehicle mission, length of flight, length of various flight phases, and the various sources of vibration excitation likely to occur.

Table 3.1 was prepared to provide an over-all picture of the various flight vehicles which are in service now, or will be in the near future. It is realized that not every single flight vehicle will fit exactly into one of the four categories shown in Table 3.1. This breakdown mainly serves a qualitative purpose to help the vibration engineer to prepare an exact chart for the particular vehicle under consideration.

3.2.1 Vibration Sources

The column headed "Vibration Sources" in Table 3.1 requires some additional discussion since, for low measurement errors, sample length is dependent upon the frequency range to be recorded (this is shown in later Sections 4, 6, 7, and 8).

The frequency content of the various sources of excitation generally ranges from 1 to 10,000 cps. Without a detailed analysis of the types of vehicles being considered, providing information as to size, structural design, weight, engine type and characteristics, velocity profiles, and launch conditions, the excitations can only be estimated. This estimation would include a prediction, in certain cases, of the probability of occurrence of various frequency ranges, choosing only those which appear the most likely. It should be noted, however, that other frequencies may also occur in practice which may cause damage even though their probability of occurrence is low.

Keeping these qualifications in mind, Table 3.2 can be used with appropriate caution to estimate the frequency ranges of various flight

Table 3.1 Types of Flight Vehicles

Category	Flight Phases	Approximate Time Duration	Vibration Sources	Relative Severity (See Note 3)
1. **Short-term Missiles** Rocket or jet powered, air to air, air to ground, ground to air, ground to ground. Total powered flight less than 20 sec.	a) Transportation (airborne only)	See Category 4, Phases a - f	See Category 4, Phases a - f	M
	b) Take-off (with or without booster)	Less than 2 Sec. Includes short hold down time to develop full thrust	Rocket Exhaust (Noise source) Rough Burning (Direct excitation)	H
	c) Powered flight	5 - 15 Sec.	Boundary layer turbulence and rocket exhaust.	H
2. **Space Vehicles, Probes, Ballistic Missiles** (IRBM, ICBM, ALBM, etc.) Total powered flight usually less than 30 minutes.	a) Transportation (airborne only)	See Category 4, Phases a - f	See Category 4, Phases a - f	M
	b) Take-off (includes ground effect period)	5 - 10 Sec. Includes 2 - 6 sec. hold-down time to develop full thrust	Rocket exhaust, Surface winds	H
	c) Subsonic flight	15 - 30 Sec.	Rocket exhaust, boundary layer turbulence, wind shear and gusts	H
	d) Transonic flight (0.8M - 1.2M approximately)	2 - 6 Sec.	Oscillating shocks, buffet	H
	e) Supersonic flight	2 - 4 Minutes Depends on Trajectory	Boundary layer turbulence	M
	f) Ballistic trajectory (or orbit)	15 Min. on up	None	L
	g) Re-entry	3 - 10 Minutes	Boundary layer turbulence, base pressure fluctuations	M

Table 3.1 Types of Flight Vehicles (Cont'd)

Category	Flight Phases	Approximate Time Duration	Vibration Sources	Relative Severity (See Note 3)
3. **Long-Term Missiles** Usually jet powered, such as Regulus, Snark, etc.	a) Transportation (airborne only)	See Category 4, Phases a - f	See Category 4, Phases a - f	M
	b) Take-off(with or without booster)	5 - 10 Seconds	Jet-exhaust, (and rocket exhaust with booster)atmospheric turbulence	F
	c) Powered flight	30 Min.- 10 Hrs.	Jet-exhaust, atmospheric turbulence, boundary layer turbulence	M
4. **Aircraft.** Rocket and jet powered, orbital bombers, etc. All long-term (See Note No. 2)	a) Warm-up	1 - 15 Minutes	Jet exhaust	M
	b) Taxi	5 - 15 Minutes	Runway roughness, jet exhaust	M
	c) Run-up	2 - 20 Minutes	Jet exhaust	H
	d) Take-off	1 - 5 Minutes	Runway roughness, Jet exhaust, atmospheric turbulence	H
	e) Climb	3 - 30 Minutes	Jet exhaust, atmospheric turbulence, Boundary larger turbulence	M-H
	f) Cruise and Mission, flight maneuvers, etc.	1 - 8 Hours (might include a transonic region)	Jet exhaust, atmospheric turbulence, boundary layer turbulence, buffet	M-H
	g) Descent	5 - 15 Minutes	Atmospheric turbulence. Boundary layer turbulence	M

Table 3.1. Types of Flight Vehicles (Cont'd)

Category	Flight Phases	Approximate Time Duration	Vibration Sources	Relative Severity (See Note 3)
4. (Continued)	h) Landing gear down, flaps down	1 - 15 Minutes	Flap buffet, gusts (atmospheric turbulence)	M
	i) Landing (runway or carrier)	5 Sec - 2 Min	Runway roughness, excitation of fundamental modes at touchdown (1 - 50 cps)	M

Notes:

1. Direct excitation due to rotating equipment within the vehicle is not listed under vibration sources and has to be taken into account as required. Also machine gun or missile firings from aircraft will have to be considered separately.

2. Flight Phases for orbital bombers or rocket-powered aircraft may differ considerably from those known.

3. L - Low (0 - 1.0 g rms)
   M - Medium (1.0 - 10.0 g's rms)
   H - High (10.0 g's rms or higher)

vehicle vibration sources. The nature of these exciting forces is predominantly random.

Table 3.2 Vibration Sources

1. Rocket Exhaust Noise, Jet Noise and Base Pressure Fluctuations	40 - 10,000 cps
Most Significant Region	100 - 2,000 cps
2. Boundary Layer Turbulence	100 - 10,000 cps
Most Significant Region	300 - 5,000 cps
3. Wind Shear	0 - 2 cps
Most Significant Region	1 - 2 cps
4. Atmospheric Turbulence	0 - 10 cps
Most Significant Region	1 - 10 cps
5. Surface Turbulence and Buffet (Oscillating Shocks)	10 - 100 cps
6. Runway Roughness	1 - 50 cps

3.2.2 End Use of Vibration Data

Another important area in any study of flight vehicle vibrations is the purpose or end use for which the data is to be obtained. Generally, there are three main fields of application:

1. Structural fatigue analysis and system reliability.
2. Establishing vibration environment for equipment.
3. Human comfort considerations.

The relationship of these fields to sample size, length, and timing, is somewhat more subtle, and numerical results cannot be obtained directly. For instance, some specific fatigue study may require special data reduction equipment, which in turn may impose certain requirements on the samples. Or, for some other applications, specific transducers may have to be used which then would again place some restrictions on how the samples should be taken.

3.3 RESPONSE OF LINEAR STRUCTURES TO PERIODIC AND RANDOM EXCITATION

Having obtained a qualitative picture of flight vehicle operating characteristics and excitation sources of the vibration environment, the question that now arises is: How does the structure within the vehicle respond to these forces and what can be expected to occur at a single point on that structure.

Advanced analyses of structural responses to periodic and random excitations are presented in Section 9 of this report. Some basic relationships will be reviewed below for the purpose of summarizing some well known results to provide the reader with a simplified description of how linear structures respond to periodic and random disturbances. For a more complete development with extensive discussions and proofs, the reader is referred to appropriate references at the end of this section (e.g., References [6], [7], [14], [16], [20], [24]).

3.3.1 Response to Single Frequency Excitation

The simplest model of a vibrating system is the single-degree-of-freedom system. With proper interpretation the response of this system can be used to estimate the response of many other more complex configurations. Within certain limitations, this also applies to the response in one of the normal modes of continuous structures.

Figures 3.1 and 3.2 below show two possible configurations for the single-degree-of-freedom system.
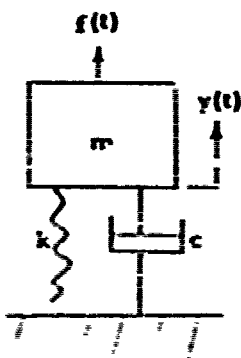
Figure 3.1 Single Degree System (Fixed Base)
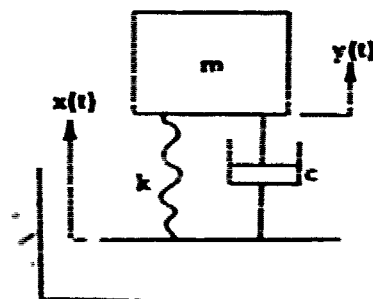
Figure 3.2 Single Degree System (Movable Base)

In Figure 3.1, the mass m is supported through a spring with spring constant
k, and dashpot with damping c, over a fixed foundation. The mass is sub-
jected to a force excitation $f(t)$. In Figure 3.2, the excitation is applied
as a motion $x(t)$ to the movable base of the system.

In standardized notation, the equations of motion for both systems
may be made identical. For Figure 3.1, a single complex periodic exciting
force $f(t) = A e^{i\omega t}$, $i = \sqrt{-1}$, results in a complex frequency response function,
(see Reference [5], p. 1-3),

$$H(\omega) = \frac{1}{1 - (\frac{\omega}{\omega_n})^2 + i 2 \zeta \frac{\omega}{\omega_n}} \tag{3.1}$$

and an output response $y(t) = H(\omega) f(t)$, where ω is the forcing frequency in
radians/sec., $\omega_n^2 = (k/m)$ is the natural frequency of free undamped oscil-
lations, and $\zeta = (c/c_{cr})$ is the damping ratio with $c_{cr} = 2\sqrt{km}$ defined as
the critical damping coefficient. For the system of Figure 3.2, $f(t)$ is
replaced by $-m \ddot{x}(t)$, and $y(t)$ by $[y(t) - x(t)]$.

A typical illustration of time-histories of a periodic input excitation
and output response record for a single-degree-of-freedom system near
resonance ($\omega = \omega_n$) is drawn in Figure 3.3 below.
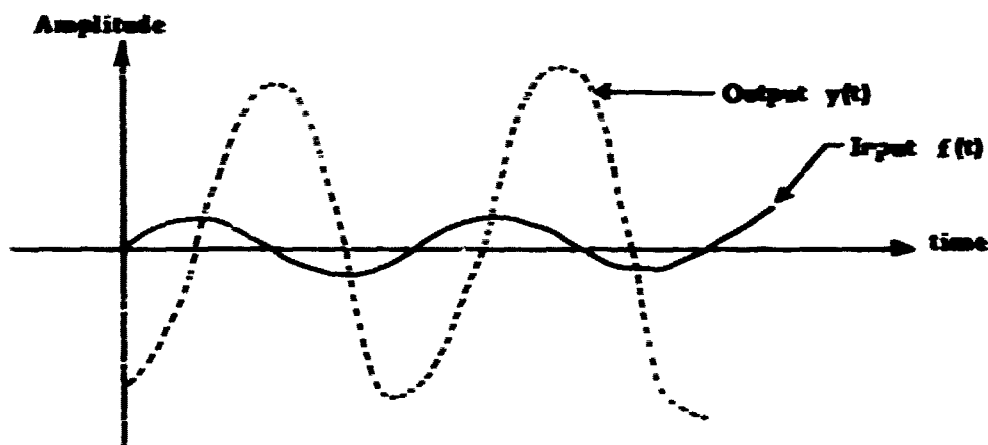
Figure 3.3 Time-History of Input and Output for Single-Degree-
of-Freedom System Near Resonance.

For Figure 3.1, the frequency response function $H(\omega)$ may be interpreted as the complex ratio of the force in the spring to the exciting force. For Figure 3.2, $H(\omega)$ may be interpreted as the complex ratio of force in the spring to the inertia force that would be imposed on the mass if it were rigidly attached to the moving foundation.

The real (Re) and imaginary (Im) parts of $H(\omega)$ and its magnitude response function $|H(\omega)|$ are plotted in Figure 3.4 for $\zeta = 0.10$.
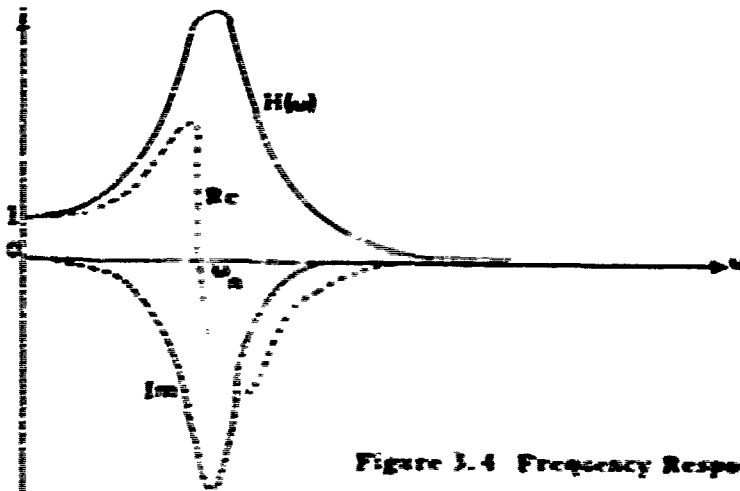
Figure 3.4 Frequency Response Function $H(\omega)$

In many applications, the magnitude response function $|H(\omega)|$ is of greatest importance since it provides a measure of the magnitude of vibration transmitted to the mass for base-excited systems.   For light damping ($\zeta < 0.05$) the resonance peak of $|H(\omega)|$ occurs approximately at ω_n where $|H(\omega_n)| = (1/2\zeta)$.   Instead of ζ the symbol Q is sometimes introduced as defined by $Q = (1/2\zeta)$.   Then $|H(\omega_n)| = Q$, and for $Q > 10$, the amplitude falls to $(Q/\sqrt{2}) = 0.707Q$ at the points P_1 and P_2 with frequencies $\omega_n \pm (\omega_n/2Q)$, respectively, as shown in Figure 3.5 below.   These points are called half-power points because the power that is absorbed by a dashpot at a given frequency is proportional to the square of the amplitude.   The frequency difference (ω_n/Q) between the half-power points P_1 and P_2 is generally referred to as the bandwidth of the system.
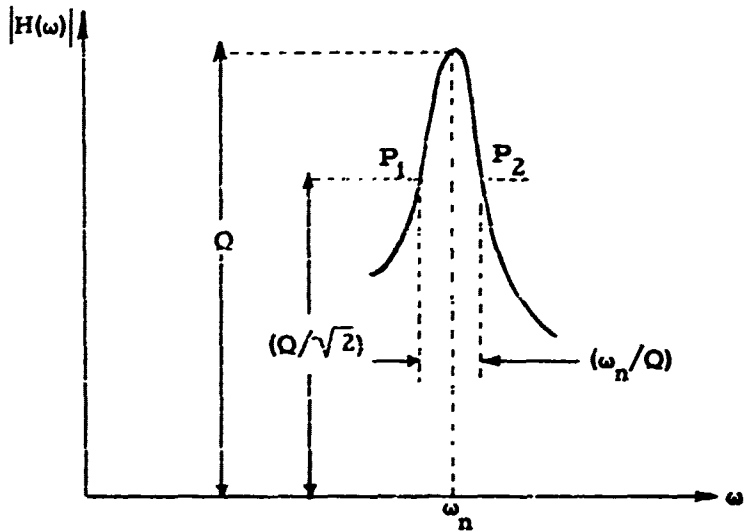
Figure 3.5 Magnitude Response Function $|H(\omega)|$

In vibration work, phase information can often be ignored and the only consideration will be amplitude. Convenient measures of amplitude are the mean-square or root mean-square values. For a time-history of vibration amplitude expressed by $y(t)$, the mean-square value over a time interval T is given by

$$\overline{y^2(t)} = \frac{1}{T} \int_0^T y^2(t) \, dt \tag{3.2}$$

It can be shown (Reference [6], p. 1-7), see also Section 9.1.1, that the ratio of a mean-square output $\overline{y^2}(t)$ of a single degree of freedom system to a mean square input $\overline{f^2}(t)$ is equal to $|H(\omega)|^2$, the square of the magnitude response function

$$\frac{\overline{y^2(t)}}{\overline{f^2(t)}} = |H(\omega)|^2 \tag{3.3}$$

3.3.2 Response to Multiple Frequency Excitation

If the input consists of a superposition of steady simple harmonic functions with different amplitudes, phases, and frequencies, it is convenient to describe its mean square value in terms of a discrete spectral density function, as defined below. By Fourier Series Analysis, it can be shown that if a real input $f(t)$ is of period P, and has no constant term (e.g., no dc component), then it can be represented by the real part of the following series, [Reference 6, pp. 1-7 through 1-11],

$$f(t) = \sum_{n=-\infty}^{\infty} a_n e^{in\omega_0 t} \quad ; \quad \omega_0 = \frac{2\pi}{P} \tag{3.4}$$

$$a_n = \frac{1}{P} \int_0^P f(t) e^{-in\omega_0 t} dt \quad (n = \pm 1, \pm 2, \ldots)$$

$$a_0 = 0$$

where the complex coefficients $a_n (n = \pm 1, \pm 2, \ldots)$ contain information about the phases of the various components. A similar representation holds for the output response function $y(t)$ when $f(t)$ is the input to a single-degree-of-freedom system.

A typical time-history of the response to multiple frequency excitation is drawn in Figure 3.6 below. In part (a), the input wave consists of the sum of two frequencies ω and 2ω. Part (b) illustrates a possible response output with a natural frequency ω_n between ω and 2ω.
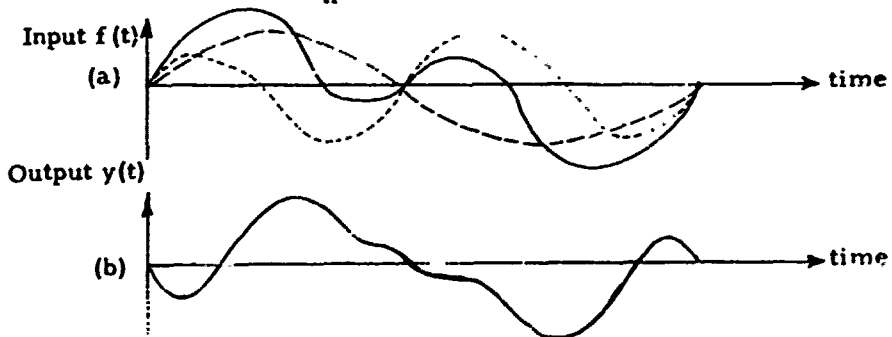
Figure 3.6 Response to Multiple Frequency Excitation

From Equation (3.4), if one deals here with mean square values, there results

$$\overline{f^2(t)} = \sum_{n=1}^{N} \frac{a_n a_n^*}{2} = \sum_{n=1}^{N} S_f(n\omega_0)\, \Delta\omega \qquad (3.5)$$

where a_n^* denotes the complex conjugate of a_n, and $\Delta\omega = (2\pi/P)$. The sum runs theoretically from $n = 1$ to $n = \infty$, but in actual practice will stop at some finite large value N. The quantity $a_n a_n^* = |a_n|^2$ contains no phase information. Equation (3.5) shows that the mean square value of a complex wave is just the sum of the mean squares of the component frequencies. The quantity $S_f(n\omega_0)$ which equals $(a_n a_n^*/2\, \Delta\omega)$ is called a <u>discrete spectral density function</u>, and is commonly measured in units such as $inch^2/rad/sec$ for displacements or g^2/cps for acceleration.

For the mean square steady-state output, one obtains by extending Equation (3.3) for this case.

$$\overline{y^2(t)} = \sum_{n=1}^{N} \frac{a_n a_n^*}{2} |H(n\omega_0)|^2 = \sum_{n=1}^{N} S_f(n\omega_0)\Delta\omega \, |H(n\omega_0)|^2 = \sum_{n=1}^{N} S_y(n\omega_0)\Delta\omega \qquad (3.6$$

where $\Delta\omega = (2\pi/P)$.

A pictorial representation of Equation (3.5) and (3.6) are given in Figures (3.7) and (3.8), respectively. In these figures, the vertical lines represent the contribution to the total mean square value of the individual frequency components. Note from Equation (3.6) that the individual spectral lines $S_f(n\omega_0)$ and $S_y(n\omega_0)$ are related by

$$S_y(n\omega_0) = |H(n\omega_0)|^2 \, S_f(n\omega_0) \qquad (3.7)$$

The total mean square values for input or output are obtained by summing all the individual spectral lines, as shown in Equations (3.5) and (3.6).

$S_f(\omega)$

**Input
Spectral
Density**

ω_0 $2\omega_0$ $N\omega_0$ ω
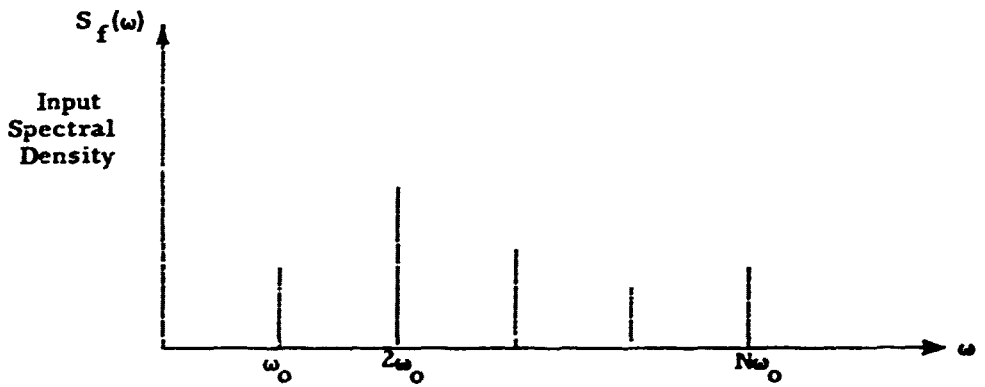
Figure 3.7 Discrete Input Spectrum

$S_y(\omega)$

**Response
Spectral
Density**

ω_0 $2\omega_0$ $N\omega_0$ ω
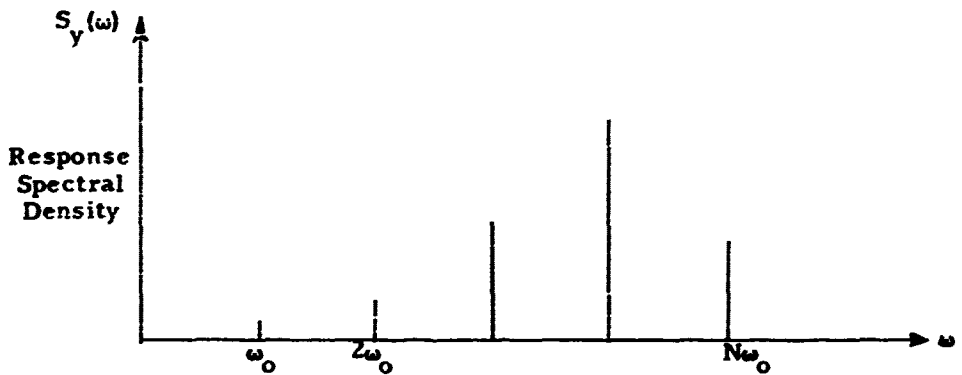
Figure 3.8 Discrete Output Spectrum

3.3.3 Response to Random Excitation

A schematic representation of the input-output relation for any linear system is shown in Figure 3.9.
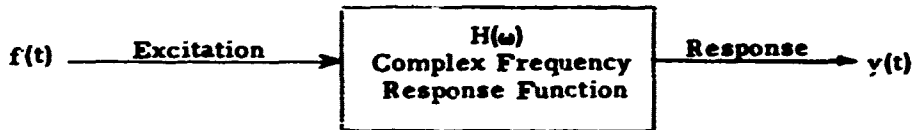
Figure 3.9 Input-Output System

The mathematical nature of $H(\omega)$ for a single-degree-of-freedom system is shown in Equation (3.1).

Important statistical properties of the excitation random process are the mean value, (which for simplicity, as well as fact, may usually be assumed to be zero), the mean square value, the power spectral density function, and the autocorrelation function. These and other topics are covered in considerable detail in Sections 4, 6, and 7. Examples of narrow-band and wide-band continuous power spectral density functions and their respective time-histories are pictured in Figure 3.10.

One very important relationship which exists between the excitation power spectral density function and response power spectral density function for random excitation [Reference 6, pp. 4-1 through 4-7], see also Sections 4.9.4 and 9.1.1 of this report, is given by

$$S_y(\omega) = \left|H(\omega)\right|^2 S_f(\omega) \tag{3.8}$$

In words, the power spectrum of the response is equal to the power spectrum of the excitation multiplied by the square of the system magnitude response function.
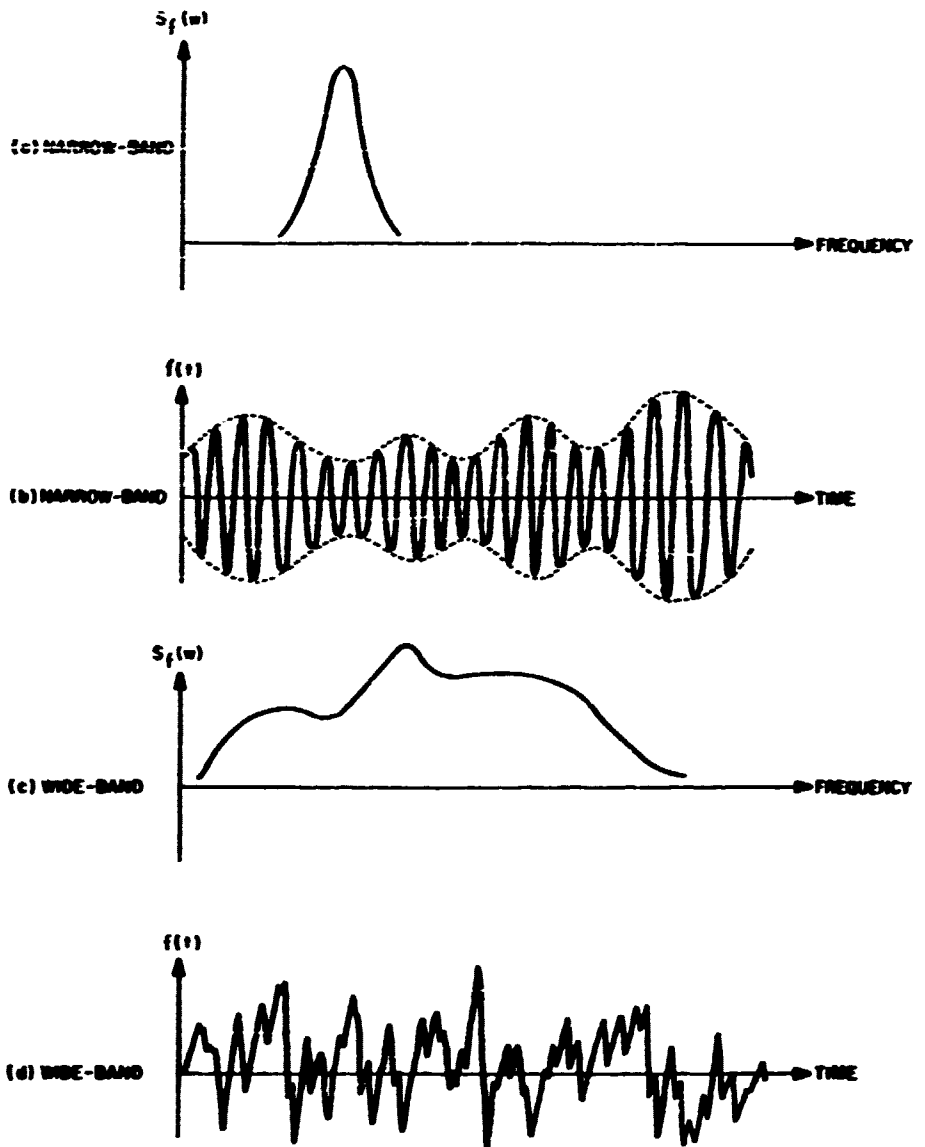
Figure 3. 10 Examples of Narrow-band and Wide-band Power-spectra
and their Respective Time-histories

For a single-degree-of-freedom system subjected to "white noise" (an input having a uniform power spectrum S_0 from zero frequency to infinity, see Figure 3.11), the mean square response is [Reference 6, p. 4-7], derived here in Section 4.9.4,

$$\overline{y^2(t)} = \frac{Q\omega_n S_0}{4} \qquad ; \quad \omega_n = 2\pi f_n \qquad (3.9)$$
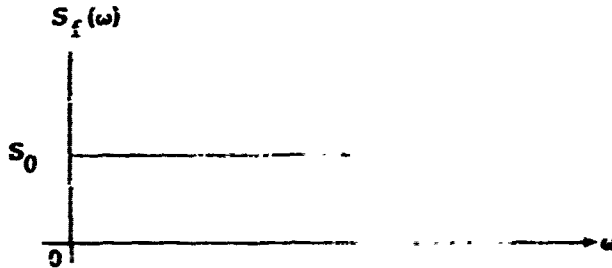
Figure 3.11 White Noise Spectrum

Even though the result of Equation (3.9) is valid only for an infinitely wide-band input, it provides a good approximation for a lightly damped system subjected to a continuous spectrum which is uniform in the vicinity of the system natural frequency. This is illustrated in Figure 3.12 below.
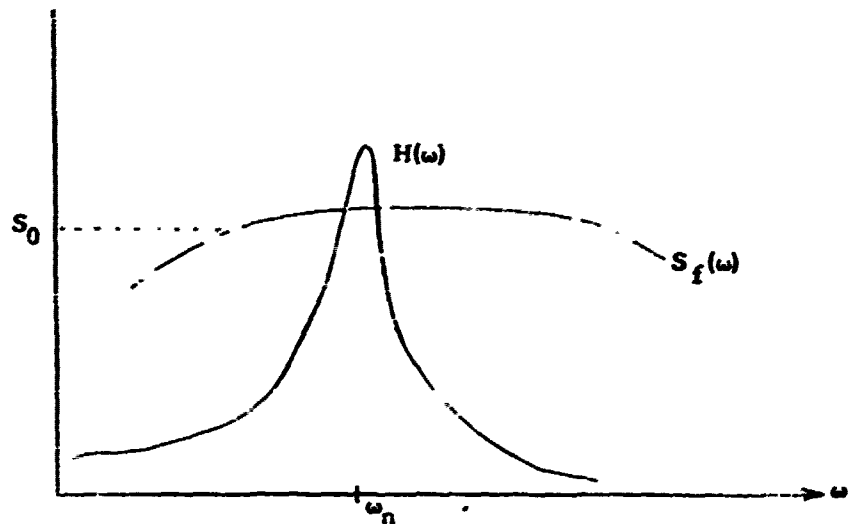
Figure 3.12 Mean Square Response of Lightly Damped System $\simeq (Q\omega_n S_0/4)$

One additional important parameter of the response is the amplitude probability density function. For a lightly damped single-degree-of-freedom system, the narrow-band response to wide-band random excitation is shown in Figure 3.13.
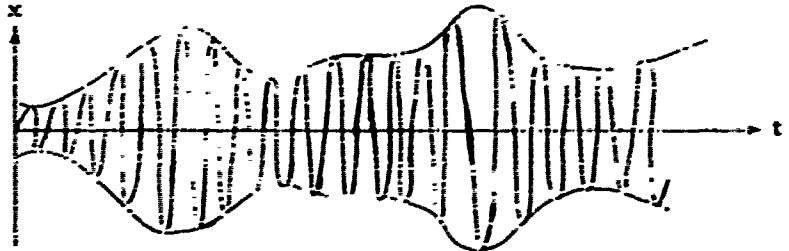
Figure 3.13 Narrow-Band Response to Wide-Band Excitation

The narrow-band response above is approximately a sine wave at frequency ω_n with a randomly varying amplitude and phase. It can be shown, (see Reference [6], pp. 4-11 through 4-13) that for many cases the probability density of the envelope is approximated closely by a Rayleigh probability density function.

$$p(A) = \frac{A}{R_x(0)} \; \epsilon^{-A^2/2R_x(0)} \quad ; \; A \geq 0 \qquad (3.10)$$

where $R_x(0)$ is the autocorrelation function of the response evaluated at zero. A theoretical treatment of this matter which extends the above result to include wide-band response is discussed in Section 4.9.3.

3.4 EMPIRICAL RESULTS FROM EXISTING FLIGHT VEHICLES

As mentioned before, a considerable amount of interpretation is required to arrive at the actual vibration environment from a knowledge of the vibration sources and analysis of the response of structures. To help bridge this gap, some actual data on frequency ranges and magnitudes will now be presented, which has been observed in present day flight vehicles.

Figures 3.14 through 3.17 show levels of maximum accelerations recorded in jet aircraft and missiles as summarized from References [7, 13, 14, 20] at the end of this section.
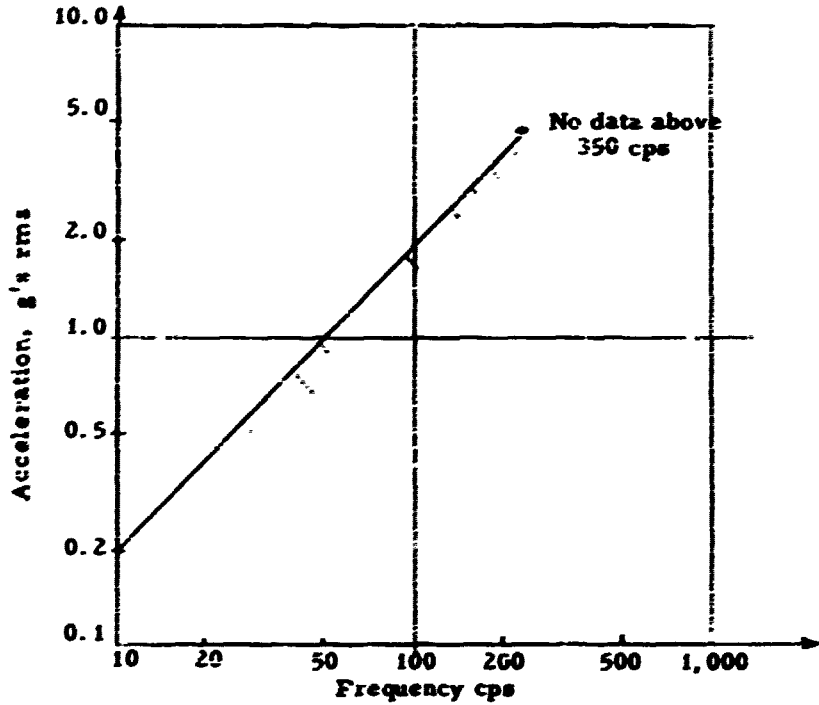
Figure 3.14 Maximum Accelerations in Jet Fighter Aircraft
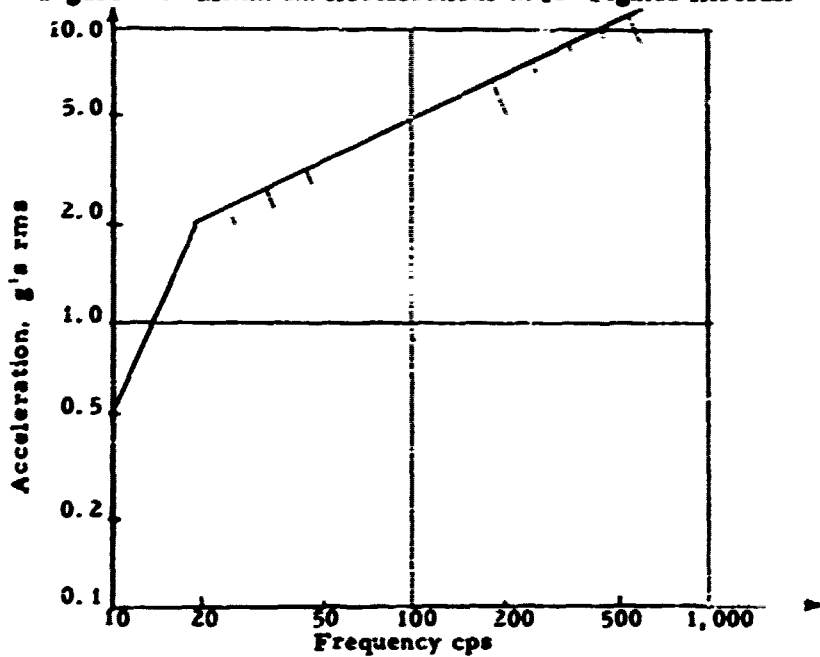
Figure 3.15 Maximum Accelerations in Jet Bomber Aircraft.
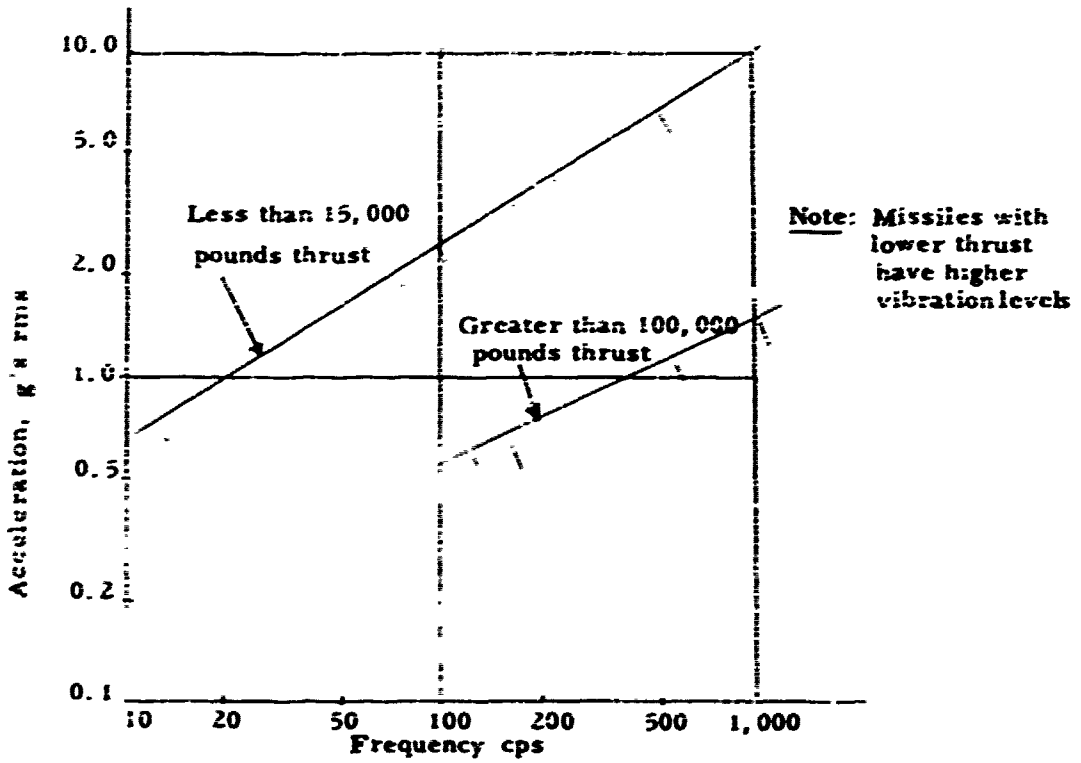
Figure 3.16 Maximum Missile Launch Accelerations
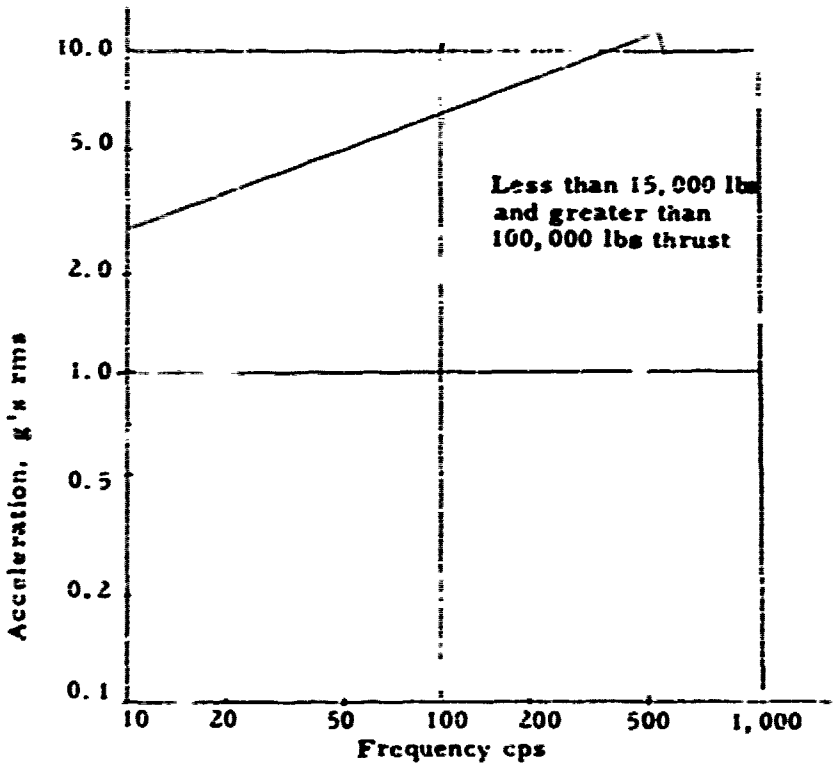
Figure 3.17 Maximum Missile Flight Accelerations

Upon examining the references, one finds very little statistical analysis and confidence in the above curves. The various authors, for the most part, do not state information regarding the data reduction methods used, location and type of transducers, lengths of records, and calibration checks. Some of the data was reduced to rms levels without regard to whether the data was random, sinusoidal or a combination of the two. Other data was processed assuming randomness, stationarity, and normality, without justifying these fundamental assumptions. Of course, on the basis of expediency, such methods may sometimes be understood, as long as these limitations are not ignored when conclusions are drawn or predictions are made. Some of the more recent reports, such as Kennard [13], have shown an awareness of this problem. information regarding methods used for obtaining the records and data reduction techniques are retained with the final presentation of the vibration data.

Figures 3. 16 and 3. 17 are representative only for certain classes of missiles and the levels shown are an "average" maximum. For specific environments the reader is referred to the literature, some of which appears in the references. For instance, there have been occurrences of acceleration levels in excess of 100 g's rms, and some of the more recent data includes frequencies above 2000 cps.

It is also interesting to note, in surveying the references that nearly all flight vehicle vibration response data are obtained in the form of acceleration levels. No direct statements have been found as to why acceleration levels are measured in preference to velocity, displacement, or strain levels. However, it is believed that acceleration data has been favored for many good reasons including the following.

Piezoelectric crystal transducers, which produce voltage signals proportional to applied acceleration, are the only commercially available vibration transducers that have a wide frequency response range (2 to over 20,000 cps) while being small and light in weight (under 1 ounce). Velocity signal generating transducers have a more limited frequency response (normally 10 to 2000 cps) and are considerably heavier (some weighing several ounces). True displacement signal generators applicable for flight test use are not commercially available. In the days of propeller airplanes, the nature

of the transducer was of little importance since the response was predominately periodic and the dynamic energy was concentrated in the frequency range below 500 cps. The transducer output signal could easily be differentiated or integrated during measurement to obtain a signal proportional to any parameter desired. In modern flight vehicles, where a broad band random response is predominate, the transducer output signals are not so easily manipulated.

To date, most of the final presentation of random vibration data has been in the form of power spectral density distributions with assumed Gaussian amplitude distributions. The actual amplitude distribution of the structural vibration response in modern vehicles has been the target of investigation by many engineers only recently with interesting results.

W. S. Shipley [21] determined the amplitude distribution of the vibration environment in the Sergeant Missile and ended up with a near perfect normal distribution. R. W. Mustain [18] investigated the amplitude distributions of the vibration environment in the SM-62 missile, but did not obtain good adherence to the normal distribution.

F. H. Eng [9] presents amplitude distribution results (distribution of peak amplitudes) from missile vibration data which deviate rather widely from the expected Rayleigh distribution. Of particular interest is a paper by Edwin Kamps [12] who investigated the distribution of peak pressures in the near-field noise generated by the exhaust of a high performance jet engine, a major source of structural vibration in piloted aircraft. The results deviated substantially from a Rayleigh distribution. This deviation is noted also by Kennard [13]

Forms of vibration data presentation and interpretation other than power spectra and amplitude distributions have been developed primarily in association with the fatigue problem. One such development considers the structural vibration environment in terms of an intensity spectrum and a structural susceptibility function, Reference [15].

It may be of interest to note that sample lengths taken for vibration measurements have usually been 2 - 10 seconds long. One reason for this is probably that magnetic tape recordings of this duration make convenient tape loops for analysis. Few references surveyed considered the statistical errors associated with sample length. Another reason for short samples may be to get around the problem of reducing data which is not stationary. One paper dealing with the problem of spectral analysis of time varying data is Reference [24].

The discussion that follows in Sections 4 through 9 of this report will provide improved methods for dealing with flight vehicle vibration problems. These methods will result in a better understanding of some of the problem areas of past measurement, and will provide the engineer with tools to increase his confidence in measurements yet to be made.

3.5 REFERENCES

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4. MATHEMATICAL BACKGROUND FOR ANALYZING
VIBRATION PHENOMENA

4.1 FORMS OF VIBRATION PHENOMENA

Steady-state vibration phenomena may be divided naturally into four main descriptive forms: (1) sinusoidal, (2) periodic, (3) complex, and (4) random. Combinations of these forms, of course, occur also. Historically the first three types of phenomena were studied mathematically quite extensively, and only in recent years have mathematical techniques been developed to properly evaluate random phenomena. This has led to many separate investigations emphasizing various limited aspects of vibration data analysis. It is the purpose of this section to look at the over-all vibration analysis problem, and to discuss in a broad way many mathematical ideas that have been found to be appropriate.

A single vibration record, or a set of records, is usually described in terms of its time behavior and its frequency characteristics. See Fig. 4.1. For example, after suitable processing, one may plot instantaneous amplitude values versus time, the frequency being fixed at a particular value. Or, one may plot mean square amplitude values (associated with a particular fixed record length) versus frequency, the time being held constant. Various other vibration properties may similarly be plotted versus time or versus frequency. Thus, a number of two-dimensional plots may be generated. This can be displayed as a three-dimensional plot provided one understands that the vertical scale represents different quantities, in general, when referred to the time and frequency axes. For a fixed value of time, frequency characteristics are displayed, while for a fixed value of frequency, time behavior is displayed.

4.1.1 Sinusoidal Vibration

Sinusoidal vibration is defined mathematically by a time-varying function satisfying the equation

$$x(t) = A \sin (2\pi f t + \theta) \tag{4.1}$$
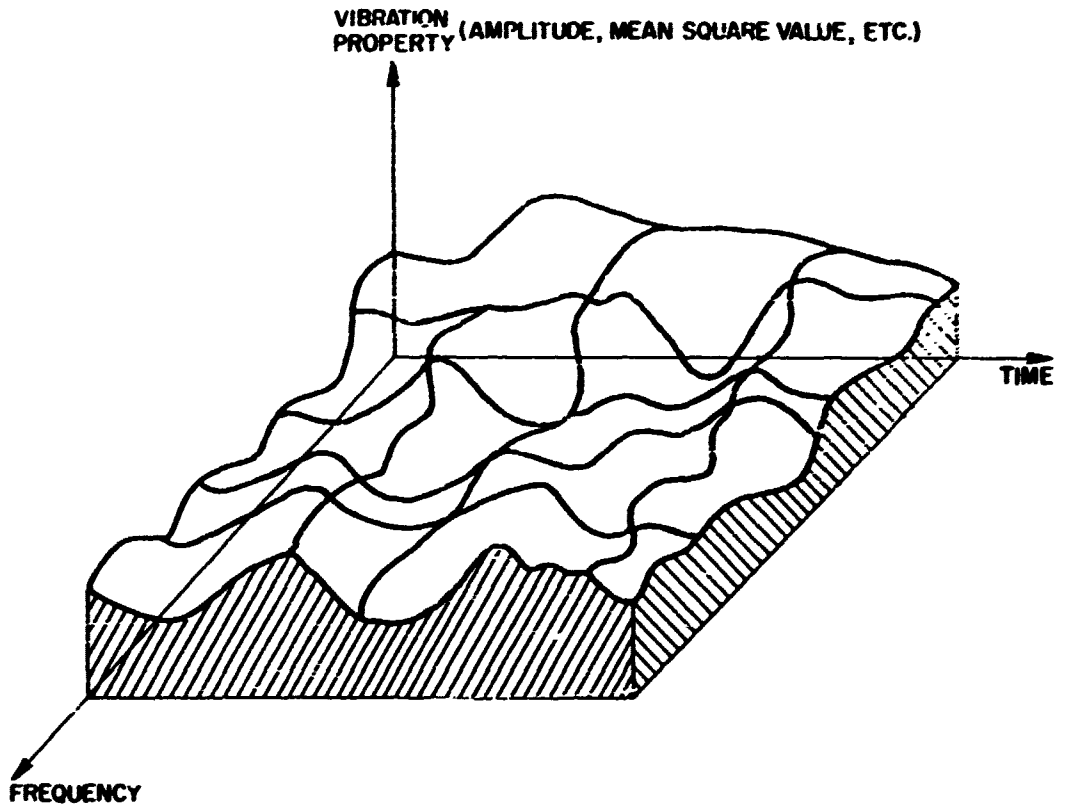
Figure 4.1 Three-Dimensional Plot of Different Vibration Properties versus Time and versus Frequency

where

A = constant amplitude factor

f = cyclical frequency (usually in cycles per second, cps)

θ = initial phase angle with respect to the time origin

x(t) = instantaneous amplitude at time t (usually in seconds),
the magnitude of x(t) itself usually being in inches or
feet or g-value (to be defined later) for vibration data.

A complete description of x(t) for all t is known once A, f and θ have
been specified. For single wave analysis, the phase angle θ is often
ignored. Thus, Eq. (4.1) may be characterized completely by A and f.
On setting θ = 0, Eq. (4.1) becomes

$$x(t) = A \sin 2\pi ft \qquad (4.2)$$

Eq. (4.2) can be pictured by an amplitude-time plot, or by an amplitude-
frequency plot. See sketch below.
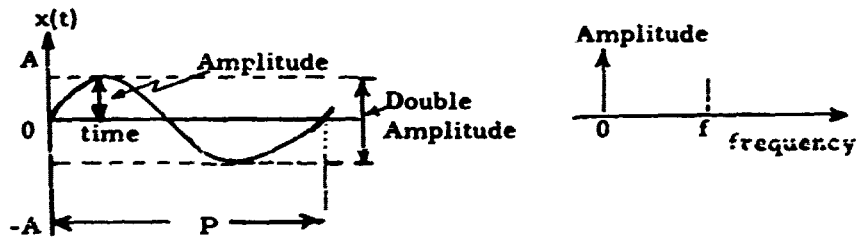
SINGLE SINE WAVE

The period P equals the time interval for one full vibration (or cycle),
usually in seconds. The frequency f equals the number of cycles per
unit time, usually cycles per second (cps), and is related to P by

$$P = 1/f \qquad (4.3)$$

The g-value equals the instantaneous acceleration magnitude expressed
in units of $g \approx 32.2 \text{ ft/sec}^2$, the acceleration due to gravity. From
Eq. (4.2)

$$\left| \ddot{x}(t) \right| = (2\pi f)^2 \left| x(t) \right|$$

Hence

$$\frac{\left| \ddot{x}(t) \right|}{g} = \frac{(2\pi f)^2 \left| x(t) \right|}{g} = g\text{-value} \qquad (4.4)$$

For example, if $x(t) = 0.012$ inches, and if $f = 100$ cps, then the associated g-value is 12.3.

In order to describe for later analysis the superposition effects of two or more sine waves, with relative phase angles which can be ignored, it is sufficient merely to plot a discrete frequency spectrum of amplitudes versus frequency. See sketch below for a graphical plot of the sum of three sine waves. The corresponding mathematical equation as a function of time t is here

$$x(t) = A_1 \sin \ 2\pi f_1 t + A_2 \sin 2\pi f_2 t + A_3 \sin 2\pi f_3 t$$
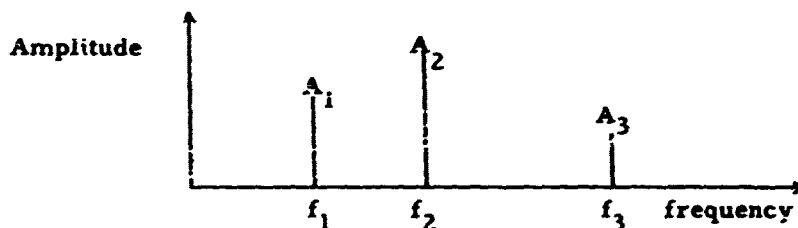
SUM OF THREE SINE WAVES
(PICTURED BY AN AMPLITUDE-FREQUENCY PLOT)

4.1.2 Periodic Vibration

Periodic vibration is defined mathematically by a time-varying function $x(t)$ whose waveform is such that there exists a fundamental period P having the property that

$$x(t) = x(t \pm nP) \quad \text{for all integers } n = 1, 2, 3, \ldots \text{ and all } t \qquad (4.5)$$

The reciprocal of P, namely $f = (1/P)$, is called the fundamental frequency of vibration. It may be shown that all frequencies present in $x(t)$ are constant multiples of this fundamental frequency.

A simple sinusoidal function is a special case of a periodic function with period $P = (1/f)$ as can be seen directly from the defining relation since

$$\sin 2\pi ft = \sin 2\pi f \left[t \pm (n/f) \right] \text{ for all } n \text{ and all } t$$

The sum of three sine waves may or may not be periodic depending on the commensurability of f_1, f_2 and f_3. For example, with three frequencies expressed by rational numbers, say $f_1 = 2/3$ cps, $f_2 = 3/4$ cps and $f_3 = 5/6$ cps, then $P = 12$ sec would be the fundamental period since 3, 4 and 6, the denominators in f_1, f_2 and f_3, respectively, would all divide into 12 an integral number of times, and no smaller number than 12 has this property. However, if any one of the frequencies in question should be expressed by an irrational number, e.g. $\sqrt{2}$ cps, while one or both of the other frequencies are expressed by rational numbers, then no fundamental period would exist.

If the time-varying function is periodic with period P, then with few exceptions in practice, it may be expanded in a Fourier series according to the formula

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos 2\pi nft + \sum_{n=1}^{\infty} b_n \sin 2\pi nft \qquad (4.6)$$

where
$$f = 1/P$$

$$a_n = \frac{2}{P} \int_0^P x(s) \cos 2\pi nfs \, ds \qquad ; \qquad n = 0, 1, 2, \ldots$$

$$b_n = \frac{2}{P} \int_0^P x(s) \sin 2\pi nfs \, ds \qquad ; \qquad n = 1, 2, 3, \ldots$$

A Fourier series is determined completely through knowledge of the amplitude coefficients a_n and b_n at frequencies nf, and may be pictured by plotting these coefficients as a function of frequency.

4.1.3 Complex Vibration

Complex vibration will be defined here as non-periodic vibration phenomena which may be described by some suitable analytic time-varying function. The availability of such an analytic function means that the entire time history of the vibration in question can be stated completely. There are no probabilistic features associated with its time behavior.

Three simple examples of complex vibration are:

$$x(t) = A_1 \sin 2\pi f_1 t + A_2 \sin 2\pi f_2 t \quad ; \quad f_1 \text{ rational, } f_2 \text{ irrational}$$

$$x(t) = \begin{cases} Ae^{-at} \sin bt & ; \quad t \geq 0 \\ 0 & ; \quad t < 0 \end{cases}$$

$$x(t) = \begin{cases} 1 & ; \quad |t| \leq C \\ 0 & ; \quad |t| > C \end{cases}$$
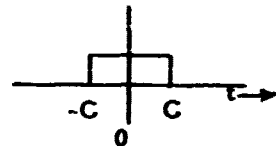
Observe that for complex vibration, as opposed to periodic vibration, there exists no fundamental period P such that $x(t) = x(t + P)$ for all t. This is the distinguishing difference between complex vibration and periodic vibration. Both types of vibration are required to have explicit analytic representations. If a time-varying function has a Fourier Integral representation, but not a Fourier Series representation, then it would belong to the class of complex vibrations rather than the class of periodic vibrations.

4.1.4 Random Vibration

Random vibration is that type of time-varying excitation which consists of randomly varying amplitudes and frequencies such that its behavior can be described only in statistical terms. No analytical representation exists for the complete vibration in question so that it cannot be classified as a complex form of vibration. The motion does not repeat itself in finite time periods. See sketch below for an example of a single random vibration record.
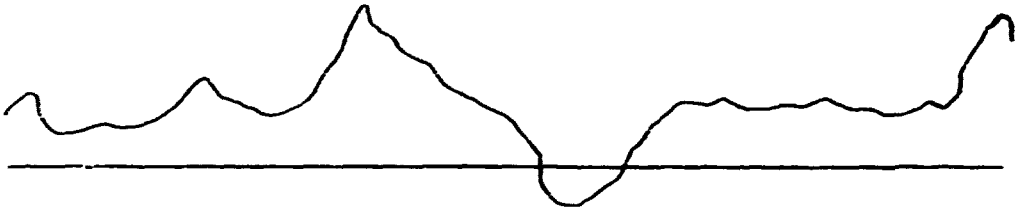
SINGLE RANDOM VIBRATION RECORD

A particular observed random vibration record is usually a unique
set of circumstances never likely to repeat exactly in all its characteris-
tics. An observed record, in actual practice, is merely a special example
out of a large set of possible records that might have occurred. In order
to analyze this single record, it is necessary, in general, to analyze
statistically the entire collection of random records of which it is a part.
This collection (also called ensemble) of records, assuming it can be
characterized statistically in ways still to be described, is known as a
random process. The sketch below is a picture of a random process in
which the individual records are laid one above the other using some
common time base.
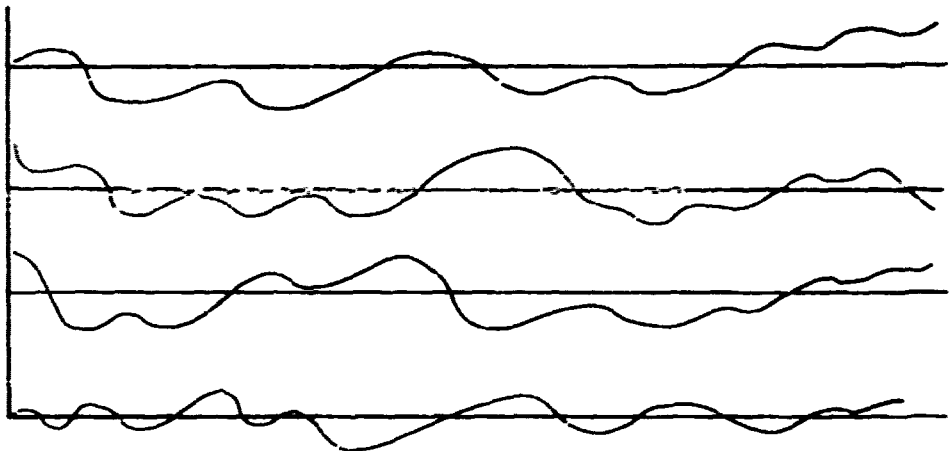
RANDOM PROCESS

In general, no individual record is representative of any other record, nor is any individual record in its time-wise behavior at fixed values of time. In general, statistical properties as averaged over the ensemble of records vary with the time at which the measurements are made. Such random processes are called "non-stationary" to distinguish them from other special "stationary" random processes, where statistical properties as averaged over the ensemble of records are invariant with respect to time. These matters will be discussed later in further analytical detail.

Present consensus of information from qualified people doing vibration data analysis for missile and space flight vehicles indicates that the overwhelming portion of data falls in the random vibration category. A limited amount of data under very special conditions is better described as (a) sinusoidal, (b) periodic, or (c) complex. For this reason, the discussion to follow will concentrate itself on analyzing random vibration phenomena, with some special attention devoted to sinusoidal vibration.

4.2 SINGLE ANALYTIC RECORDS

The <u>mean value</u> of a single analytic record x(t) of length T will be defined by

$$\bar{x} = \frac{1}{T} \int_0^T x(t) \, dt \tag{4.7}$$

The average absolute value of x(t) is defined by

$$\overline{|x|} = \frac{1}{T} \int_0^T |x(t)| \, dt \tag{4.8}$$

A function (not identically zero) with zero mean value will have a non-zero average absolute value.

The <u>mean square</u> value of x(t) is defined by

$$\overline{x^2} = \frac{1}{T} \int_0^T x^2(t) \, dt \tag{4.9}$$

By definition, the root mean square (rms) value, (usually denoted by σ if the mean value is zero) is the positive square root of the mean square value.

Example: Sine Wave.

$$x(t) = A \sin 2\pi ft \quad : \quad 0 \leq t \leq T$$

(4.10)

$$T = nP = n/f \quad \text{where } n \cdot \text{integer} \quad ; \quad P = \text{period}$$

The mean value, average absolute value, and mean square value, of a sine wave; averaged over a length T equal to an integral number of periods, are given by:

$$\bar{x} = \frac{A}{T} \int_0^T \sin 2\pi ft \, dt = 0$$

$$\overline{|x|} = \frac{A}{T} \int_0^T |\sin 2\pi ft| \, dt = \frac{A}{(1/2f)} \int_0^{(1/2f)} \sin 2\pi ft \, dt = \frac{2A}{\pi} \approx 0.636A$$

(4.11)

$$\overline{x^2} = \frac{A^2}{T} \int_0^T \sin^2(2\pi ft) \, dt = \frac{A^2}{(1/f)} \int_0^{(1/f)} \sin^2(2\pi ft) \, dt = A^2/2$$

The rms value, σ, for a sine wave is given by

$$\sigma = \sqrt{\overline{x^2}} = \frac{A}{\sqrt{2}} \approx 0.707A$$

(4.12)

Note that in terms of σ, the average absolute value

$$\overline{|x|} = \frac{2A}{\pi} = (2/\pi) \sqrt{2} \, \sigma \approx 0.900 \, \sigma$$

(4.13)

For a single analytic record $x(t)$, the (cumulative) probability distribution function $P(x_0)$ defines the fraction of time, on the average,

that the magnitude of $x(t)$ will be below x_0. Clearly, $P(-\infty) = 0$ and $P(+\infty) = 1$, since the magnitude will never be below $(-\infty)$ and will always be below $(+\infty)$. For all other values of x_0, $P(x_0)$ will be a non-decreasing function of x_0 between zero and unity.

The fraction of time, on the average, that the magnitude of $x(t)$ lies between x_1 and x_2, where $x_1 < x_2$, is given by $P(x_2) - P(x_1)$. Define the quantity

$$p(x_1, x_2) = \frac{P(x_2) - P(x_1)}{x_2 - x_1}$$

and take the limit of this quantity as $|\Delta x| = |x_2 - x_1|$ approaches zero. Assuming this limit to exist, there results the derivative

$$p(x) = \frac{dP(x)}{dx} \tag{4.14}$$

with

$$P(x) = \int_{-\infty}^{x} p(x)\, dx \tag{4.15}$$

The quantity $p(x)$ is called the <u>probability density function</u>. For small Δx, the probability that $x(t)$ lies between x_0 and $x_0 + \Delta x$ is given by $p(x_0) \Delta x$.

$$\text{Prob}\left[x_0 < x(t) \leq x_0 + \Delta x \right] = \int_{x_0}^{x_0 + \Delta x} p(x)\, dx \approx p(x_0)\, \Delta x \tag{4.16}$$

<u>Example:</u> Sine Wave.

$$x(t) = A \sin 2\pi ft$$

The probability distribution function for the instantaneous amplitude values of the above sine wave is given by, see Ref.[1, p.116].

$$P(x) = 1/2 + (1/\pi)\sin^{-1}(x/A) \qquad (4.17)$$
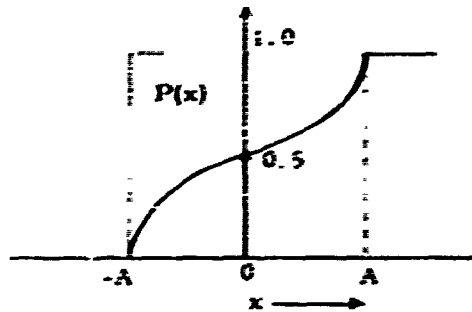
The corresponding probability density function for the instantaneous amplitude values of the sine wave is given by $p(x) = 0$ for $|x| > A$, and

$$p(x) = \frac{dP(x)}{dx} = \frac{1}{\pi\sqrt{A^2 - x^2}} \qquad ; \qquad |x| \leq A \qquad (4.18)$$
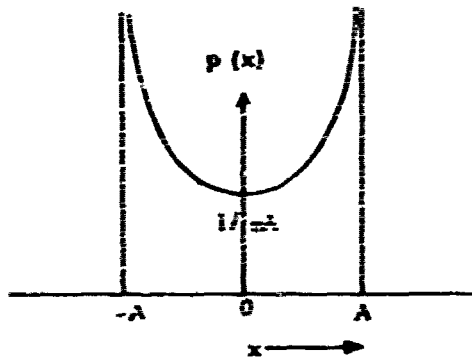
These results for a single sine wave should be well understood because of their considerable difference to a Gaussian (normal) distribution function which is associated frequently with random phenomena. The Gaussian (cumulative) probability distribution and probability density functions will be discussed in a later section.

The power spectral density function $S_x(f)$ associated with a single analytic record $x(t)$ is defined as the limiting value of the mean square

value $\overline{x^2}(t)$ contained in an ideal bandpass filter with center frequency f, divided by the bandwidth B, as the bandwidth approaches zero. Thus, the power spectral density function indicates the rate of change of the mean square value with frequency. The total mean square value in x(t) is obtained by integrating (summing) $S_x(f)$ over all frequencies from $f = -\infty$ to $f = \infty$, (equivalent to integrating $2S_x(f)$ from $f = 0$ to $f = \infty$), while the mean square value of x(t) between the frequencies f_a and f_b is obtained by integrating $2S(f)$ from f_a to f_b. (The factor of 2 results from mathematical inclusion of negative frequencies.) In equation form

$$\sigma_x^2(f_a \leq f \leq f_b) = 2 \int_{f_a}^{f_b} S_x(f)\, df \tag{4.19}$$

$$\sigma_x^2 = \overline{x^2}(t) = \int_{-\infty}^{\infty} S_x(f)\, df = 2 \int_0^{\infty} S_x(f)\, df$$

Example: Sine Wave

$$x(t) = A \sin 2\pi f_0 t$$

Since x(t) contains power only at one frequency f_0, and since the total mean square value in x(t) is given by

$$\overline{x^2}(t) = \frac{A^2}{2} = \int_{-\infty}^{\infty} S_x(f)\, df \tag{4.20}$$

it follows that

$$S_x(f) = (A^2/2)\, \delta(f - f_0) \tag{4.21}$$

where $\delta(f - f_0)$ is the usual Dirac delta function as defined by

$$\delta(f - f_0) = 0 \qquad \text{for} \qquad f \neq f_0 \tag{4.22}$$

$$\int_{-\infty}^{\infty} \delta(f - f_0)\, df = 1$$

In words, the spectral density function of a sine wave of amplitude A at frequency f_0 is a delta function at f_0 multiplied by the power factor $(A^2/2)$

The _autocorrelation function_ $R_x(\tau, T)$ associated with a single analytic record $x(t)$ of fixed finite length T will be defined by

$$R_x(\tau, T) = \frac{1}{T - \tau} \int_0^{T-\tau} x(t) \, x(t+\tau) \, dt \qquad 0 \le \tau \le T \qquad (4.23)$$

$$\approx (1/T) \int_0^T x(t) \, x(t+\tau) \, dt \qquad \text{if} \qquad \tau \ll T$$

For fixed τ, as T approaches infinity, the autocorrelation function $R_x(\tau)$ is defined by

$$R_x(\tau) = \lim_{T \to \infty} (1/T) \int_0^T x(t) \, x(t+\tau) \, dt \qquad (4.24)$$

In particular, the above definition (without passage to the limit) applies to a periodic function of period P equal to T

Example: Sine Wave

$$x(t) = A \sin 2\pi f t \qquad , \qquad \text{Period } T = (1/f)$$

$$R_x(\tau) = (1/T) \int_0^T A^2 \sin 2\pi f t \, \sin 2\pi f(t+\tau) \, dt$$

$$= (A^2/2) \cos 2\pi f \tau \qquad (4.25)$$

In words, the autocorrelation function of a sine wave of amplitude A at frequency f is a cosine wave of amplitude $(A^2/2)$ and same frequency f. Thus, periodicities present in a function are retained in its autocorrelation function. Note from Eqs. (4.25) and (4.20) that

$$R_x(0) = (A^2/2)' = \int_{-\infty}^{\infty} S_x(f) \, df \qquad (4.26)$$

One may, in fact, prove a more general result

$$R_x(\tau) = \int_{-\infty}^{\infty} S_x(f) \cos 2\pi f \tau \, df \qquad (4.27)$$

The above results may be extended easily to sums of sine waves and to other analytic functions. The discussion to follow will now concern itself with establishing various fundamental probability and statistical concepts which extend the above treatment on single analytic records to the more important physical situations of random phenomena.

4.3 PROBABILITY FUNDAMENTALS FOR RANDOM RECORDS

The underlying concept in probability theory is that of a set, namely, a collection of objects such that it is possible to determine of any particular object whether or not it is a member of the set. In particular, the possible outcomes of an experiment (or a measurement) represent a set of points called the sample space. These points may be grouped together in various ways, called events, and under suitable conditions probability measures may be assigned to each event. These probabilities always lie between zero and one, the probability of an impossible event being zero, of a certain event being one. For sample spaces containing a finite number of points, the probability of a particular event is simply the ratio of the number of points in the event to all possible points. For sample spaces containing an infinite number of points, a more sophisticated approach is required.

The union of two events A and B is the set of sample points which belong to A or to B or to both, and is denoted by

$$A \cup B \qquad \text{or} \qquad B \cup A$$

The intersection of two events A and B consists of the set of sample points which belong both to A and to B, and is denoted by

$$A \cap B \qquad \text{or} \qquad B \cap A$$

A probability measure may be assigned to the set of events of a sample space if the following axioms hold:

(1) To each event A, a probability $P(A)$ is defined by a non-negative real number depending on A such that $0 \leq P(A) \leq 1$.

(2) If A and B are events, then $A \cup B$ and $A \cap B$ are events.

(3) There is a most general event, say G, which includes all other possible events.

(4) For the most general event G, $P(G) = 1$.

(5) If A and B are different events with no point in common, then

$$P(A \cup B) = P(A) + P(B) \tag{4.28}$$

For the above case, the probability measure is said to be additive. For the general case where A and B may have overlapping points,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \tag{4.29}$$

4.3.1 One Random Variable

A random variable $R(k)$ is defined as a real-valued point function of k, where k is a point from the sample space. That is, a random variable $R(k)$ represents a real number between $-\infty$ and $+\infty$ which is associated to each sample point k that might occur. Stated another way, the outcome of an experiment, namely k, can be represented by a real number, namely $R(k)$. For historical reasons, this numerical random outcome is called a random variable.

Let $R(k)$ denote a certain random variable, then for any fixed number α, the random event, $R(k) \leq \alpha$, is the set of possible outcomes k such that $R(k) \leq \alpha$. In terms of the underlying probability measure in the sample space, one may define a (first-order) <u>cumulative distribution function</u> $P_R(\alpha)$ as the probability which is assigned to the set of points k satisfying the desired inequality $R(k) \leq \alpha$. Observe that the set of points k satisfying $R(k) \leq \alpha$ is a subset of the totality of all points k which satisfy $R(k) \leq \infty$. In notation form

$$P_R(\alpha) = \Prob_{k} \left[R(k) \leq \alpha \right] \tag{4.30}$$

Clearly

$$P_R(a) \leq P_R(b) \qquad \text{if} \qquad a \leq b$$

$$P_R(-\infty) = 0 \qquad ; \qquad P_R(\infty) = 1$$

(4.31)

For example, let the sample space consist of tosses of a single coin where the two possible outcomes, called heads and tails, are assumed to occur with equal probability (1/2). The random variable R(k) for this example takes on only two discrete values, R(heads) and R(tails), to which arbitrary real numbers may be assigned, e. g., let R(heads) = a and R(tails) = b where a and b are real numbers with, say, b > a. With these choices for R(k), it follows that the distribution function

$$P_R(\alpha) = \begin{cases} 0 & ; \quad \alpha < a \\ 1/2 & ; \quad a \leq \alpha < b \\ 1 & ; \quad \alpha \geq b \end{cases}$$
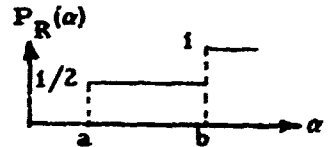
As a second example, let the sample space consist of choosing a point at random in the interval [0, 1], including the end points. A continuous random variable R(k) for this example may be defined by the numerical value of the chosen point. The corresponding distribution function becomes

$$P_R(\alpha) = \begin{cases} 0 & ; \quad \alpha < 0 \\ \alpha & ; \quad 0 \leq \alpha \leq 1 \\ 1 & ; \quad \alpha \geq 1 \end{cases}$$
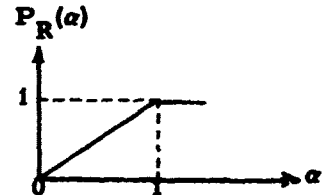
If the random variable assumes a continuous range of values (which will be assumed hereafter) then a (first-order) probability density function $p_R(\alpha)$ may be defined by the differential relation

$$p_R(\alpha) \, d\alpha = \text{Prob}\left[\alpha < R(k) \leq \alpha + d\alpha\right]$$

(4.32)

Note that

$$p_R(\alpha) \geq 0$$

(4 33)

$$\int_{-\infty}^{\infty} p_R(\alpha)\, d\alpha = 1$$

$$P_R(\alpha) = \int_{-\infty}^{\alpha} p_R(\alpha)\, d\alpha \quad ; \quad \frac{dP_R(\alpha)}{d\alpha} = p_R(\alpha)$$

The probability density function $p_R(\alpha)$ should not be confused with the (cumulative) probability distribution function $P_R(\alpha)$.

Suppose $R = R(k)$ takes on values in the range $-\infty$ to $+\infty$. Then the mean value (also called expected value. average value) of R is given by the limit of the sum of assumed values when each value is multiplied by its appropriate probability of occurrence. That is,

$$E(R) = \lim_{N \to \infty} \sum_{i=1}^{N} \alpha_i \, \text{Prob}\left[R(k) = \alpha_i\right] = \int_{-\infty}^{\infty} \alpha\, p_R(\alpha)\, d\alpha = \overline{R}$$

(4.34)

Similarly. the expected value of any real single-valued continuous function $g(R)$ of the random variable R is given by

$$E\left[g(R)\right] = \int_{-\infty}^{\infty} g(\alpha)\, p_R(\alpha)\, d\alpha = \overline{g(R)}$$

(4.35)

In particular. for $g(R) = R^2$, the mean square value of R is given by

$$E\left[R^2\right] = \int_{-\infty}^{\infty} \alpha^2\, p_R(\alpha)\, d\alpha = \overline{R^2}$$

(4.36)

and the variance of R is defined by the mean square value of R about its mean value. namely by.

$$\sigma^2(R) = E\left[(R - \overline{R})^2\right] = \overline{R^2} - (\overline{R})^2$$

(4.37)

By definition, the underlined standard deviation of R, denoted by σ, is the positive square root of the variance. The standard deviation is measured in the same units as the mean value.

4.3.2 Two Random Variables

Consider next two random variables $R = R(u)$ and $S = S(v)$ where u and v are points in a suitable sample space. Let $P_R(\alpha)$ and $P_S(\beta)$ be the distribution functions associated with R and S respectively. The joint (second-order) cumulative distribution function $P_{R,S}(\alpha, \beta)$ is defined to be the probability which is associated with the subset of combined points (u, v) in the sample space satisfying the inequalities $R(u) \leq \alpha$ and $R(v) \leq \beta$. The total set of combined points (u, v) satisfies the inequalities $R(u) \leq \infty$ and $R(v) \leq \infty$. In notation form,

$$P_{R,S}(\alpha, \beta) = \text{Prob} \left[R(u) \leq \alpha \quad \text{and} \quad S(v) \leq \beta \right] \tag{4.38}$$

Clearly,

$$P_{R,S}(-\infty, \beta) = 0 = P_{R,S}(\alpha, -\infty)$$

$$P_{R,S}(-\infty, \infty) = 1$$

As before, assuming the random variables to be continuous, the joint (cumulative) probability distribution function $P_{R,S}(\alpha, \beta)$ should not be confused with the joint (second-order) probability density function $p_{R,S}(\alpha, \beta)$ which is defined by the differential relation

$$p_{R,S}(\alpha, \beta) \, d\alpha \, d\beta = \text{Prob} \left[\alpha < R(u) \leq \alpha + d\alpha \quad \text{and} \quad \beta < S(v) \leq \beta + d\beta \right] \tag{4.39}$$

Note that

$$p_{R,S}(\alpha, \beta) \geq 0$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_{R,S}(\alpha, \beta) \, d\alpha \, d\beta = 1$$

$$P_{R, S}(\alpha, \beta) = \int_{-\infty}^{\alpha} \int_{-\infty}^{\beta} p_{R, S}(\alpha, \beta) \, d\alpha \, d\beta$$

Also

$$p_R(\alpha) = \int_{-\infty}^{\infty} p_{R, S}(\alpha, \beta) \, d\beta$$

$$p_S(\beta) = \int_{-\infty}^{\infty} p_{R, S}(\alpha, \beta) \, d\alpha$$

Two random variables R and S are said to be (statistically) independent if

$$p_{R, S}(\alpha, \beta) = p_R(\alpha) \, p_S(\beta) \tag{4.40}$$

It follows that

$$P_{R, S}(\alpha, \beta) = P_R(\alpha) \, P_S(\beta)$$

The expected value of any real single-valued continuous function $g(R, S)$ of the two random variables $R = R(u)$ and $S = S(v)$ is given by

$$E[g(R, S)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\alpha, \beta) \, p_{R, S}(\alpha, \beta) \, d\alpha \, d\beta = \overline{g(R, S)} \tag{4.41}$$

For example, if $g(R, S) = (R - \bar{R})(S - \bar{S})$ where $\bar{R}$ and $\bar{S}$ are the mean values of R and S, respectively, this defines the covariance $\rho(R, S)$ between R and S, namely,

$$\rho(R, S) = E[(R - \bar{R})(S - \bar{S})] = E[RS] - E[R] \, E[S]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\alpha - \bar{R})(\beta - \bar{S}) \, p_{R, S}(\alpha, \beta) \, d\alpha \, d\beta \tag{4.42}$$

A simple relation exists between the covariance of R and S and the standard deviations of R and S as expressed by the inequality

$$\left| \rho(R, S) \right| \leq \sigma(R) \, \sigma(S) \tag{4.43}$$

Thus, the magnitude of the covariance between R and S is less than or equal to the product of the standard deviation of R multiplied by the standard deviation of S.

It follows from the above result that the normalized quantity

$$\Gamma(R, S) = \frac{\rho(R, S)}{\sigma(R) \, \sigma(S)} \tag{4.44}$$

known as the correlation coefficient, will lie between -1 and +1. Random variables R and S whose correlation coefficient is zero are said to be uncorrelated. This concept should not be confused with the previous definition of independent random variables. Note that if R and S are independent random variables, then

$$E[RS] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \alpha\beta \, p_{R,S}(\alpha, \beta) \, d\alpha \, d\beta$$

$$= \int_{-\infty}^{\infty} \alpha \, p_R(\alpha) \, d\alpha \int_{-\infty}^{\infty} \beta \, p_S(\beta) \, d\beta = E[R] \, E[S] \tag{4.45}$$

Hence $\rho(R, S)$ and, in turn, $\Gamma(R, S)$ equal zero so that independent random variables are also uncorrelated. The converse statement is not true in general; that is to say, uncorrelated random variables are not necessarily independent. However, for physically important situations involving two or more normally (Gaussian) distributed random variables (to be defined later), being mutually uncorrelated does imply independence.

The conditional probability density function of R, given that $S = \beta$, (i.e., given that S is between β and $\beta + d\beta$ for small $d\beta$), is defined by

$$p_R(\alpha \mid S = \beta) \, d\alpha = \text{Prob}\left[\alpha < R(u) \leq \alpha + d\alpha \mid \text{given that } \beta < S(v) \leq \beta + d\beta \right]$$

or

$$P_R(\alpha \,|\, S = \beta) = \frac{P_{R,S}(\alpha,\beta)}{P_S(\beta)} \qquad \text{assuming} \qquad P_S(\beta) \neq 0 \qquad (4.46)$$

For independent random variables, this simplifies to

$$P_R(\alpha \,|\, S = \beta) = \frac{P_R(\alpha)\, P_S(\beta)}{P_S(\beta)} \cdot P_R(\alpha) \qquad (4.47)$$

In words, the conditional probability density function for R, given S, is now the same as the original probability density function for R alone.

These ideas may be extended in a straight-forward manner to handle situations of three or more random variables where higher-order probability distributions would be involved.

4.3.3 Special Probability Distributions

By way of illustration, as well as because of their importance to physical problems, some special probability distributions will now be described.

(a) Uniform (Rectangular) Distribution

A random variable R is said to follow a <u>uniform (or rectangular)</u> distribution over the interval $(a \leq \alpha \leq b)$ if its probability density function is given by, Ref. $\left[1. \quad p \quad 86 \right]$,

$$P_R(\alpha) = \frac{1}{b - a} \qquad ; \qquad a \leq \alpha \leq \beta$$
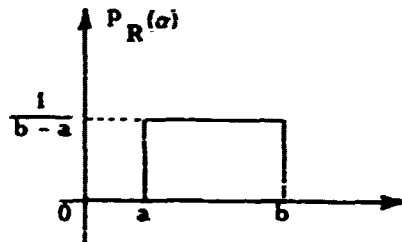
$$= 0 \qquad ; \qquad \text{otherwise} \qquad (4.48)$$

The corresponding (cumulative) uniform distribution function is

$$P_R(\alpha) = \int_{-\infty}^{\alpha} P_R(\alpha)\, d\alpha = \frac{\alpha - a}{b - a} \qquad ; \qquad a \leq \alpha \leq b$$

$$= 0 \qquad ; \qquad \text{otherwise}$$
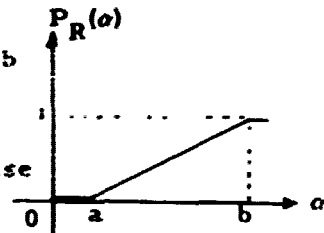
The mean value μ and the variance σ^2 (standard deviation = σ) of the random variable R become

$$\mu = E[R] = \int_{-\infty}^{\infty} \alpha\, p_R(\alpha)\, d\alpha = \frac{1}{b-a} \int_a^b \alpha\, d\alpha = \frac{a+b}{2}$$

$$\sigma^2 = E[(R-\mu)^2] = \int_{-\infty}^{\infty} (\alpha-\mu)^2\, p_R(\alpha)\, d\alpha = \frac{(b-a)^2}{12}$$

<u>Problem:</u> Find the probability that R lies in the range $[\mu - \lambda\sigma,\ \mu + \lambda\sigma]$ where λ is a positive numerical constant. The answer here is

$$\text{Prob}[\mu - \lambda\sigma < R \le \mu + \lambda\sigma] = \lambda(\sigma/\mu) \qquad (4.49)$$

For example, if R is uniformly distributed in the range (0, b), then $\mu = (b/2)$ and $\sigma = (b/\sqrt{12})$. Now, if $\lambda = 1.0$, then

$$\text{Prob}[\mu - \sigma < R \le \mu + \sigma] = 57.7\%$$

On the other hand, the value of λ such that the probability will be 95% is obtained from

$$0.95 = \lambda\,(\sigma/\mu) = \lambda(2/\sqrt{12}) \quad \text{or} \quad \lambda \approx 1.65$$

Thus

$$\text{Prob}[\mu - 1.65\,\sigma \le R \le \mu + 1.65] = 0.95$$

These numerical values for λ are quite different from values appropriate to the normal distribution which follows.

(b) Normal (Gaussian) Distribution

A random variable R is said to follow a <u>normal</u> (or <u>Gaussian</u>) distribution if its probability density function is given by, Ref. $[1 \quad p.93]$.

$$P_R(\alpha) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-(\alpha - \mu)^2/2\sigma^2\right] \quad (4.50)$$
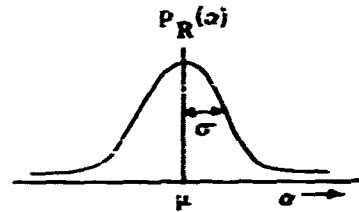
where μ is any real constant and σ is any positive constant. It is verified easily that μ and σ constitute the mean value and standard deviation of the random variable R since

$$E[R] = \int_{-\infty}^{\infty} \alpha P_R(\alpha)\, d\alpha = \mu$$

$$E[(R - \mu)^2] = \int_{-\infty}^{\infty} (\alpha - \mu)^2\, P_R(\alpha)\, d\alpha = \sigma^2$$

The normal distribution function is by definition

$$P_R(\alpha) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\alpha} \exp\left[-(\alpha - \mu)^2/2\sigma^2\right]\, d\alpha$$
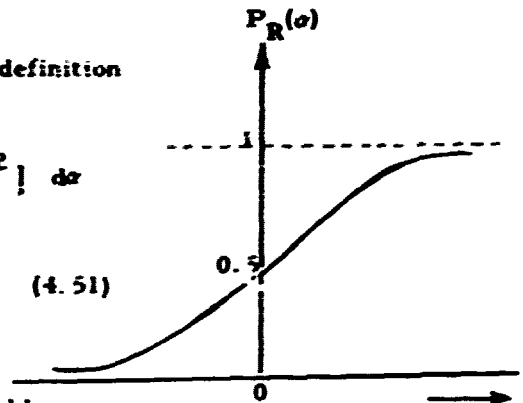
$$= \text{Prob}[R \le \alpha] \qquad (4.51)$$

and is readily available in statistical tables

Using simple numerical methods, or from tables, it is now convenient to determine the probability that the random variable R will assume values in any desired range. In particular

$$P_R(\mu + \lambda\sigma) - P_R(\mu - \lambda\sigma) = \text{Prob}[\mu - \lambda\sigma < R \le \mu + \lambda\sigma] \qquad (4.52)$$

represents the probability (i.e., confidence level) that R will be within plus and minus λ standard deviations of the mean value. For $\lambda = 1$, 2, and 3, the confidence results are 68.3%, 95.4% and 99.7%, respectively. Working the other way, for 80% confidence $\lambda = 1.3$.

For a large positive constant λ, one may show for a Gaussian distribution that

$$\text{Prob}\left[R > \mu + \lambda\sigma\right] \approx \frac{1}{\lambda\sqrt{2\pi}} e^{-\lambda^2/2}$$

In particular, for $\lambda = 3$,

$$\text{Prob}\left[R > \mu + 3\sigma\right] \approx 0.002.$$

The importance of the normal distribution in physical problems may be attributed in part to the <u>Central Limit Theorem</u> which asserts that this distribution is approximated closely by the distribution of the sum random variable of a large number of independent small random variables acting together.

To be a bit more specific, let R_1, R_2, ..., R_N be N mutually independent random variables whose individual distributions are not specified. Denote by μ_i and σ_i^2 the mean value and variance of each R_i, $i = 1, 2, ..., N$. Consider the sum random variable

$$R = \sum_{i=1}^{N} a_i R_i$$

where a_i are arbitrary fixed constants and assume that none of the $a_i R_i$ contributes significantly to the sum. Now, the mean value μ and variance σ^2 of R become

$$\mu = E[R] = E\left(\sum_{i=1}^{N} a_i R_i\right) = \sum_{i=1}^{N} a_i E[R_i] = \sum_{i=1}^{N} a_i \mu_i$$

$$\sigma^2 = E[(R - \mu)^2] = E\left[\sum_{i=1}^{N} a_i(R_i - \mu_i)\right]^2 = \sum_{i=1}^{N} a_i^2 \sigma_i^2$$

the last result due to the mutual independence of R_i with R_j for $i \neq j$. The Central Limit Theorem states that under fairly common conditions, the sum random variable R will be normally distributed as $N \rightarrow \infty$ with the above mean value μ and variance σ^2, see Ref.[1, p. 97].

(c) Truncated Normal Distribution

A random variable R is said to follow a truncated normal distribution in the range $a < A$ if its probability density function is given by

$$p_R(a) = \frac{1}{\sigma C \sqrt{2\pi}} \exp\left[-(a-\mu)^2/2\sigma^2\right] \quad , \quad a < A$$

$$= 0 \qquad\qquad\qquad\qquad ; \quad a \geq A$$

(4.53)

where

$$C = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{A} \exp\left[-(a-\mu)^2/2\sigma^2\right] da$$

The reason for introducing the constant C is to satisfy the requirement

$$\int_{-\infty}^{\infty} p_R(a)\, da = 1$$

Observe that to the left of the value $a = A$, except for the scale factor C, the truncated normal distribution has the same shape as the original untruncated normal distribution. The parameters μ and σ here do not represent the mean value and standard deviation of the truncated distribution, but refer back to the underlying untruncated distribution.

(d) Rayleigh Distribution

A random variable R which is restricted to non-negative values is said to follow a Rayleigh distribution if its probability density function is given by, Ref [1, p.136],

$$p_R(a) = (a/v^2)\exp(-a^2/2v^2) \quad ; \quad a \geq 0$$

$$= 0 \qquad\qquad\qquad\qquad ; \quad a < 0$$

(4.54)

The Rayleigh distribution should not be confused with a Gaussian distribution where the random variable may take on both positive and negative values.

The corresponding Rayleigh distribution function is given by

$$P_R(a) = \text{Prob}[\ R \leq a\ ] = 1 - \exp\{-a^2/2v^2\}$$

$$(4.55)$$
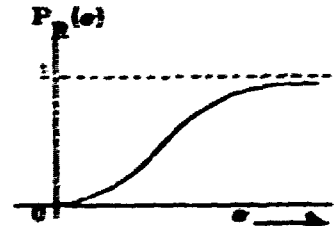
For a Rayleigh distribution, the mean value and mean square value are

$$E[R] = \int_0^\infty a\, p_R(a)\, da = (\pi/2)^{1/2}\, v \approx 1.25\, v$$

$$E[R^2] = \int_0^\infty a^2\, p_R(a)\, da = 2\, v^2$$

Hence the variance is now expressed by

$$\sigma^2 = E[R^2] - (E[R])^2 = \left(\frac{4 - \pi}{2}\right) v^2 \approx 0.43 v^2$$

(e) N-Dimensional Normal Distribution

Let $R_1, R_2, \ldots, R_n$ be N random variables defined over the same sample space. Denote their mean values, variances, and covariances by

$$\mu_i = E[R_i]$$

$$\sigma_i^2 = E[(R_i - \mu_i)^2]$$

$$\rho_{ij} = E[(R_i - \mu_i)(R_j - \mu_j)] \qquad . \qquad \rho_{ii} = \sigma_i^2$$

Their combined distribution is said to be an N-dimensional normal distribution if the associated N-fold probability density function is given by, Ref [1, p. 118].

$$p(a_1, a_2, \ldots, a_N) = \frac{\exp\left(\frac{-1}{2|M|} \sum_{i,j=1}^{N} M_{ij}(a_i - \mu_i)(a_j - \mu_j)\right)}{(2\pi)^{N/2} |M|^{1/2}} \tag{4.56}$$

where $\|M\|$ is the covariance matrix of the ρ_{ij}, $|M|$ is the determinant of $\|M\|$, and M_{ij} is the cofactor of ρ_{ij} in determinant $|M|$. To be more explicit,

$$\|M\| = \begin{vmatrix} \rho_{11} & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{21} & \rho_{22} & \cdots & \rho_{2n} \\ \rho_{n1} & \rho_{n2} & \cdots & \rho_{nn} \end{vmatrix} \qquad \rho_{ii} = \sigma_i^2 \tag{4.57}$$

and the cofactor M_{ij} of any element ρ_{ij} is defined to be the determinant of order N-1 formed by omitting the ith row and jth column of $\|M\|$, multiplied by $(-1)^{i+j}$.

The outstanding feature of the N-dimensional normal distribution is that all of its properties are determined solely from knowledge of the various mean values μ_i and covariances ρ_{ij}. For N = 1, the above reduces to

$$p(a_1) = \frac{1}{\sigma_1 \sqrt{2\pi}} \exp\left[-(a_1 - \mu_1)^2 / 2\sigma_1^2\right] \tag{4.58}$$

which is the usual one-dimensional normal distribution defined previously. For N = 2, there results

$$p(a_1, a_2) = \frac{\exp\left\{\frac{-1}{2(1 - r_{12}^2)}\left[\left(\frac{a_1 - \mu_1}{\sigma_1}\right)^2 - 2r_{12}\left(\frac{a_1 - \mu_1}{\sigma_1}\right)\left(\frac{a_2 - \mu_2}{\sigma_2}\right) + \left(\frac{a_2 - \mu_2}{\sigma_2}\right)^2\right]\right\}}{2\pi\sigma_1\sigma_2\sqrt{1 - r_{12}^2}}$$

$$\tag{4.59}$$

where $\Gamma_{12} = \dfrac{\rho_{12}}{\sigma_1 \sigma_2}$ is the correlation coefficient between R_1 and R_2.

Observe that when R_1 and R_2 are uncorrelated so that $\Gamma_{12} = 0$, one obtains

$$p(\alpha_1 \alpha_2) = p(\alpha_1) \, p(\alpha_2) \tag{4.60}$$

which shows that R_1 and R_2 are also independent.

Similar formulas may be written down for higher order cases where $N = 3, 4, 5, \ldots$ etc. For arbitrary N, it follows quite easily that if all different pairs of normally distribution random variables are mutually uncorrelated, (i. e., $\Gamma_{ij} = 0$ whenever $i \neq j$), then these random variables are mutually independent in the probability sense, that is

$$p(\alpha_1 \, \alpha_2, \ldots, \alpha_N) = p(\alpha_1) \, p(\alpha_2) \cdots p(\alpha_N) \tag{4.61}$$

The importance of the N-dimensional normal distribution in physical problems, analogous to the common one-dimensional normal distribution, is due in part to the <u>Multidimensional Central Limit Theorem</u>. This theorem yields the result that the vector sum of a large number of mutually independent N-dimensional random variables approaches an N-dimensional normal distribution under fairly general conditions. Particular applications of this theorem, relative to zero crossing properties of random records and expected number of maxima values, for example, may be used to justify an assumption that a random record $x(t)$ and its succeeding time derivative $\dot{x}(t)$ will follow a two-dimensional normal distribution, and that $x(t)$, $\dot{x}(t)$ and $\ddot{x}(t)$ will follow a three-dimensional normal distribution.

(f) Distribution of Sums of Independent Random Variables

Suppose R_1 and R_2 are independent random variables with probability density functions $p_{R_1}(\alpha_1)$ and $p_{R_2}(\alpha_2)$ respectively. Let

$$R = a_1 R_1 + a_2 R_2 \tag{4.62}$$

be a typical composite sum record of R_1 with R_2 where $a_1 \neq 0$ and $a_2 \neq 0$ are arbitrary fixed constants (usually known in engineering problems). Then the probability density function $p_R(\alpha)$ associated with R is given by

$$p_R(\alpha) = \int_{-\infty}^{\infty} p_{R_1}(\alpha_1) \, p_{R_2}[(\alpha - a_1 \alpha_1)/a_2] \, d\alpha_1 \qquad (4.63)$$

For the special case where $R = R_1 + R_2$, that is, $a_1 = a_2 = 1$, one obtains

$$p_R(\alpha) = \int_{-\infty}^{\infty} p_{R_1}(\alpha_1) \, p_{R_2}(\alpha - \alpha_1) \, d\alpha_1 \qquad (4.64)$$

From the above relation, one may verify that the sum of two independent uniform distributions is no longer a uniform distribution. However, the sum of two independent normal distributions remains a normal distribution, with mean and variance equal to the sum of the individual means and variances. The latter result may be extended to the sum of N independent normally distributed random variables.

4.4 RANDOM PROCESSES

A random process $\{^k x(t)\}$, $-\infty < t < \infty$, $k = 1, 2, 3, \ldots$, is an ensemble of functions of time which can be characterized through its statistical properties. See Figure 4.2. In the physical world, each particular $^k x(t)$, t variable, k fixed, represents the result of a single observation or experiment, and constitutes a sample function of the random process.

For example, each $^k x(t)$ might represent vertical wing loads on an airplane as a function of time t, the superscript index k denoting different airplanes; or each $^k x(t)$ might represent runway roughness at different locations k as a function of distance, the distance variable taking the place of time.

A particular sample function, in general, would not be suitable for representing the entire random process to which it belongs. Under certain conditions to be described later, however, it turns out that for the class or ergodic random processes, it is possible to derive desired statistical information about the entire random process from appropriate analysis
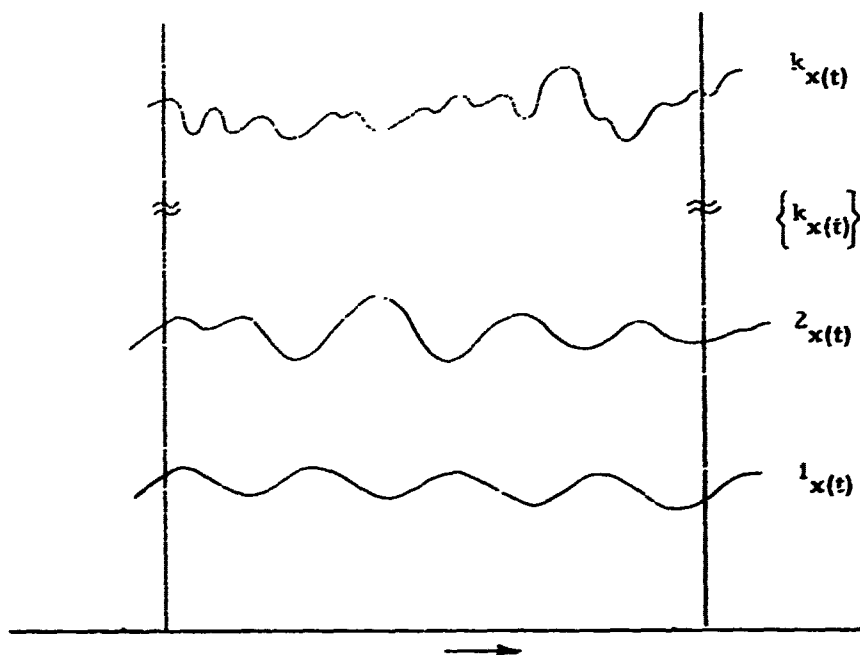
Figure 4.2 Random Process

of a single arbitrary sample function. For the situation of a pair of random processes $\left\{{}^{k}x(t)\right\}$ and $\left\{{}^{k}y(t)\right\}$, the corresponding problem is to estimate joint statistical properties of the two random processes from proper analysis of an arbitrary pair of sample functions ${}^{k}x(t)$ and ${}^{k}y(t)$.

Consider two random processes $\left\{{}^{k}x(t)\right\}$ and $\left\{{}^{k}y(t)\right\}$. The first statistical quantities of interest are the <u>mean values</u> at arbitrary fixed values of t, denoted by

$$\mu_x(t) = \left\langle {}^{k}x(t) \right\rangle_{\text{Av over } k} \quad ; \quad \text{t fixed}$$

$$\mu_y(t) = \left\langle {}^{k}y(t) \right\rangle_{\text{Av over } k} \quad ; \quad \text{t fixed}$$

(4.65)

Note that k is averaged out in computing these ensemble averages which are indicated by angular brackets. In general, these mean values are different at different times, that is,

$$\mu_x(t_1) \neq \mu_x(t_2) \quad \text{if} \quad t_1 \neq t_2$$

$$\mu_y(t_1) \neq \mu_y(t_2) \quad \text{if} \quad t_1 \neq t_2$$

The next statistical quantities of interest are the <u>covariance functions</u> at arbitrary fixed values of τ and t,

$$\rho_x(\tau, t) = \left\langle \left[{}^k x(t) - \mu_x(t) \right] \left[{}^k x(t + \tau) - \mu_x(t + \tau) \right] \right\rangle_{\text{Av over } k}$$

$$\rho_y(\tau, t) = \left\langle \left[{}^k y(t) - \mu_y(t) \right] \left[{}^k y(t + \tau) - \mu_y(t + \tau) \right] \right\rangle_{\text{Av over } k} \qquad (4.66)$$

$$\rho_{xy}(\tau, t) = \left\langle \left[{}^k x(t) - \mu_x(t) \right] \left[{}^k y(t + \tau) - \mu_y(t + \tau) \right] \right\rangle_{\text{Av over } k}$$

In general, these quantities are different for different combinations of τ and t. Observe that at $\tau = 0$, (omitting the index k for simplicity in notation, but still retaining angular brackets to imply ensemble averages)

$$\rho_x(0, t) = \left\langle \left[x(t) - \mu_x(t) \right]^2 \right\rangle_{\text{Av}} = \sigma_x^2(t)$$

$$\rho_y(0, t) = \left\langle \left[y(t) - \mu_y(t) \right]^2 \right\rangle_{\text{Av}} = \sigma_y^2(t) \qquad (4.67)$$

$$\rho_{xy}(0, t) = \left\langle \left[x(t) - \mu_x(t) \right]\left[y(t) - \mu_y(t) \right] \right\rangle_{\text{Av}} = \rho_{xy}(t)$$

Thus the covariance functions $\rho_x(0, t)$ and $\rho_y(0, t)$ represent the ordinary variances of $\{x(t)\}$ and $\{y(t)\}$ at a fixed value of t, while $\rho_{xy}(0, t)$ represents the covariance between $\{x(t)\}$ and $\{y(t)\}$. As before, different results would generally be obtained for different values of t.

Other statistical quantities can be defined over the ensemble which involve fixing three or more times, and in this way, the random processes can be described in finer and finer detail. However, if $\{x(t)\}$, $\{y(t)\}$ form a two-dimensional Gaussian distribution at a fixed value of t, then $\{x(t)\}$ and $\{y(t)\}$ are separately Gaussian. The mean values and covariance functions listed above then provide a complete description of the underlying probability structure. For this reason, the main emphasis in this section is concerned only with these two statistical quantities and their relationships to power spectral density functions.

If the mean values $\mu_x(t)$ and $\mu_y(t)$, together with the covariance functions $\rho_x(\tau, t)$, $\rho_y(\tau, t)$, $\rho_{xy}(\tau, t)$, yield the same results for all fixed values of t (that is, are independent of time translations), then the random processes $\{x(t)\}$ and $\{y(t)\}$ are said to be weakly stationary. If all possible probability distributions involving $\{x(t)\}$ and $\{y(t)\}$ are independent of time translations, then the processes are said to be strongly stationary. Since the mean values and covariance functions are consequences only of the first-order and second-order probability distributions, it follows that the class of strongly stationary random processes is a subclass of the class of weakly stationary random processes. For Gaussian random processes, however, weak stationarity implies strong stationarity since all possible probability distributions may be derived from the mean values and covariance functions. Thus, for Gaussian random processes, these two stationary concepts coincide. Random processes which are not stationary are said to be nonstationary.

4.4.1 Correlation (Covariance) Structure of Weakly Stationary Random Processes

For weakly stationary random processes, $\{x(t)\}$, $\{y(t)\}$, which will be considered from henceforth, the mean values become constants independent of t, namely,

$$\mu_x = \langle x(t) \rangle$$

$$\mu_y = \langle y(t) \rangle$$

(4.68)

For simplicity, and without loss of generality, it will be assumed from henceforth (unless stated otherwise) that these mean values are zero.

The covariance functions for weakly stationary random processes are also independent of t, and with zero mean values, may be designated by

$$R_x(\tau) = \left\langle x(t)\, x(t + \tau) \right\rangle \quad ; \quad R_x(0) = \sigma_x^2$$

$$R_y(\tau) = \left\langle y(t)\, y(t + \tau) \right\rangle \quad ; \quad R_y(0) = \sigma_y^2 \qquad (4.69)$$

$$R_{xy}(\tau) = \left\langle x(t)\, y(t + \tau) \right\rangle \quad ; \quad R_{xy}(0) = \rho_{xy}$$

where R is introduced instead of ρ to agree with engineering usage. For non-zero mean values, R is different from ρ. The quantities $R_x(\tau)$ and $R_y(\tau)$ are called the <u>autocorrelation functions</u> of $\left[x(t)\right]$ and $\left[y(t)\right]$, respectively, while $R_{xy}(\tau)$ is called the <u>cross-correlation function</u> between $\left[x(t)\right]$ and $\left[y(t)\right]$.

For arbitrary values of μ_x and μ_y, the covariance functions are related to the correlation functions by the equation·

$$\rho_x(\tau) = R_x(\tau) - \mu_x^2$$

$$\rho_y(\tau) = R_y(\tau) - \mu_y^2 \qquad (4.70)$$

$$\rho_{xy}(\tau) = R_{xy}(\tau) - \mu_x\, \mu_y$$

Thus, correlation functions are iderical with covariance functions when the mean values are zero. The reader should be careful not to confuse these un-normalized correlation functions with the correlation coefficient defined in Equation (4.44).

From the stationary hypothesis, it follows that the autocorrelation functions $R_x(\tau)$ and $R_y(\tau)$ are even functions of τ, that is

$$R_x(-\tau) = R_x(\tau)$$

$$R_y(-\tau) = R_y(\tau) \tag{4.71}$$

while the cross-correlation function is neither odd nor even, but satisfies the relation

$$R_{xy}(-\tau) = R_{yx}(\tau) \tag{4.72}$$

An upper bound for the cross-correlation (autocorrelation) function is given by the inequality

$$\left| R_{xy}(\tau) \right|^2 \leqslant R_x(0) \, R_y(0) \tag{4.73}$$

the equal sign occurring only if the two processes are identically equal to each other and to a constant, a trivial case. Hence, a normalized correlation coefficient $\Gamma_{xy}(\tau)$ may be defined by

$$\Gamma_{xy}(\tau) = \frac{R_{xy}(\tau)}{\sqrt{R_x(0) \, R_y(0)}} \tag{4.74}$$

such that $\Gamma_{xy}(\tau)$ lies between -1 and $+1$. The coefficient $\Gamma_{xy}(\tau)$ measures the degree of linear dependence between $\left\{ x(t) \right\}$ and $\left\{ y(t) \right\}$ for a displacement τ in $\left\{ y(t) \right\}$ relative to $\left\{ x(t) \right\}$. The reader is cautioned not to confuse this concept with the previous definition in Equation (4.44).

In summary, the covariance structure of weakly stationary random processes $\left\{ x(t) \right\}$, $\left\{ y(t) \right\}$, assuming zero mean values,

may be described by the four correlation functions $R_x(\tau)$, $R_y(\tau)$, $R_{xy}(\tau)$ and $R_{yx}(\tau)$. These need be calculated only for values of $\tau > 0$ since symmetry properties listed above, Equations (4.71) and (4.72), yield results for $\tau < 0$.

4.4.2 Spectral Decomposition of Stationary Random Processes

The spectral decomposition of arbitrary random processes $\{x(t)\}$, a collection of time functions, (the superscript index k omitted for simplicity in notation), depends upon the requirement that each particular member of the random process $x(t)$, a sample time function, have a complex Fourier transform $X(f)$, where f denotes the frequency, (usually cps), such that

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\, dt \qquad\qquad j = \sqrt{-1} \qquad\qquad (4.75a)$$

and conversely,

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}\, df \qquad\qquad (4.75b)$$

A sufficient set of conditions for this to occur is that $x(t)$ and its derivative $\dot{x}(t)$ be piecewise continuous in every finite interval (a, b) and that $|x(t)|$ be integrable over $(-\infty, \infty)$.

Similarly, every $y(t)$ from an arbitrary random process $\{y(t)\}$ must have a complex Fourier transform $Y(f)$ where

$$Y(f) = \int_{-\infty}^{\infty} y(t)e^{-j2\pi ft}\, dt \qquad\qquad (4.76a)$$

*

$$y(t) = \int_{-\infty}^{\infty} Y(f)e^{j2\pi ft}df \qquad (4.76b)$$

Thus, the original pair of real random processes $\{x(t)\}$, $\{y(t)\}$ may be described in terms of two new complex random processes $\{X(f)\}$, $\{Y(f)\}$.

Example: Special cases of a constant and a periodic function. If $x(t) = 1$, a constant, then its inverse Fourier transform relation

$$1 = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df$$

implies that

$$X(f) = \delta(f)$$

where $\delta(f)$ is the usual Dirac delta function. Also, if $x(t) = e^{j2\pi f_0 t}$, a complex-valued periodic function, then

$$e^{j2\pi f_0 t} = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df$$

Hence

$$1 = \int_{-\infty}^{\infty} X(f)e^{j2\pi(f-f_0)t}df$$

$$= \int_{-\infty}^{\infty} X(\mu + f_0)e^{j2\pi\mu t}d\mu \qquad \mu = f - f_0$$

which yields

$$X(\mu + f_0) = \delta(\mu)$$

or

$$X(f) = \delta(f - f_0)$$

Now for

$$x_c(t) = \cos 2\pi f_0 t = (1/2)\left[e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right]$$

it follows that

$$X_c(f) = (1/2)\left[\delta(f - f_0) + \delta(f + f_0)\right]$$

Similarly

$$x_s(t) = \sin 2\pi f_0 t = (1/2)\left[e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}\right]$$

and

$$X_s(f) = (1/2)\left[\delta(f - f_0) - \delta(f + f_0)\right]$$

This concludes the example

If $\{x(t)\}$ and $\{y(t)\}$ have zero mean values, which is assumed here, it follows that the ensemble averages

$$\langle X(f)\rangle = 0 = \langle Y(f)\rangle$$

Since $x(t)$ is real, it may be expressed in terms of the complex conjugate $\overline{X(f)}$ by

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{-j2\pi ft}\, df$$

Now, the square value

$$x^2(t) = \int_{-\infty}^{\infty} \overline{X(f)}e^{-j2\pi ft}\, df \int_{-\infty}^{\infty} X(g)e^{j2\pi gt}\, dg$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{X(f)}\, X(g)e^{j2\pi t(g-f)}\, dg\, df$$

using a variable of integration g instead of f in the second integral to avoid confusion. The ensemble averaged value of $x^2(t)$ is thus given by

$$\langle x^2(t)\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle\overline{X(f)}\, X(g)\rangle\, e^{j2\pi t(g-f)}\, dg\, df \tag{4 77}$$

From separate considerations, the power spectral density function $S_x(f)$ associated with the random process $\{x(t)\}$, where f ranges over

$(-\infty, \infty)$, may be defined by the relation

$$\left\langle x^2(t) \right\rangle = \int_{-\infty}^{\infty} S_x(f) \, df \qquad (4.78)$$

which indicates how $\left\langle x^2(t) \right\rangle$ is distributed over the doubly-infinite frequency range $(-\infty, \infty)$. In particular, $S_x(f) \, df$ represents the amount of "power" in $\left\langle x^2(t) \right\rangle$ lying in the frequency range $(f, f + df)$ so that $S_x(f)$ is real and non-negative for all f.

Since these last two equations for $\left\langle x^2(t) \right\rangle$ must be equivalent, one obtains

$$S_x(f) = \int_{-\infty}^{\infty} \left\langle \overline{X(f)} \, X(g) \right\rangle e^{j2\pi(g-f)} \, dg \qquad (4.79)$$

which is satisfied by the requirement that

$$\left\langle \overline{X(f)} \, X(g) \right\rangle = S_x(f) \, \delta(f - g) \qquad (4.80)$$

where $\delta(f - g)$ is a delta function defined by

$$\delta(f - g) = 0 \text{ when } f \neq g$$

$$\int_{-\infty}^{\infty} \delta(f - g) \, dg = 1 \qquad (4.81)$$

The above discussion helps to justify the fact that one may prove from a deeper direct analysis (not developed here) that pairs of complex random variables $X(f)$, $X(g)$, $Y(f)$, $Y(g)$ satisfy the relations

$$\left\langle \overline{X(f)}\, X(g) \right\rangle = S_x(f)\, \delta(f - g)$$

$$\left\langle \overline{Y(f)}\, Y(g) \right\rangle = S_y(f)\, \delta(f - g) \qquad (4.82)$$

$$\left\langle \overline{X(f)}\, Y(g) \right\rangle = S_{xy}(f)\, \delta(f - g)$$

where $S_x(f)$ and $S_y(f)$ are called the power spectral density functions of the random processes $\left[x(t)\right]$ and $\left[y(t)\right]$, respectively, while $S_{xy}(f)$ is called the cross-power spectral density function of $\left[x(t)\right]$ to $\left[y(t)\right]$. The frequency variable f ranges over $(-\infty, \infty)$.

It is now quite simple to derive the correspondence between these spectral density functions and the stationary correlation functions $R_x(\tau)$, $R_y(\tau)$, $R_{xy}(\tau)$. The results are

$$R_x(\tau) = \int_{-\infty}^{\infty} S_x(f) e^{j2\pi f\tau}\, df$$

$$R_y(\tau) = \int_{-\infty}^{\infty} S_y(f) e^{j2\pi f\tau}\, df \qquad (4.83)$$

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} S_{xy}(f) e^{j2\pi f\tau}\, df$$

proving that the concepts are Fourier transforms pairs. Consequently, the inverse relations yield

$$S_x(f) = \int_{-\infty}^{\infty} R_x(\tau) e^{-j2\pi f\tau}\, d\tau$$

$$\qquad (4.84a)$$

$$S_y(f) = \int_{-\infty}^{\infty} R_y(\tau) e^{-j2\pi f\tau}\, d\tau$$

$$S_{xy}(\tau) = \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-j2\pi f\tau} d\tau \qquad (4.84b)$$

No such simple relationships exist for nonstationary correlation functions.

From the symmetry properties of stationary correlation functions, it follows that

$$S_x(-f) = S_x(f)$$

$$S_y(-f) = S_y(f) \qquad (4.85)$$

$$S_{xy}(-f) = \overline{S_{xy}(f)} = S_{yx}(f)$$

Since $S_x(f)$ and $S_y(f)$ are also real and non-negative for all f, this proves that power spectral density functions are real, non-negative, even functions of f, while cross-power spectral density functions are complex-valued functions of f.

The above relations for the real-valued power spectral density functions $S_x(f)$ and $S_y(f)$ may be simplified to

$$S_x(f) = \int_{-\infty}^{\infty} R_x(\tau) \cos 2\pi f\tau \, d\tau = 2\int_0^{\infty} R_x(\tau) \cos 2\pi f\tau \, d\tau$$

$$\qquad (4.86)$$

$$S_y(f) = \int_{-\infty}^{\infty} R_y(\tau) \cos 2\pi f\tau \, d\tau = 2\int_0^{\infty} R_y(\tau) \cos 2\pi f\tau \, d\tau$$

while

$$R_x(\tau) = 2\int_0^{\infty} S_x(f) \cos 2\pi f\tau \, df$$

$$\qquad (4.87)$$

$$R_y(\tau) = 2\int_0^{\infty} S_y(f) \cos 2\pi f\tau \, df$$

This last result shows that for the physically realizable positive frequency range where f varies only over $(0, \infty)$, the quantities $G_x(f)$ and $G_y(f)$ defined by

$$G_x(f) = 2S_x(f) \quad : \quad 0 < f < \infty$$

$$G_y(f) = 2S_y(f) \quad : \quad 0 < f < \infty$$

(4.88)

represent the physically realizable power spectral density functions associated with $\{x(t)\}$ and $\{y(t)\}$ respectively. For mathematical calculations, the use of two-sided power spectral density functions $S_x(f)$, $S_y(f)$, defined over $(-\infty, \infty)$, and exponentials with imaginary exponents, frequently simplifies the analysis. In actual practice, one measures $G_x(f)$ and $G_y(f)$, defined over $(0, \infty)$. The reader is cautioned not to confuse these quantities. In terms of the physically realizable power spectral density functions $G_x(f)$ and $G_y(f)$, the correspondance with the stationary correlation functions $R_x(\tau)$ and $R_y(\tau)$ becomes

$$G_x(f) = 4 \int_0^\infty R_x(\tau) \cos 2\pi f \tau \, d\tau$$

(4.89)

$$G_y(f) = 4 \int_0^\infty R_y(\tau) \cos 2\pi f \tau \, d\tau$$

while

$$R_x(\tau) = \int_0^\infty G_x(f) \cos 2\pi f \tau \, df$$

(4.90)

$$R_y(\tau) = \int_0^\infty G_y(f) \sin 2\pi f \tau \, df$$

For the complex-valued cross-power spectral density function $S_{xy}(f)$, and the cross-correlation function $R_{xy}(\tau)$, one finds

$$S_{xy}(f) = \int_{-\infty}^{\infty} R_{xy}(\tau) \cos 2\pi f\tau \, d\tau - j \int_{-\infty}^{\infty} R_{xy}(\tau) \sin 2\pi f\tau \, d\tau$$

$$(4.91)$$

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} S_{xy}(f) \cos 2\pi f\tau \, df + j \int_{-\infty}^{\infty} S_{xy}(f) \sin 2\pi f\tau \, df$$

Now, define the real and imaginary parts of $S_{xy}(f)$ by

$$S_{xy}(f) = C_{xy}(f) - j \, Q_{xy}(f) \tag{4.92}$$

where $C_{xy}(f)$ is called the <u>co-spectrum</u> of x to y and $Q_{xy}(f)$ is called the <u>quad-spectrum</u> of x to y. Observe that

$$C_{xy}(f) = \int_{-\infty}^{\infty} R_{xy}(\tau) \cos 2\pi f\tau \, d\tau = C_{xy}(-f)$$

$$(4.93)$$

$$Q_{xy}(f) = \int_{-\infty}^{\infty} R_{xy}(\tau) \sin 2\pi f\tau \, d\tau = -Q_{xy}(-f)$$

so that $C_{xy}(f)$ is a real-valued even function of f, while $Q_{xy}(f)$ is a real-valued odd function of f. From the symmetry relation $R_{xy}(\tau) = R_{yx}(-\tau)$, it follows that

$$C_{xy}(f) = \int_{0}^{\infty} \left[R_{xy}(\tau) + R_{yx}(\tau) \right] \cos 2\pi f\tau \, d\tau = \frac{1}{2}\left[S_{xy}(f) + S_{xy}(-f) \right]$$

$$(4.94)$$

$$Q_{xy}(f) = \int_{0}^{\infty} \left[R_{xy}(\tau) - R_{yx}(\tau) \right] \sin 2\pi f\tau \, d\tau = \frac{j}{2}\left[S_{xy}(f) - S_{xy}(-f) \right]$$

These relations will be needed later in Section 6.2.

From the above discussion, one sees that the spectral decomposition of the stationary random processes may be described by the three functions $S_x(f)$, $S_y(f)$, $S_{xy}(f)$, or by the four functions $S_x(f)$, $S_y(f)$, $C_{xy}(f)$ and $Q_{xy}(f)$, which need be calculated only for $f > 0$, since the symmetry properties, Equations (4.85) and (4.93), yield results for $f < 0$.

Analogous to the definition of the normalized correlation coefficient, Equation (4.74), the <u>coherence function</u> $\gamma_{xy}^2(f)$ is defined by

$$\gamma_{xy}^2(f) = \frac{\left| S_{xy}(f) \right|^2}{S_x(f)\, S_y(f)} \tag{4.95}$$

Since the cross-power spectral density function $S_{xy}(f)$ may be shown to satisfy the inequality

$$\left| S_{xy}(f) \right|^2 < S_x(f)\, S_y(f) \tag{4.96}$$

it follows that the coherence function lies between zero and one, and is a measure of the linear relationship between $\left\{ x(t) \right\}$ and $\left\{ y(t) \right\}$ at frequency f.

Certain authors prefer to use angular frequencies $\omega = 2\pi f$ in place of cyclical frequencies as is being followed in this report. This can lead to considerable confusion in factors of (2π). A desire to preserve

$$R_x(0) = \int_{-\infty}^{\infty} S_x(f)\, df = \int_{-\infty}^{\infty} \bar{S}_x(\omega)\, d\omega$$

shows that for consistency, one must have

$$S_x(f) = 2\pi\, S_x(\omega) \tag{4.97}$$

Similarly,

$$S_y(f) = 2\pi S_y(\omega)$$

$$S_{xy}(f) = 2\pi S_{xy}(\omega)$$

4.4.3 Ergodic Stationary Random Processes

Consider two weakly stationary random processes $\left\{ {}^k x(t) \right\}$ and $\left\{ {}^k y(t) \right\}$, k variable. and two arbitrary sample functions ${}^k x(t)$ and ${}^k y(t)$, k fixed. These stationary random processes are said to be ergodic if the mean values and covariance (correlation) functions, (or spectral density functions), which are defined by certain ensemble averages, see Section 4.4.1,may be calculated by performing corresponding time averages on the arbitrary sample pair of functions. In this way, the underlying statistical structure of the weakly stationary random processes may be determined quite simply from an available sample pair without the need for collecting a considerable amount of data.

To be more specific, it is necessary to introduce some mathematical notation. The mean values of ${}^k x(t)$ and ${}^k y(t)$, k fixed, when computed by a time average are given by

$$^k\mu_x = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} {}^k x(t)\ dt$$

$$(4.98)$$

$$^k\mu_y = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} {}^k y(t)\ dt$$

Observe that the answer is no longer a function of t since t has been averaged out. In general, however, the answer is a function

of the particular record chosen — hence, the use of the parameter k.

The cross-covariance function between $^k x(t)$ and $^k y(t + \tau)$ when computed by a time average is defined by the expression

$$^k \rho_{xy}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \left[{}^k x(t) - {}^k \mu_x \right] \left[{}^k y(t + \tau) - {}^k \mu_y \right] dt$$

(4.99)

while the autocovariance functions are defined by

$$^k \rho_x(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \left[{}^k x(t) - {}^k \mu_x \right] \left[{}^k x(t + \tau) - {}^k \mu_x \right] dt$$

(4.100)

$$^k \rho_y(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \left[{}^k y(t) - {}^k \mu_y \right] \left[{}^k y(t + \tau) - {}^k \mu_y \right] dt$$

These quantities should now be compared with the previously defined ensemble mean values μ_x, μ_y, and ensemble covariance functions $\rho_x(\tau)$, $\rho_y(\tau)$, $\rho_{xy}(\tau)$ for stationary random processes, Equations (4.68) and (4.70). If it turns out that independent of k, (with the possible exception of a set of sample functions of zero probability)

$$^k \mu_x = \mu_x$$

(4.101)

$$^k \mu_y = \mu_y$$

$$^k\rho_x(\tau) = \rho_x(\tau)$$

$$^k\rho_y(\tau) = \rho_y(\tau) \qquad\qquad (4.102)$$

$$^k\rho_{xy}(\tau) = \rho_{xy}(\tau)$$

then the random processes $\{x(t)\}$ and $\{y(t)\}$ are said to be weakly ergodic. If all ensemble averaged statistical properties of $\{x(t)\}$, $\{y(t)\}$, not just the means and covariances, are deducible from corresponding time averages, then the random processes are said to be strongly ergodic. Thus, strong ergodicity implies weak ergodicity, but not conversely. No distinction between these concepts exists for Gaussian random processes.

For an arbitrary random process to be ergodic, it must first of all be stationary. Secondly, each sample function must be representative of all the others in the sense described above so that it doesn't matter which particular sample function is used in the time averaging calculations. This restriction serves to eliminate many stationary random processes from being ergodic. To repeat, a stationary random process may or may not be ergodic.

There are two important classes of random processes which one can state in advance will be ergodic. The first ergodic class is the class of stationary Gaussian random processes whose power spectral density functions are continuous. That is, no sharp lines (delta functions) appear in the power spectra corresponding to finite amounts of power at discrete frequencies. The second ergodic class (a special case of the first class) is the class of stationary Gaussian Markoff processes, a Markoff process being defined as one whose relationship to the past does not extend beyond the immediately preceding observation. The autocorrelation function of a stationary Gaussian Markoff process may be shown to be of a simple exponential form.

4.5 STATISTICAL PROPERTIES OF ESTIMATES

Consider, first of all, the statistical properties that should be possessed by any set of estimates. Let $\{^k x(t)\}$, $-\infty < t < \infty$, be a real stationary random process where $k = 1, 2, 3, \ldots$, (perhaps even uncountable) denotes the different numbers of the random process. Suppose that z is the true value of an unknown parameter of the random process $\left[^k x(t)\right]$, e.g., its mean value or its power spectrum. Suppose that $^k z(T)$ is an estimate of z obtained from a measurement made on a particular finite sample $^k x(t)$ of the random process extending only over a finite time period from $0 \leqslant t \leqslant T$. How should these different possible $^k z(t)$ be related to z. For ease of notation, the superscript index k will henceforth be omitted, and expected values (averages) are tacitly understood to be taken over this index.

It seems fairly obvious to start with that, on the average, $z(T)$ should yield the true value z. In other words, for fixed T, if one takes an ensemble average over all of the possible $z(T)$ that might occur, then this procedure should give z without any error. A set of estimates having this property is said to be __unbiased.__ To be precise,

__Definition 1__. A set of estimates $\left[z(T)\right]$ is said to be __unbiased__ if, independent of T, the expected value is the true value, that is,

$$E\, z(T) = z \qquad\qquad (4.103)$$

When this occurs, $z(T)$ is also said to be an __unbiased__ estimate of z.

For a fixed T, the mere fact that the expected value of a set of estimates $\left[z(T)\right]$ is unbiased does not imply that any particular $z(T)$ will lie close to the true value z. There may in fact be widespread deviations from the true value. Furthermore, it may

happen that these deviations do not decrease as T is increased. To analyze these cases, for a fixed value of T, the mean square error is defined as the expected value over the set of estimates of the square of the deviations from the true value, namely,

$$E\left[z(T) - z\right]^2 \tag{4.104}$$

The expected value above represents an ensemble average over the various possible $\left[z(T) - z\right]^2$ occurring from different finite samples $z(T)$ of the random process. It appears highly desirable, from a physical point of view, to require that this mean square error should approach zero as T becomes large. Then, for large T, any particular estimate of $z(T)$ would necessarily tend to closely approximate the true value z. Estimates having this desired property are said to be consistent. In more precise terms, one writes

Definition 2. A set of estimates $\left[z(T)\right]$ is said to be consistent if

$$\lim_{T \to \infty} E\left[z(T) - z\right]^2 = 0 \tag{4.105}$$

When this occurs, $z(T)$ itself is also said to be a consistent estimate of z.

Observe that the mean square error

$$E\left[z(T) - z\right]^2 = E\left[z(T) - E\,z(T) + E\,z(T) - z\right]^2$$

$$= E\left[z(T) - E\,z(T)\right]^2 + \left[E\,z(T) - z\right]^2 \tag{4.106}$$

since the usual middle term has a factor equal to zero, namely,

$$E\left[z(T) - E\,z(T)\right] = 0$$

Thus, the mean square error is the sum of two parts: the variance (or square of the random error) of the estimate as given by

$$\sigma^2 \left[z(T) \right] = E\left\{ z(T) - E \ z(T) \right\}^2 = E\left[z^2(T) \right] - \left[E \ z(T) \right]^2 \qquad (4.107)$$

and the square of the bias (or systematic error) of the estimate as given by

$$\left[b \ z(T) \right]^2 = \left[E \ z(T) - z \right]^2 \qquad (4.108)$$

It will be demonstrated in Sections 4.82 and 4.83 that great care is required to insure that both the variance and the bias will approach zero as T becomes large when estimating (i.e., measuring) the power spectrum and cross-power spectrum.

4.6 MEASUREMENT OF MEAN VALUES

The following discussion is now concerned with estimating the mean values μ_x, μ_y of a pair of (weakly) stationary random processes $\left\{ x(t) \right\}$, $\left\{ y(t) \right\}$ by performing a finite time average on arbitrary continuous sample functions x(t) and y(t), which are known only for a finite time interval $0 \le t \le T$. By assuming certain commonly satisfied conditions for the autocorrelation functions of the random processes, it is shown that the estimates in question are unbiased and consistent.

The same analysis covers measurements of either μ_x or μ_y. Consider z(t) as representing x(t) or y(t). Suppose that a single sample record z(t) from a stationary random process $\left\{ z(t) \right\}$ is averaged only over a finite time T. Define the sample mean value estimate by

$$m(T) = \frac{1}{T} \int_0^T z(t) \ dt \qquad (4.109)$$

Then

$$E\,m(T) = E\left[\frac{1}{T}\int_0^T z(t)\,dt\right] = \frac{1}{T}\int_0^T E\,z(t)\,dt$$

$$= \frac{1}{T}\int_0^T \mu\,dt = \mu \tag{4.110}$$

where the true mean value $\mu = E\,z(t)$ is independent of t. Hence, $m(T)$ is an unbiased estimate of μ.

For simplicity and without loss of generality, it will be assumed unless stated otherwise that the mean value $\mu = 0$. Now, the mean square error over the set of estimates $\left[m(T)\right]$ becomes

$$E\left[m(T)-\mu\right]^2 = E\left[m(T)\right]^2 - \mu^2 = E\left[m(T)\right]^2 \tag{4.111}$$

where

$$E\left[m(T)\right]^2 = \frac{1}{T^2}\int_0^T\int_0^T E\left[z(u)\,z(v)\right]\,du\,dv \tag{4.112}$$

By definition, the autocorrelation function $R(\tau)$ for a stationary random process $\left[z(t)\right]$ is defined by

$$R(\tau) = E\left[z(t)\,z(t+\tau)\right] \tag{4.113}$$

From the stationary hypothesis, it follows that $R(\tau)$ is independent of τ, and an even function of τ with a maximum at $\tau = 0$. It will be assumed that $R(\tau)$ is continuous and finite for all values of τ, and that all periodic components in $R(\tau)$ have been removed at the outset.

additional integrability properties satisfied by $R(\tau)$, which will be needed later, are assumed to be

$$\int_{-\infty}^{\infty} |R(\tau)|\, d\tau < \infty \quad ; \quad \int_{-\infty}^{\infty} R^2(\tau)\, d\tau < \infty$$

$$\int_{-\infty}^{\infty} |\tau|\, |R(\tau)|\, d\tau < \infty \quad ; \quad \int_{-\infty}^{\infty} |\tau|\, R^2(\tau)\, d\tau < \infty \tag{4.114}$$

These conditions are generally satisfied in nature, e.g., consider the exponential function $R(\tau) = \exp(-b|\tau|)$ where $b > 0$.

In terms of an arbitrary autocorrelation function $R(\tau)$, the mean square error, Equations (4.111) and (4.112), becomes

$$E\left[m(T)\right]^2 = \frac{1}{T^2} \int_0^T \int_0^T R(u - v)\, du\, dv$$

$$= \frac{1}{T} \int_{-T}^{T} \left(1 - \frac{|\tau|}{T}\right) R(\tau)\, d\tau \tag{4.115}$$

The second expression occurs from the first by substituting $\tau = u - v$, $d\tau = du$, and reversing the orders of integration between τ and v. Now,

$$\lim_{T \to \infty} T\, E\left[m(T)\right]^2 = \int_{-\infty}^{\infty} R(\tau)\, d\tau < \infty \tag{4.116}$$

provided that $R(\tau)$ and $\tau R(\tau)$ are absolutely integrable over $(-\infty, \infty)$ to justify passage to the limit inside the integral sign. In particular, Equation (4.115) shows that, for large T, the mean square error

$$E[m(T)]^2 \to \frac{1}{T} \int_{-\infty}^{\infty} R(\tau)\, d\tau \qquad (4.117)$$

Hence $E[m(T)]^2$ approaches zero as T approaches infinity, proving that $m(T)$ is a __consistent__ estimate of μ.

Example: Quantizing Amplitude Levels.

Suppose $\mu = 0$; $R(\tau) = e^{-b|\tau|}$ where $b = 2\pi B > 0$ $\qquad (4.118)$

Then

$$\sigma^2 = E[m(T)]^2 \to \frac{1}{T} \int_{-\infty}^{\infty} e^{-b|\tau|}\, d\tau = \frac{2}{bT} = \frac{1}{\pi BT} \qquad (4.119)$$

The parameter B can be shown (Ref. [1], Chapter 7) to be the positive realizable bandwidth in cycles per second of a low-pass (RC type) filter, while $R(\tau)$ can be interpreted as the normalized output autocorrelation function from this filter to a bandwidth-limited white noise input.

From knowledge of the normalized standard deviation σ, one may now decide to what degree of refinement amplitude levels should be quantized. In particular, one can estimate how accurately various amplitude values should be measured. With 68% confidence, (assuming a normal distribution), measurements of a desired mean value will be within $\pm \sigma$ units on either side of the true mean value. Hence, the plus and minus allowance equals 2σ for 68% confidence. Thus, for 68% confidence, the number of perceptible amplitude values would be given by $(1/2\sigma)$ where σ is the normalized standard deviation. For this example where $\sigma^2 = (1/\pi BT)$, assume $BT \approx (10^4/\pi)$. Then $\sigma^2 \approx (1/10^4)$ and $\sigma \approx (1/100)$. Here then, the number of quantized amplitude levels should be $(1/2\sigma) \approx 50$. This concludes the example.

4.7 MEASUREMENT OF AUTOCORRELATION AND CROSS-CORRELATION FUNCTIONS

The next statistical quantities of interest are the autocorrelation functions $R_x(\tau)$, $R_y(\tau)$, and the cross-correlation function $R_{xy}(\tau)$. As in Section 4.6 the mean values μ_x and μ_y are assumed to be zero. For continuous data $x(t)$, $y(t)$ which exists only for $0 \leq t \leq T$, the sample cross-correlation function estimate $R_{xy}(\tau, T)$ will be defined by

$$
R_{xy}(\tau, T) =
\begin{cases}
\dfrac{1}{T-\tau} \displaystyle\int_0^{T-\tau} x(t)\, y(t+\tau)\, dt & ; \quad 0 \leq \tau < T \\[4mm]
\dfrac{1}{T-|\tau|} \displaystyle\int_{|\tau|}^{T} x(t)\, y(t+\tau)\, dt & ; \quad -T < \tau \leq 0
\end{cases}
\tag{4.120}
$$

To avoid use of absolute value signs, τ will be considered positive from henceforth and the reader should supply required similar separate proofs for negative τ. The sample autocorrelation function estimates $R_x(\tau, T)$ and $R_y(\tau, T)$ are merely special cases when the two records coincide, namely,

$$
R_x(\tau, T) = \frac{1}{T-\tau} \int_0^{T-\tau} x(t)\, x(t+\tau)\, dt \quad ; \quad 0 \leq \tau < T
$$

$$
R_y(\tau, T) = \frac{1}{T-\tau} \int_0^{T-\tau} y(t)\, y(t+\tau)\, dt \quad ; \quad 0 \leq \tau < T
\tag{4.121}
$$

Thus, by analyzing the cross-correlation function estimate, one derives results also for the autocorrelation function estimates. In particular, measurement of mean square values are present in Eq. (4.121) when $\tau = 0$, namely,

$$
R_x(0, T) = \frac{1}{T} \int_0^T x^2(t)\, dt
\tag{4.122}
$$

An ensemble average over the set of possible estimates $\left\{R_{xy}(\tau, T)\right\}$ yields

$$E\left\{R_{xy}(\tau, T)\right\} = \frac{1}{T-\tau}\int_0^{T-\tau} E\left[x(t)\, y(t + \eta)\right] dt$$

$$= \frac{1}{T-\tau}\int_0^{T-\tau} R_{xy}(\eta)\, dt = R_{xy}(\eta) \qquad (4.123)$$

Hence $R_{xy}(\tau, T)$ is an <u>unbiased</u> estimate of $R_{xy}(\eta)$.

The mean square error here is given by

$$E\left\{R_{xy}(\tau, T) - R_{xy}(\eta)\right\}^2 = E\left\{R_{xy}^2(\tau, T)\right\} - R_{xy}^2(\eta)$$

$$= \frac{1}{(T-\eta)^2}\int_0^{T-\tau}\int_0^{T-\tau}\left[E\left\{x(u)\, y(u + \eta)\, x(v)\, y(v + \eta)\right\} - R_{xy}^2(\eta)\right] du\, dv$$

$$(4.124)$$

At this point, in order both to simplify the later mathematical analysis and also to agree with many physical cases of interest, it will be assumed that the random processes $\left\{x(t)\right\}$, $\left\{y(t)\right\}$ are jointly Gaussian for any set of fixed times. This restriction may be removed by substituting certain integrability conditions on the non-Gaussian pr   of the random processes without altering in any essential way the results to be derived. When $\left\{x(t)\right\}$, $\left\{y(t)\right\}$ are jointly Gaussian, it follows that $\left\{x(t)\right\}$ and $\left\{y(t)\right\}$ are separately Gaussian.

For Gaussian stationary random processes with zero mean values, the statistical expression

$$E\left[x(u)\,y(u+\tau)\,x(v)\,y(v+\tau)\right]=R_{xy}^{2}(\tau)+R_{x}(v-u)\,R_{y}(v-u)$$

$$+\,R_{xy}(v-u+\tau)\,R_{yx}(v-u-\tau) \qquad (4.125)$$

Hence, the mean square error

$$E\left[R_{xy}(\tau,T)-R_{xy}(\tau)\right]^{2}=\frac{1}{(T-\tau)^{2}}\int_{0}^{T-\tau}\int_{0}^{T-\tau}\left[R_{x}(v-u)\,R_{y}(v-u)\right.$$

$$\left.+\,R_{xy}(v-u+\tau)\,R_{yx}(v-u-\tau)\right]dv\,du$$

$$=\frac{1}{T-\tau}\int_{-(T-\tau)}^{T-\tau}\left(1-\frac{|\gamma|}{T-\tau}\right)\left[R_{x}(\gamma)\,R_{y}(\gamma)+R_{xy}(\gamma+\tau)\,R_{yx}(\gamma-\tau)\right]d\gamma$$

$$(4.126)$$

The second expression occurs from the first by letting $\gamma = v - u$, $d\gamma = dv$, and then reversing the order of integration between γ and u. Now,

$$\lim_{T\to\infty}T\,E\left[R_{xy}(\tau,T)-R_{xy}(\tau)\right]^{2}$$

$$=\int_{-\infty}^{\infty}\left[R_{x}(\gamma)\,R_{y}(\gamma)+R_{xy}(\gamma+\tau)\,R_{yx}(\gamma-\tau)\right]d\gamma<\infty \qquad (4.127)$$

assuming $R_{x}(\gamma)\,R_{y}(\gamma)$ and $\gamma R_{x}(\gamma)\,R_{y}(\gamma)$ are absolutely integrable $(-\infty, \infty)$. This proves that $R_{xy}(\tau,T)$ is a <u>consistent</u> estimate of $R_{xy}(\tau)$ which for large T has a mean square error given by

$$E[R_{xy}(\tau, T) - R_{xy}^2(\tau)]^2 \approx \frac{1}{T} \int_{-\infty}^{\infty} [R_x(\gamma)R_y(\gamma) + R_{xy}(\gamma + \tau)R_{yx}(\gamma - \tau)]d\gamma \qquad (4.128)$$

Example: Output Autocorrelation Fuction of Low-Pass Filter to White Noise Input

As shown in Chapter 7 of Ref. [1], the normalized output auto-correlation function $R_x(\tau)$ of a low-pass (RC type) filter to a white noise (bandwidth-limited constant power spectrum) input is given by an exponential function

$$R_x(\tau) = e^{-b|\tau|} \qquad ; \qquad b = 2\pi B = (1/RC) \qquad (4.129)$$

The parameter b is in angular frequency units of ω (rad/sec), while B is a realizable measurable positive frequency bandwidth in units of cycles per second (cps).

From Equation (4.126), one can prove for large T that

$$\sigma_x^2(\tau, T) = E[R_x(\tau, T) - R_x(\tau)]^2 \approx \frac{k}{bT} \qquad ; \qquad 1 \le k \le 2 \qquad (4.130)$$

Thus, the output signal-to-noise amplitude ratio, as defined by the ratio of the mean value to the standard deviation, becomes here

$$\frac{R_x(\tau)}{\sigma_x(\tau, T)} \approx \left(\frac{bT}{k}\right)^{1/2} e^{-b|\tau|}$$

At $\tau = 0$, where $R_x(0)$ equals the mean square value, this ratio is

$$\frac{R_x(0)}{\sigma_x(0, T)} \approx \left(\frac{bT}{k}\right)^{1/2} \qquad ; \qquad b = 2\pi B \qquad ; \qquad 1 \le k \le 2 \quad (4.131)$$

To consider another important idea, suppose many measurements of $R_x(\tau, T)$ are made at a given fixed value of τ, using different sampl of length T from a stationary random process $\{x(t)\}$. Then, to a reason le order of approximation, these different measurements may be assumed

to be normally distributed about the mean value $R_x(\tau)$. Hence, 95% of these values will, on the average, lie within $\pm 2\sigma$ of the mean value. Thus, for 95% certainty that an arbitrary measurement lies within p percent of the true mean value, it is necessary that

$$0.01 \, p \, R_x(\tau) = 2\sigma_x (\tau, T)$$

or

$$\frac{R_x(\tau)}{\sigma_x (\tau, T)} = \frac{200}{p} \qquad (4.132)$$

Note that if p = 5% or 10%, then (R/σ) must equal 40 and 20, respectively.

For the low-pass filter example described above, one can now determine the size of (bT) required to guarantee in advance that an arbitrary correlation measurement will be within p% of the true mean value. To illustrate, letting p = 5% and k = 2, one obtains for the maximum point $\tau = 0$,

$$bT = 2\pi BT \approx 3200 \qquad \text{or} \qquad BT \approx 500$$

This concludes the example.

To summarize the work thus far, the analysis in Sections 4.5 and 4.6 has stated commonly satisfied mathematical conditions such that:

(a) A set of estimates $\{m(T)\}$, see Eq. (4.109), for measuring the mean value from continuous data is both unbiased and consistent, with mean square error for large T given by Eq. (4.117).

(b) A set of estimates $\{R_{xy}(\tau, T)\}$, see Eq. (4.120), for measuring the cross-correlation function from continuous data is both unbiased and consistent, with mean square error for large T given by Eq. (4.128). Autocorrelation function estimates are obtained by merely letting $x(t) = y(t)$, and mean square values by then setting $\tau = 0$.

Power spectra and cross-power spectra measurements will now be developed in the next section.

4.8 MEASUREMENT OF POWER SPECTRA AND CROSS-POWER SPECTRA

For stationary random processes with zero mean values, the real-valued cross-correlation function $R_{xy}(\tau)$ and the two-sided complex-valued cross-power spectral density function $S_{xy}(f)$, which is defined for $-\infty \leqslant f \leqslant \infty$, are related by Equations (4.83), (4.84) and (4.92), namely,

$$S_{xy}(f) = \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-j2\pi f \tau} \, d\tau = C_{xy}(f) - jQ_{xy}(f)$$

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} S_{xy}(f) e^{j2\pi f \tau} \, df$$

(4.133)

As special cases of the above, the real-valued autocorrelation functions $R_x(\tau)$, $R_y(\tau)$ yield the real-valued two-sided power spectral density functions $S_x(f)$, $S_y(f)$ through the relations

$$S_x(f) = \int_{-\infty}^{\infty} R_x(\tau) e^{-j2\pi f \tau} \, d\tau$$

$$S_y(f) = \int_{-\infty}^{\infty} R_y(\tau) e^{-j2\pi f \tau} \, d\tau$$

(4.134)

The problem at hand is to estimate $S_{xy}(f)$, $S_x(f)$ and $S_y(f)$ from data which is known only for a finite time interval. In order to estimate in a physical device the complex-valued function $S_{xy}(f)$, it is necessary to estimate its real-valued components, namely, the co-spectrum $C_{xy}(f)$ and the quad-spectrum $Q_{xy}(f)$. Since $S_x(f)$ and $S_y(f)$ are real-valued functions, their estimation is easier to

accomplish and to explain than $S_{xy}(f)$. Consequently, the discussion to follow begins with power spectrum measurements, after which cross-power spectrum measurements will be taken up in Section 4.8.7.

4.8.1 Power Spectra Measurements

A schematic picture of a general filter device for estimating the power spectral density function associated with a single random record, say x(t), is displayed in Figure 4.3 below.

$$x(t) \longrightarrow \boxed{\begin{array}{l}\text{Tunable Narrowband}\\\text{Filter Centered at } f_c\\\text{(Bandwidth B)}\end{array}} \xrightarrow{a(t)} \boxed{\begin{array}{l}\text{Squarer,}\\\text{Integrator}\\\text{and}\\\text{Averager}\end{array}} \longrightarrow S_x(f_c, T, B)$$
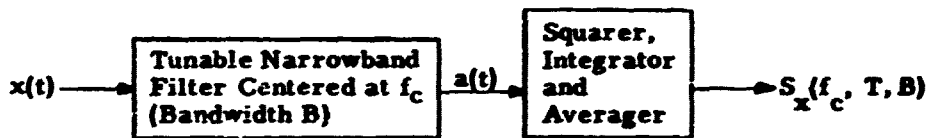
Figure 4.3 Constant Bandwidth Filter Device for Measuring Power Spectrum

The input random record x(t) is assumed to be of finite time-length T, and to be drawn from a stationary random process with zero mean value. The tunable narrowband discriminating filter is assumed to have a finite nonzero constant bandwidth B centered at a frequency f_c which may be varied over the frequency range of interest. It turns out that in order to obtain a consistent estimate of $S_x(f)$, one must introduce a filtering procedure which averages over a band of frequencies. The final filter output quantity $S_x(f, T, B)$ describes the time average of $x^2(t)$ in terms of its frequency components lying inside the frequency band $f_c - (B/2)$ to $f_c + (B/2)$, divided by the bandwidth B. Analog equipment of this type appears in Reference[14].

The output quantity $S_x(f_c, T, B)$ is a smoothing-over-frequency estimate of the true power spectral density function $S_x(f)$ at $f = f_c$ which would be associated with input records of infinite length and

bandwidths of zero width. The quantity $S_x(f)\, df$ is the infinitely long time average of the product $x^2(t)$ from frequencies lying between f and $(f + df)$. The total time average of $x^2(t)$ over all frequencies is obtained by integrating $S_x(f)\, df$ from $-\infty$ to $+\infty$. Because of its relationship to power dissipated in a unit resistance by a current $x(t)$, the time average of $x^2(t)$ may be considered as the "average power" in $x(t)$, and is the main justification for calling $S_x(f)$ a power spectral density function.

In an actual physical device, the bandwidth B is not zero and the record lengths T are not infinite. It is important to be able to predict within established levels of confidence how closely an actual measurement $S_x(f, T, B)$ will approximate the desired true measurement $S_x(f)$. This problem will now be discussed and some of the main analytical results will be stated.

Let the frequency response function of the narrowband filter centered at f_c be of an idealized form (see sketch below)

$$\mathcal{H}(f) = \sqrt{1/2B} \quad \text{for} \quad \left| f \pm f_c \right| \leqslant B/2$$

$$= 0 \qquad \text{otherwise} \qquad\qquad (4.135)$$

where the full bandwidth $B = B(T)$ is a function of T to be specified later. For definiteness, assume that $f_c > (B/2)$.
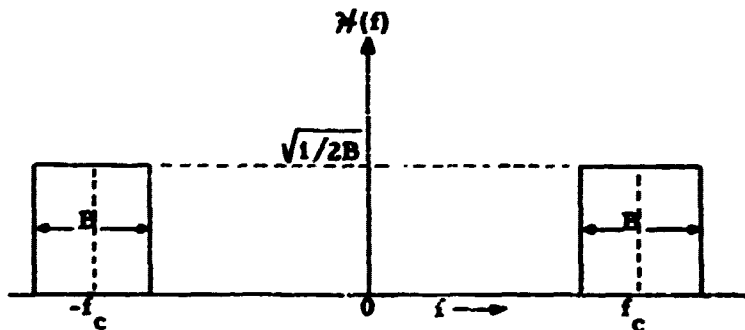
Ideal Narrowband Filter

The inverse Fourier transform of $\mathcal{H}(f)$ is the weighting function of the filter denoted by h(t), that is,

$$h(t) = \int_{-\infty}^{\infty} \mathcal{H}(f) e^{j2\pi ft} \, df$$

For a filter to be physically realizable, it is necessary that h(t) = 0 for t < 0.

In terms of an arbitrary h(t), the corresponding frequency response function

$$\mathcal{H}(f) = \int_{-\infty}^{\infty} h(t) e^{-j2\pi ft} \, dt = H(2\pi f)$$

When using $\omega = 2\pi f$, $\mathcal{H}(f)$ becomes $\mathcal{H}(\omega/2\pi) = H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} \, dt$.

It follows for real h(t) that $\mathcal{H}(-f) = \overline{\mathcal{H}(f)}$, the complex conjugate of $\mathcal{H}(f)$. This is the reason why the ideal narrowband filter has a theoretical pass-band in the negative frequency region with $\mathcal{H}(-f) = \overline{\mathcal{H}(f)}$. Note also that $\mathcal{H}(f)$ is defined so that

$$\int_{-\infty}^{\infty} |\mathcal{H}(f)|^2 \, df = 1$$

In words, $|\mathcal{H}(f)|^2$ has unit area over the doubly infinite frequency range for any bandwidth B.

For an input x(t), the filter output a(t) is given by

$$a(t) = \int_{0}^{T} x(u) \, h(t-u) \, du = \int_{-\infty}^{\infty} x(u) \, h(t-u) \, du \quad ; \quad t \geq 0$$

since x(u) is zero outside the range (0, T). This output is now squared, then integrated over all positive time and average by T, to yield as a smoothed estimate for the time average of $x^2(t)$ in the bandwidth B,

$$S_x(f_c, T, B) = \frac{1}{T} \int_0^\infty a^2(t)\, dt \approx \frac{1}{T} \int_0^T a^2(t)\, dt \text{ for large } T \tag{4.136}$$

By tuning the filter to different center frequencies f_c, one obtains $S_x(f, T, B)$ for all f.

Further analysis (not given here) shows that

$$S_x(f_c, T, B) = \int_{-\infty}^\infty |\tilde{H}(f)|^2 S_x(f, T)\, df \tag{4.137}$$

where

$$S_x(f, T) = \int_{-\infty}^\infty \tilde{R}_x(\tau, T) e^{-j2\pi f\tau}\, d\tau = \int_{-T}^T \tilde{R}_x(\tau, T) e^{-j2\pi f\tau}\, d\tau \tag{4.138}$$

$$\tilde{R}_x(\tau, T) = \int_\infty^\infty S_x(f, T) e^{j2\pi f\tau}\, df$$

$$= \frac{1}{T} \int_0^{T-|\tau|} x(t)\, x(t + |\tau|)\, dt \quad \text{for } |\tau| \leq T \tag{4.139}$$
$$\text{(otherwise zero)}$$

On setting $\tau = 0$, one derives the relations

$$\tilde{R}_x(0, T) = \int_{-\infty}^\infty S_x(f, T)\, df = \frac{1}{T} \int_0^T x^2(t)\, dt \tag{4.140}$$

which shows how $S_x(f, T)$ distributes the "power" in $x^2(t)$ over the doubly infinite frequency range from $-\infty$ to $+\infty$.

4.8.2 Analysis of Bias

For the bias problem, one may prove that $S_x(f_c, T, B)$, (as obtained from Figure 4.3), provides an asymptotically unbiased estimate of $S_x(f_c)$ as $T \to \infty$ if the bandwidth $B \to 0$ as $T \to \infty$. This condition on B as a function of T is therefore assumed.

From Equations (4.137) and (4.138),

$$S_x(f_c, T, B) = \int_{-\infty}^{\infty} |H(f)|^2 \left[\int_{-T}^{T} \tilde{R}_x(\tau, T) e^{-j2\pi f \tau} \, d\tau \right] df$$

(4.141)

By definition, Equation (4.108), the bias term is given by

$$b\left[S_x(f_c, T, B)\right] - E\left[S_x(f_c, T, B)\right] - S_x(f_c)$$

(4.142)

At this point, some detailed mathematical analysis must be carried out which is not developed here. The final result is expressed in the important asymptotic formula below which assumes that $R_x(\tau)$, $\tau R_x(\tau)$ and $\tau^2 R_x(\tau)$ are all absolutely integrable functions over $(-\infty, \infty)$. This result is that at any frequency f,

$$\lim_{T \to \infty} B^{-2} \left| b\left[S_x(f, T, B)\right] \right| \approx \frac{1}{24} \left| S_x''(f) \right|$$

(4.143)

where $S_x''(f)$ is the second derivative of $S_x(f)$ with respect to f as given by

$$S_x''(f) = -4\pi^2 \int_{-\infty}^{\infty} \tau^2 R_x(\tau) e^{-j2\pi f \tau} \, d\tau$$

(4.144)

From the above, it follows that the bias term approaches zero as T approaches infinity provided that B approaches zero.

In fact, for large T,

$$\left| b\left[S_x(f, T, B)\right] \right| \approx \frac{B^2}{24} \left| S_x''(f) \right|$$ (4.145)

No apparent limitation exists from the above formula on how rapidly B should approach zero as T approaches infinity. It will be shown in the next section that B may not approach zero too rapidly if the variance in the estimate is to be small.

4.8.3 Analysis of Variance

For the variance problem, one may prove that the estimate $S_x(f_c, T, B)$, (as obtained from Figure 4.3), has a variance which approaches zero as $T \to \infty$ if the product $BT \to \infty$. Since the bias approaches zero as $T \to \infty$ only if $B \to 0$, these two statements taken together imply that B should approach zero slower than T approaches infinity.

By definition, Equation (4.107) the variance is given by

$$\sigma^2 \left[S_x(f_c, T, B) \right] = E \left[S_x(f_c, T, B) - E\, S_x(f_c, T, B) \right]^2$$ (4.146)

After a considerable amount of careful mathematical analysis, one may derive the following important result. At any frequency f,

$$\lim_{T \to \infty} BT\, \sigma^2 \left[S_x(f, T, B) \right] \approx S_x^2(f) \quad : \quad f \neq 0$$

$$\approx 2 S_x^2(0) : \quad f = 0$$ (4.147)

Thus, for large T.

$$\sigma^2\left[S_x(f,T,B)\right] \approx (1/BT)\, S_x^2(f) \quad ; \quad f \neq 0$$

$$\approx (2/BT)\, S_x^2(0) \quad ; \quad f = 0$$

(4.148)

These equations show that the variance approaches zero as $T \to \infty$ provided that $BT \to \infty$. This result combined with the previous result for the bias term gives the two parts required for a mean square error analysis of power spectrum (and cross-power spectrum) measurements. Observe that at the zero frequency point, $f = 0$, the right-hand side is increased by a factor of two over the general result which is valid for $f \neq 0$. In the sequel, formulas will refer to cases where $f \neq 0$, and should be modified by this factor of two if $f = 0$.

4.8.4 Mean Square Error

The mean square error of the power spectrum estimate $S_x(f,T,B)$ at any frequency $f \neq 0$ is given by the expression, see Equation (4.106).

$$E\left[S_x(f,T,B) - S_x(f)\right]^2 = \sigma^2\left[S_x(f,T,B)\right] + b^2\left[S_x(f,T,B)\right]$$

$$\approx \frac{S_x^2(f)}{BT} + \left(\frac{B^2|S_x''(f)|}{24}\right)^2 \qquad \text{for large } T \qquad (4.149)$$

using Equations (4.145) and (4.148). It is clear that the mean square error approaches zero as $T \to \infty$ if B, considered as a function of T, is restricted so that $B \to 0$ and $BT \to \infty$.

For example, suppose

$$B = cT^{\alpha-1} \qquad ; \qquad c > 0, \qquad 0 < \alpha < 1 \qquad (4.150)$$

Here, $B \to 0$ as $T \to \infty$, and $BT \to \infty$ as $T \to \infty$.

Equation (4.149) above is one of the more important statistical results in this report since it indicates the mean square error to be expected in estimating $S_x(f)$ [or $S_y(f)$] using any given finite B and finite T. Further analysis of this result will be taken up in Section 4.9.

4.8.5 Frequency Resolution

Another important practical question in power spectrum measurements is to determine how closely estimates should be taken along any frequency range of interest. It is clear that if these points are spaced too closely together, the results would be highly correlated and considerable extra unnecessary work would be involved. On the other hand, for points spaced too far apart, considerable information may be lost. It is important to determine the smallest frequency interval Δf that can be resolved in power spectrum measurements in the sense that estimates taken at this frequency interval apart will be essentially uncorrelated. For idealized narrowband filters, a choice of $\Delta f = B$ represents the minimum resolution attainable. Two different peaks in a power spectrum which are less than B cps apart may be blurred together and not distinguished from one another. Two peaks which are further than B cps apart, however, would be separated.

In actual practice, since realizable filters do not have sharp cut-off edges, a more reasonable figure to use for the resolution is 2B cps.

Thus, for high resolution, the bandwidth B should be made as small as possible. This is also desirable, as stated earlier, in order for the estimates to have a low bias. Low bias and high resolution

are consequently complementary properties, both being consequences of narrowband filtering. From the point of view of reducing the variance in the estimates, however, for a given record length T, the bandwidth B should be made as large as possible since the variance is inversely proportional to the BT product. Thus, the choice of B is quite critical. If T is not restricted in length, then it is possible to attain arbitrarily high resolution and small bias as well as arbitrarily low variance. To accomplish this objective, one should let B approach zero and T approach infinity, but in such a way that B approaches zero at a slower rate than T approaches infinity.

The quantity

$$n = 2BT \tag{4.151}$$

represents the <u>number of statistical degrees of freedom</u> associated with a finite record T seconds long and restricted to a frequency bandwidth (0, B) cps wide, in the sense that the record can be reconstructed from its samples taken (1/2B) seconds apart on the time scale. Thus, 2BT numbers completely determine such a record. Ref. $\left[1. \text{ p. } 57\right]$.

4.8.6 Correction for Mean and Linear Trend

The previous analysis has assumed that the input random record x(t) is a sample member from a stationary random process with zero mean value. If the mean value is not zero, then the power spectral density function will exhibit a large peak (theoretically infinite) at zero frequency. Considerable distortions will occur in measurements of the power spectra curve at low frequencies by feeding the record directly into the analog device of Section 4.8.1 without correcting for this non-zero mean value.

A second correction may be needed to subtract out a slowly varying linear trend (i.e. non-zero slope of x(t) with respect to time) about which the random record may be oscillating. This may be due to the recording equipment, or to an actual change in the random record over a long observation time Whatever the cause,

it is clear that a better estimate of the power spectra curve can be obtained by taking proper account of this linear trend in the data.

Let $\xi(t)$ represent an input random record from a random process $\left\{\xi(t)\right\}$ which may need to be corrected for a non-zero mean value and for a linear trend. In particular, suppose that

$$\xi(t) = m_x + a_x \left(t - \frac{T}{2}\right) + x(t) \quad : \quad 0 < t < T \qquad (4.152)$$

where m_x denotes the measured mean value of $\xi(t)$ over its length $[0, T]$, the parameter a_x denotes the average slope of the record $\xi(t)$ with respect to time t, and the final term $x(t)$ represents a sample record from a stationary random process $\left\{x(t)\right\}$ with zero mean value and zero slope. Observe that if m_x and a_x equal zero, then $\xi(t)$ becomes $x(t)$.

The parameters m_x and a_x may be estimated from $\xi(t)$ by the easily derived formulas

$$m_x = \frac{1}{T} \int_0^T \xi(t)\, dt \qquad (4.153)$$

$$a_x = \frac{1}{(T/3)(2T/3)} \left[\int_{(2T/3)}^T \xi(t)\, dt - \int_0^{(T/3)} \xi(t)\, dt \right] \qquad (4.154)$$

These relations lead to a simple analog device for determining $x(t)$ from $\xi(t)$ as sketched in Figure 4.4. This output $x(t)$ can now be fed into the power spectral analyzer circuit of Figure 4.3 so as to yield estimates of $S_x(f)$.
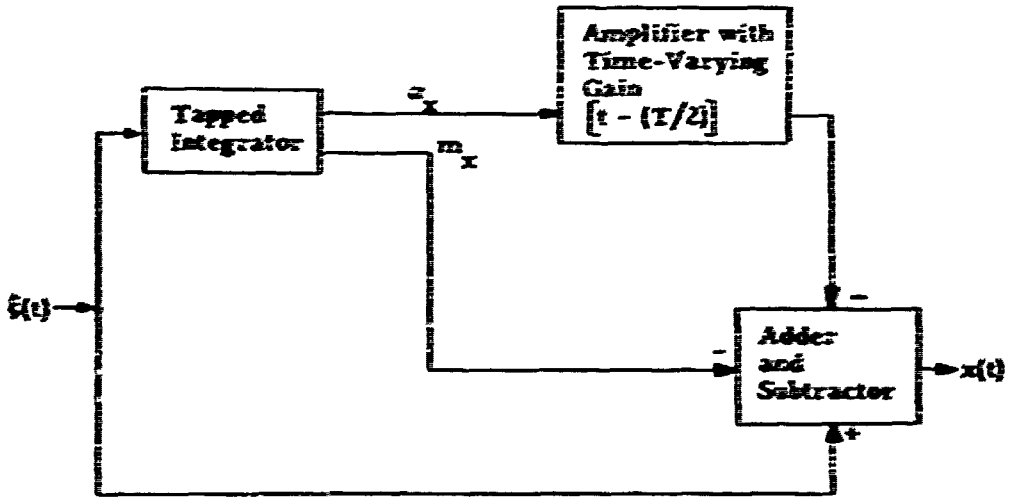
Figure 4.4 Circuit for Removing Mean Value and Linear Trend

The autocorrelation function estimate now becomes,

$$R_x(\tau, T) = \frac{1}{T-\tau} \int_0^{T-\tau} x(t) \, x(t + \tau) \, dt \quad ; \quad 0 < \tau < T$$

$$= \frac{1}{T-\tau} \int_0^{T-\tau} \xi(t) \, \xi(t + \tau) \, dt - m_x^2 - \frac{1}{12} \lambda(\tau, T) \, a_x^2 \qquad (4.155)$$

where

$$\lambda(\tau, T) = \tau^2 \left[1 - 2(\tau/T) - 2(\tau/T)^2 \right] \qquad (4.156)$$

Similarly, let $\eta(t)$ represent an input random record from a second random process $\left\{\eta(t)\right\}$ such that

$$\eta(t) = m_y + \alpha_y \left(t - \frac{T}{2}\right) + y(t) \quad : \quad 0 \leqslant t \leqslant T \tag{4.157}$$

where m_y denotes the measured mean value of $\eta(t)$ over its length $[0, T]$, the parameter α_y denotes the average slope of the record $\eta(t)$ with respect to time t, and the final term $y(t)$ represents a sample record from a stationary random process $\left\{y(t)\right\}$ with zero mean value and zero slope. As before,

$$m_y = \frac{1}{T} \int_0^T \eta(t)\, dt$$

$$\alpha_y = \frac{1}{(T/3)(2T/3)} \left[\int_{(2T/3)}^T \eta(t)\, dt - \int_0^{(T/3)} \eta(t)\, dt \right] \tag{4.158}$$

$$R_y(\tau, T) = \frac{1}{T-\tau} \int_0^{T-\tau} y(t)\, y(t + \tau)\, dt \quad ; \quad 0 \leqslant \tau \leqslant T$$

$$= \frac{1}{T-\tau} \int_0^{T-\tau} \eta(t)\, \eta(t + \tau)\, dt - m_y^2 - \frac{1}{12} \lambda(\tau, T)\, \alpha_y^2 \tag{4.159}$$

where $\lambda(\tau, T)$ is given by Equation (4.156).

Finally, a cross-correlation function estimate $R_{xy}(\tau, T)$ is given by

$$R_{xy}(\tau, T) = \frac{1}{T-\tau} \int_0^{T-\tau} x(t) \, y(t + \tau) \, dt \quad ; \quad 0 < \tau < t$$

$$= \frac{1}{T-\tau} \int_0^{T-\tau} \xi(t) \, \eta(t + \tau) \, dt - m_x m_y - \frac{\tau}{2}(m_x \alpha_y - m_y \alpha_x) - \lambda(\tau, T) \alpha_x \alpha_y$$

$$(4.160)$$

This formula includes the autocorrelation function estimates $R_x(\tau, T)$ and $R_y(\tau, T)$ as special cases, and reduces to the usual expression involving calculation of only the first term on the right-hand side when the quantities m_x, m_y, α_x and α_y equal zero. Other situations when some but not all of these quantities equal zero are also readily obtainable.

4.8.7 Cross-Power Spectra Measurements

A schematic picture of a filter device for estimating the cross-power spectral density function associated with two random records $x(t)$ and $y(t)$ is displayed in Figure 4.5 below. Physically realizable real-valued estimates are obtained of the co-spectrum $C_{xy}(f)$ and the quad-spectrum $Q_{xy}(f)$ which can later be combined to yield the cross-power spectrum $S_{xy}(f)$ from the defining relation

$$S_{xy}(f) = C_{xy}(f) - jQ_{xy}(f) \tag{4.161}$$

The input random records $x(t)$ and $y(t)$ are assumed to be of finite time-length T and to be drawn from stationary random processes with zero mean values. The two separate identical tunable narrowband filters are assumed to have a finite nonzero constant bandwidth B centered at a frequency f_c which may be varied over the frequency range of interest. To estimate the co-spectral density

function, $C_{xy}(f)$, the in-phase frequency components in the filter outputs are multiplied together, then integrated and averaged. This is completely analogous to what was done previously in individual power spectra measurements, the multiplier circuit now performing the same role as the previous squarer circuit. To estimate the quad-spectral density function, $Q_{xy}(f)$, one of the filter outputs is passed through a 90° phase shifter before being multiplied by the output of the other filter. The product is then integrated and averaged as before. This yields the average product of the 90° out-of-phase frequency components in the two random functions, a proper physical interpretation of the quad-spectrum. The absolute value and phase angle of the cross-spectrum may be determined by vectorially combining the co-spectrum and quad-spectrum.
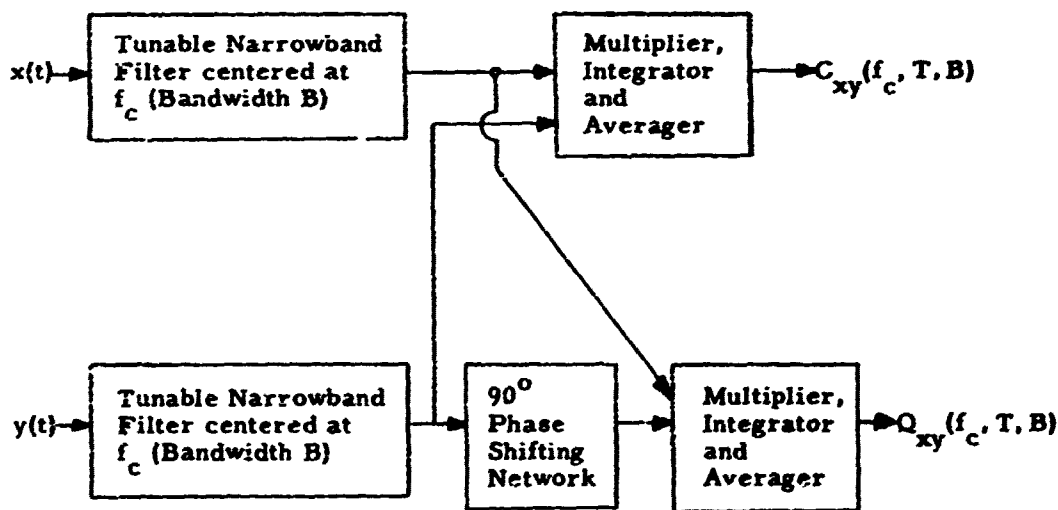
Figure 4.5 Cross-Power Spectral Density Analyzer

A complete analysis can now be carried out which will indicate the bias and variance to be associated with the estimates $C_{xy}(f, T, B)$ and $Q_{xy}(f_c, T, B)$ that would be obtained using Figure 4.5 Many parts of the analysis are quite similar to what was sketched previously for individual power spectra estimates. Since a detailed mathematical analysis of this type is not deemed to be appropriate for this report, only the main conclusions will be summarized below.

The bias terms for any frequency f are bounded above by

$$b\left[C_{xy}(f, T, B)\right] \leq \frac{B^2}{24} \left|S''_{xy}(f)\right|$$

$$(4.162)$$

$$b\left[Q_{xy}(f, T, B)\right] \leq \frac{B^2}{24} \left|S''_{xy}(f)\right|$$

where $S''_{xy}(f)$ is the second derivative of $S_{xy}(f)$ with respect to f, and is related to $R_{xy}(\tau)$ by the expression

$$S''_{xy}(f) = -4\pi^2 \int_{-\infty}^{\infty} \tau^2 R_{xy}(\tau) e^{-j2\pi f\tau} d\tau \qquad (4.163)$$

The variance terms for any frequency $f \neq 0$ are bounded above by

$$\sigma^2\left[C_{xy}(f, T, B)\right] \leq \frac{S_x(f) S_y(f)}{BT}$$

$$(4.164)$$

$$\sigma^2\left[Q_{xy}(f, T, B)\right] \leq \frac{S_x(f) S_y(f)}{BT}$$

At $f = 0$, the right-hand sides above should be multiplied by a factor of two.

Thus, one finds that a mean square error analysis for co-spectrum and quad-spectrum estimates is closely analogous to a mean square error analysis of individual power spectrum. In particular, the conflicting demands on B to be small for low bias (and high resolution), and to be large for low variance are the same as previously.

Furthermore, if the actual available records are not $x(t)$ and $y(t)$, but $\xi(t)$ and $\eta(t)$, respectively, where $\xi(t)$ and $\eta(t)$ have non-zero mean values and non-zero slopes during the time of observation then prior corrections must be made as indicated in Section 4.8.6.

This completes the main discussion on how to estimate power spectra and cross-power spectra from continuous data, and how to evaluate the expected mean square error of the measurements. Some further statistical error analysis will be developed in the next section.

4.8.8 Confidence Limits and Design Relations

From Eq. (4.149) for continuous data, the mean square error of the estimate $S(f, T, B)$, (which will be taken as representative of S_x, S_y, C_{xy} and Q_{xy} as well), is given by

$$E\left[S(f, T, B) - S(f)\right]^2 \approx \frac{|S(f)|^2}{BT} + \left(\frac{B^2}{24}\right)^2 |S''(f)|^2 \qquad (4.165)$$

The mean square percentage error of the estimate, denoted by ϵ^2, is defined by the mean square error divided by the square of the true value. Hence

$$\epsilon^2 = \epsilon^2\left[S(f, T, B)\right] = \frac{E\left[S(f, T, B) - S(f)\right]^2}{S^2(f)}$$

$$\approx \frac{1}{BT} + \frac{B^4}{576} \left|\frac{S''(f)}{S(f)}\right|^2 \qquad (4.166)$$

Let the quantity

$$\lambda(f) = \left|\frac{S(f)}{S''(f)}\right|^{1/2} \qquad (4.167)$$

Then $\lambda(f)$ has units of frequency (cps), and is called the "spectral bandwidth" of the random process $\{x(t)\}$ under consideration. In terms of $\lambda(f)$.

$$\epsilon^2 \approx \frac{1}{BT} + \frac{1}{576}\left[\frac{B}{\lambda(f)}\right]^4 \qquad (4.168)$$

The quantity ϵ itself is called also the "standard error". This equation enables one to make quantitative statements about the mean square percentage error ϵ^2 in measuring a power spectrum $S(f)$ for given values of B, T and $\lambda(f)$. The latter quantity $\lambda(f)$ demands some apriori knowledge of the spectrum which one is trying to measure.

If ϵ^2 is large, then any particular individual measurement $S(f, T, B)$ would not be likely to fall close to the true value $S(f)$. However, if ϵ^2 is small, then all individual measurements of $S(f, T, B)$ would tend to closely approximate $S(f)$. Thus, to guarantee in advance that an arbitrary measurement represents well the true measurement, one should try to make ϵ^2 as small as possible through prior choice of B and T.

Returning to Eq. (4.168), suppose that the "spectral bandwidth" $\lambda(f)$ is known (or can be reasonably estimated) for the random process under consideration. Suppose also that the bandwidth B of the discriminating filter, and the record length T, can be set to any desired design values. Then, in order to nearly always be able to separate peaks in the true spectrum $S(f)$ which may be a spectral bandwidth $\lambda(f)$ apart, it appears reasonable to select B so that

$$B \leq \frac{\lambda(f)}{2} \qquad (4.169)$$

This choice of B (together with a proper T as found below) will then guarantee, with a low probability of error, that if $S(f)$ has two distinct peaks which are $\lambda(f)$ cps apart, then these two peaks can be resolved by taking measurements of $S(f)$ at intervals of B cps apart. Another way of looking at this statement is to say that measurements of $S(f)$ at intervals of B cps apart will practically always distinguish peaks which are $2B$ cps apart.

Assuming B to satisfy $B \leq \lambda(f)/2$, the second term in Eq. (4.168) becomes negligible and Eq. (4.168) reduces to the simple relation

$$\epsilon^2 \approx \frac{1}{BT} \tag{4.170}$$

In particular, for $\epsilon = 0.10$, corresponding to a <u>root mean square percentage error</u> of 10%, the value of the product BT should be

$$BT \approx 100 \qquad \text{or} \qquad T \approx (100/B) \tag{4.171}$$

To illustrate these last formulas, suppose that $\lambda(f) \geq 40$ cps. First, from Eq. (4.169), choose B = 20 cps. Then from Eq. (4.171), choose T = 5 seconds. It follows that different measurements of S(f) taken 20 cps apart will now resolve peaks which are 40 cps apart, and the rms percentage error in the measurements will be at most 10 percent.

If B is not small compared to $\lambda(f)$, then the original formula of Eq. (4.168) must be used to calculate the rms percentage error. For example, suppose that $B = 2\lambda(f)$ at a particular value of f. Then, for B = 20 cps and T = 5 sec, the same two values considered in the previous paragraph, it now follows that $\epsilon^2 \approx 0.038$ and $\epsilon \sim 19.5$ percent. It is clear from this example how important it is to have $B \leq 0.?? \lambda(f)$ for all f, if this is possible.

Suppose that the tunable filter (see Fig. 4.3) is tuned in a uniform continuous fashion over some wide frequency interval B in the time T. Then, the average amount of time T that the input record x(t) spends within the narrow discriminating filter bandwidth B (for any center frequency f_c) satisfies the relation

$$\text{S. R.} = \frac{df_c}{dt} = (\tilde{B}/\tilde{T}) = (B/T) \tag{4.172}$$

where S. R. is the sweep rate (cps/sec). Solving for T, and substituting in $\epsilon^2 = (1/BT)$, one obtains

$$\epsilon^2 = \frac{\text{S. R.}}{B^2} \tag{4.173}$$

which indicates how the mean square error ϵ^2 varies as a function of S. R. and B. Observe that for an rms error $\epsilon \leq 10\%$, S. R. should satisfy

$$\text{S. R.} \leq 0.01 \; B^2 \text{ cps/sec} \qquad \text{(B in cps)} \qquad (4.174)$$

For example, if B = 20 cps, then S. R. ≤ 4 cps/sec in order to keep the rms error below 10%.

4.8.9 Constant Percentage Q Filters

The previous analysis involved using a constant bandwidth filter. For purposes of comparison, as well as for its own physical interest, similar results will be written down for constant percentage Q filters. By definition, a constant percentage Q filter is defined by the relation

$$Q = \frac{f_c}{B} = \text{constant} \qquad (4.175)$$

Thus, as the center frequency f_c increases, the bandwidth B must increase also to maintain Q constant.

As shown earlier, the spectral resolution is proportional to B. Hence, the fractional resolution for different center frequencies f_c is proportional to (B/f_c). For a constant bandwidth filter (i. e., B = constant), the fractional resolution will decrease as f_c increases. However, for a constant Q filter, the fractional resolution will not change as f_c increases since $(B/f_c) = (1/Q) = $ constant. The actual spectral resolution will be poorer for the constant Q filter as f_c increases.

For constant Q filters, the mean square error ϵ^2 becomes

$$\epsilon^2 = \frac{1}{BT} = \frac{Q}{Tf_c} \qquad (4.176)$$

and, hence, decreases as f_c increases, for constant Q and T.

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The maximum scan rate S.R. becomes

$$S.R. = \frac{df_c}{dt} = e^2 B^2 = (e/Q)^2 f_c^2 \qquad (4.177)$$

One may now solve for f_c as a function of time for the two situations where B = constant or Q = constant. Let f_1 = minimum frequency of interest and f_2 = maximum frequency of interest.

Case 1: B = constant

$$\frac{df_c}{dt} = e^2 B^2$$

$$\int_{f_1}^{f_2} df_c = \int_0^t e^2 B^2 dt$$

$$f_2 - f_1 = e^2 B^2 t \qquad (4.178)$$

Now, the number of filters n_B required to cover the frequency range $(f_2 - f_1)$ in a constant bandwidth system is given by

$$n_B = \frac{f_2 - f_1}{B} = \frac{f_1}{B}\left(\frac{f_2}{f_1} - 1\right) \approx Q\left(\frac{f_2}{f_1}\right) \qquad (4.179)$$

$$\text{if } Q = \frac{f_1}{B} \quad \text{and} \quad \frac{f_2}{f_1} \gg 1$$

Case 2: Q = constant

$$\frac{df_c}{dt} = \left(\frac{e}{Q}\right)^2 f_c^2$$

$$\int_{f_1}^{f_2} \frac{df_c}{f_c^2} = \int_0^t \left(\frac{e}{Q}\right)^2 dt$$

$$\qquad (4.180)$$

$$f_2 - f_1 = (e/Q)^2 f_1 f_2 t = e^2 B^2 (f_2/f_1)t \quad \text{if} \quad Q = (f_1/B)$$

Thus the frequency range is covered more quickly by a set of constant Q filters. The number of filters n_q required to cover the frequency range may be estimated from the formula

$$n_q \approx Q \ln\left(\frac{f_2}{f_1}\right) \qquad \text{if} \qquad Q \gg 1 \qquad \text{and } f_2 \gg f_1 \qquad (4.181)$$

This result shows that a considerable reduction in the number of required filters may be achieved by using constant Q filters.

4.9 FURTHER MATHEMATICAL ANALYSIS

4.9.1 Instantaneous Amplitude Distribution

Consider a random vibration record $x(t)$ which is a representative member of a stationary ergodic Gaussian random process with zero mean value. From the ergodic property, the time-wise behavior of $x(t)$ over a long period of time will exhibit the same statistical characteristics as corresponding ensemble averages at various fixed times. As a consequence, it follows that the probability density function associated with the instantaneous amplitude values of $x(t)$ that will occur over a long time interval is given here by the Gaussian formula for zero mean value, namely,

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \, e^{-x^2/2\sigma^2} \qquad (4.182)$$

where the variance σ^2 equals

$$\sigma^2 = \left\langle x^2(t) \right\rangle_{Av} \qquad ; \qquad \text{independent of } t$$

$$\approx \frac{1}{T} \int_0^T x^2(t) \, dt \quad \text{for large } T$$

$$= \int_{-\infty}^{\infty} S_x(f) \, df = 2 \int_0^{\infty} S_x(f) \, df \qquad (4.183)$$

Note that

$$p(0) = \frac{1}{\sigma\sqrt{2\pi}} \approx \frac{0.399}{\sigma}$$

$$p(\sigma) = p(0) e^{-1/2} \approx 0.607 \, p(0)$$

the quantity $S_x(f)$ denoting the two-sided power spectral density function of $x(t)$ as defined over $(-\infty, \infty)$. Statistical procedures for estimating $S_x(f)$ from finite data were developed in the previous section.

Thus, the probability density function $p(x)$ is completely characterized through knowledge of $S_x(f)$ since $S_x(f)$ alone determines σ. This important result places knowledge of $S_x(f)$ at the forefront of much work in analysis of random records obeying a normal distribution.

If the mean value of $x(t)$ is not zero, then the underlying probability density function is given by the general Gaussian formula

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} \tag{4.184}$$

where the mean value

$$\mu = \left\langle x(t) \right\rangle_{Av} \quad ; \quad \text{independent of } t$$

$$\approx \frac{1}{T}\int_0^T x(t)\, dt \quad \text{for large } T \tag{4.185}$$

and the variance

$$\sigma^2 = \left\langle \left[x(t) - \mu \right]^2 \right\rangle_{Av}$$

$$= \int_{-\infty}^{\infty} \tilde{S}_x(f)\, df \tag{4.186}$$

the quantity $\tilde{S}_x(f)$ denoting the power spectral density associated with the zero mean value portion of $x(t)$.

Suppose that a long record $x(t)$ of length T may be separated into distinct independent phases $x_k(t)$, $k = 1, 2, \ldots, n$, each lasting for a time T_k which is long enough to exhibit stationary statistical properties for each phase. Suppose, also, for the sake of simplicity, that each $x_k(t)$ is normally distributed with mean $\mu_k = 0$ and variance σ_k^2 which may differ from one phase to another. How may one estimate an overall distribution for the entire record $x(t)$?

One approach to this problem may be formulated as the superposition of a number of independent random variables x_k (Section 4.3b) provided each random variable is weighted according to its relative time of occurrence (T_k/T). Thus, to consider a specific case, even though say

$$x(t) = x_1(t) \qquad 0 \leq t \leq T_1$$
$$= x_2(t) \qquad T_1 \leq t \leq T_1 + T_2 = T \tag{4.189}$$

one may consider $x(t)$ to be given by the sum of suitable portions of both $x_1(t)$ and $x_2(t)$ spread out over the entire time T. The resulting $x(t)$ would not reflect the actual time behavior of the original $x(t)$, but it would reflect the relative proportions of time that $x(t)$ spends in various amplitude levels. From this point of view, by the superposition theorem for independent random variables, if $x_1(t)$ and $x_2(t)$ are each normally distributed as hypothesized above, then $x(t)$ will also be normally distributed, with mean zero, and variance

$$\sigma^2 = \left(\frac{T_1}{T}\right) \sigma_1^2 + \left(\frac{T_2}{T}\right) \sigma_2^2 \tag{4.190}$$

Note that the final variance σ^2 weights each individual variance σ_k^2 according to its relative time interval (T_k/T). This result is generalized easily to many variables.

To check this result for an obvious case, suppose $x_1(t) = x_2(t)$ and $T_1 = T_2 = (T/2)$. Thereupon $\sigma^2 = \sigma_1^2 = \sigma_2^2$ which agrees with the formula.

4.9.2 Measurement of Amplitude Probability Density Function

Consider a stationary random signal, x(t). The probability that x(t) assumes particular amplitude values between x and x + Δx in a total time T seconds may be estimated by,

$$\hat{P}(x,\ x + \Delta x) = \frac{1}{T} \sum_{i=1}^{k} t_i = \frac{\Delta \tau}{T} \qquad (4.191)$$

where τ_i is the time spent by the signal in the range (x, x + Δx) during the ith entry to the range. See sketch below.
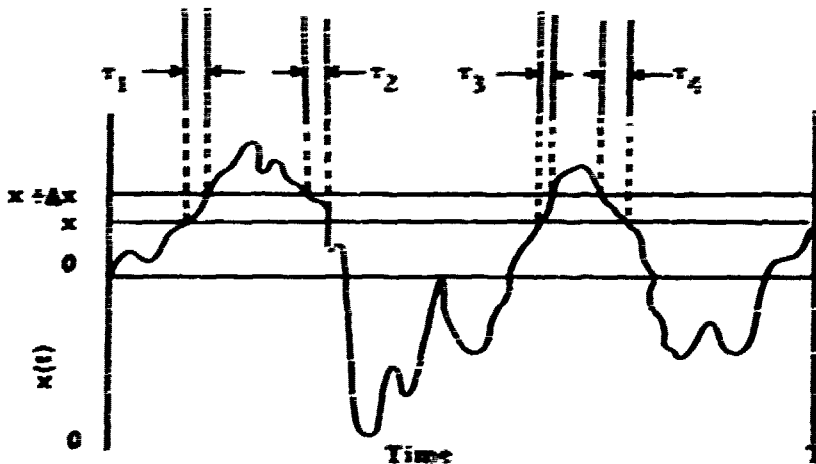
Note that Δτ/T is the total fractional portion of the time spent by the signal in the range (x, x + Δx). The hat over P, namely ($\hat{P}$), in Eq. (4.191) signifies the equation is only an estimate of probability since the total sampling time T will always be finite in practice. Eq. (4.191) will approach an exact probability statement as T approaches infinity.

The probability density function is given by,

$$p(x) = \lim_{\Delta x \to 0} \frac{P(x,\ x + \Delta x)}{\Delta x} \qquad (4.192)$$

The probability density function, p(x) then defines the probability, P, of amplitudes occurring between any two amplitude limits, x_1 and x_2, as follows:

$$P(x_1 < x < x_2) = \int_{x_1}^{x_2} p(x)\, dx \qquad (4.193)$$

In Eq. (4.192), the procedure of taking the limit as Δx approaches zero is beyond the capability of physical instruments. However, the probability density may be approximated as follows:

$$\hat{p}(x) \approx \frac{P(x, x + \Delta x)}{\Delta x} \qquad \text{for small } \Delta x \qquad (4.194)$$

Substituting Eq. (4.191) into Eq. (4.194),

$$\hat{p}(x) \approx \frac{1}{T}\left(\frac{\Delta \tau}{\Delta x}\right) \qquad \text{for small } \Delta x \qquad (4.195)$$

The quantity $\hat{p}(x)$ is an estimate of the true probability density function p(x) because the sampling time T is not infinite and the amplitude window Δx is not infinitesimal.

Consider now the statistical accuracy of probability density estimates obtained by physically accomplishing the functions of Eq. (4.195). Towards this goal assume that repeated measurements of $\Delta \tau$ for a given amplitude window Δx over a fixed record length of T will be distributed about the expected value of $\Delta \tau$ by some distribution function not yet defined. The equivalent number of statistical degrees of freedom (the effective number of observations) for each measurement is $n = 2 \Delta \tau B_N$. Eq.(4.151)where B_N is defined below: From Eq. (4.195). $\Delta \tau = \Delta x\, T\, \hat{p}(x)$. Then, the number of degrees of freedom for a measurement of p(x) will be.

$$n = 2(\Delta x)\, T\, B_N\, \hat{p}(x) \qquad (4.196)$$

The factor B_N is the equivalent noise bandwidth of the input signal and is given by,

$$B_N = \int_0^\infty \left| \frac{\mathcal{H}(f)}{\mathcal{H}_{max}} \right|^2 \, df \qquad (4.197)$$

where $\mathcal{H}(f)$ is the frequency response function of a filter which may be associated with the input signal.

In accordance with common statistical practice, the mean square percentage error ϵ^2 of the measurement $\hat{p}(x)$ will be defined as follows:

$$\epsilon^2 = \frac{\sigma_{\hat{p}}^2}{\hat{p}^2(x)} \qquad (4.198)$$

where $\sigma_{\hat{p}}^2$ is the measurement variance. In terms of the true population variance σ_p^2,

$$\epsilon^2 = \frac{\sigma_p^2}{n \hat{p}^2(x)} \qquad \text{since} \qquad \sigma_{\hat{p}}^2 = \frac{\sigma_p^2}{n} \qquad (4.199)$$

Assuming $\sigma_p^2 \approx \hat{p}^2(x)$, a conservative estimate, one now obtains

$$\epsilon \approx \frac{1}{n} = \left[\frac{1}{2(\Delta x) \, TB_N \, \hat{p}(x)} \right]^{1/2} \qquad (4.200)$$

The normalized standard deviation, ϵ in Eq. (4.200), is often called the standard error of the measurement.

It will be worthwhile to discuss in more detail the meaning of ϵ. It has been stated that the measurement of $\Delta \tau$, now reduced to $\hat{p}(x)$, will be distributed in some manner about the true probability density $p(x)$. It is often assumed the distribution of $\hat{p}(x)$ will be normal with a standard error of ϵ. If the original random variable that was sampled to determine $\hat{p}(x)$ had been normally distributed, then indeed $\hat{p}(x)$ must be normally distributed about a mean of $p(x)$. But it is obvious that the original random variable, namely time τ_i in Eq. (4.191), cannot be normally distributed since time cannot physically take on negative values. A

normally distributed random variable must theoretically be able to attain very large positive and negative values about the mean with equal probability.

However, from the central limit theorem, under fairly general conditions, the distribution of $\hat{p}(x)$ will approach a normal distribution about $p(x)$ as the number of degrees of freedom, n, of the measurement becomes large, regardless of the distribution of the original random variable. This is to say that if the value of e is much less than one, then $\hat{p}(x)$ may be considered to be normally distributed about a mean of $p(x)$ with a normalized standard deviation of e.

Specific statistical confidence statements can now be associated with e by using a standardized normal distribution table. For example, in any given measurement of $\hat{p}(x)$,

$$\text{Prob}\left[p(x) - \sigma_{\hat{p}(x)} < \hat{p}(x) < p(x) + \sigma_{\hat{p}(x)} \right] = 0.68$$

(4.201)

$$\text{where } \sigma_{\hat{p}(x)} = e\hat{p}(x)$$

Then, if a value of $\hat{p}(x)$ were measured with a e of 0.10, one would expect with 68% confidence that the true $p(x)$ is between $0.90\,\hat{p}(x)$ and $1.10\,\hat{p}(x)$. In other words, if the measurement of $\hat{p}(x)$ were repeated many times, it would be expected that $\hat{p}(x)$ would fall within 10% of the true value $p(x)$ approximately 68% of the time.

Note in Eq. (4.200) that the standard error of the measurement becomes larger as $\hat{p}(x)$ becomes smaller. It should be no surprise if this equation fails for small values of $\hat{p}(x)$, as will be demonstrated later in numerical examples. A good rule of thumb to determine if Eq. (4.200) is a valid estimate of a normal distribution standard error is that e should never exceed 0.3.

The standard error of an estimate $\hat{p}(x)$ is defined in Eq. (4.200) in terms of the noise bandwidth B_N for the signal being analyzed, which in turn is a function of the power spectral density of the signal. The power spectral density of the signal may be thought of as the shaping of white noise (flat power spectrum) by a particular filter with a given transfer function $\mathcal{H}(f)$. B_N is then determined directly from Eq. (4.197).

Consider three cases as follows:

1. Rectangular (Idealized) Low Pass Filter
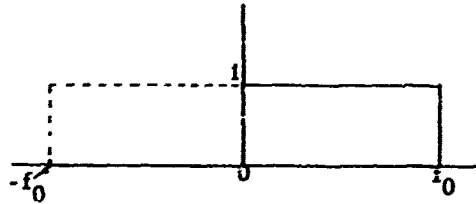
$$\left|\mathcal{H}(f)\right|^2 = \begin{cases} 1 & |f| \le f_0 \\ 0 & \text{otherwise} \end{cases}$$

$$B_N = \int_0^{f_0} df = f_0 \qquad\qquad (4.202)$$

2. Rounded Low Pass Tuned Filter
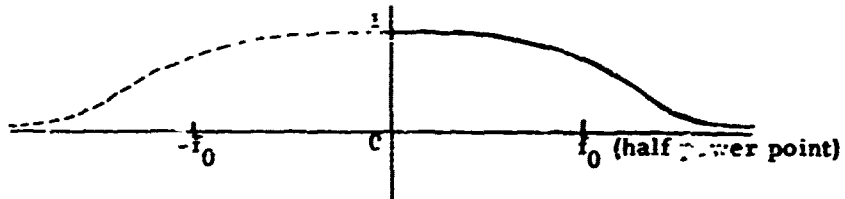
$$\left|\mathcal{H}(f)\right|^2 = \frac{1}{1 + (f/f_0)^2}$$

$$B_N = \int_0^{\infty} \frac{df}{1 + (f/f_0)^2} = \frac{\pi}{2} f_0 \qquad (4.203)$$

3. Gaussian Low Pass Filter
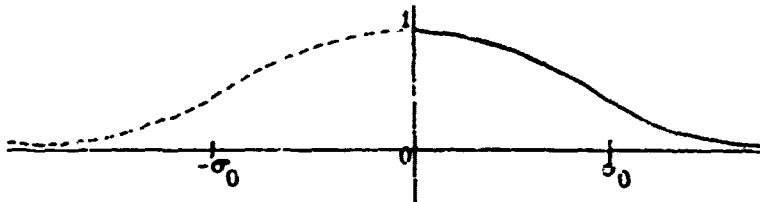
$$\left| N(f) \right|^2 = e^{-f^2/2\sigma_0^2}$$

$$B_N = \int_0^\infty e^{-f^2/2\sigma_0^2} \, df = \sqrt{\frac{\pi}{2}} \, \sigma_0 \qquad (4.204)$$

where σ_0 is the standard deviation of the Gaussian filter characteristic.

The above three examples define the noise bandwidth in Eq. (4.200) for three very simple frequency spectra. It is obvious that the noise bandwidth for signals with complex power spectral density functions would be very difficult to determine. For simplicity, it will frequently be necessary to assume the signal being analyzed is band limited white noise with a sharp cut off at f_0 cps as in case 1. The standard error ϵ of a probability density estimate $\hat{p}(x)$ is then simply,

$$\epsilon \sim \left[\frac{1}{2 f_0 T \, \hat{p}(x) \, (\Delta x)} \right]^{1/2} \qquad (4.05)$$

Numerical calculations may now be carried out as desired for physical examples. This is done for a hypothetical example in Section 7.5.5 of this report.

4.9.3 Threshold Crossings and Peak Value Distribution

Consider a random record $x(t)$ whose behavior over a long period of time exhibits many random oscillations. The expected number of zero crossings per unit time (usually seconds) of the record, denoted by N_0, gives an indication of its "apparent frequency". For example, a 60 cps sine wave has 120 zeros per second. For a random record, the situation is, of course, more complex but still knowledge of N_0, in addition to other quantities, helps to characterize the random record. This type of information and certain of its extensions discussed below is particularly useful for fatigue analysis and reliability prediction of structures under random loading and vibration.

At an arbitrary level, say, $x = \alpha$, the expected number of crossings per unit time through the interval $(\alpha, \alpha + d\alpha)$, where $d\alpha$ is arbitrary small, will be denoted by N_α. It follows that the expected number of times per unit time that $x(t)$ exceeds the value α (i.e., crosses the line $x = \alpha$ with positive slope) is given by $(1/2)N_\alpha$, since $x(t)$, on the average, passes the value α half of the time with positive slope and half of the time with negative slope. When $\alpha = 0$, N_α reduces to N_0, the expected number of zero crossings per unit time; the quantity $(1/2)N_0$ represents the expected number of zero crossings per unit time with positive or negative slopes.

General probability formulas may be written down for evaluating N_α. In practice, however, useful simple formulas have been obtained which apply only to situations where the random record $x(t)$ is assumed to be a sample member from a stationary ergodic Gaussian random process with zero mean value, governed by Equation(4.182). Analysis of this important case is due to Rice [Ref.13], and yields the following result:

$$N_\alpha = 2(\sigma_{\dot{x}}/\sigma_x) e^{-\alpha^2/2\sigma_x^2} \tag{4.206}$$

where

$$\sigma_x^2 = \int_{-\infty}^{\infty} S_x(f)\, df = 2 \int_0^{\infty} S_x(f)\, df \tag{4.207}$$

$$\sigma_{\dot{x}}^2 = \int_{-\infty}^{\infty} f^2 S_x(f)\, df = 2 \int_0^{\infty} f^2 S_x(f)\, df \tag{4.208}$$

Physically, σ_x represents the rms value of x(t), and $\sigma_{\dot{x}}$ represents the rms value of $\dot{x}(t)$. Thus, setting $\alpha = 0$,

$$N_0 = 2(\sigma_{\dot{x}}/\sigma_x) \qquad (4.209)$$

These formulas depend upon knowledge of the power spectrum $S_x(f)$ in a surprisingly simple manner.

For example, for an ideal band-pass filter whose pass band extends from f_a to f_b cps, the expected number of zeros per second for a "white" random noise input is shown in Ref. [13. p. 61] to be given by

$$N_0 = 2 \left[\frac{f_b^3 - f_a^3}{3(f_b - f_a)} \right]^{1/2}$$

As special cases,

$$N_0 \approx 1.55\, f_b \qquad \text{if} \qquad f_a = 0 \quad \text{(low-pass filter)}$$

$$N_0 \to 2f_b \qquad \text{if} \qquad f_a \to f_b \quad \text{(extreme narrow-band filter)}$$

By an analogous but more complicated analysis, R-- [13. p. 79] derives a further property about the expected number of maxima (or minima) of x(t) per second, denoted by $\mathcal{M}$. Since the quantity $\mathcal{M}$ represents either the number of positive peaks or the number of negative peaks, which may be expected to occur equally often on the average, the expected number of both positive and negative peaks per second is given by $2\mathcal{M}$. The expression for $\mathcal{M}$ turns out to be simply

$$\mathcal{M} = (\sigma_{\ddot{x}}/\sigma_{\dot{x}}) \qquad (4.210)$$

where

$$\sigma_{\ddot{x}}^2 = \int_{-\infty}^{\infty} f^4 S_x(f)\, df = 2\int_0^{\infty} f^4 S_x(f)\, df \qquad (4.211)$$

Physically, $\sigma_{\ddot{x}}$ represents the rms value of $\ddot{x}(t)$.

The probability that a positive peak will fall between $(\alpha, \alpha + d\alpha)$ can also be calculated. In terms of a standard variable z with zero mean and unit variance,

$$z = (x/\sigma) \quad ; \quad \sigma^2 = \int_{-\infty}^{\infty} S_x(f)\, df \qquad (4.212)$$

the probability density function $w(z)$ that a positive peak will fall between z and $z + dz$ is expressed by the formula, Ref. [8, p. 23],

$$w(z) = \frac{k_1}{\sqrt{2\pi}}\, e^{-z^2/2k_1^2} + \left(\frac{N_0}{2m}\right) z\, e^{-z^2/2}\, [1 - P_n(z/k_2)] \qquad (4.213)$$

where

$$k_1 = \sqrt{1 - (N_0/2m)^2}$$

$$k_2 = \frac{k_1}{(N_0/2m)} \qquad\qquad \frac{N_0}{2m} = \frac{\sigma_{\ddot{x}}^2}{\sigma_x\, \sigma_{\ddot{x}}} \qquad (4.214)$$

and

$$P_n(z/k_2) = \frac{1}{\sqrt{2\pi}}\, \int_{z/k_2}^{\infty} e^{-y^2/2}\, dy \qquad (4.215)$$

Note that $P_n(z/k_2)$ is the probability for a standard normal distribution with zero mean and unit variance that the value (z/k_2) will be exceeded. This integral is readily available in statistical tables.

The shape of $w(z)$ is determined by the parameter $(N_0/2m)$. It can be shown from basic considerations that $(N_0/2m)$ always falls between zero and unity.

This results from the fact that $(N_0/2m) = (\sigma_{\ddot{x}}^2/\sigma_x\, \sigma_{\ddot{x}})$, and from the Schwartz inequality

$$\sigma_{\ddot{x}}^2 = \int_{-\infty}^{\infty} f^2\, S_x(f)\, df \le \left[\int_{-\infty}^{\infty} S_x(f)\, df\right]^{1/2} \left[\int_{-\infty}^{\infty} f^4\, S_x(f)\, df\right]^{1/2} = \sigma_x\, \sigma_{\ddot{x}}$$

Hence

$$0 \leq (N_0/2\mathcal{M}) = (\sigma_{\dot{x}}^2/\sigma_x \sigma_{\ddot{x}}) \leq 1 \qquad (4.216)$$

If $(N_0/2\mathcal{M}) = 0$, then $w(z)$ reduces to a standardized <u>normal (Gaussian) probability density function</u>,

$$w(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \qquad \text{when} \qquad (N_0/2\mathcal{M}) = 0 \qquad (4.217)$$

This case occurs in practice <u>for wide-band noise</u> where the expected number of maxima and minima per second, $2\mathcal{M}$, is much larger than the expected number of zero crossings per second, N_0, so that $(N_0/2\mathcal{M})$ approaches zero.

If $(N_0/2\mathcal{M}) = 1$, then $w(z)$ becomes a standardized <u>Rayleigh prob-ability density function</u>,

$$w(z) = z e^{-z^2/2} \qquad \text{when} \qquad (N_0/2\mathcal{M}) = 1 \qquad (4.218)$$

This case occurs in practice <u>for narrow-band noise</u> where the expected number of maxima and minima per second, $2\mathcal{M}$, is approximately equal to the expected number of zero crossings per second, N_0, so that $(N_0/2\mathcal{M})$ approaches unity. The general form of $w(z)$ from Eq. (4.213) is thus something between a Gaussian and a Rayleigh probability density function, and is plotted in Figure 4.6 below as a function of z for three values of the dimensionless parameter $(N_0/2\mathcal{M})$ equal to 0, 0.5 and 1.0.

In terms of $w(z)$ the probability $P_p(z)$ that a positive peak chosen at random from among all the possible positive peaks will exceed the value z is given by the formula

$$P_p(z) = \int_z^\infty w(z)\, dz$$

$$= P_n(z/k_1) + \frac{N_0}{2\mathcal{M}} e^{-z^2/2} [1 - P_n(z/k_2)] \qquad (4.219)$$
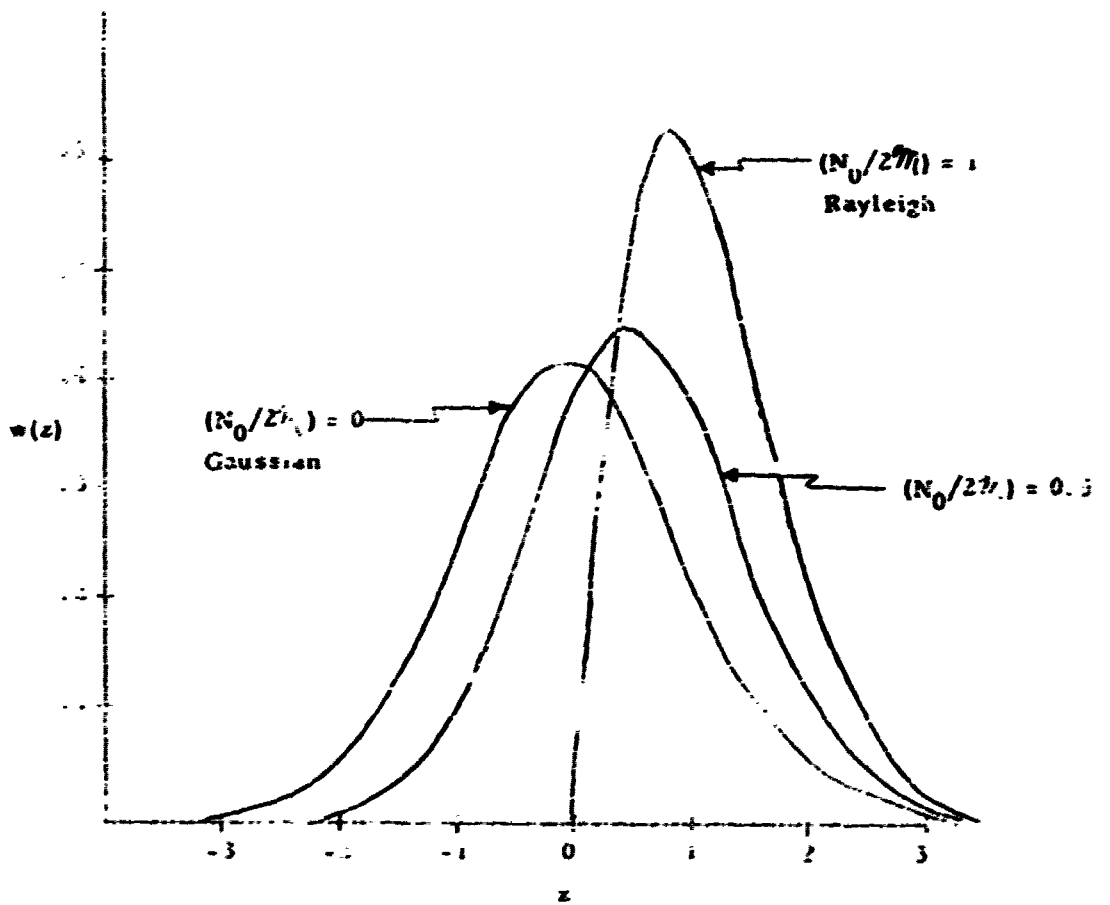
Figure 4.6 Peak Probability Density Function w(z) versus z

using the P_n of Eq. (4.215). A graph of $P_p(z)$ as a function of z is plotted in Figure 4.7 below for three fixed values of $(N_0/2\bar{n})$ equal to 0, 0.5 and 1 0.
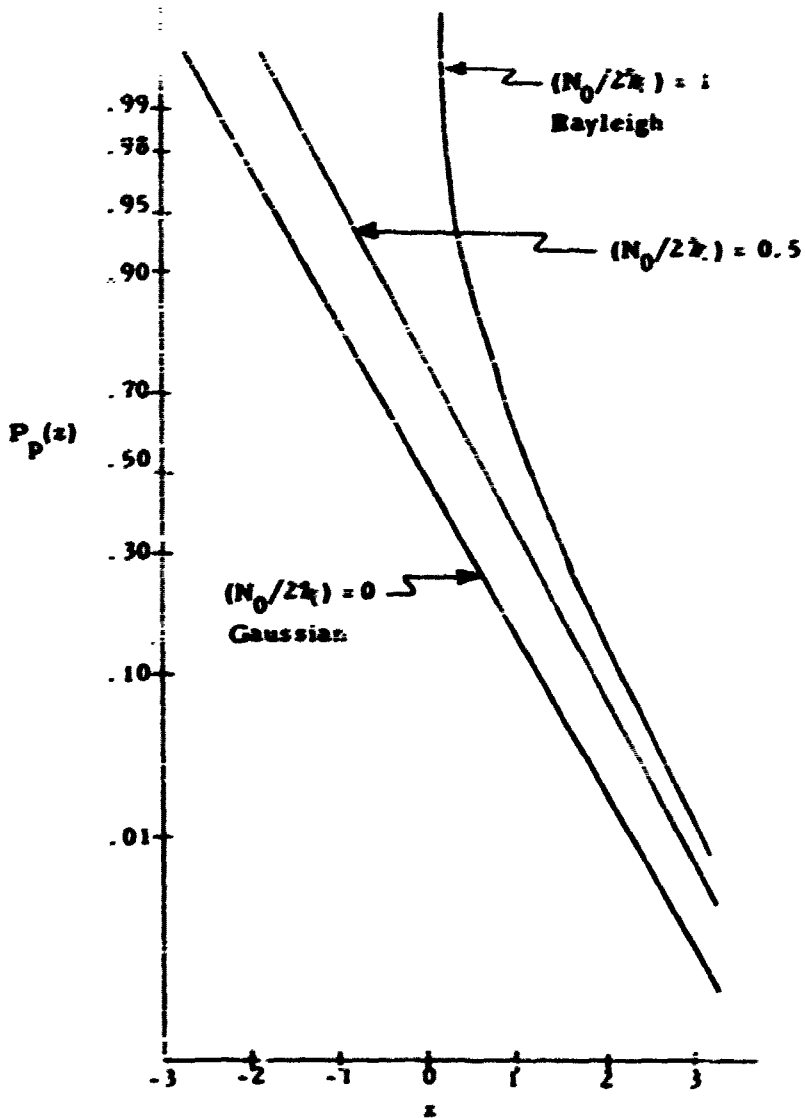
Figure 4.7 Graph of $P_p(z) = \int_z^{\infty} w(z)\, dz$ versus z

From the above, it should be noted that the actual number of positive peaks per second which would exceed the value $a = z\sigma$, denoted by M_a, may be estimated by the formula

$$M_a = \mathfrak{M} P_p(a/\sigma) = \mathfrak{M} P_p(z) \tag{4.220}$$

For large values of a relative to σ, one may verify

$$M_a \approx (N_0/2) e^{-a^2/2\sigma^2} \tag{4.221}$$

showing that for large a, the expected number of maxima per second lying above the line $x = a$ is equal to the expected number of times per second that $x(t)$ crosses the line $x = a$ with positive slope.

The expected number of peaks which exceed the value a in time T_1 is given by

$$M_a T_1 = \mathfrak{M} T_1 P_p(a/\sigma_1) \tag{4.222}$$

This can be set equal to the expected number of peaks which exceed the value a in time T_2 by introducing a different mean square value σ_2^2 such that

$$M_a T_2 = \mathfrak{M} T_2 P_p(a/\sigma_2) = \mathfrak{M} T_1 P_p(a/\sigma_1)$$

Now,

$$\frac{T_2}{T_1} = \frac{P_p(a/\sigma_1)}{P_p(a/\sigma_2)} \tag{4.223}$$

Suppose the mean square value is such that σ_1^2 occurs for time T_1 followed by σ_2^2 for time T_2. What should be the equivalent σ^2 for time $T = T_1 + T_2$ if equivalence is based on having the same number of peaks exceeding a? The expected number of peaks which exceed the value a in time T_1 and the value a in time T_2 is given by

$$M_e \, T_1 + M_e \, \bar{T}_2 = \Re T_1 \, P_p(e/\sigma_1) + \Re T_2 \, P_p(e/\sigma_2)$$

The above should now be set equal to

$$M_e T = \Re T \, P_p(e/\sigma)$$

yielding the relation

$$T = \left[\frac{P_p(e/\sigma_1)}{P_p(e/\sigma)}\right] T_1 + \left[\frac{P_p(e/\sigma_2)}{P_p(e/\sigma)}\right] T_2 \qquad (4.224)$$

In general, for N distinct mean square values in N time periods, one should set

$$T = \sum_{i=1}^{N} c_i \bar{T}_i \quad \text{where } c_i = \frac{P_p(e/\sigma_i)}{P_p(e/\sigma)} \qquad (4.225)$$

The general solution for $P_p(e/\sigma)$ is

$$P_p(e/\sigma) = \sum_{i=1}^{N} (T_i/T) \, P_p(e/\sigma_i) \qquad (4.226)$$

Thus, knowledge of all quantities on the right-hand side of Eq. (4.226) enables one to solve for $P_p(e/\sigma)$, and in turn for the parameters (e/σ) and σ. If $e = 0$, however, Eq. (4.226) becomes an identity and σ can not be determined.

4.9.4 Measurement of Linear System Frequency Response Function

For a constant parameter linear system, it is well-known that such a system can be characterized by a weighting function $h(\tau)$, which, by definition, yields the response of a system to a unit impulse function τ time units after the impulse occurs. See sketch.
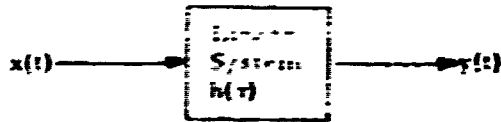
Linear System

For the system to be physically realizable it is necessary that

$$h(\tau) \quad 0 \quad \text{for} \quad \tau < 0 \qquad (4.227)$$

since a system can not respond to an impulse before it occurs.

If $x(t)$ is an input to this system, and $y(t)$ the resulting output, then the output is given as a weighted linear sum over the entire (infinite) past history of the input as expressed by

$$y(t) \quad \int_0^\infty h(\tau) \, x(t - \tau) \, d\tau \qquad (4.228)$$

If the system operates on $x(t)$ only for a finite fixed ti. T, then

$$y(t \quad \int_0^T h(\tau) \, x(\cdot - \tau) \, d\tau \qquad (4.229)$$

If $x(t)$ exists only for $t > 0$, then

$$y(t) \quad \int_0^t h(\tau) \, x(t - \tau) \, d\tau \qquad (4.230)$$

Instead of using $(h(\tau))$, the system may be characterized by its frequency response function $|H(\omega)|$ which is defined as the Fourier

transform of $h(\tau)$, namely,

$$H(f) = \int_0^\infty h(\tau) e^{-j2\pi f\tau} d\tau , \quad H(2\pi f)$$

(4.231)

assuming $h(\tau) = 0$ for $\tau < 0$ as needed for physical realizability. The frequency response function should not be confused with the transfer function of the system as defined by the Laplace transform of $h(\tau)$, and denoted by

$$H(p) = \int_0^\infty h(\tau) e^{-p\tau} d\tau$$

(4.232)

Note that $H(f)$ differs formally from $H(p)$ by merely replacing p in $H(p)$ by $j2\pi f$.

The frequency response function is a complex-valued function of f such that

$$H(f) = |H(f)| e^{j\phi(f)}$$

(4.233)

where $|H(f)|$, the absolute value of $H(f)$ measures the amplitude response (gain) of the system to an input sinusoidal exciting frequency f, while $\phi(f)$ indicates the corresponding phase shift.

Consider a complex-valued sinusoidal input

$$x(t) = e^{j2\pi f_0 t}$$

(4.234)

From Equations (4.228) and (4.231), the response $y(t)$ is

$$y(t) = H(f_0) e^{j2\pi f_0 t}$$

(4.235)

For a real-valued sinusoidal input

$$x(t) = \sin(2\pi f_o t + d) \tag{4.236}$$

the real-valued output is given by

$$y(t) = |\mathcal{H}(f)| \sin\left[2\pi f_o t + d + \phi(f_o)\right] \tag{4.237}$$

These relations show how knowledge of both the gain factor and the phase shift term are needed to describe the system's operation.

From physical realizability requirements, the frequency response function, the gain factor, and the phase shift term satisfy the symmetry properties

$$\mathcal{H}(-f) = \overline{\mathcal{H}(f)}$$

$$|\mathcal{H}(-f)| = |\mathcal{H}(f)| \tag{4.238}$$

$$\phi(-f) = -\phi(f)$$

If one linear system, described by $\mathcal{H}_1(f)$, is followed by a second linear system, described by $\mathcal{H}_2(f)$, then the overall system may be described by $\mathcal{H}(f)$ where

$$\mathcal{H}(f) = \mathcal{H}_1(f)\,\mathcal{H}_2(f) \tag{4.239}$$

Hence

$$|\mathcal{H}(f)| = |\mathcal{H}_1(f)|\,|\mathcal{H}_2(f)|$$

$$\phi(f) = \phi_1(f) + \phi_2(f)$$

Thus, on cascading two linear systems, the gain factors multiply and the phase shift factors add together.

Assume now that the input x(t) is a representative member from a stationary random process with zero mean value. Then, the same property is true for y(t), and the ordinary power spectral density functions $S_x(f)$ and $S_y(f)$ are related to $|\mathcal{H}(f)|$ by the simple (real-valued) formula, Ref. [1, p. 72],

$$S_y(f) = |\mathcal{H}(f)|^2 S_x(f) \qquad (4.240)$$

This is an important result and is frequently quoted. At any fixed frequency f, knowledge of two of these quantities determines the third. The phase shift term $\phi(f)$, however, is still in doubt since this formula involves only the gain factor.

Example: Single-Degree-of Freedom System Output Response

For the frequency response function governing a single-degree-of-freedom system, see Eq. (3.1),

$$\mathcal{H}(\omega/2\pi) = H(\omega) = \frac{1}{1 - (\omega/\omega_n)^2 + j\, 2(\omega/\omega_n)} \quad ; \quad j = \sqrt{-1}, \ \omega = 2\pi f$$

$$\mathcal{f} = (1/2Q)$$

Suppose that the input power spectral density function is "white noise:, see Fig. 3.11, where $S_x(f) = S_0$, a constant, $0 \le f < \infty$. From Eq. (4.88), this corresponds to $S_x(f) = (S_0/2)$ when $-\infty \le f \le \infty$. Now, from Eq. (4.97), on changing to angular frequency ω,

$$S_x(\omega) = [S_x(f)/2\pi] = (S_0/4\pi) \qquad ; \qquad -\infty \le \omega \le \infty$$

Analogous to Eq. (4.240), the output power spectral density function $S_y(\omega)$, in terms of angular frequency ω, is now given by

$$S_y(\omega) = |H(\omega)|^2 S_x(\omega) = \frac{(S_0/4\pi)}{\left[1 - \left(\dfrac{\omega}{\omega_n}\right)^2\right]^2 + 4\mathcal{f}^2 \left(\dfrac{\omega}{\omega_n}\right)^2}$$

Finally, similar to Eq. (4.19) or (4.183), in terms of angular frequency ω, the mean square value of the output becomes

$$\sigma_y^2 = \overline{y^2(t)} = \int_{-\infty}^{\infty} S_y(\omega)\, d\omega = 2\int_0^{\infty} S_y(\omega)\, d\omega$$

$$= \frac{S_0}{2\pi} \int_0^{\infty} \frac{d\omega}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2} = \frac{\omega_n S_0}{8\zeta} = \frac{Q\omega_n S_0}{4}$$

This is the derivation of Eq. (3.9) in Section 3.3.3. End of example.

Recall from Section 4.9.1 the fact that a stationary Gaussian random process with zero mean value is known completely from its power spectral density function. It can be shown that the response of a linear system to a stationary Gaussian input is also stationary and Gaussian. Thus, the gain factor $|H(f)|$ of the frequency response function of a linear system characterizes the output system response to any stationary Gaussian input. This provides one of the main physical motivations for measuring as accurately as possible the gain factor of system frequency response function of a linear system. However, if the system response to an arbitrary given input is desired, it is required to measure not only the gain factor of the system but also its phase shift term.

By a straight-forward approach, one may verify that the entire frequency response function $H(f)$ is related to the input power spectral density function $S_x(f)$, and to the cross-power spectral density function $S_{xy}(f)$ between the input and the output, by another simple (complex-valued) formula, Ref. [1, p. 75],

$$S_{xy}(f) = H(f)\, S_x(f) \tag{4.241}$$

Thus, if

$$S_{xy}(f) = |S_{xy}(f)|\, e^{j\theta(f)} \quad ; \quad H(f) = |H(f)|\, e^{j\theta(f)} \tag{4.242}$$

one obtains two relations which involve both the gain factor and the phase shift.

$$|S_{xy}(f)| = |H(f)| \, S_x(f) \tag{4.243}$$

$$\theta(f) = \varphi(f) \tag{4.244}$$

Note that the coherence function becomes here, Eq. (4.95)

$$\gamma_{xy}^2(f) = \frac{|S_{xy}(f)|^2}{S_x(f) \, S_y(f)} = 1 \tag{4.245}$$

indicating complete linear dependence between x and y at every frequency. The value of the coherence function will be less than unity if additive noise occurs either in the input or output of the linear system.

The main limitations in applying the above formulas are due to failing to satisfy requirements that the system is of a constant parameter linear type, and that the input random process is stationary. No such simple relation exists for time varying linear systems, for non-linear systems, or for nonstationary random processes.

4.9.5 Confidence Limits Based on Coherence Function

For cases where additive noise occurs in the input or output of the linear system, an estimate of the true frequency response $H(f)$ may be obtained by measuring $S_{xy}(f)$ and $S_x(f)$. To distinguish between the true value of $H(f)$ and a particular estimate of the true value which would be measured in practice, let

$$\hat{H}(f) = \frac{\hat{S}_{xy}(f)}{\hat{S}_x(f)} = |\hat{H}(f)| e^{j\hat{\phi}(f)} \tag{4.246}$$

represent the estimate in question where $|\hat{H}(f)|$ denotes the estimate of the true gain factor $|H(f)|$, and $\hat{\phi}(f)$ denotes the estimate of the true phase shift term $\phi(f)$.

Results of Goodman Ref. [6], quoted by Katz Ref. [9] and partially displayed graphically by Press Ref. [3], demonstrate that to a very close approximation,

$$\text{Prob}\left[\left|\frac{|\hat{\mathcal{H}}(f)| - |\mathcal{H}(f)|}{|\mathcal{H}(f)|}\right| < \sin \epsilon \text{ and } \left|\hat{\phi}(f) - \phi(f)\right| < \epsilon\right]$$

$$\approx 1 - \left[\frac{1 - \gamma_{xy}^2(f)}{1 - \gamma_{xy}^2(f)\cos^2\epsilon}\right]^n \qquad (4.247)$$

where $\gamma_{xy}^2(f)$ is the coherence function and n is the number of degrees of freedom, see Section 4.8.5.

$$n = 2BT \qquad (4.248)$$

with terms as defined previously.

Formula (4.247) is of considerable practical importance in determining the confidence level at which the gain and phase can be estimated to within a desired error, for a given value of γ^2 and n. For example, suppose one wants to estimate the phase ϕ to within 0.1 radian ($\epsilon = 0.10$) and the gain $|\mathcal{H}(f)|$ to within 10% error ($\sin \epsilon \approx 0.10$) at a 90% confidence level (Prob = 0.90). The table below shows the required number of degrees of freedom n corresponding to various assumed values for the coherence function γ^2.

γ^2	0.9	0.8	0.7	0.6	0.5
n	27	58.5	100	156	232

$\epsilon = 0.10$

Prob = 0.90

In practice, one will not know in advance what the coherence function will be, and can only roughly estimate it from the measurements. This restricts the application of the above work to some extent. However, a conservative choice is usually desirable, and the above formula (4.247) shows clearly that accurate measurements of a frequency response function is strongly dependent upon the value of the coherence function.

4.9.6 Statistics for Extreme Vibration Amplitudes

Consider samples of size n, drawn independently and at random from an underlying distribution described by its probability density function $f(x)$, or alternatively by its distribution function $F(x) = \int_{-\infty}^{x} f(y)dy$. Let the observations drawn be $\{x_1, x_2, \ldots, x_n\}$.

Define the statistic

$$z_n = \max\{x_1, x_2 \ldots, x_n\} \tag{4.249}$$

The problem is to find the probability density and distribution function of z_n, and to consider their asymptotic behavior when n becomes large. A good discussion of these matters appears in Ref. [5].

The distribution of z_n is obtained as follows:

Define

$$H_n(z) = \text{Prob}(z_n \leq z) = \text{Prob}(\text{all } x_i \leq z)$$

$$= \text{Prob}(x_1 \leq z, x_2 \leq z, \ldots, x_n \leq z)$$

$$= \left[F(z)\right]^n \tag{4.250}$$

The quantity $H_n(z)$ is the distribution function of z_n. The corresponding density function $h_n(z)$ is found by differentiating $H_n(z)$. This gives

$$h_n(z) = H_n'(z) = nf(z)\left[F(z)\right]^{n-1} \tag{4.251}$$

Thus $H_n(z)$ and $h_n(z)$ are determined entirely from n, $f(z)$ and $F(z)$

A special case for which one can obtain an exact distribution of z_n for any n is provided by an underlying uniform distribution.

Example 1: Underlying Uniform Distribution over (0, A).

$$F(x) = 0 \quad ; \quad x < 0$$

$$= \frac{x}{A} \quad ; \quad 0 \le x \le A \tag{4.252}$$

$$= 1 \quad ; \quad x > A$$

Now

$$H_n(z) = [F(z)]^n = 0 \quad ; \quad z < 0$$

$$= \left(\frac{z}{A}\right)^n \quad ; \quad 0 \le z \le A \tag{4.253}$$

$$= 1 \quad ; \quad z > A$$

and

$$h_n(z) = H_n'(z) = \frac{nz^{n-1}}{A^n} \quad ; \quad 0 \le z \le A$$

$$\tag{4.254}$$

$$= 0 \quad ; \quad z < 0 \text{ or } z > A$$

Note that for any value of n,

$$\bar{z}_n = \int_{-\infty}^{\infty} z h_n(z) dz = \frac{n}{A^n} \int_0^A z^n dz = \left(\frac{n}{n+1}\right) A$$

$$\overline{z_n^2} = \int_{-\infty}^{\infty} z^2 h_n(z) dz = \frac{n}{A^n} \int_0^A z^{n+1} dz = \left(\frac{n}{n+2}\right) A^2$$

$$\sigma_n^2 = \overline{z_n^2} - (\bar{z}_n)^2 = \frac{n}{(n+1)^2 (n+2)} A^2 \tag{4.255}$$

Hence, for large n,

$$\bar{z}_n \approx A$$

$$\sigma_n \approx \frac{A}{n} \tag{4.256}$$

In words, for large n, the expected value of the maximum from a
sample of size n equals the right-hand end point of the underlying
uniform distribution governing the original samples.   The standard
deviation in this estimate approaches zero inversely with n.

Example 2:   Underlying Normal Distribution with Zero Mean and Unit
             Standard Deviation.

The case of an underlying normal distribution does not lead to
closed-form answers.   Numerical methods must be employed to approxi-
mate desired results.

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} \, dy \qquad (4.257)$$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \qquad (4.258)$$

Here x is a normal random variable with unit standard deviation.

Let $y = \mu + \sigma x$ and $dy = \sigma \, dx$

Then $g(y) \, dy = f(x) \, dx = \frac{1}{\sigma\sqrt{2\pi}} e^{-(y-\mu)^2/2\sigma^2} dy$

Hence $g(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(y-\mu)^2/2\sigma^2}$ $\qquad (4.259)$

Now y is a normal random variable with mean value μ and standard
deviation σ.   This procedure shows how to modify the underlying
distribution to cover an arbitrary mean value and standard deviation.

Similarly

$$G(y) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{y} e^{-(y-\mu)^2/2\sigma^2} \, dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{y-\mu}{\sigma}} e^{-v^2/2} \, dv = F\left(\frac{y-\mu}{\sigma}\right)$$

Hence, letting $x = \frac{y-\mu}{\sigma}$,

$$G(y) = F\left(\frac{y-\mu}{\sigma}\right) = F(x) \tag{4.260}$$

Thus, the distribution function $F(x)$, associated with zero mean value and unit standard deviation, may be generalized to the distribution function $G(y)$, where the mean value is μ and the standard deviation is σ, by replacing x by $(y-\mu)/\sigma$. For simplicity, $F(x)$ will be used instead of $G(y)$ in further calculations.

From Eq. (4.250), for samples of size n from an underlying normal population, with zero mean and unit standard deviation,

$$H_n(z) = \left[F(z)\right]^n = \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-y^2/2} \, dy\right]^n \tag{4.261}$$

and

$$h_n(z) = nf(z)\left[F(z)\right]^{n-1} = \frac{n}{\sqrt{2\pi}} e^{-z^2/2}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-y^2/2} \, dy\right]^{n-1} \tag{4.262}$$

Clearly, these equations can now be analyzed by referring to readily available statistical tables of the normal distribution. See Table 5.1 at end of next Section 5. Figure 4.8 at end of this section displays $H_n(z)$ as a function of the normal variate z for fixed values of n equal to 5, 10, 100 and 1000. The case where $n = 1$ corresponds to $H_1(z) = F(z)$.

To illustrate how to apply Figure 4.8, consider the curve where $n = 10$. The mode value of z is defined as that value of z for which $H_n(z) = 0.50$. For $n = 10$, the mode value is seen to be approximately 1.50. In words, there is a 50% probability that in samples of size 10, the largest value z will be less than or equal to 1.5. This result assumes that $\mu = 0$ and $\sigma = 1$. Note that for $n = 1$, the mode value is equal to zero, while for $n = 1000$, the mode value becomes 3.2. At the 95% probability point, for $n = 10$, the largest value will be at most equal to 2.7, while for $n = 1000$, the largest value will be at most equal to 4.0.

For n - 1 there is a 95% probability that the largest value will be at most equal to 1.7. These examples show how the extreme value statistics for an underlying normal population change with increases in sample size.

For arbitrary μ and σ, there is a 50% probability that in samples of size 10, the largest value z will be less than or equal to $\mu + 1.5\sigma$, and a 95% probability that the largest value will be less than or equal to $\mu + 2.7\sigma$. Similar statements apply to other sample sizes for arbitrary μ and σ.

It should also be realized that in samples of small size, considerable uncertainty exists in estimates of μ and σ. As an illustration of material to be explained in Section 5 of this report, for samples of size n, the true mean value μ is bounded by

$$\text{Prob}\left[\mu \leq \bar{x} + t_\alpha \, s/\sqrt{n}\right] = 1 - \alpha \tag{4.263}$$

where α = desired level of significance (e g., 1% or 5%) and t_α is taken from the "t" distribution with (n-1) degrees of freedom, see Sections 5.2.3 and 5.3.3. The quantity $(1 - \alpha)$ is a confidence coefficient of $100(1-\alpha)$ percent (e.g., 99% or 95%). For samples of size n, the true standard deviation σ is bounded by

$$\text{Prob}\left[\sigma \leq s\sqrt{n/\chi_\alpha^2}\right] = 1 - \alpha \tag{4.264}$$

where χ_α^2 is taken from the chi-square distribution with (n-1) degrees of freedom at the α level of significance, see Sections 5.2.2 and 5.4.1. In particular, for $\alpha = 5\%$,

$$\text{Prob}\left[\mu \leq \bar{x} + t_{5.0} \, s/\sqrt{n}\right] = 0.95$$

$$\text{Prob}\left[\sigma \leq s\sqrt{n/\chi_{5.0}^2}\right] = 0.95 \tag{4.265}$$

Then, since both extremes will not occur simultaneously, in general, for λ standard deviations from the mean value,

$$\text{Prob}\left[\mu + \lambda\sigma \leq \bar{x} + s\left[t_{5.0}/\sqrt{n} + \lambda\sqrt{n/\chi_{5.0}^2}\right]\right] \geq 0.95 \tag{4.266}$$

For example, for $n = 10$, corresponding to 9 degrees of freedom, one finds from Tables 5.3 and 5.4 at end of Section 5 that $t_{5.0} = 1.8$ and $x_{5.0}^2 = 15$. Hence, for $n = 10$,

$$\text{Prob}\left[\mu + \lambda\sigma \leq \bar{x} + s(0.57 + 0.82\lambda)\right] \geq 0.95$$

Also, as shown earlier, for samples of size 10, if z equals the maximum value, then

$$\text{Prob}\left[z \leq \mu + 2.7\sigma\right] = 0.95$$

Hence,

$$\text{Prob}\left[z \leq \bar{x} + 2.8s\right] \geq 0.95$$

This completes the discussion in this section of the report.
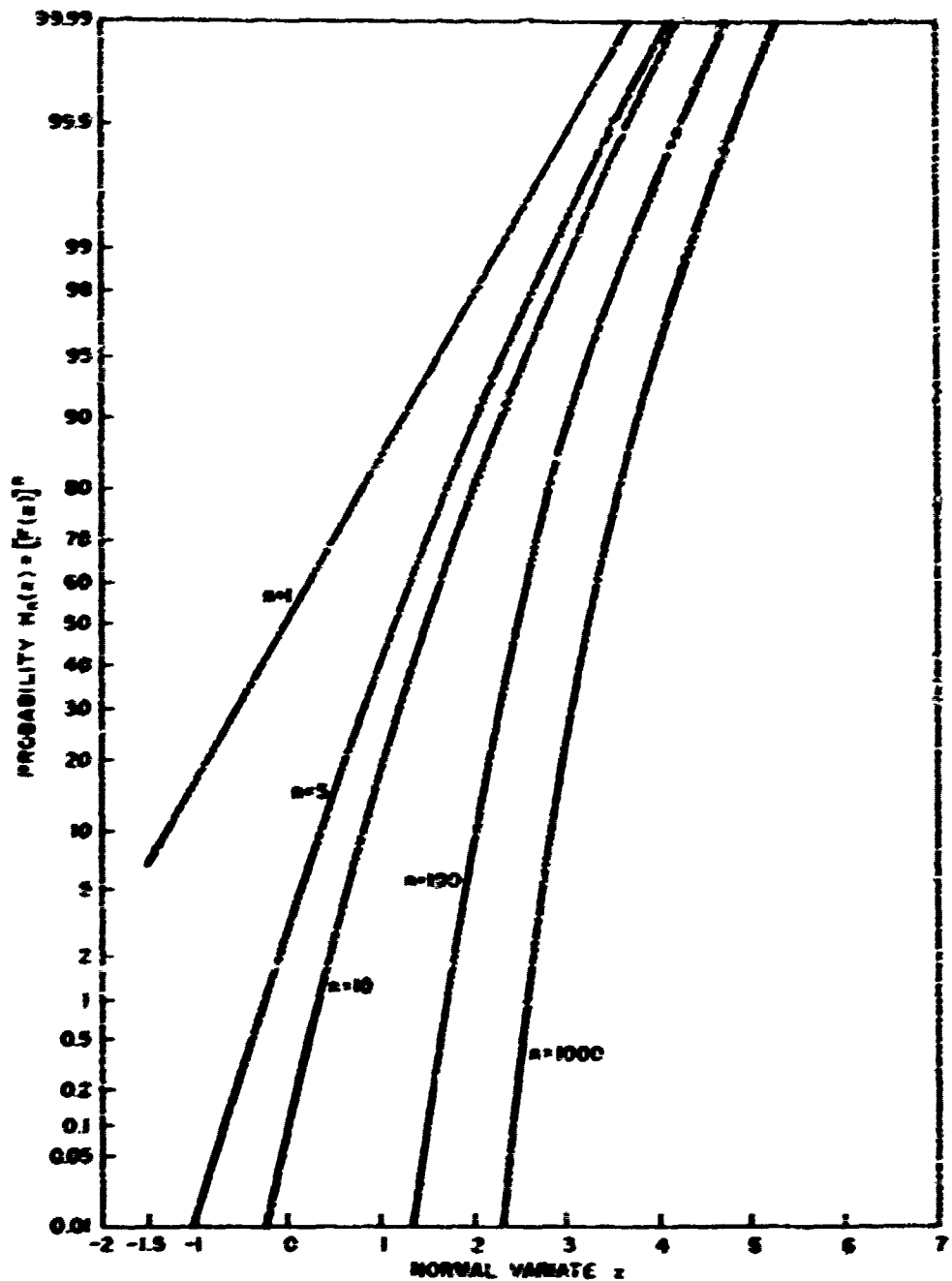
Figure 4.8. Probabilities of Normal Extremes

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5. STATISTICAL TECHNIQUES FOR EVALUATING DATA

5.1 THE ESTIMATION PROBLEM AND HYPOTHESIS TESTING

Two broad areas exist in the general area of statistics which are problems of estimation and testing of statistical hypotheses. Estimation can be further broken down into that of point estimates and interval estimates of the parameters of a distribution of interest.

5.1.1 Estimation Theory

The general theory of estimation is quite an involved subject and no attempt will be made to discuss it in any detail here. However, there are three qualities desirable to have in any estimate. First, one wants an estimate to be unbiased, that is, the expected value of the estimate should be the true value. Second, the variance of the estimate should be a minimum, as compared to other possible estimates, and third, the variance of the estimate should approach zero for large sample size. An estimate with the second property is said to be efficient; an estimate with the third property is said to be consistent. The first and third concepts have been defined and discussed in Section 4.5.

Examples of efficient estimates are the arithmetic mean of a sample of N observation

$$\bar{x} = \frac{\sum_{i=1}^{N} x_i}{N} \tag{5.1}$$

as an estimate of the population mean μ, and the sample variance

$$s^2 = \frac{\sum_{i=1}^{N} (x_i - \bar{x})^2}{N} \tag{5.2}$$

as an estimate of the population variance σ^2 The expected value of the sample mean is

$$E(\bar{x}) = \mu \tag{5.3}$$

and $\bar{x}$ is an unbiased estimate of μ. However, in Section (5.3.1) it will be shown that

$$E(s^2) = \left(\frac{N-1}{N}\right)\sigma^2 \tag{5.4}$$

Therefore, s^2 is a biased estimate of σ^2. The corrected sample variance to remove the bias would be

$$\left(\frac{N}{N-1}\right)s^2 = \frac{\sum_{i=1}^{N}(x_i - \bar{x})^2}{N-1} \tag{5.5}$$

Several methods exist for obtaining formulas to compute parameter estimates from sample data. One procedure commonly used is the method of maximum likelihood (see Reference [2], pp. 68-71). This method gives estimates which are often biased, but are consistent, asymptotically efficient and asymptotically normal under general conditions. Also the bias may often be removed easily as for the sample variance in Equation (5.5). Most of the statistical estimates in this report are of this type.

Besides the point estimates of a parameter, it is useful to be able to give an interval in which the parameter of interest probably lies. That is, an indication of the precision of the estimate is desirable. This leads to the concept of a confidence interval. To be specific, given a parameter θ, and a small predetermined probability α, an interval (θ_1^*, θ_2^*) is desired such that

$$\text{Prob } (\theta_1^* < \theta < \theta_2^*) = 1 - \alpha \tag{5.6}$$

The probability $(1 - \alpha)$ is called the confidence coefficient of the confidence interval (θ_1^*, θ_2^*). The confidence intervals so computed in a series of repeated experiments could be expected to contain θ in $(1 - \alpha)$ percent of the cases.

5.1.2 Hypothesis Testing

The simplest situation is the test of a hypothesis against a single alternative. To be more specific, suppose that the variable in question has a probability density function $p(x, \theta)$, while under the alternative hypothesis it has a probability density function $p(x, \theta_0)$. That is, the density functions are completely specified and if the parameter of interest is not θ then it is θ_0. This is known as a **simple hypothesis**. If any other parameters exist in the density functions which are not known, or if the alternative to θ is not completely specified, then the hypothesis is called a **composite hypothesis**. This is the more common situation and would be illustrated by the hypothesis that the mean of a normal distribution is μ with the alternative hypothesis being that it is either $\mu_0 > \mu$ or $\mu_0 \neq \mu$. That is, μ_0 is not completely specified.

The hypothesis tested is often the **null hypothesis**. This would be something such as: The parameters θ and θ_0 are the same. Stated in another equivalent way, there is no conclusive evidence that parameters θ and θ_0 are not equal. After selecting a null hypothesis, the next step is to select a region such that if the observation falls within this region the hypothesis is accepted. This is the **region of acceptance.** The complementary region where the hypothesis is rejected is the **critical region** or **region of rejection.**

To choose these regions one first decides upon a small probability α such that the hypothesis will be rejected α percent of the time when it is really true. This error α which can be made is known as the **Type I error.** The numerical value α is known as the **level of significance** of the test and is usually chosen to be 0.05 or 0.01 (5% or 1%).

Suppose the random variable under consideration has a probability density function $p(x, \theta)$, associated with a hypothesized parameter θ. Then the critical region would be to the right of a value x_c determined by

$$\int_{x_c}^{\infty} p(x, \theta) \, dx = \alpha \tag{5.7}$$

Figure 5.1 below illustrates these concepts.

$p(x, \theta)$

Area = $1 - \alpha$

Area = α = Level of significance

x

Region of Acceptance

x_c
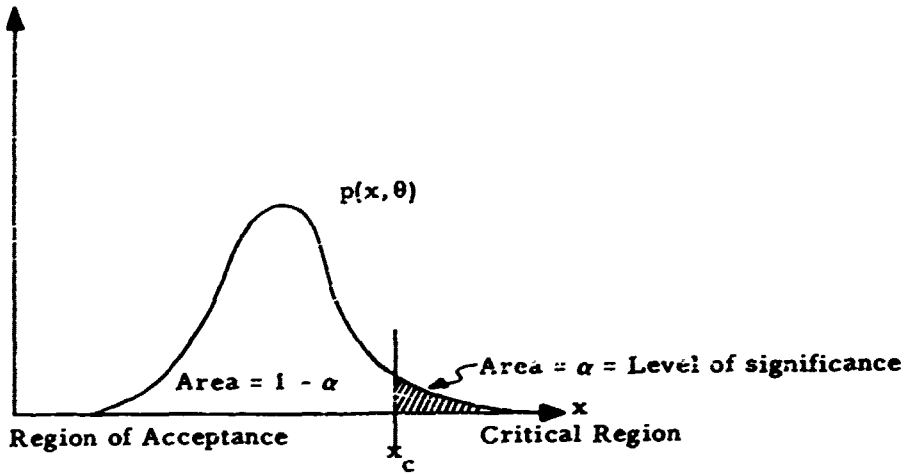
Critical Region

Figure 5.1 Illustration of Critical Region and Level of
Significance (Type I Error)

Another type of error may be committed when testing a hypothesis.
Namely, if the hypothesis is really false, it still might be accepted. This
error is known as the Type II error. For instance, suppose instead of
$p(x, \theta)$, the random variable actually has a probability density function
$p(x, \theta_0)$, where $\theta_0 \neq \theta$, located somewhat to the right of $p(x, \theta)$ as illus-
trated in Figure 5.2. Then there is a certain probability β that the
observation may fail to the left of x_c and be accepted. This Type II
error β, see Figure 5.2, is given by

$$\int_{-\infty}^{x_c} p(x, \theta_0) \, dx = \beta \qquad (5.8)$$

The probability $(1 - \beta)$ is called the power of the test. Clearly, for
a given value of α, it is desirable to have β very small. However, β is
generally made small only at the expense of increased sample sizes.
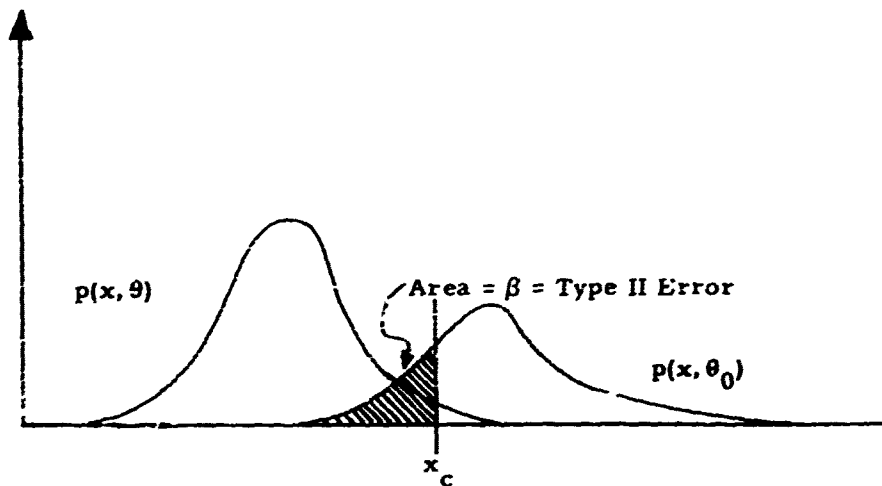
Figure 5.2 Illustration of Type II Error

Two variations for a statistical test often arise. A test in which
the critical region is located only at one tail of the distribution as in
Figure 5.1 is referred to as a one-tailed test. In many situations,too
large a deviation in either the positive or negative directions would be
damaging to the hypothesis. In these cases the critical region would be
divided into two parts, one at each tail of the distribution, and the region
of acceptance would be the interval between these two points. The left
and right critical points would then be selected such that the area under
the probability density function for each critical region is $\alpha/2$. This
maintains the level of significance at α, and is referred to as a two-tailed
test.

5.2 SPECIAL PROBABILITY DISTRIBUTIONS FOR STATISTICAL TESTS

Four probability distributions arise very frequently in the application of classical statistical techniques. These are the normal distribution, the chi-square distribution, the "t" distribution, and the F distribution. They will be discussed below in this order. Section 5.2 will emphasize their mathematical properties and Section 5.3 their physical applications.

5.2.1 The Normal Distribution

The most important distribution arising in both applied and theoretical statistics is given by the probability density function

$$\phi\left(\frac{x - \mu}{\sigma}\right) = \frac{1}{\sigma\sqrt{2\pi}} \, e^{-\frac{(x - \mu)^2}{2\sigma^2}} \tag{5.9}$$

or the corresponding distribution function

$$\Phi\left(\frac{x - \mu}{\sigma}\right) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{(t - \mu)^2}{2\sigma^2}} \, dt \tag{5.10}$$

The distribution was originally deduced in 1733 by the mathematician DeMoivre as the limit of the binomial distribution. The distribution is often credited to Gauss who did much work with it at a later date. As a result, a random variable having a distribution function given by Eq. (5.10) is often referred to as having a Gaussian or normal distribution, see Section 4.3.3(b).

The mean and variance of the distribution are

$$E(x) = \int_{-\infty}^{\infty} x \phi\left(\frac{x - \mu}{\sigma}\right) dx = \mu \tag{5.11}$$

$$\text{Var}(x) = \int_{-\infty}^{\infty} (x - \mu)^2 \phi\left(\frac{x - \mu}{\sigma}\right) dx - \sigma^2 \tag{5.12}$$

When a random variable has a normal distribution with mean μ and standard deviation (s. d.) σ, it is often spoken of as being normal (μ, σ) for the sake of brevity. The standardized variable

$$z = \frac{x - \mu}{\sigma} \tag{5.13}$$

then is normal $(0, 1)$ with the distribution function

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2} \, dt \tag{5.14}$$

The density function $\phi(z) = (1/\sqrt{2\pi}) e^{-z^2/2}$ is unimodal and symmetric having points of inflection at ± 1.

Equation (5.14) is well tabulated for many values of z. However, due to the symmetry, $\Phi(z) = \Phi(-z)$, the tabulation is frequently given only for positive values of z. Also, the tabulation often only goes up to $z = 3$ since the probabilities of exceeding this value is very small. See Table 5.1 at the end of Section 5 for a fairly comprehensive tabulation of the normal distribution in this way. In Table 5.1, for $z \geq 0$, the area of the normal curve is defined by

$$\text{Area} = (1/\sqrt{2\pi}) \int_{0}^{z} e^{-t^2/2} \, dt$$

For $z \geq 0$, the distribution function $\Phi(z)$, as defined in Eq. (5.14), is now given by $\Phi(z) = 0.50 + \text{Area}$.

A useful definition is that of the p percent value of the normal distribution. This is the unique root λ_p of the equation.

$$\text{Prob} \left(|x - \mu| > \lambda_p \sigma \right) = p \tag{5.15}$$

In words, Eq. (5.15) yields the probability p, in percent, that a normally distributed variable will deviate from its mean in either direction by more than λ_p times its s.d. See Table 5.2 at end of Section 5 for a tabulation of this type.

For later applications, one should note that a linear function $ax + b$ of a variable x which is normal (μ, σ) will be normally distributed $(a\mu + b, |a|\sigma)$.

The normal distribution derives much of its usefulness because of the well known "Central Limit Theorem", see Section 4.3.3(b). This essentially states that sums of independent random variables under fairly general conditions will be approximately normally distributed, regardless of the underlying distributions, when the sample size is large. Since many physically observed variables may actually be the sums of many less obvious variables, or when the means of large numbers of observations are considered, the normal distribution often applies in unexpected areas.

5.2.2 The Chi-Square Distribution

Let $x_1, x_2, \ldots, x_n$ be n independent random variables, each of which has the same normal distribution with zero mean and unit variance. The sum of their squares

$$y = \chi^2 = x_1^2 + x_2^2 + \ldots + x_n^2 \tag{5.16}$$

is called chi-square with n degrees of freedom (d.f.) The number of d.f., n, represents the number of independent or "free" squares entering into the expression for χ^2.

The variable y has the probability density function

$$p_n(y) = \frac{1}{2^{n/2} \Gamma(n/2)} \, y^{[(n/2)-1]} \, e^{-(y/2)} \qquad ; y > 0$$

$$\tag{5.17}$$

$$= 0 \qquad ; y < 0$$

where $\Gamma(n/2)$ is the well known Gamma function. (See Reference [4] for discussion of the Gamma Function). For $n = 1$ and $n = 2$, $p_n(y)$ is a monotonically decreasing function for positive y. For $n > 2$, the function is unimodal and non-symmetric.

The mean and variance of this distribution are

$$E(\chi^2) = n \tag{5.18}$$

$$\text{Var}(\chi^2) = 2n \tag{5.19}$$

The probability that the random variable $y = \chi^2$ assumes a value exceeding a given quantity $y_0 = \chi_0^2$ is given by

$$\text{Prob}(y > y_0) = \int_{y_0}^{\infty} p_n(y)\, dy \tag{5.20}$$

Conversely, if one wants to find a quantity $y_p = \chi_p^2$ such that the probability takes a given value, say p, then the following equation must be solved:

$$\text{Prob}(y > y_p) = \int_{y_p}^{\infty} p_n(y)\, dy = p \tag{5.21}$$

The unique root $\chi_p^2 = y_p$ of this equation is called the p percent value of χ^2 for n d.f. Essentially the same definitions will be used for the t and F distributions which are discussed below. A tabulation of some of these values of y_p for the chi-square distribution appears in Table 5.3.

5.2.3 The Student "t" Distribution

Let Y and Z be independent random variables such that Y is normal (0, 1) and Z has a χ^2 distribution with n d.f. The variable Z will therefore always be positive, and a variable t can be defined as

$$t = \sqrt{n}\ \frac{Y}{\sqrt{Z}} \tag{5.22}$$

where the positive square root is taken. The distribution of the variable t is known as Student's "t" distribution with n d.f. It was first studied by the statistician W. Gosset who wrote under the pseudonym "Student", hence the name.

The probability density function of t is given by

$$P_n(t) = \frac{C_n}{\left(1 + \frac{t^2}{n}\right)^{\frac{n+1}{2}}} \tag{5.23}$$

where

$$C_n = \frac{1}{\sqrt{n\pi}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \tag{5.24}$$

The mean is finite for $n > 1$ and the variance is finite for $n > 2$. These are given by

$$E(t) = 0 \qquad\qquad n > 1 \tag{5.25}$$

$$Var(t) = \frac{n}{n-2} \qquad\qquad n > 2 \tag{5.26}$$

As in the previous section a p percent value of the t distribution is defined, where p is expressed in percent, by the root t_p of the equation

$$Prob(t > t_p) = \int_{t_p}^{\infty} p_n(t) \, dt = p \tag{5.27}$$

Also, as for χ^2, the parameter n is the number of d.f. A tabulation of values of t as a function of n and p is given in Table 5.4. Some care must be taken in using other available tables of the "t" distribution in the same manner as Table 5.4. Some texts will have $Prob(|t| > t_p) = p$ instead of $Prob(t > t_p) = p$, Eq. (5.27), since the "t" distribution is symmetric, i.e. $p_n(t) = p_n(-t)$. In this case a value for p in Table 5.4 would correspond to a value for $2p$ in other tables constructed considering deviations in both tails of the probability density function.

5.2.4 The F Distribution

Given two independent variables U and V which have χ^2 distributions with m and n d.f. respectively, the distribution of the variable

$$F = \frac{U/m}{V/n} = \frac{n}{m} \cdot \frac{U}{V} \qquad (5.28)$$

is known as the F distribution with (m, n) d.f. Since F is positive, the density function of F is zero for $F < 0$, while for $F > 0$, the density function is given by

$$P_{m,n}(F) = C \frac{F^{\left(\frac{m}{2} - 1\right)}}{(mF + n)^{\left[\frac{m+n}{2}\right]}} \qquad ; \qquad F > 0 \qquad (5.29)$$

where C is an appropriate constant depending only on m and n.

As for χ^2 and t, where p is expressed in percent, a p percent value of the F distribution is defined by the root F_p of the equation

$$\text{Prob}(F > F_p) = \int_{F_p}^{\infty} P_{m,n}(F)\,dF = p \qquad (5.30)$$

Some values of F as a function of m and n and p are given in Table 5.5.

5.3 SAMPLING THEORY AND APPLICATIONS

The distributions defined and discussed in Section 5.2 will now be illustrated as sampling distributions. They will be discussed in the context of their most usual applications, with additional areas of application indicated which are particularly appropriate for analysis of vibration data.

A sampling distribution may be defined in general as follows: Let X be a random variable with a distribution function $F(x)$. Let

x_1, x_2, ..., x_N be a sample of N observed values of X. Any well defined function of these variables, say $g(x_1, x_2, ..., x_N)$ will be a sample of the random variable $g(X_1, X_2, ..., X_N)$ where each X_i has the same distribution function $F(x)$. The probability distribution of $g(X_1, X_2, ..., X_N)$ is called the sampling distribution of the quantity $g(x_1, x_2, ... x_N)$.

If samples of N values are repeatedly drawn, and if the characteristic $g(x_1, x_2, ..., x_N)$ is computed for each sample, a sequence is obtained of the observed values of the random variable $g(X_1, X_2, ..., X_N)$. In this way every sample characteristic is associated with a certain random variable. One may then talk of the sampling distributions of quantities such as the arithmetic mean of a sample given by Eq. (5.1) and the sample variance given by Eq. (5.2).

In principle, the sampling characteristic may be determined by the distribution function $F(x)$ of the basic random variable X. In practice, however, it may be difficult to find an explicit expression for the result.

5. 3. 1 Estimates of the Mean and Variance

As has been implied in preceding discussions, μ and σ will denote the true mean and standard deviation (s. d.) of a distribution corresponding to a random variable. The estimates of these values as computed from a sample of size N will be denoted by $\bar{x}$ (or m) and s respectively. The Greek letters μ and σ will often be referred to as the population or universe values as opposed to their sample estimates $\bar{x}$ and s. It is important to understand and keep in mind this distinction between true values and sample estimates. Any true population value is some fixed real number whereas its estimate computed from a sample will have a sampling distribution.

Consider first the sample mean

$$\bar{x} = \frac{\sum_{i=1}^{N} x_i}{N}$$

(5. 31)

By observing that the sample mean is a linear combination of N variables x_i, and applying the well known addition theorem for means and variances of independent random variables, one deduces

$$E(\bar{x}) = \mu_{\bar{x}} = \frac{1}{N} \sum_i E(x_i) = \frac{1}{N} (N\mu) = \mu \qquad (5.32)$$

Also,

$$\text{Var}(\bar{x}) = \sigma^2_{\bar{x}} = \sum_i \frac{1}{N^2} \sigma^2 = \frac{1}{N^2} (N\sigma^2) = \frac{\sigma^2}{N} \qquad (5.33)$$

or

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{N}}$$

Thus, the expected value of the sample characteristic $\bar{x}$ is equal to the true mean value μ. Moreover, since the standard deviation is inversely proportional to $\sqrt{N}$, the sample mean $\bar{x}$ is a very precise estimate of μ for large N. That is, the distribution of $\bar{x}$ will be concentrated in the vicinity of μ since it will have a very small s.d. for large N. Also, it follows from the central limit theorem, that for large N the distribution of $\bar{x}$ will be approximately normal $(\mu, \sigma/\sqrt{N})$.

Consider next the sample variance

$$s^2 = \frac{1}{N} \sum_i (x_i - \bar{x})^2 = \frac{1}{N} \sum_i (x_i - \mu + \mu - \bar{x})^2$$

$$= \frac{1}{N} \sum_i (x_i - \mu)^2 - \frac{2}{N} (\bar{x} - \mu) \sum_i (x_i - \mu) + \frac{1}{N} \sum_i (\bar{x} - \mu)^2$$

$$= \frac{1}{N} \sum_i (x_i - \mu)^2 - \frac{2}{N} (\bar{x} - \mu) N(\bar{x} - \mu) + \frac{1}{N} N(\bar{x} - \mu)^2$$

Thus

$$s^2 = \frac{1}{N} \sum_i (x_i - \mu)^2 - (\bar{x} - \mu)^2 \qquad (5.34)$$

Since

$$E(x_i - \mu)^2 = \sigma^2 \qquad \text{and} \qquad E(\bar{x} - \mu)^2 = \frac{\sigma^2}{N} \qquad (5.35)$$

it follows that

$$E(s^2) = \frac{1}{N}(N\sigma^2) - \frac{\sigma^2}{N} = \left(\frac{N-1}{N}\right)\sigma^2 \qquad (5.36)$$

This is the result mentioned in Section 5.1.1. This bias may be removed as indicated by Equation (5.5), but notice that for large N the bias is insignificant.

The standard deviation of s^2 is small for large N. (See Ref. [5], p. 183). Therefore, s^2 will be a precise estimate of σ^2 for large N.

In the above discussion, the observations used in calculating the mean are implicitly assumed to all be taken from the same population in the same way, and therefore, would have identical variances. Suppose a different situation exists where the mean of a population is to be estimated from observations having different variances. This could arise if a mean was to be estimated from several sample means each having different sample sizes, or if the observations were possibly taken with instruments of varying precision. It is then desirable to know the manner in which these observations should be weighted in order to give an estimate of μ with minimum variance. The derivation of the result is fairly involved and the reader is referred to Ref. [2] for a proof. However, the final result is to weight the observations inversely to their variances. That is, for

$$\bar{x} = \sum_i a_i x_i \qquad (5.37)$$

choose

$$a_i = \frac{w_i}{\sum_1 w_i} \qquad (5.38)$$

where

$$w_i = \frac{1}{\sigma_i^2} \qquad (5.39)$$

The variance of the distribution of $\bar{x}$ then is

$$\text{Var}(\bar{x}) = \frac{1}{\sum_i w_i} \qquad (5.40)$$

5.3.2 Tne Chi-Square Goodness of Fit Test as a Test for Normality

(a) General Remarks

In certain situations such as comparing a normal distribution with a frequency histogram of some observed data, it is desirable to be able to evaluate the discrepancy between the observed and expected frequencies. It is customary in such problems to compute a statistic which measures this discrepancy and study its sampling distribution. A good statistic at hand for this problem is χ^2, or chi-square (see Section 5.2.2).

For subsequent discussion, the following notation will be adopted:

f_i is the frequency observed in the ith class,

F_i is the expected frequency in the ith class.

Then define the statistic χ^2 by the sum of weighted squares

$$\chi^2 = \sum_{i=1}^{k} \frac{(f_i - F_i)^2}{F_i} \qquad (5.41)$$

where there are k class intervals.

Since $\sum f_i = \sum F_i$, the quantity $\sum (f_i - F_i)$ cannot be used as a
measure, and the square must be used. It is apparent that χ^2 will
give some measure of the difference in frequencies as compared to
the expected frequencies. Obviously, a large value for χ^2 indicates
a larger probability of genuine difference from the expected distribu-
tion.

The limiting distribution for χ^2 depends on one other parameter,
the number of independent squared variables in χ^2, called the number
of "degrees of freedom (d.f.)." There exists a family of curves, one
for each number of degrees of freedom. Each independent linear
restriction imposed on the observations decreases this parameter by
one. For instance, with k class intervals and a sample size N, one
can always compute the frequency in the last class interval after the
first k - 1 are known. So in this case one is left with (k - 1) d.f.
Also in the present case of interest, the normal distribution must be
fitted to a frequency histogram which required the computation of
the mean and variance. This imposes 2 additional restrictions re-
ducing the number of d.f. to (k - 3) d.f. For each parameter esti-
mated from the observations, an additional restriction is imposed,
and in general there are (k - a - 1) d.f., where a is the number of
estimated parameters.

Tables for the χ^2 distribution normally give the value of χ^2
that will be exceeded p% of the time for (n) d.f. The percentage p
is often selected as 5% and the null hypothesis (i.e., there is no
conclusive evidence that the observed values were selected from a
non-normal distribution) is accepted if the computed χ^2 value is
less than the appropriate table value.

(b) Applying the Test

The null hypothesis, as mentioned above, is always considered.
If this hypothesis can be accepted at the p% level of significance, then
one may reasonably conclude that the parent population is indeed
normal, if supported by other evidence. There is at most a p% prob-
ability of rejecting the hypothesis when it is true (Type I error).

The first step is to group the observations in some selected class intervals. These may or may not have been previously selected, but may have to be modified for this test. The choice of the number of intervals affects the sensitivity of the test and common practice has indicated 10-25 equal class intervals is a desirable selection, subject to certain restrictions.

For the 5% level of significance the following tables below have been developed to assist in selecting the number (k) of class intervals as a function of the sample size N and the expected frequency in each class, see Ref. [10].

Minimum Optimum Number (k) of Class Intervals (5% Level) for Sample Size N

N	200	400	600	800	1000	1500	2000
k	16	20	24	27	30	35	39

Recommended Expected Frequencies for Each Class Interval

N = Sample Size	200	400	600	800	1000	1500	2000
* Minimum F_i	5-10	5-10	5-10	5-10	5-10	5-10	5-10
** Maximum F_i	12	20	25	29	30	40	49

*The end intervals may have an expected frequency as small as one, pool (if necessary) to obtain $F_i \geqslant 1$. Do not pool to obtain $F_i > 5$.

**Maximum F_i may be exceeded slightly but in no case should F_i exceed 50.

Experience has also suggested the following practical rules of thumb.

1. If there are two or more d. f. and the expectation in each cell is more than 5, the chi-square table assures a good approximation to the exact probabilities.

2. If more approximate probabilities are acceptable, an expectation of 2 in each cell is acceptable.

3. With more than 2 d.f. an expectation of one in the tails of the distribution is satisfactory if the interior intervals have an expectation of 5 or more.

The sample mean

$$\bar{x} = \sum_{i=1}^{N} \frac{x_i}{N} \tag{5.42}$$

and the standard deviation

$$s = \sqrt{\frac{\sum (x_i - \bar{x})^2}{N}} \tag{5.43}$$

must be computed and the normal distribution can now be "fitted" to the histogram. The class interval end points should now be converted into standard deviations from the mean, i.e., where x_i is the ith end point compute $z_i = (x_i - \bar{x})/s$. The leftmost interval should be considered to be from $(-\infty, z_i)$ and the rightmost (z_{k-1}, ∞) when reading tables of areas under the normal density function to compute the expected frequencies. Proportions of N expected to lie in each class interval may now be found in a table of areas under the normal density function. Merely multiply these by N to obtain the expected frequencies. Some intervals may have to be pooled in order to meet the previously suggested expected frequency minimums. Now compute

$$\chi^2 = \sum_{1}^{k} \frac{(f_i - F_i)^2}{F_i} \tag{5.44}$$

Then compare χ^2 with the chi-square table value, χ_p^2, under $n = (k - 3)$ d.f. at the 5% (normally used) level of significance. If χ^2 is less than χ_p^2 one may conclude there is no good reason for believing the data comes from a non-normal distribution, and a hypothesis of normality

is therefore accepted. If $\chi^2 > \chi^2_p$, the hypothesis of normality would be rejected at the p% level of significance.

It is sometimes convenient to remember that the expected value of χ^2 is n, so if the computed value for χ^2 is less than the number of d.f. it is unnecessary to consult the χ^2 table and it may be concluded that the discrepancies between observed and expected frequencies is negligible.

When n > 30, χ^2_p may be computed via the normal distribution since $\sqrt{2\chi^2}$ is approximately normal with mean $\sqrt{2n - 1}$ and standard deviation 1. Then

$$\chi^2_p = \frac{1}{2}\left(\sqrt{2n - 1} + \lambda_{2p}\right)^2 \tag{5.45}$$

where λ_{2p} is the 2p% value for the normal distribution, i.e., $\pm\lambda_{2p}$ would be the number of standard deviations containing $(100 - 2p)\%$ of the area under the normal density function. As before, the probability that $\chi^2 > \chi^2_p$, a one-sided interval, is given by p%.

(c) Computational Examples

Case 1. Less than 30 d. f.

Corresponding to the frequency histogram (Fig. 5.3) there are the following set of hypothetical voltages. (See Fig. 5.4.) Computations may be made from the grouped data with a negligible loss of accuracy, i.e., consider all observations in an interval to be located at the midpoint.

The formulas used for $\bar{x}$ and s are equivalents of the definitions used for computational purposes. "fx" indicates f observations at the point x. The formula for s is obtained as follows.

$$s^2 = \frac{\sum f(x - \bar{x})^2}{N} = \frac{\sum fx^2}{N} - \frac{\sum 2\bar{x} fx}{N} + \frac{N\bar{x}^2}{N}$$

$$= \frac{\sum fx^2}{N} - 2\bar{x}^2 + \bar{x}^2 = \frac{\sum fx^2}{N} - \bar{x}^2 \tag{5.46}$$
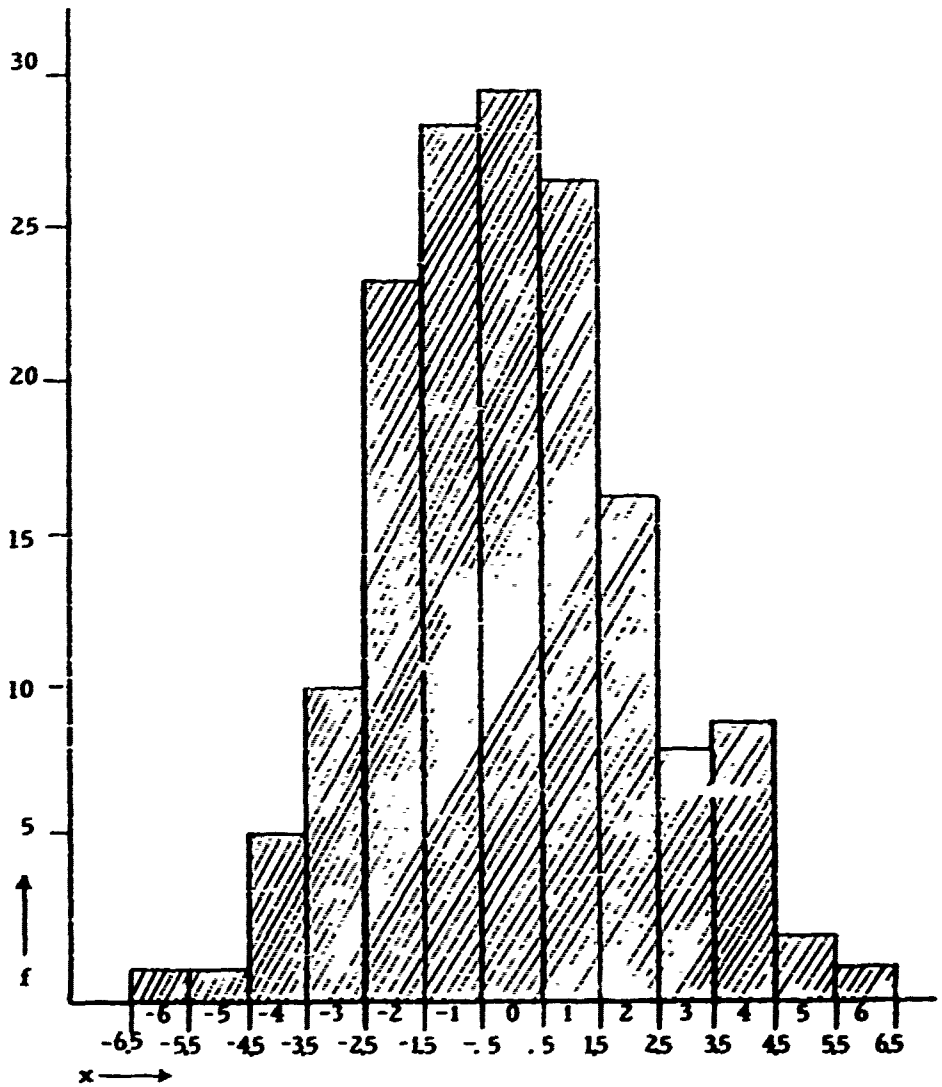
Figure 5.3. Frequency Histogram of Hypothetical Data

Figure 5.4.   Tabulated Values of Figure 5.3

Interval $x_i - x_{i+1}$	Midpoint x	f	fx	x^2	fx^2	$x_i = \dfrac{x_i - \bar{x}}{s}$		ΔA	$F = N(\Delta A)$	$F - f$	$(F-f)^2/F$
-6.5 / -5.5	-6	-1	-6	36	36	-∞	-2.04	.021	3.3	2.3	1.603
-5.5 / -4.5	-5	1	-5	25	25	-2.04	-1.66	.027	4.3	3.3	2.532
-4.5 / -3.5	-4	5	-20	16	80	-1.66	-1.30	.048	7.8	2.8	1.005
-3.5 / -2.5	-3	10	-30	9	90	-1.30	-.92	.082	13.3	3.3	.818
-2.5 / -1.5	-2	23	-46	4	92	-.92	-.56	.109	17.2	-5.8	1.955
-1.5 / -0.5	-1	28	-28	1	28	-.56	-.18	.141	22.3	-5.7	1.456
-0.5 / 0.5	0	29	0	0	0	-.18	.18	.143	22.6	-6.4	1.812
0.5 / 1.5	1	26	26	1	26	.18	.56	.141	22.3	-3.7	.613
1.5 / 2.5	2	16	32	4	64	.56	.92	.109	17.2	1.2	.083
2.5 / 3.5	3	8	24	9	72	.92	1.30	.082	13.3	5.3	2.112
3.5 / 4.5	4	9	36	16	144	1.30	1.66	.048	7.8	-1.2	.184
4.5 / 5.5	5	2	10	25	50	1.66	2.04	.027	4.3	2.3	1.230
5.5 / 6.5	6	1	6	36	36	2.04	∞	.021	3.3	2.3	1.603
		159	-1		743				159		17.006

$$N = \sum f = 159 \qquad n = k - 3 = 10 \qquad \bar{x} = \left(\frac{\sum fx}{N}\right) = -0.006 \qquad s = \sqrt{\frac{\sum fx^2}{N} - \bar{x}^2} = 2.7$$

After the z_i are computed using the formula $z_i = (x_i - \bar{x})/s$, the proportion of the area, ΔA, under the normal density function lying in each interval is found from Table 5.1. Note that since the normal curve is symmetric the table gives only the values for $0 < z < \infty$ and the areas for the left half are found using the same table. For example, for the interval $(1.5, 2.5)$ the corresponding z values are $(.56, .92)$. From Table 5.1, one finds $.3212$ and $.2123$ for $.92$ and $.56$ respectively. Subtracting and rounding, $\Delta A = .109$ is obtained.

As shown in Fig. 5.4, the expected frequencies F_i are calculated by multiplying these areas by $N = 159$. The computed value of chi-square comes out to be 17.0.

Looking at the chi-square table (Table 5.3) under $n = 10$ degrees of freedom, one finds

$$x^2_5 = 18.3 \quad \text{and} \quad x^2_{10} = 16.0$$

If the 5% level of significance is being used, the data would be accepted as being from a normal distribution. However, if one had decided to work at a 10% level the null hypothesis would be rejected and non-normality assumed. In this case, other factors probably should be considered when deciding whether or not to assume normality.

Case 2. More than 30 d.f.

To illustrate the computation of approximate values of x^2_p which may be used when $n > 30$ use the above value $n = 10$. Then for the 5% level, one has from Eq. (5.45),

$$x^2_5 = \frac{1}{2}\left(\sqrt{20 - 1} + 1.64\right)^2 = 18.0$$

where $1.64 = \lambda_{2p}$ is found from Table 5.2 for $2p = 10$. In this case the agreement is quite close to the previously calculated $x^2_5 = 18.3$, but the method is not generally reliable unless $n > 30$.

(d) Other Applications of χ^2

    Chi-square tests may be employed to compare any set of observed frequencies with some set of theoretical frequencies. The distribution to be compared against need not be the normal distribution but may be any one of interest. For example, this test may be applied to a Rayleigh distribution, or to a combined Rayleigh-Gaussian distribution as illustrated by Eq. (4. 213) in Section 4.9.3.

(e) Limitations of the Test

    The χ^2 curve is only an approximation to the true distribution so care must be exercised that the χ^2 test is employed only when this approximation is good. The previously mentioned rules for minimum expectations and class intervals should be adhered to, and sample sizes should be of the order of 200 or larger.

    Since there can exist distributions other than the normal that would give similar expected frequencies for intervals, it must be borne in mind that desired results with the χ^2 test do not completely justify assuming the parent population to be normal. If possible, the χ^2 test should be supported by other conclusions.

5. 3. 3 Applications of Student's "t" Distribution

(a) Introduction

    In making inferences about the mean μ of a population, it is necessary to take into account the standard deviation (s. d.) σ of that population. In most practical situations neither μ nor σ are known and their sample estimates $\bar{x}$ for μ and s for σ as calculated by

$$\bar{x} = \frac{\sum_1^N x_i}{N} \tag{5.47}$$

and

$$s = \sqrt{\frac{\sum_1^N (x_i - \bar{x})^2}{N}} \tag{5.48}$$

must be used. Relationships of $\bar{x}$ and s to μ and σ will be investigated in this section.

(b) Special Case Where σ is Known

In the special case where σ is known and μ is unknown, one can apply the normal distribution, rather than the "t"-distribution to be described, in making inferences about μ from sample means. Using Eq. (5.32) and (5.33) from Section 5.3.1, the mean of the distribution of the sample means is

$$\mu_{\bar{x}} = \mu \qquad (5.49)$$

and the variance of the distribution of the sample means is

$$\sigma_{\bar{x}}^2 = \frac{\sigma^2}{N} \qquad (5.50)$$

The s.d. of the distribution of the sample means is then

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{N}} \qquad (5.51)$$

One can now "standardize" the observations by subtracting the mean and dividing by the standard deviation which gives the following normally distributed variable, z, with mean zero and s.d. of unity, namely,

$$z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{N}}} = \frac{(\bar{x} - \mu)\sqrt{N}}{\sigma} \qquad (5.52)$$

Thus, given a sample of N observations from a normal population with s.d. σ with which to test the hypothesis that the mean is μ, one computes $\bar{x}$ from Eq. (5.47) and then z from Eq. (5.52). Next, consult the table of the normal distribution to obtain the percent of time the

sample value will lie outside the range $\pm z$. If this percentage is greater than some arbitrarily selected level of significance α, say 5%, one would accept the hypothesis that the mean is μ.

For example, assume a sample of size $N = 9$ from which $\bar{x} = 12.3$ is computed. Suppose further that the population s.d. is $\sigma = 2.0$ and one wants to test the hypothesis $\mu = 13$. Computing z from Eq. (5.52), one obtains

$$z = \frac{12.3 - 13}{2.0} \sqrt{9} = -1.05$$

In a table of the normal distribution (see Table 5.1) one finds that ± 1.05 contains 70.6% of the area under the normal density function. Alternatively, the sample value would lie outside the range ± 1.05 approximately 29.4% of the time. Therefore, working at the 5% level of significance one accepts the hypothesis that $\mu = 13$.

(c) General Case Where σ is Unknown

For the more common case when both σ and μ are unknown, consider the following statistic of the general form

$$t = \sqrt{n}\,\frac{Y}{\sqrt{Z}} \qquad\qquad (5.53)$$

where Y and Z are independent random variables. The variable Y is normally distributed with mean zero and s.d. of unity while Z has a χ^2 distribution with n degrees of freedom. The distribution of "t" is known as the "Student" or "t" distribution. (See Section 5.2.3.)

The "Student-t" distribution is actually a family of curves depending on a parameter n, the number of "degrees of freedom" (d.f.). A graph of the t-distribution even for small n, closely resembles the normal density function although actually having much "thicker" tails. In other words, the probability of large deviations from the mean is higher than in the case of the normal distribution. The expectation of t is zero for $n > 1$ and the variance is $n/(n - 2)$ for $n > 2$. For $n \leq 1$

the mean is not finite, while the variance is not finite for $n \leqslant 2$. However, probabilities can still be computed for $n = 1$ and tables of the t-distribution usually range from $n = 1$ to $n = 120$. (See Table 5.4.)

The variable t is asymptotically normal with mean zero and unit s.d. for large n. In most practical situations $n > 30$ is a sufficiently large value to justify use of the normal approximation.

Some specific forms of Eq. (5.53) will now be given for testing hypotheses about means. In samples of size N from a normal population with mean μ and s.d. σ, the estimates $\bar{x}$ and s^2 are independent. The sample estimate $\bar{x}$ is normal with mean μ and s.d. $\sigma/\sqrt{N}$, while Ns^2/σ^2 has a χ^2 distribution with $(N - 1)$ d.f. A somewhat intuitive justification for Ns^2/σ^2 having a χ^2 distribution is as follows: Chi-square is essentially the sum of squared deviations from some fixed number - the expected frequency - then these squares divided by some fixed number - again the expected frequency.

For Chi-square with k class intervals one has

$$\chi^2 = \sum_{i=1}^{k} \frac{(f_i - F_i)^2}{F_i} \tag{5.54}$$

For the variance, by rewriting Eq. (5.48), one obtains

$$Ns^2 = \sum_{i=1}^{N} (x_i - \bar{x})^2 \tag{5.55}$$

Dividing both sides by the population variance, a fixed number,

$$\frac{Ns^2}{\sigma^2} = \frac{\sum_{i=1}^{N} (x_i - \bar{x})^2}{\sigma^2} \tag{5.56}$$

The resemblance between Eq. (5.54) and Eq. (5.56) is apparent. As in χ^2, Eq. (5.56) is a sum of N squared variables, but with an additional restriction. In this case

$$\sum_{1}^{N} (x_i - \bar{x})^2 = 0 \qquad (5.57)$$

Hence Ns^2/σ^2 contains only $(N - 1)$ independent squared variables and follows a χ^2 distribution with $(N - 1)$ d.f.

Consider the two independent variables

$$Y = \sqrt{N}\left(\frac{\bar{x} - \mu}{\sigma}\right) \qquad (5.58)$$

and

$$Z = \frac{Ns^2}{\sigma^2} \qquad (5.59)$$

Note that Y is normal with mean zero and unit s.d., while Z has a χ^2 distribution with $(N - 1)$ d.f. which fulfills the requirements for the variable t in Eq. (5.59). One can now write

$$t = \sqrt{N - 1}\,\frac{Y}{\sqrt{Z}} = \frac{\sqrt{N}\left(\frac{\bar{x} - \mu}{\sigma}\right)}{\sqrt{N}\,\frac{s}{\sigma}}\,\sqrt{N - 1} = \frac{\bar{x} - \mu}{s}\,\sqrt{N - 1} \qquad (5.60)$$

which has the t distribution with $(N - 1)$ d.f. and provides a test for comparing a sample mean $\bar{x}$ against some hypothetical population mean μ.

Another important application of the t distribution is as follows. Consider two independent samples of size N_1 and N_2 with means $\bar{x}$ and $\bar{y}$ and variances s_1^2 and s_2^2 respectively. If the two parent populations have means μ_1 and μ_2 and a common s.d. of σ, the following holds.

The variable

$$\bar{x} - \bar{y} - (\mu_1 - \mu_2) \qquad (5.61)$$

is normal with mean zero and s.d.

$$s = \sqrt{\sigma^2_{\bar{x}} + \sigma^2_{\bar{y}}} = \sqrt{\frac{\sigma^2}{N_1} + \frac{\sigma^2}{N_2}} = \sigma \sqrt{\frac{1}{N_1} + \frac{1}{N_2}} \qquad (5.62)$$

which is obtained using the previously mentioned addition theorem for variance and the fact that the s. d. of the distribution of means is $\sigma / \sqrt{N}$.

Also, the variable

$$\frac{N_1 s_1^2 + N_2 s_2^2}{\sigma^2} \qquad (5.63)$$

has a χ^2 distribution with $(N_1 + N_2 - 2)$ d. f.

One can write

$$t = \sqrt{\frac{N_1 N_2 (N_1 + N_2 - 2)}{N_1 + N_2}} \cdot \frac{\bar{x} - \bar{y} - (\mu_1 - \mu_2)}{\sqrt{N_1 s_1^2 + N_2 s_2^2}} \qquad (5.64)$$

which has the t-distribution with $(N_1 + N_2 - 2)$ d. f. This provides a test for comparing two sample means. The usual case will be the hypothesis $\mu_1 = \mu_2$ or $(\mu_1 - \mu_2) = 0$, and the test will be for equality of $\bar{x}$ and $\bar{y}$. There are some slight variations to the above case which will be noted in the section on applications below.

(d) Applications

Several slightly different situations exist in applying the t-distribution to tests about means. Each case has a different approach for "large" or "small" samples. The main cases are:

Case 1.   Tests about μ when σ is known.

Case 2.   Tests about $\mu_1 - \mu_2$

    i) When σ_1 and σ_2 are known.

    ii) When σ_1 and σ_2 are unknown but presumed equal.

iii) When σ_1 and σ_2 are unknown and unequal.[1]

Case 3.   Confidence intervals for estimates of μ.

The general procedure for all cases is as follows. First, the mean $\bar{x}$ and standard deviation s, as given by Eq. (5.47) and (5.48), are computed from a sample of N observations. Then t is computed as given by Eq. (5.60) or (5.64), whichever is appropriate. Assuming one is working at a given level of significance, say $\alpha = 5\%$, one then looks in Table 5.4 under the appropriate number of d.f. for t_p, where $p = (\alpha/2)$. This is for a "two-tailed" test, which is correct assuming the sample mean can vary to either side of the hypothesized population mean. In other words, 2.5% of the t-distribution lies to the right of $t_{2.5}$ and also 2.5% lies to the left of $-t_{2.5}$ giving a total of 5% of the area in the two tails. Therefore, if $|t| > t_{2.5}$ it can be concluded that only 5% of the time will a value this large be obtained if the means are really equal and therefore there exists a statistically significant difference, i.e., there is a 5% chance of rejecting the hypothesis of equality when it is really true (Type I error).

A "one-tailed" test is acceptable if the logic of the problem indicates the difference in the means could be in only one direction. Then for the 5% level of significance, one looks in Table 5.4 under $t_{5.0}$ since one is concerned about a deviation in only one direction. Note that values of t are tabulated in the same manner as χ^2. For n degrees of freedom, t_p gives the proportion of area p under the density function for t in the interval (t_p, ∞). However, the t distribution is symmetric whereas χ^2 is not. The entries for $n = \infty$ are values for the normal distribution. In most practical situations as previously mentioned, for $n > 30$, the value for t is close enough to that of the normal to justify using normal tables, and $n > 30$ will be considered a "large" sample.

(e) Computational Examples

Case 1. Tests about μ when σ is unknown.

Suppose a sample of $N = 6$ observations has $\bar{x} = 12.3$ and $s^2 = 4.8$, and it is desired to test the hypothesis that $\mu = 15$ at the 5% level of significance with a two-tailed test. Compute t by Eq. (5.60).

$$t = \sqrt{5} \frac{12.3 - 15}{\sqrt{4.8}} = -2.76$$

In Table 5.4 for 5 d.f. one finds $t_{2.5} = 2.57$. This value implies that ± 2.57 would contain 95% of the area under the density function. Therefore, working at the 5% level of significance, the hypothesis is rejected since $2.76 > 2.57$.

For the case when N is large, say 37, compute t as in the above example but now refer to tables of the normal distribution rather than the t-table. Assume the same mean and variance. To test the hypothesis $\mu = 13$, compute

$$t = \sqrt{36} \frac{12.3 - 13}{\sqrt{4.8}} = -1.91$$

The 2.5% value for the normal distribution is 1.96 so in this case the hypothesis $\mu = 13$ is accepted.

Case 2. i) When σ_1 and σ_2 are known.

Here is a special case similar to that described in Section 5.5.3(b). As explained previously, one need only refer to the table of the normal distribution after computing

$$z = \frac{\bar{x}_1 - \bar{x}_2 - (\mu_1 - \mu_2)}{\sqrt{\dfrac{\sigma_1^2}{N_1} + \dfrac{\sigma_2^2}{N_2}}} \qquad (5.65)$$

to find the percentage of time the variable lies outside
the range $\pm z$.

ii) When σ_1 and σ_2 are unknown but presumed equal.

Assume two samples of size $N_1 = 11$ and $N_2 = 12$
with means 82.1 and 72.6 and variances 276.16 and
320.24 respectively. The hypothesis to be tested is
that of equality of means. Letting $(\mu_1 - \mu_2) = 0$,
compute from Eq. (5.64),

$$ t = \sqrt{\frac{(12)(11)(21)}{23}} \cdot \frac{82.1 - 72.6}{\sqrt{(11)(276.16) + (12)(320.24)}} = 1.26 $$

Looking in Table 5.4 under $(N_1 + N_2 - 2) = 21$ d.f.
one finds $t_{2.5} = 2.08$. The value 1.26 is within the
region of acceptance so accept the hypotheses that
$\mu_1 = \mu_2$.
It should be noted that when samples are small and
variability is large the observed difference must be
very large to appear significant. The failure to find
a significant difference may be due to the small
number of cases examined rather than to the equality
of population means.
For large sample sizes compute t from the same
formula, Eq. (5.64), but one may use tables of the
normal distribution rather than tables of t if con-
venient.

iii) When σ_1 and σ_2 are unknown and unequal.

Discussion of this case for small samples is fairly
complicated and will be left for outside study.

Case 3.   Confidence intervals for estimates of μ.

Given a sample estimate $\bar{x}$ of a population mean μ
it is possible to compute an interval about $\bar{x}$ which

has a given probability of containing the population mean μ. Choosing some small level of significance α in percent, usually 5% or 1%, a confidence coefficient $(1 - \alpha)$ is obtained. Now consider the expression

$$\text{Prob}\left[-t_{(\alpha/2)} < \frac{\bar{x} - \mu}{s}\sqrt{N - 1} < t_{(\alpha/2)}\right] = 1 - \alpha \quad (5.66)$$

This is read as the probability of the expression in the parenthesis is equal to $(1 - \alpha)$. By a simple manipulation of the above inequality one finds

$$P\left[\bar{x} - \frac{s}{\sqrt{N - 1}}\, t_{(\alpha/2)} < \mu < \bar{x} + \frac{s}{\sqrt{N - 1}}\, t_{(\alpha/2)}\right] = (1 - \alpha)$$

$$(5.67)$$

A confidence interval corresponding to the sample estimate $\bar{x}$ of μ has now been obtained. For example, choosing $\alpha = 5$ there is a 95% probability that μ is included in the interval

$$\bar{x} \pm \frac{s}{\sqrt{N - 1}} \cdot t_{2.5} \quad\quad (5.68)$$

where $t_{2.5}$ is obtained from Table 5.4 under $(N - 1)$ d.f.

Using the hypothetical values from Case 1: $N = 6$, $\bar{x} = 12.3$, $s = 2.19$ and $t_{2.5} = 2.57$ one can compute a confidence interval using Eq. (5.68). The result is

$$\bar{x} + \frac{s}{\sqrt{N - 1}}\, t_{2.5} = 12.3 + \frac{2.19}{5}\, 2.57 = 13.4$$

and

$$\bar{x} - \frac{s}{\sqrt{N - 1}}\, t_{2.5} = 12.3 - \frac{2.19}{5}\, 2.57 = 11.2$$

There is a 95% confidence that the true population mean μ lies in the interval (11.2, 13.4).

(f) Other Applications

The t-distribution arises frequently in statistics. Another important application is for testing regression coefficients for significance (see Section 5.6.2D).

5.3.4 Applications of the F Distribution

(a) Introduction

In applying the "t" distribution as a test for equivalence of means from two samples, it is necessary to know whether or not the variances of the two samples should be considered equal or not. The F distribution, described in Section 5.2.4, exists for making this test based on computing the ratio of the two variances.

Consider two independent variables U and V which have χ^2 distributions with m and n degrees of freedom respectively. The F distribution as defined by the variable

$$F = \frac{U/m}{V/n} = \left(\frac{n}{m}\right) \cdot \left(\frac{U}{V}\right) \tag{5.69}$$

leads to a useful test about the ratio of two variances.

From previous considerations pertinent to the t-distribution one recalls that in samples from a normal distribution, the random variable Ns^2/σ^2 has a χ^2 distribution with $(N - 1)$ d.f. Let s_1^2 and s_2^2 be sample variances based upon random samples of size N_1 and N_2 respectively. Since $N_1 s_1^2/\sigma_1^2$ and $N_2 s_2^2/\sigma_2^2$ possess independent χ^2 distributions with $(N_1 - 1)$ and $(N_2 - 1)$ degrees of freedom respectively,

$$\frac{U}{m} = \frac{N_1 s_1^2}{(N_1 - 1)\sigma_1^2} \tag{5.70}$$

and

$$\frac{V}{n} = \frac{N_2 s_2^2}{(N_2 - 1)\sigma_2^2} \tag{5.71}$$

satisfy the requirements for Eq. (5.69).

In general the hypothesis $\sigma_1^2 = a\sigma_2^2$ can be made where a is some constant. Eq. (5.69) would then become

$$F = \frac{N_1 s_1^2 / a(N_1 - 1)}{N_2 s_2^2 / (N_2 - 1)} \qquad (5.72)$$

which possesses the F distribution with $(N_1 - 1)$ and $(N_2 - 1)$ d.f. A more usual situation is the hypothesis $\sigma_1 = \sigma_2$ and Eq. (5.69) is then

$$F = \frac{N_1 s_1^2 / (N_1 - 1)}{N_2 s_2^2 / (N_2 - 1)} \qquad (5.73)$$

which also possess the F distribution with $(N_1 - 1)$ and $(N_2 - 1)$ d.f. This distribution is well known and tabulated, and provides a test for comparing sample variances. (See Table 5.5.) For the special case of equal sample sizes N. Eq. (5.73) reduces to a simpler form

$$F = \frac{s_1^2}{s_2^2} \qquad (5.74)$$

with $(N - 1, N - 1)$ d.f.

The general random variable F defined by Eq. (5.69) has a mean value and variance given by

$$E(F) = \frac{n}{n - 2} \quad \text{for } n > 2, \quad \text{independent of } m$$

$$Var(F) = \frac{2n^2(m + n - 2)}{m(n - 2)^2 (n - 4)} \quad \text{for } n > 4$$

For the special case of Eq. (5.74), where $n = m = (N - 1)$, the above becomes

$$E(F) = \frac{N - 1}{N - 3} \approx 1 \text{ for large } N$$

$$Var(F) = \frac{4(N - 1)(N - 2)}{(N - 3)^2 (N - 5)} \approx \frac{4}{N} \text{ for large } N$$

Thus, for large N, the value of F should be close to unity for equivalences of variances.

The unique root F_p of the equation

$$\text{Prob}(F > F_p) = p \tag{5.76}$$

where p is a given percentage, is called the p percent value of the F distribution, and is tabulated in Table 5.5 for different percentage values of p. From the definition of F, it may be shown directly that

$$F_{1-p} = \frac{1}{F_p} \tag{5.77}$$

so that given the value of F_p for one side of the distribution, one can immediately find the value for the opposite side. Thus, the $(1 - p)$ percent value of the F distribution equals the $(1 - p)$ percent value of the $(1/F)$ distribution.

The F distribution also bears interesting relationships to the t and χ^2 distributions under certain conditions. When $n = 1$, for example,

$$F_\alpha = t_{\alpha/2}^2 \tag{5.78}$$

where F has $(1, m)$ d.f. and t has m d.f. This relation is not easily shown but can be found in the available literature. When $m \to \infty$, $s_2^2 \to \sigma^2$ and F will approach s_1^2/σ^2. But ns_1^2/σ^2 has a χ^2 distribution with n d.f. Hence,

$$\frac{ns_1^2}{\sigma^2} = \chi^2 \tag{5.79}$$

and

$$F = \frac{s_1^2}{\sigma^2} = \frac{\chi^2}{n} \qquad (5.80)$$

Therefore, when m approaches infinity, F has (n, ∞) d.f. and F follows a χ^2/n distribution.

(b) Applications

Tabulating the F distribution is more complex than t and χ^2 since there are two degree-of-freedom (d.f.) parameters. This would require a three way table so in the interests of economy of space, the different percent values given are limited. Table 5.5 is in four parts for the 5, 2.5, 1.0, and 0.5 percent values with n and m d.f. for numerator and denominator respectively. By the reciprocal relation of Eq. (5.77) one always has the 95, 97.5, 99, and 99.5 percent values also.

To test for the equality of variances from two samples of size N_1 and N_2 one must first compute s_1^2 and s_2^2. It is customary to compute F from Eq. (5.73) with the larger of s_1^2 and s_2^2 in the numerator. This is allowable due to the reciprocal property. Under normal circumstances, a two-tailed test must be used. Since the hypothesis is that the variance estimates s_1^2 and s_2^2 come from populations with the same variance, too great a discrepancy between s_1^2 and s_2^2 is damaging to this hypothesis regardless of which is larger. The region of rejection must include both tails of the distribution so that the hypothesis will be rejected if either

$$s_1^2/s_2^2 \text{ is very small or } s_1^2/s_2^2 \text{ is very large}$$

which is the same as if

$$s_2^2/s_1^2 \text{ is very large or } s_2^2/s_1^2 \text{ is very small.}$$

Therefore, working at some given level of significance α, reject the hypothesis of variance-equality if $F > F_{(\alpha/2)}$.

Conceivably, the logic of the problem could indicate a one-tailed test. In this case one should reject if $F > F_{\alpha}$.

(c) Computational Example:

Assume two samples of size $N_1 = 21$ and $N_2 = 11$ with $s_1^2 = 250.6$ and $s_2^2 = 63.8$ and one wants to test the hypothesis $\sigma_1 = \sigma_2$ at the 5% level of significance using a two-tailed test. Computing from Eq. (5.73), one obtains

$$F = \frac{\dfrac{20(250.6)}{20}}{\dfrac{11(63.8)}{10}} = 3.75$$

In Table 5.5b under $n = 20$ and $m = 10$ d.f. one finds $F_{2.5} = 3.42$. Therefore, one can discard the hypothesis of $\sigma_1 = \sigma_2$ at the 5% level of significance.

In a simpler case of equal sample sizes, say $N_1 = N_2 = 31$, with variances $s_1^2 = 260.5$ and $s_2^2 = 217.1$ one computes from Eq. (5.74)

$$F = \frac{260.5}{217.1} = 1.2$$

Working again at the 5% level of significance, one finds from Table 5.5b under (30, 30) d.f., $F_{2.5} = 2.07$. In this case, accept the hypothesis of equality of variances.

Application of Eq. (5.72) under the hypothesis $\sigma_1 = a\sigma_2$ would be identical to the above computational procedures except for including the factor a.

(d) Other Applications

The F distribution is applied extensively in the area of analysis of variance. Variances for several interacting factors can be studied simultaneously using these techniques. (See Sections 5.4.2 and 5.5.7(b).)

5.4 STATISTICAL RESULTS FROM REPEATED EXPERIMENTS

In samples of vibration data taken from a single flight of some vehicle, one can only make estimates of parameters of interest concerning that one flight. Some confidence, in the form of a probability statement, may be attached to these estimates based upon certain statistical considerations such as the form of a probability density function and sample sizes. It is desirable to be able to make some statements as to how representative one flight is of other flights of the same or similar vehicles under similar conditions. This cannot be done without repeating the experiment, i.e., the flight, which in turn can then give some estimates of variation between flights as opposed to variation within a flight. Statistical tests can be performed to determine whether or not the variability is the same from flight to flight, and whether or not an average vibration level (e.g. rms acceleration) is the same from flight to flight. Given the same variability and average vibration level from flight to flight (within the limits of allowed random variation), estimates can be obtained of the over-all mean and variance (with a specific confidence) of the entire population of these flights.

This section of the report discusses certain increased information and confidence to be gained in taking vibration samples from several different flights as opposed to only one flight. Statistical analysis of the data from a single flight is described in detail as well as statistical analysis for data samples from several flights. The distinction is explained between a large sample estimate from a single flight, as opposed to an estimate obtained from combining a number of smaller samples from several flights. It is shown that meaningful results may be obtained from repeated experiments due to simultaneous verification of important assumptions concerning the whole class of flights.

5.4.1 Analysis for Single Flight

Assume N observations of some parameter exist for a single flight (experiment). One can then compute, for this "ith" flight, the mean

$$\bar{x}_i = \frac{\sum_{j=1}^{N} x_{ij}}{N} \tag{5.81}$$

and the variance

$$s_i^2 = \frac{\sum_{j=1}^{N} (x_{ij} - \bar{x}_i)^2}{N} \tag{5.82}$$

Assuming the distribution is "normal", a confidence interval around $\bar{x}_i$ can be computed which has a given probability, for example 95% of containing the population mean μ. This interval is given by

$$\text{Prob}\left[\bar{x}_i \pm t_{(\alpha/2)} \frac{s_i}{\sqrt{N-1}}\right] = 1 - \alpha \tag{5.83}$$

where α is some small proportion ($0 \leq \alpha \leq 1$) and t is taken from the tables of the "t" distribution with $(N - 1)$ degrees of freedom (see Sections 5.2.3 and 5.3.3). For example, choose α = .05. Then the above expression would give an interval having 100 $(1 - .05)$ = 95% probability of containing the population mean μ. It must be realized that this confidence interval is based on the assumption that many samples of size N will be drawn from the <u>same</u> population. However, there is only one sample (one flight) and care must be taken in attaching significance to this interval until more flights have been conducted and it is verified that they do indeed come from the same population. That is, there should be no statistically significant difference between the variance and means from flight to flight.

A "tolerance" interval may also be computed for the individual values. That is, an interval is desired that will contain a given proportion P of the population of individuals $(1 - \alpha)$ percent of the time. If the population mean μ and standard deviation σ were known, then the interval would merely be

$$\mu \pm \lambda_\alpha \sigma \qquad (5.84)$$

where λ_α is obtained from Table 5.2 using α instead of p. Note that the p-percent value definition for the normal distribution considers the absolute value of deviations from the mean. However, the definitions for χ^2, t, and F only consider deviations in one direction. This results in changing from $t_{\alpha/2}$ to λ_α rather than $\lambda_{\alpha/2}$ in large sample approximations.

However, in general, only the estimates $\bar{x}_i$ and s_i of the population parameters exist so it is intuitively clear that the interval must be somewhat wider than

$$\bar{x}_i \pm \lambda_\alpha s_i \qquad (5.85)$$

and should in fact depend on the sample size N. Equation (5.85) should then become

$$\bar{x}_i \pm K s_i \qquad (5.86)$$

and tabulated values of tolerance factors K as a function of N, P and $(1 - \alpha)$ may be found in Table 2.1 of Ref. [6] (see Fig. 5.5 and Table 5.6 of this report). As an illustration for $\alpha = 5\%$, $(\mu \pm 1.96\sigma)$ contains 95% of a normal distribution. However, when μ and σ are not known, for example where $N = 10$, $P = .95$, and a confidence coefficient of $(1 - \alpha) = .95$ is desired, one has from Table 5.6 that $K = 3.38$. Thus the interval $(\bar{x}_i \pm 3.38 s_i)$ contains at least 95% of the population 95% of the time, for samples of size 10.

A confidence interval may also be computed for the variance. For small samples, $N < 30$, recalling that Ns^2/σ^2 has a χ^2 distribution, the desired relation is

$$\text{Prob}\left[\frac{Ns^2}{\chi^2_{\alpha/2}} < \sigma^2 < \frac{Ns^2}{\chi^2_{1-(\alpha/2)}}\right] = 1 - \alpha \qquad (5.87)$$

where χ^2 is obtained from tables of the χ^2 distribution with $(N - 1)$ degrees of freedom. For $N > 30$, one can make use of the fact that s is approximately normally distributed with mean σ and standard deviation $\sigma/\sqrt{2N}$. Then

$$s \pm \lambda_\alpha \frac{s}{\sqrt{2N}} \tag{5.86}$$

has a probability of $(1 - \alpha)$ of containing the population standard deviation σ.

5.4.2 Analysis for Several Flights

The statistical analysis involved comes under the general heading of "analysis of variance". The hypothesis to be tested is that the mean values of some particular parameter from flight to flight are not significantly different. It will be assumed that N observations arise of this particular parameter in each of k different flights, a total of Nk observations in all. The hypothesis of equal means will be tested for cases where it has been determined first that the populations are "normal" and have equal variances. A statistical test for normality has been described in Section 5.3.2, based on the Chi-square distribution. Another test is described in Section 6.1.10. Assuming normality, the present discussion will proceed by testing for equality of variances, and then performing an analysis to test for equality of means.

(1) Variance Equality Test.

A simple test for homogeneity of k variances σ_1^2, σ_2^2, ..., σ_k^2 computed from samples of size N is due to Hartley, Ref.[9], and considers the ratio of the largest to the smallest variance. This is denoted by

$$F_{max} = \frac{s_{max}^2}{s_{min}^2} \tag{5.87}$$

Table 5.7 gives values for the 5% and 1% levels of significance for F_{max} in a set of k mean squares all based on $(N - 1) = n$ degrees of

freedom.   The application of the test is simple: compute the variance estimators s_1^2, s_2^2, ..., s_k^2 and then select the largest and smallest values and compute F_{max} from Eq. (5.39). If F_{max} is larger than the appropriate 5% (1%) value in Table 5.6, the hypothesis $\sigma_1^2 = \sigma_2^2 = \ldots = \sigma_k^2$ is rejected at the 5% (1%) level of significance, otherwise accepted. An example at the end of this section illustrates this test.

(2) Equality of Means Test.

The hypothesis of equal means will be tested on the assumption of equal variances from "normal" populations. To do this, two estimates of the total population variances are calculated:

    (i) The first based upon the pooled variances of the several flights, called the "mean square within groups".

    (ii) The second based upon the variation of the means from flight to flight, called the "mean square between groups".

Since these two estimates have the property of independence, the "F" test described in Sections 5.2.4 and 5.3.4 can be applied.

    (a) **Basis for Test of Hypothesis:** The hypothesis is $\mu_1 = \mu_2 = \ldots = \mu_k$ and the test is based on the following considerations. Note that both variance estimates are computed from the same data. The estimate of the variance from pooling and averaging the within flight variances will be an unbiased estimate of the over-all population variance σ^2 whether or not the hypothesis of equal means is true. If the hypothesis of equal means is true the estimate of σ^2 from the flight means will also be unbiased. However, if the flights have different means, the mean square between flights will have an expectation larger than σ^2. This would be due to the inclusion of the additional variation due to the different means. This procedure yields a "one-tailed" F-test since the mean square between groups must always be the larger, and therefore only a too-large value is damaging to the hypothesis.

Assume N observations are obtained from each of k flights and one wishes to test the hypothesis of equals means.

That is

$$\mu_1 = \mu_2 = \ldots = \mu_k = \mu \tag{5.90}$$

If this hypothesis is true and the assumption of normal populations with equal variances is justified, then all the observations may be regarded as random observations from a single normal population with mean μ and variance σ^2. The means of the samples would then be random observations from a normal population of sample means for which

$$E(\bar{x}) = \mu \tag{5.91}$$

and

$$\sigma_{\bar{x}}^2 = \frac{\sigma^2}{N} \tag{5.92}$$

The unknown variance σ^2 can be estimated from the variation between the k sample means (between groups). It can also be estimated from the within flight variances (within groups). The ratio of these two estimates provides the statistic for testing the hypothesis.

(b) Estimate of σ^2 from Variation Between Groups:
Let $\bar{x}_i$ be the estimate of the mean from the ith sample. First compute the mean of the means

$$\bar{x} = \frac{\displaystyle\sum_{i=1}^{k} \bar{x}_i}{k} \tag{5.93}$$

and then

$$N s_{\bar{x}}^2 = \frac{N \displaystyle\sum_{i=1}^{k} (\bar{x}_i - \bar{x})^2}{k} \tag{5.94}$$

since $N\sigma_{\bar{x}}^2 = \sigma^2$. Equation (5.94) is the estimate of σ^2 from the between group variation. Note that this estimate is based on the random variation expected in sample means of size N plus any additional variation due to real differences in these means if the hypothesis of equal means is false.

(c) <u>Estimate of σ^2 from Within Group Variations:</u>
Next pool and average the variance estimates from each flight to obtain the second estimate of σ^2, the mean square within groups. In general, weight each sample variance according to its number of degrees of freedom and divide by the total number of degrees of freedom which gives

$$s^2 = \frac{(N_1 - 1) s_1^2 + (N_2 - 1) s_2^2 + \ldots + (N_k - 1) s_k^2}{N_1 + N_2 + \ldots + N_k - k} \qquad (5.95)$$

In the case of equal sample sizes Eq. (5.95) reduces to

$$s^2 = \frac{(N - 1) \sum_{i=1}^{k} s_i^2}{Nk - k} = \frac{\sum_{i=1}^{k} s_i^2}{k} \qquad (5.96)$$

Alternatively, one may write

$$s^2 = \frac{\sum_{i=1}^{k} \sum_{j=1}^{N} (x_{ij} - \bar{x}_i)^2}{Nk} \qquad (5.97)$$

for computing directly from the original data. Note that this estimate will not be affected by any variation due to different means.

(d) <u>The Variance Ratio:</u> Two independent estimates of the population variance have now been computed, namely, $N s_{\bar{x}}^2$, which has $(k - 1)$ d.f. and s^2 which has $(Nk - k)$ d.f. To form the desired

ratio recall that $k(Ns_{\bar{x}}^2)/\sigma^2$ and $Nk(s^2)/\sigma^2$ have χ^2 distributions with $(k-1)$ and $(Nk-k)$ d.f. respectively. Taking for the numerator the term with the greatest expectation, $k(Ns_{\bar{x}}^2)/\sigma^2$, the F ratio then is

$$F = \frac{\dfrac{k(Ns_{\bar{x}}^2)/\sigma^2}{(k-1)}}{\dfrac{Nk(s^2)/\sigma^2}{(Nk-k)}} = \frac{\left(\dfrac{k}{k-1}\right)(Ns_{\bar{x}}^2)}{\left(\dfrac{Nk}{Nk-k}\right)(s^2)} = \frac{s_{\bar{x}}^2}{s^2} \cdot \frac{(Nk-k)}{(k-1)} \tag{5.98}$$

or

$$F = \frac{N\displaystyle\sum_{i=1}^{k}(\bar{x}_i - \bar{x})^2}{\displaystyle\sum_{i=1}^{k}\sum_{j=1}^{N}(x_{ij} - \bar{x}_i)^2} \cdot \frac{(Nk-k)}{(k-1)}$$

Since a one-tailed test can be applied, F is compared with F_α with $\left[(k-1), (Nk-k)\right]$ d.f. The hypothesis is rejected at the α percent level of significance if $F > F_\alpha$.

(3) Parameter Estimation.

(a) Mean Vibration Confidence Interval: Assuming, that at a given level of significance, equal variances and means from flight to flight exist, statements can be made about the probability of exceeding certain values within a flight, and the over-all mean vibration level can be located more precisely. For instance, suppose an interval with a certain probability of containing the true mean μ is desired. Assuming k flights of N observations each with means $\bar{x}_i$ and variances of s_i^2, the over-all sample mean $\bar{x}$ can be computed by Eq. (5.93). Then compute the variance of the distribution of means of samples of size Nk by computing the population variance estimate

from Eq. (5.96) and dividing by the total sample size Nk. This
gives for the standard deviation of the means

$$s_{\bar{x}} = \frac{s}{\sqrt{Nk}} \qquad \text{where} \qquad s^2 = \frac{\sum_{1}^{k} s_i^2}{k} \qquad (5.99)$$

To obtain a confidence interval that has a probability of $(1 - \alpha)$ of
including the true mean μ, use the following equation:

$$\text{Prob}\left[\bar{x} - t_{(\alpha/2)} \frac{s}{\sqrt{Nk - k}} < \mu < \bar{x} + t_{(\alpha/2)} \frac{s}{\sqrt{Nk - k}}\right] = (1 - \alpha) \qquad (5.100)$$

or

$$\bar{x} \pm t_{(\alpha/2)} \frac{s}{\sqrt{Nk - k}} \qquad (5.101)$$

where the value $t_{(\alpha/2)}$ is obtained from the tables of the t distribution
with $(Nk - k)$ d.f.; that is, the value defined by the equation

$$\text{Prob}\left[t > t_{(\alpha/2)}\right] = (\alpha/2) \qquad (5.102)$$

If $(Nk - k) > 30$ the t distribution approximates the normal distribution
closely enough so that one may use

$$\bar{x} \pm \lambda_{\alpha} \frac{s}{\sqrt{Nk}} \qquad (5.103)$$

where λ_{α} is obtained from the tables of the normal distribution.
To illustrate the two cases, suppose the sample size is Nk = 16
and one seeks an interval that has a probability of $(1 - \alpha) = .99$ of
containing μ. If k = 2, it is seen from the tables of the t distribution
under 14 d.f. that $t_{0.5} = 2.98$. Using Eq. (5.101)

$$\bar{x} \pm t_{0.5} \frac{s}{\sqrt{Nk - k}} = \bar{x} \pm \left(\frac{2.98}{3.7}\right) s = \bar{x} \pm (.8)s$$

With Nk = 36 and using Eq. (5.103), $\lambda_{1.0}$ is found to be 2.58 and the interval is

$$\overline{x} \pm \lambda_{1.0} \frac{s}{\sqrt{Nk}} = \overline{x} \pm \left(\frac{2.58}{6}\right) s = \overline{x} \pm (.43) s$$

(b) Individual Vibration Confidence Interval: To make a similar statement about the individual values rather than the mean value of a flight, a tolerance interval may be computed using Eq. (5.86) and the procedure described in that paragraph. Note that now the sample size is effectively considerably increased from N to Nk and as a result the interval will be significantly smaller. Figure 5.5 gives an indication of the dependence of K on the sample size for a constant proportion P and confidence coefficient (1 - α). The data in Fig. 5.5 is a plot of the special case in Table 5.6 (at end of section) where P = 0.95.
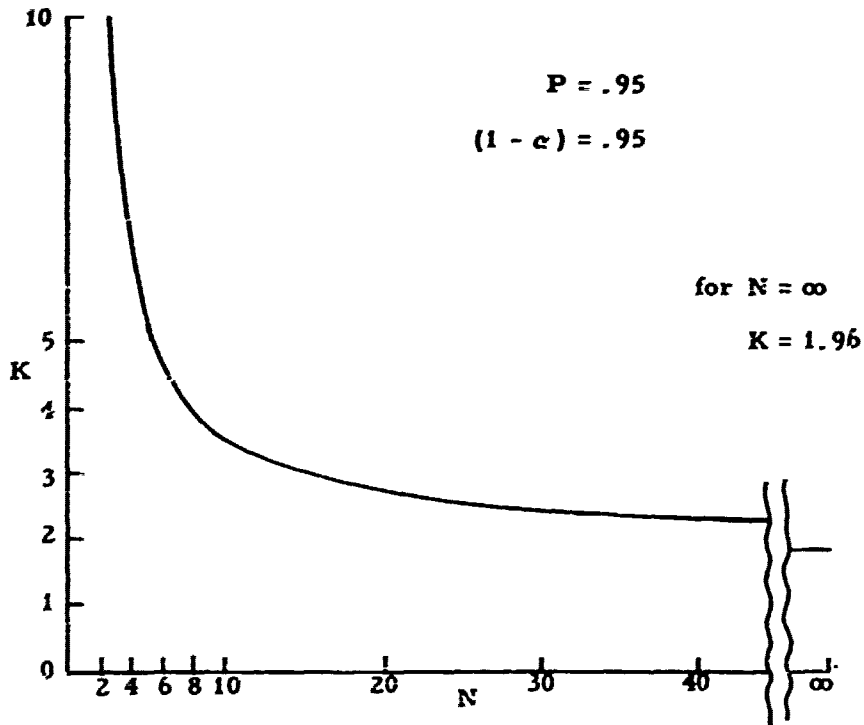
Figure 5.5. Tolerance Factor Curve

(c) Population Variance Confidence Interval: One can also obtain an interval estimate of the population variance. An estimate of s^2 with $(Nk - k)$ d.f. may be computed from Eq. (5.96). Then the procedure described in the analysis for a single flight will give the confidence interval for the population variance. Depending on the sample size, either Eq. (5.87) or (5.88) is applied.

An estimate of the population variance with $(Nk - 1)$ rather than $(Nk - k)$ d.f. may be obtained at the expense of recomputation of an s^2 directly from the original data pooled, that is

$$s^2 = \frac{\sum_1^{Nk} (x_i - \bar{x})^2}{Nk} \tag{5.104}$$

The danger in using this method is that the flights may actually not have the same mean but by chance the differences failed to be caught by the test. In this case this estimate of the variance will contain the flight to flight variance and will be larger than it should be.

(d) Results if Means are Unequal: Equation (5.96) still gives a legitimate unbiased estimate of the within flight variance whether or not the means from flight-to-flight are equal. Therefore tolerance intervals may still be computed for the individual values within a flight in the manner described in (b) above.

One can also estimate the variance of the distribution of the flight-to-flight means in the following manner. The between group variance estimate $Ns_{\bar{x}}^2$ is an estimate of the within group population variance plus the additional variance due to the flight-to-flight variation. Letting $s_M^2 = Ns_{\bar{x}}^2$,

$$s_M^2 \approx \sigma^2 + N\sigma_\mu^2 \quad ; \quad \sigma_\mu^2 = E(\mu_i - \mu)^2 \tag{5.105}$$

Since s^2 as computed from Eq. (5.96) is an estimate of σ^2, the relation

$$\frac{s_M^2 - s^2}{N} = \text{Estimate of } \sigma_\mu^2 \qquad (5.106)$$

is obtained. If the flight means are normally distributed, one may again compute tolerance intervals using k as the sample size. Note the means are being considered as individual observations from some population in this situation. Averages of N measurements of the heights of k men would be an analogous situation.

For the over-all mean $\bar{x}$ as computed from Eq. (5.93), a confidence interval can be computed with aid of the normal distribution by the following relation:

$$\text{Prob}\left[\bar{x} \pm \lambda_\alpha \frac{s_M}{\sqrt{Nk - 1}}\right] = 1 - \alpha \qquad (5.107)$$

where λ_α is taken from Table 5.2.

5.4.3 Selection of Sample Size and Number of Flights

Methods exist for the selection of sample size N and number of flights k so as to minimize the total number of observations Nk if some additional assumptions are made. Equation (5.105) indicates two components of variance in the estimate of variance determined from the variation between means. If the second of these, σ_μ^2, is zero, then the means are equal and the hypothesis is accepted (1 - α) percent of the time when working at the α level of significance. In this case the Type I error is α percent. That is, the hypothesis is rejected α percent of the time when it is really true. For selection of N and k one must also consider the Type II error β. That is, the probability of accepting the hypothesis when it is really false. Tables 8.3 and 8.4 in Ref. [6], based on pages 311-314 of the text, determine N and k as a function of α and β.

An alternative way of considering the problem is given on pages 527, 528 of Ref. [5]. Here the "operating characteristic" (O. C.) curve of the test is considered. To obtain on O. C. curve the probability of acceptance of the hypothesis is plotted versus some measure of the deviation from the desired value. Then, the N and k that gives the O. C. curve with the steepest slope, consistent with economically feasible sizes of N and k, is selected. This would be the test that discriminates most effectively against values that are considered acceptable and values that are considered unacceptable. These O. C. curves, or "power" curves as they are sometimes called, will be discussed more fully in the next section on quality control procedures.

There is no real advantage in taking k samples of size N over one large sample of size Nk if it is known for sure that all the flights will have the same vibration levels. However, if one has any reservations as to the vibration levels from flight to flight (as one certainly should), several flights must be sampled to verify the assumptions. This is the real worth of the repeated experiments while at the same time allowing the computation of better estimates of the mean and variance of the population due to the effectively increased sample size. When the hypothesis of equal means holds, one is still able to pool samples while having the assurance of a given probability that some of the assumptions have been verified.

Estimating vibration levels for another flight from a sample of a single flight, even though the sample of data is large, is at best a somewhat tenuous procedure. However, collecting smaller amounts of data from several flights will still allow the use of the final, large, combined sample size with the corresponding narrower confidence intervals while having much greater assurance that the estimates to other flights are reasonable.

5.4.4 Computational Example

Assume the hypothetical data in Fig. 5.6 below has been obtained from 5 flights with sample sizes of 15.

x	1 f	2 f	3 f	4 f	5 f
3			1		
2	2	2	1	1	1
1	2	5	2	3	2
0	3	3	5	5	2
-1	4	3	4	4	4
-2	3	1	2	2	4
-3	1	1			2
-4					
$\sum f = N$	15	15	15	15	15

$(N = 15, k = 5)$

Figure 5.6. Computational Example

First, compute the estimates of the mean and variance, x_i and s_i^2, from Eq. (5.81) and (5.82), for each flight.

$$\bar{x}_1 = -.47 \qquad \bar{x}_2 = .07 \qquad \bar{x}_3 = .07 \qquad \bar{x}_4 = -.07 \qquad \bar{x}_5 = -.93$$

$$s_1^2 = 2.109 \qquad s_2^2 = 1.925 \qquad s_3^2 = 1.525 \qquad s_4^2 = .995 \qquad s_5^2 = 2.065$$

Also compute the over-all mean $\bar{x}$ from Eq. (5.93).

$$\bar{x} = \frac{\sum \bar{x}_i}{k} = -.27$$

(1) Variance Equality Test

The data will be assumed to be taken from a normal population so the first hypothesis to be tested is $\sigma_1^2 = \sigma_2^2 = \sigma_4^2 = \sigma_5^2 = \sigma^2$. For this, use Eq. (5.89) and compute

$$F_{max} = \frac{s_{max}^2}{s_{min}^2} = \frac{2.065}{.995} = 2.08$$

In Table 5.7 for $k = 5$, and $n = N - 1 = 14$ d.f., $F_{max} \approx 4.7$ is found for the 5% level of significance. Since the computed value is 2.08, accept the hypothesis of a common variance at the 5% level of significance.

(2) Equality of Means Test

The next hypothesis to test is $\mu_1 = \mu_2 - \ldots = \mu_5 = \mu$. To do this first compute the estimate of the variance from the between group variation from Eq. (5.94).

$$N s_{\bar{x}}^2 = \frac{N \sum_1^k (\bar{x}_i - \bar{x})}{k} = \frac{15 (.7468)}{5} = 15 (.149) = 2.24$$

The second estimate of the variance from within group variation is computed from Eq. (5.96). This gives

$$s^2 = \frac{\sum_1^k s_i^2}{k} = \frac{2.109 + 1.925 + 1.525 + .995 + 2.065}{5} = 1.72$$

Next the F ratio is computed from Eq. (5.97):

$$F = \frac{(.15)\ 70}{(1.72)\ 4} = 1.53$$

In tables of the F distribution under $(k - 1) = 4$ and $(Nk - k) = 70$ d.f. one finds for the 5% level of significance $F_{.05}\ (4,\ 70) = 2.50$. Since the computed value is less, accept the hypothesis of equal means.

(3) Mean Vibration Confidence Interval

To obtain a confidence interval that has a probability of $(1 - \alpha) = .99$ of containing the true over-all mean μ, Eq. (5.103) is used since $Nk > 30$.

$$\bar{x} \pm \lambda_{1.0} \frac{s}{\sqrt{Nk}} = (-.27) \pm 2.58 \frac{1.31}{8.65} = (-.27) \pm (.39)$$

where $\bar{x}$ and s are computed from Eq. (5.93) and (5.96) respectively. The desired interval is then, $(-.66,\ .12)$.

(4) Individual Vibration Confidence Interval

A tolerance interval may be computed for the individual values for a flight. The interval that will contain $P = 99\%$ of the values $(1 - \alpha) = 95\%$ of the time is obtained using Eq. (5.86). In Table 5.6 the factor K for a sample size of $Nk = 75$ is 3.002. Using $\bar{x}$ from Eq. (5.93) and s from Eq. (5.96).

$$(\bar{x} \pm Ks) = (-.27) \pm (3.00)(1.31) = (-.27) \pm (3.9)$$

That is, the interval $(-4.20,\ 3.66)$ will contain 99% of the values in 95 out of 100 such intervals that are computed.

(5) Population Variance Confidence Interval

For an interval estimate of the population standard deviation σ Eq. (5.88) is used with Nk as the sample size. For a confidence of 99%

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$$\bar{s} \pm \lambda_{1.0} \frac{s}{2N\bar{x}} = 1.31 \pm (2.58)\frac{(1.31)}{12.2} = 1.31 \pm .28$$

and thus the interval is (1.03, 1.59). This concludes the example.

5.5 QUALITY CONTROL PROCEDURES

Some areas of statistics which are prominent mainly in quality control work may have useful applications to the statistical analysis of vibration data. "Control charts" which are often applied to industrial processes will be discussed here. These yield results similar to the analysis of variance techniques discussed in the previous section on repeated experiments. They have the advantage of being extremely easy to apply, and, although the theory of their construction is based on samples from a normal universe, have proven to be effective in sampling from non-normal distributions.

"Inspection sampling plans" which originated in the area of quality control of manufactured items can be and have been extended to other areas such as control of typing mistakes in office paperwork. These sampling plans in their simplest form will be discussed here. This will require a description of the binomial, Poisson, and hypergeometric distributions which in themselves have many other applications. A particular one of these sampling plans is usually selected for a particular job on the basis of its "operating characteristic" (O.C.) curve.

An O.C. curve is a plot of the probability of accepting an incoming "lot" of some product versus the percentage of defective items in that lot. These O.C. curves also exist for statistical tests such as the "t" test for the difference between two means. For this application, the probability of accepting the hypothesis of equal means is plotted against the actual existing difference in the means. This curve then indicates in a graphical form that large actual differences lead to a smaller probability of accepting the hypothesis of equal means. These O.C. curves are often called "power" curves, and

are obtained in a similar manner for other tests such as the "F" test applied to detecting differences between two variances. A more detailed discussion of these curves will appear below.

5.5.1 Control Charts

(a) Theory and Construction

A control chart is a plot of sample means $\bar{x}_i$ of N observations taken from some continuing process. In quality control work N is usually small, of order 4 or 5, and the sample range R (the difference between the greatest and least values in the sample) is used as a measure of the dispersion. For small samples such as these the range serves as a reasonably efficient estimate of the process standard deviation. After several samples are taken from the process the over-all mean

$$\bar{\bar{x}} = \frac{\sum_{i=1}^{k} \bar{x}_i}{k} \qquad (5.108)$$

is computed and plotted on the chart. An estimate of the process standard deviation σ is then computed from the average of the sample ranges in the following manner. P. B. Patnaik (see Ref. [5]) has shown that the square of the average range from a normal distribution has a distribution that is approximately of the form of x^2. Patnaik has worked out conversion factors d_2 (see Table 5.8) for various sample sizes N based on an average of k ranges, such that the standard deviation may be computed from the formula

$$s = \frac{\bar{R}}{d_2} \qquad (5.109)$$

An "upper control limit" (UCL) and "lower control limit" (LCL) are then obtained for the control chart as follows. It is well known that the distribution of the sample means from any population approaches normality with a standard deviation

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{N}} \qquad\qquad (5.110)$$

and mean μ as N becomes large. The control limits are normally taken as

$$UCL = \bar{x} + 3\frac{s}{\sqrt{N}} \qquad\qquad (5.111)$$

and

$$LCL = \bar{x} - 3\frac{s}{\sqrt{N}} \qquad\qquad (5.112)$$

These are 3 standard deviations away from the mean and should be exceeded only approximately .3% of the time. On this basis, if a sample mean is plotted and falls outside the control limits it is assumed that some cause other than random variation is the effect. Of course for these small samples the assumption of the normal distribution must be seriously questioned and .3% is probably not an accurate estimate of the probability of a deviation but in fact should be somewhat larger. The fact remains however, that experience has definitely proven control charts to be a valuable indication of trouble in a process even for small sample sizes of 4 and 5. Figure 5.7 illustrates a typical control chart.

The range is chosen rather than the standard deviation for quality control work due to the ease of computation. However, the range loses efficiency rapidly as the sample size becomes as large as 10 or 12, so that the standard deviation should be computed directly for samples of that size. The larger sample sizes are valuable also in that the distribution of the means will become closer to normal. In this case of a large sample size, the individual sample standard deviations for several samples are just averaged to obtain s to use for the control limits.

-2											
1											
0											
-1											
-2											

UCL = 2.1

$\bar{X}$ = 0.1

LCL = -1.9

				SAMPLE NUMBER							
1	2	3	4	5	6	7	8	9	10	11	12
				MEASUREMENTS							
-2	0	0	1	0	0	2	-1	0	-2	0	2
0	-1	-1	3	3	3	0	1	-3	-1	-1	0
0	0	0	-2	-3	-2	2	-1	2	-1	1	0
1	0	-1	1	0	-3	-1	2	1	-1	2	-1
1	-1	0	-1	0	1	-1	2	1	1	1	1
$\bar{x}_i$ 0	-.4	-.4	.4	0	-.2	.4	.6	.2	.8	.6	.4
R_i 3	1	1	5	6	6	3	3	5	3	3	3
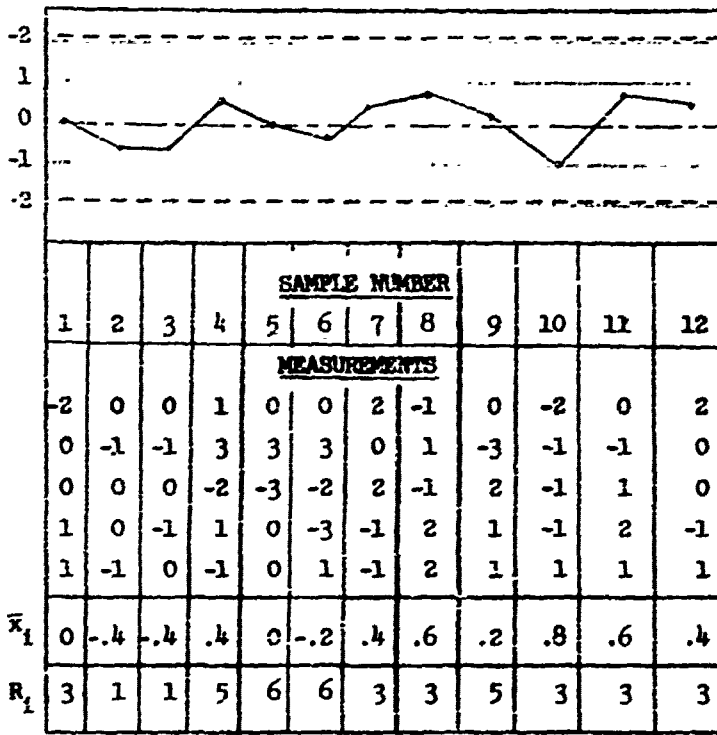
$$\bar{x} = \frac{\sum x_i}{12} = \frac{1.2}{12} = .1$$

$$\bar{R} = \frac{\sum R_i}{12} = \frac{42}{12} = 3.5$$

$$d_2 = 2.34$$

$$s = \bar{R}/d_2 = 1.5$$

$$UCL = \bar{x} + 3\frac{s}{\sqrt{5}} = 2.1$$

$$LCL = \bar{x} - 3\frac{s}{\sqrt{5}} = -1.9$$

Figure 5.7. Control Chart for Mean Values (k = 12, N = 5)

(b) Application to Flight Vibration Data

The control chart's application to flight vibration data should be as a rough visual aid to obtaining indications of a drift of the mean vibration level over a flight phase or as an indication of some unusual occurrence if a measured mean value were to exceed the control limits. Flight-to-flight means could also be plotted this way, but it is doubtful if enough "flights" would be conducted other than in laboratory experiments.

The control chart may also be used as a rough test for randomness of data, since non-random effects should be caught as point outside of the control limits on the chart. It should be noted that with fairly large sample sizes, say $N > 12$ or preferably $N > 30$, the control charts

are safely applied whether or not the underlying distribution is normal. On this basis the control chart may often be useful when other techniques are not.

The control chart for mean vibration measurements is only one of many different variations of this technique. See Ref. [5] for a more exhaustive discussion.

(c) Comparison with Analysis of Variance

The analysis of variance technique described in Section 5.4 concerning repeated experiments is comparable with a mean and standard deviation control chart in many ways. They are not exactly equivalent; however, they both give a test for equality of several means. The control chart has some advantages in that it gives a visual picture, it more or less pinpoints the exact position of the offending mean values, and is of course somewhat simpler to apply and understand.

However, the control chart is most useful where observations are easy to obtain; that is, not time consuming, reasonably inexpensive, and plentiful. The analysis of variance technique is much more useful when the maximum amount of information must be obtained from the minimum amount of data. This indicates that for limited flight vehicle vibration data, the analysis of variance will probably be the necessary technique.

5.5.2 Inspection Sampling

The situation may arise where it is convenient to measure some vibration parameter in terms of whether or not it exceeds some safe or acceptable level. In this situation the observation may be interpreted either as "acceptable" or as a "defect". Under these conditions samples of N observations could be considered as coming from an acceptable or unacceptable population having a certain percentage of defects p. This is assuming for instance, that if some vibration level is exceeded only a certain small percentage of the time, then no damage will be done.

Under these conditions the application of inspection sampling techniques may be useful. These techniques basically consist of inspecting the sample of N items, and rejecting the population as exceeding some percent defective if the number of defective items is larger than some predetermined number of allowable defects c. With the sample size N and "acceptance number" c determined, an O. C. curve may be plotted to indicate exactly the probabilities of accepting or rejecting a population with a given percent defective. To discuss these sampling plans and consider methods of computing various O. C. curves, it is necessary to first review briefly three important discrete probability distributions: the binomial, Poisson, and hypergeometric distributions.

5. 5. 3 Binomial Distribution

The binomial distribution is a discrete distribution arising from an expansion of $(p + q)^N$ where p may be thought of as the probability of a success and $q = 1 - p$ the probability of a failure. The "experiment" having these two possible results is repeated N times, and the outcome of the experiment is always independent of the past results. These repetitions are known as Bernoulli trials. That is, repeated independent trials with only two possible outcomes for each trial and their probabilities remain the same throughout the trials. Writing S for success and F for failure with p the probability of S and q the probability of F, it is clear, since the trials are independent, that the probability of any given sequence of successes and failures is

$$\text{Prob}\left[(SSFS \ldots FS)\right] = ppqp \ldots qp \qquad (5.113)$$

where S and F are just replaced with p and q respectively.

If N is the total number of trials and k the number of successes, the probability of any one sequence of k successes in N trials is

$$p^k q^{N-k} \qquad (5.114)$$

To consider the probability of the total number of successes k out of N trials, the number of different ways of distributing k letters S in N places must be computed. This is just the number of combinations of N things taken k at a time. That is,

$$\binom{N}{k} = \frac{N!}{k!(N - k)!}$$ (5.115)

Then the probability b(k; N, p) that N Bernoulli trials with probabilities p for success and q for failure result in k successes $(0 < k < N)$ is

$$b(k; N, p) = \binom{N}{k} p^k q^{N-k}$$ (5.116)

That is, multiply the total number of possible favorable events by the probability of one favorable event. In particular, the probability of no successes in N trials is q^N, and the probability of at least one success is $(1 - q^N)$. Equation (5.116) represents the kth term of the binomial expansion of $(q + p)^N$, therefore the name.

In the application of the binomial distribution to inspection sampling it is convenient to rewrite Eq. (5.116). Let $P(c/N)$ stand for the probability of c defective items in a sample of size N when sampling from a population with a fraction defective p. Then Eq.(5.116) can be written, replacing q by 1 - p and k by c,

$$P\left(\frac{c}{N}\right) = \binom{N}{c} p^c (1 - p)^{N-c} = \frac{N!}{c!(N - c)!} p^c (1 - p)^{N-c}$$ (5.117)

If p' = c/N is taken as the variable in the binomial distribution the expectation can be shown to be (see Ref. [5]),

$$E(p') = p$$ (5.118)

and the standard deviation is

$$\sigma_{p'} = \sqrt{\frac{p(1 - p)}{N}}$$ (5.19)

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5.5.4 Poisson Distribution

Calculations using the binomial distribution become quite cumbersome for large N. Therefore, approximations to the binomial are quite helpful in applications. In inspection sampling, where N is large, p is usually quite small, and the condition

$$\lambda = Np \quad , \quad \lambda = \text{constant} \tag{5.120}$$

is often reasonably satisfied. In such cases an approximation of $b(k; N, p)$ due to Poisson is convenient. Note that

$$b(0; N, p) = (1 - p)^N \tag{5.121}$$

Now substitute Eq. (5.120) which gives

$$b(0; N, p) = (1 - \frac{\lambda}{N})^N \approx e^{-\lambda} \tag{5.122}$$

for large N. By simple algebraic manipulation the following relation is obtained:

$$\frac{b(k; N, p)}{b(k - 1; N, p)} = \frac{p(N - k + 1)}{qk} \tag{5.123}$$

From the assumption of Eq. (5.120), Eq. (5.123) may be rewritten in the form

$$\frac{b(k; N, p)}{b(k - 1; N, p)} = \frac{\lambda - (k - 1)p}{qk} \approx \frac{\lambda}{k} \tag{5.124}$$

since when p is close to zero, q is close to one, giving the above relation. For k = 1, Eq. (5.124) and (5.122) yield that $b(1; N, p) \approx \lambda e^{-\lambda}$. For k = 2, one finds $b(2; N, p) \approx (\lambda^2 e^{-\lambda})/2$, and generally by induction

$$b(k; N, p) \approx \frac{\lambda^k}{k!} e^{-\lambda} \tag{5.125}$$

This is the desired Poisson approximation and is usually designated by

$$p(k; \lambda) = \frac{\lambda^k}{k!} e^{-\lambda} \quad ; \quad \lambda = Np \qquad (5.126)$$

For inspection sampling it is more convenient to let $P(c/N)$ stand for the probability of c defective items in a sample of size N, and write here

$$P\left(\frac{c}{N}\right) = \frac{\lambda^c e^{-\lambda}}{c!} \quad ; \quad \lambda = Np \qquad (5.127)$$

In words, $P\left(\frac{c}{N}\right)$ is the probability of c defects in a large sample of size N with a small fraction defective p such that $Np \approx \lambda$. For example, suppose $N = 500$ and $p = .02$. Then $Np = 10$ and the probability of c defects in a sample of size 500 is

$$P\left(\frac{c}{500}\right) = \frac{10^c e^{-10}}{c!}$$

The mean value of the number of defects c in N items is λ and the standard deviation is $\sqrt{\lambda}$. (See Ref. [5]) In terms of the variable $p' = c/N$ introduced previously, Eq. (5.118) above,

$$E(p') = \lambda \quad ; \quad \sigma_{p'} = \sqrt{\lambda} \qquad (5.128)$$

5.5.5 Hypergeometric Distribution

A different distribution is sometimes useful in computing probabilities of finding the numbers of defects in a sample from a small, finite population. Suppose the total population size is S and it contains m defects. Assume a sample of size N is taken

and the probability of finding c defects is desired. First consider the total number of combinations of N units each that may be made from S units. This is the total number of different samples of size N that may be drawn from a population of size S, namely,

$$\binom{S}{N} = \frac{S!}{N!(S-N)!} \qquad (5.129)$$

Second, note the total number of combinations of N units that may be made from (s - m) units, i.e., the number of possible samples from the "good" units in the population. This would give the probability P(0/N) of a sample of size N with zero defects as

$$P\binom{0}{N} = \frac{\binom{S-m}{N}}{\binom{S}{N}} = \frac{\frac{(S-m)!}{N!(S-m-N)!}}{\frac{S!}{N!(S-N)!}} = \frac{(S-m)!(S-N)!}{(S-m-N)!S!} \qquad (5.130)$$

In other words, P(0/N) is the ratio of the number of possible ways a sample with no defective items can occur to the total number of ways all samples can occur.

To proceed, consider the total number of ways a sample of size (N - 1) may be drawn from the (S - m) good units. Multiply this by the number of ways 1 item may be drawn from the m defective units. This would give the total number of ways a sample with (n - 1) good units and one defect may be drawn from the population. Then the probability of a sample with one defect is the ratio of this number of combinations to the total number, or

$$P\binom{1}{N} = \frac{\binom{S-m}{N-1}\binom{m}{1}}{\binom{S}{N}} = \frac{\frac{(S-m)!}{(N-1)!(S-m-N+1)!} \cdot \frac{m!}{1!(m-1)!}}{\frac{S!}{N!(S-N)!}}$$

$$(5.131)$$

The above relation illustrates the value of the notation $\binom{a}{b}$ for combinations, and the complete expansion on the right in Eq. (5.131) will be avoided henceforth.

In general, if a population contains S units with m defects, the probability $P(c/N)$ that a random sample of size N will contain c defects, where c of course is less than the total number of defects m, is

$$P\left(\frac{c}{N}\right) = \frac{\binom{S-m}{N-c}\binom{m}{c}}{\binom{S}{N}}$$

(5.132)

This gives what is known as the hypergeometric distribution since it is the general term in the series by that name. It can be seen that computations might become quite cumbersome with Eq. (5.132).

If the ratio of the sample size to the population size is less than 0.10, the binomial is usually used as an approximation to the hypergeometric. If the sample and population size are both large and the number of defects in the population is not too small, a normal approximation may be used. The mean (see Ref. [5]) of the normal approximation would then be the fraction defective p, and the standard deviation would be

$$\sigma_p = \sqrt{\frac{p(1-p)}{N}\left(1 - \frac{N}{S}\right)}$$

(5.133)

The probability of c or less defects in a sample of size N then is given by the value of the standardized Gaussian distribution function

$$\Phi(z) = \int_{-\infty}^{z} (1/\sqrt{2\pi}) \exp(-t^2/2)\, dt$$

where z is

$$z = z(c + 0.5) = \frac{\left(\dfrac{c + 0.5}{N}\right) - p}{\sigma_p} \qquad (5.134)$$

The factor $(c + 0.5)/N$ rather than c/N is to account for dividing the distribution into intervals having c as a midpoint; that is, for the probability of exactly c defects, compute $\phi\left[z(c + .5)\right] - \phi\left[z(c - .5)\right]$ to obtain the area in the interval $(c - 0.5, c + 0.5)$.

The three previously described distributions all occur in sampling plan computations with the Poisson often being the most convenient. However, various situations arise where it is necessary to apply either the binomial or the hypergeometric for reasonably accurate computations.

5.5.6 A Sampling Plan and its O.C. Curve

Several types of sampling plans are used in quality control work. However, for purposes of applications to vibration data, only the simplest will be considered: The "single sampling plan." Their employment, as previously indicated in Section 5.5.2, will probably be limited to areas where vibration data is judged as being either acceptable or unacceptable. The O.C. curves associated with these sampling plans, and their generalization to other statistical tests in Section 5.5.7, are applicable also to many other physical problems.

A sampling plan might be stated as follows; select a random sample of $N = 100$ from a given population, accept the lot if the number of defects c is 2 or less, reject if 3 or more. The O.C. curve for this plan is shown in Fig. 5.8 below where P_a is the probability of accepting the population as satisfactory, and p is the population fraction defective.

For $p = .01$, note that $P_a = .92$. Therefore, if a fraction defective of $p = .01$ was considered as being satisfactory, this sampling plan could be interpreted as a test of the hypothesis $p = .01$ at the 8% level of significance. That is, the probability of rejecting
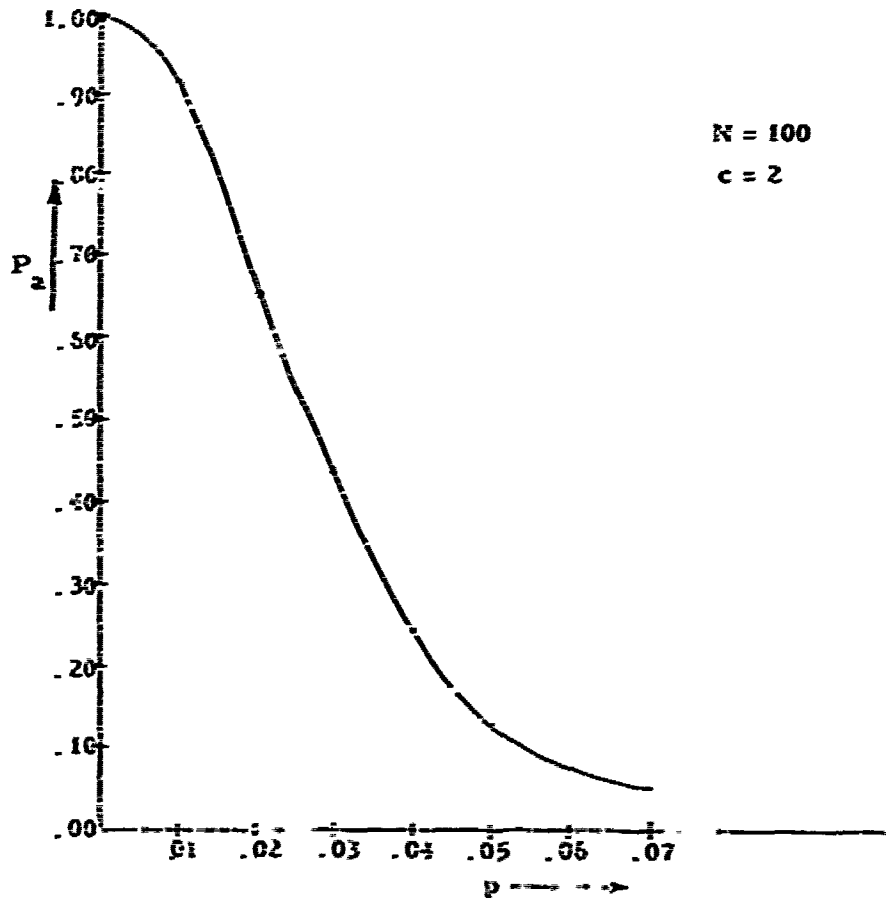
Figure 5.2. O.C. Curve for Single Sampling Plan

the hypothesis when it is really true (Type I error) is $\alpha = .08$. The interpretation for the fraction defective in vibration data analysis could be something such as the percentage of time some vibration parameter is exceeded, and that exceeding this level one percent of the time is not considered harmful.

If a fraction defective of .05 was considered unacceptable (see Fig. 5.8), the Type II error β would be 12%. That is, the probability of accepting the hypothesis when it is really false (for the value $p = .05$) is $\beta = .12$. When the acceptable value p_{α} and the unacceptable value p_{β}

of the fraction defective are decided upon, and values of α and β
selected the desired sampling plan would be one having an O.C.
curve passing through these two points. The ideal case is to have
$\alpha = 0$ and $\beta = 0$ for some value of p, but of course this is im-
possible. As α and β are decreased and p_α and p_β are brought
closer together, the sample size must be increased to obtain the
desired O.C. curve.

Sampling plan O. .. .. .ve. are usually computed with aid
of the Poisson distribution. However, occasionally the hyper-
geometric or binomial distribution might be required depending on
the conditions. Table 5.9 gives some values based on the Poisson
distribution for selecting a desired sampling plan when $\alpha = .05$ and
$\beta = .10$. Since integral values of N and c are required, an exact
solution may not always exist but one can usually obtain a satis-
factory approximation. Table 5.9 appears at end of Section 5.

Application of Table 5.9 is as follows. Suppose a plan is de-
sired with $p_\alpha = .01$, $p_\beta = .08$, $\alpha = .05$, and $\beta = .10$. First compute
the ratio p_β/p_α which in this case is 8. In the right hand column
of the table it is found that 8 lies between the value 10.96 for $c = 1$
and 6.50 for $c = 2$. For a fixed value of N, choosing $c = 1$ will in
effect give a lower O.C. curve, that is, decrease P_a for all values
of p. Choosing $c = 2$ for the same fixed value of N will in turn
raise the curve. In any case it must now be decided whether to have
the curve pass precisely through the point $(p_\alpha, 1 - \alpha)$ in which case
the second column for $p_\alpha N$ is used, or (p_β, β) in which case the third
column for $p_\beta N$ is used. T in N is computed from either

$$\frac{(p_\alpha)N}{p_\alpha} = N \qquad \text{or} \qquad \frac{(p_\beta)N}{p_\beta} = N \qquad (5.135)$$

where $p_\alpha N$ or $p_\beta N$ is obtained from the appropriate column in Table 5.9.

The chart below gives the results from the various choices mentioned in the above example.

c = 1		c = 2	
(p_β, β) held	$(p_\alpha, 1 - \alpha)$ held	(p_β, β) held	$(p_\alpha, 1 - \alpha)$ held
$N = p_\beta N / .08$	$N = p_\alpha N / .01$	$N = p_\beta N / .08$	$N = p_\alpha N / .01$
$= 3.89 / .08$	$= .355 / .01$	$= 5.32 / .08$	$= .818 / .01$
≈ 49	≈ 36	≈ 67	≈ 82

The actual α's and β's as computed from the Poisson distribution given by Eq. (5.127) are shown below. For these computations $\lambda = Np_\alpha$ or $\lambda = Np_\beta$ as appropriate and the terms for $c = 0, 1$ or $c = 0, 1, 2$ as appropriate are summed. For instance, the probability of acceptance at $p = p_\alpha$ for the plan $N = 49$, $c = 1$ would be

$$P_a = \text{Prob}\left[c \leqslant 1\right] = \sum_{c=0}^{1} \frac{(49)(.01)^c e^{-(49)(.01)}}{c!}$$

The Poisson distribution has been well tabulated so performing these computations is a relatively simple matter. Results are:

Plan	α	β
1. $N = 49$, $c = 1$	.08	.10
2. $N = 36$, $c = 1$	.05	.23
3. $N = 67$, $c = 2$	.03	.10
4. $N = 82$, $c = 2$	.05	.04

Figure 5.9 below shows the actual O.C. curves for these four plans. Note that plan 4 with the largest sample size gives the desired level

of significance $P_a = .95$ at $p = 0.01$, and then P_a drops more rapidly to give lower P_a values as the fraction defective increases. Thus, plan 4 is superior to the other plans in rejecting samples having a large number of defects.
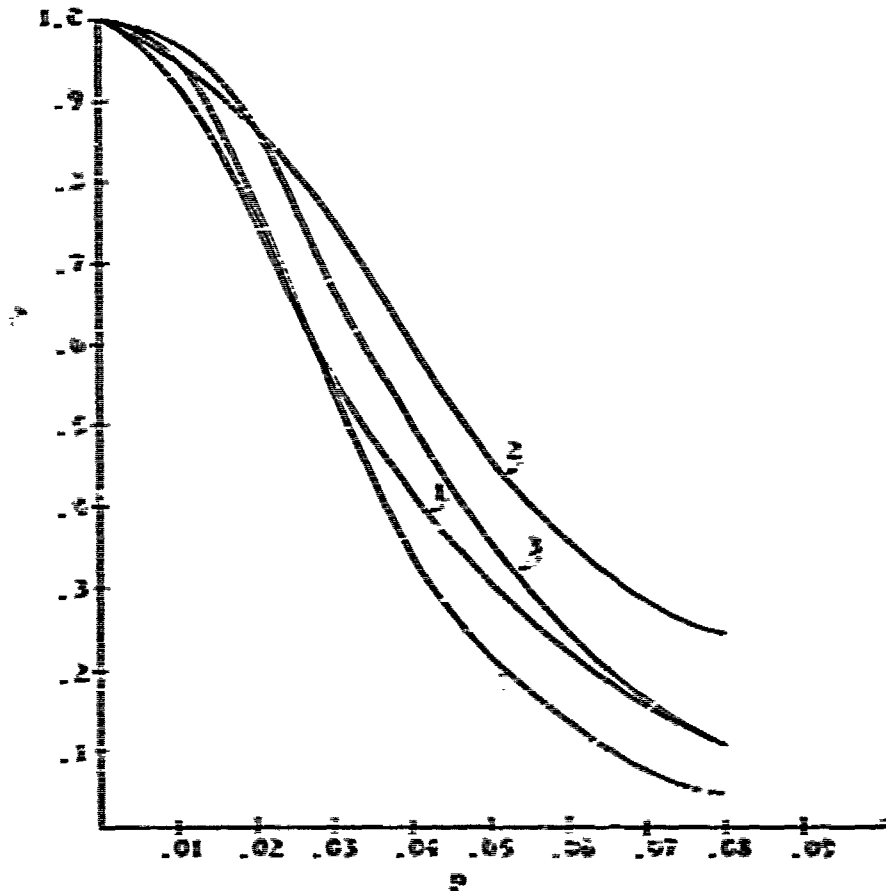
Figure 5.9. O.C. Curves for Sampling Plans Computed in Text

5.5.7 Operating Characteristic Curves

The O.C. curves shown for the sampling plans in the previous section have counterparts for any statistical test of a hypothesis. This curve for a given test would be a plot of the probability of

accepting the hypothesis versus some measure of deviation from the hypothesis under test. For instance, the t-distribution may be used to test the hypothesis that two population means are equal, based on sample estimates of these means. In this case, if the means are really equal, that is $\mu_1 - \mu_2 = 0$, there is a probability of $(1 - \alpha)$ of accepting the hypothesis when working at the α level of significance. Now if the means are really different, namely $|\mu_1 - \mu_2| > 0$, the probability of accepting the hypothesis is somewhat less than $(1 - \alpha)$. This probability of acceptance is different for each value of $|\mu_1 - \mu_2|$ and a plot of these values would give the O.C. curve for the t-test. It is intuitively clear that larger sample sizes will increase the slope of this curve so that the probability of accepting the hypothesis for deviations of $|\mu_1 - \mu_2|$ from zero decreases rapidly as $|\mu_1 - \mu_2|$ increases. In fact, if it is desired to have a specific probability of accepting the hypothesis for some arbitrary value of $|\mu_1 - \mu_2|$, a certain minimum sample size is required and may be calculated.

(a) O.C. Curve for t-test

Figure 5.10 shows the O.C. curve for the t-test of the hypothesis $\mu_0 = 50$ when $N = 25$, $\alpha = .05$, and the population standard deviation is known to be $\sigma = 10$.
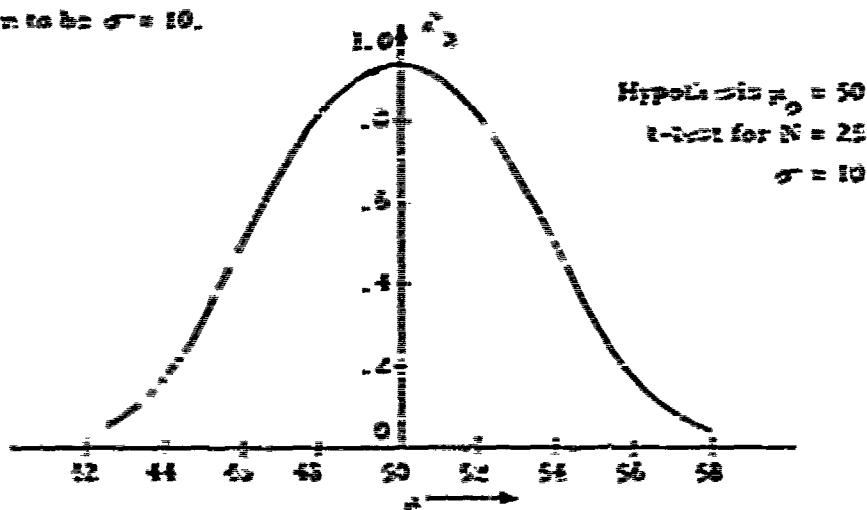
Figure 5.10. O.C. Curve for Two-Tailed t-Test

This curve goes to either side of the hypothesized value $\mu_o = 50$ since both positive and negative deviations are possible. Note that to calculate this curve the population standard deviation σ is assumed to be known. This of course is not usually the case; however, σ may often be estimated closely enough to indicate what the test is really accomplishing.

The curve in Fig. 5.10 would be constructed in the following manner. First, the $(1 - \alpha)\%$ interval around the hypothesized mean value μ_o is computed from

$$\text{Prob}\left[\mu_o - z_{(1-\alpha/2)}\frac{\sigma}{\sqrt{N}} < \bar{x} < \mu_o + z_{(1-\alpha/2)}\frac{\sigma}{\sqrt{N}}\right] = 1 - \alpha$$

(5.136)

where α is the level of significance. The values for Fig. 5.10 give the interval (46.1, 53.9). Next, assume the true mean value μ is different from 50, say 52. For a true mean value of 52, the probability for a sample mean exceeding 53.9 is clearly larger than $\alpha = 2.5\%$ In fact, the probability will be $(1 - p)$ as found from the equation

$$\text{Prob}\left[z > z_p\right] = 1 - p$$

(5.137)

where z_p is a normal variate as given by

$$z_p = \frac{\mu_\alpha - \mu}{\sigma/\sqrt{N}}$$

(5.138)

μ = true mean value

$$\mu_\alpha = \mu_o + z_{(1-\alpha/2)}\frac{\sigma}{\sqrt{N}}$$

(5.139)

In the example $N = 25$, $\sigma = 10$, $\mu_o = 50$, $\mu = 52$, and $\mu_\alpha = 53.9$. The method for finding P_a is now as follows. From Eq. (5.138),

compute $(53.9 - 52)/2 = .95 = z_p$. In Table 5.1 of the normal probability distribution the area to the right of $z_p = .95$ is found to be .17. The area to the left of the other end of the interval, 46.1, from a mean value of 52, will be negligible. Therefore, the probability of finding a value $\bar{x}$ inside the limits (46.1, 53.9) is .83 when $\mu = 52$. This gives the desired value for P_a when $\mu = 52$. Other values are computed similarly.

(b) O.C. Curve for Analysis of Variance

The same type of curves may be computed for the F test for variances. As a particular example, some O.C. curves will be illustrated for the analysis of variance technique described in Section 5.4 on repeated experiments. Here the test is for equal means from samples of size N from k flights. A variance estimate is computed from the variation of the sample mean of these flights. The estimate s_M^2 contains two components:

$$s_M^2 = \sigma^2 + N\sigma_\mu^2 \tag{5.140}$$

where σ^2 is the population variance and σ_μ^2 is the variance due to any real difference in the means. For equal means $\sigma_\mu^2 = 0$, otherwise σ_μ^2 becomes larger as the difference in the means increase. Using the ratio

$$\frac{\sigma_\mu^2}{\sigma^2} = \Delta^2 \tag{5.141}$$

as a measure of the real differences in the means, an O.C. curve may be constructed. Of course, σ^2 will probably be unknown, and must be estimated as well as possible.

Different curves can be computed for various sizes of N and k, and an O.C. curve may be selected which discriminates most highly in the region of interest with Nk made approximately a minimum. Minimum Nk implies smaller number of samples and flights. Fig. 5.11

below shows some O.C. curves for a few values of N and k where the probability of accepting the hypothesis $\mu_1 = \ldots = \mu_k = \mu$ is plotted against Δ.
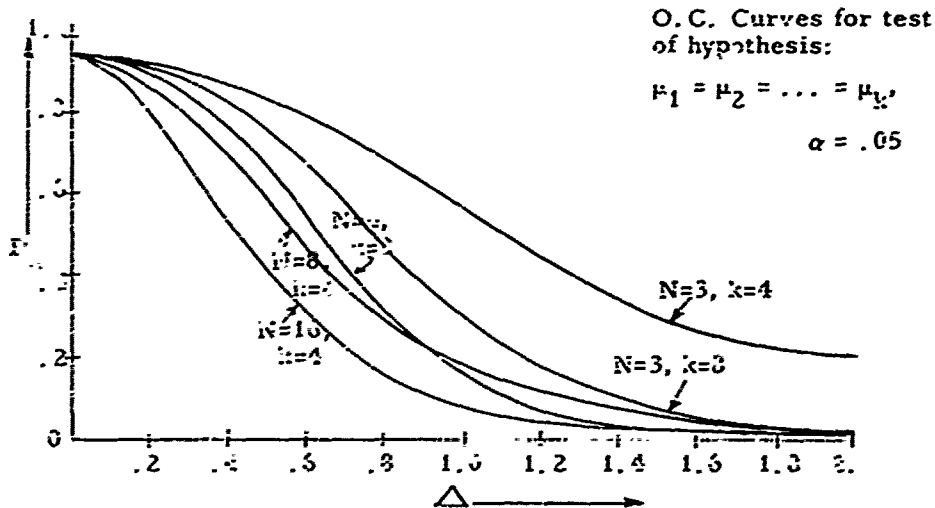
Figure 5.11. Selected O.C. Curves for Analysis of Variance Test

A comparison of the curves for (N = 4, k = 8) and (N = 8, k = 4) reveals that the second of these is more "powerful" for smaller values of Δ but less "powerful" for large values of Δ. That is, for values of Δ between zero and approximately 0.9 the second curve has smaller probabilities of accepting the hypothesis when it is really false (Type II error). However, for values of Δ larger than 0.9, the first curve has smaller probabilities of accepting the false hypothesis. All cases have the same probability of accepting the hypothesis of equal means if it is really true. That is, P_a = 0.95 when Δ = 0, or there exists a Type I error here of 5% independent of N and k. This comparison gives an indication that the same total number of observations can perform different functions. If Δ could be expected to be small, choose N = 8 and k = 4; however if Δ is expected to be large, N = 4 and k = 8 would give a more powerful test. Also notice that for values

of Δ larger than approximately 1.4, (N = 4, k = 8) gives about the same P_a's as (N = 16, k = 4). Therefore, if very large values of Δ were expected, 32 observations could do the job as well as 64.

The curves in Fig. 5.11 may be constructed using Tables 8.3 and 8.4 in Ref. [6] and the procedures described there on pp. 311-313. These tables give values of a function ϕ such that

$$\Delta^2 = \frac{\sigma_\mu^2}{\sigma^2} = \frac{\phi[\alpha, \beta, k - 1, k(N - 1)] - 1}{N} \qquad (5.142)$$

where β would be the Type II error corresponding to the real value for Δ^2. Therefore, when values for α, N, and k are selected, Δ may be plotted against β. This value for β equals P_a.

(c) O.C. Curve for F Test

These same tables in Ref. [6] may also be used for computing O.C. curves for the F test as a test for detecting a difference in two variances σ_1^2 and σ_2^2 in the following manner. For this case, when α, N_1, and N_2 are selected, the quantity

$$\phi(\alpha, \beta, N_1 - 1, N_2 - 1) = \frac{\sigma_1^2}{\sigma_2^2} \qquad (5.143)$$

Therefore, β may be plotted against values of σ_1^2 / σ_2^2. For example, if $\alpha = .05$ and $N_1 = N_2 = 16$, $\phi = 1.685$ is found in Table 8.3 of Ref. [6] for $\beta = .75$. Figure 5.12 below is the O.C. curve for the above values of α, N_1, and N_2 when a "one-tailed" F test is used.
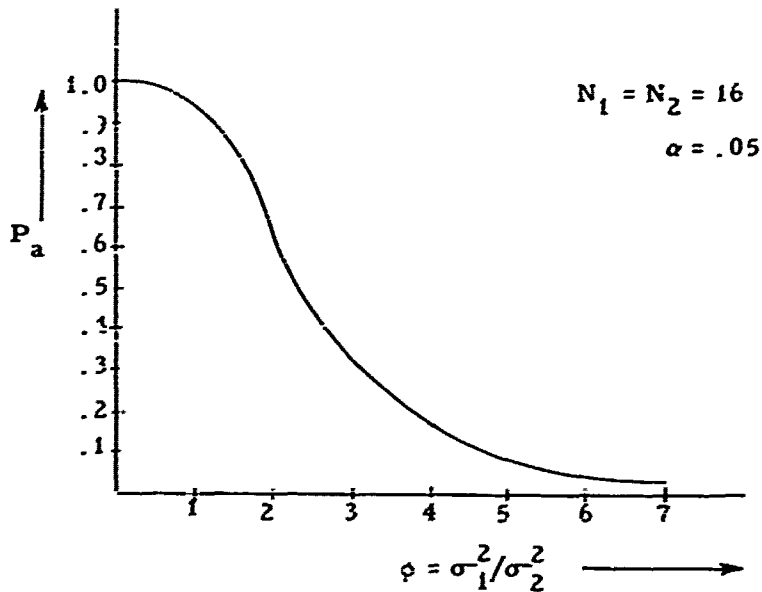
Figure 5.12. O. C. Curve for One-Tailed F Test

5.6 MULTIPLE REGRESSION TECHNIQUES

This section discusses basic multiple regression techniques which may be useful in correlating input parameters to output vibration levels. A derivation is presented of the least squares equations for obtaining an optimum linear fit to k variables. The solution of the equations and the computation of important statistics are carried out in detail for the important special case of three variables. A brief generalization to k variables is then discussed, followed by methods for fitting to nonlinear models by appropriate linear methods. Since an electronic computer becomes an important tool in cases for many variables due to the large amount of computations involved, an alternative calculating scheme is presented without justification which is helpful for computer programming.

Multiple regression is a statistical technique whereby it is determined to what extent one variable may be related mathematically to several others. A general relation between several variables is

usually assumed which involves unknown coefficients, and the co-
efficients of the variables are then calculated by some suitable method,
such as the method of "least squares". This method minimizes the
sums of the squares of the deviations of the observed values from the
predicted values.

The assumed mathematical relation is quite often chosen to be
linear. For the linear case, if it is desired to find a relation between
k variables $x_i(i = 1, 2, \ldots, k)$, the coefficients $a_i(i = 0, 2, 3, \ldots, k)$
of the following equation are determined:

$$x_1' = a_0 + a_2 x_2 + a_3 x_3 + \ldots + a_k x_k \qquad (5.144)$$

where x_1' would be thought of as the predicted dependent variable, and
$x_2, \ldots, x_k$ as the independent variables. It must be noted that inde-
pendent and dependent are not defined here in the statistical sense, but
rather more as a concession to common usage. In geometry, Eq. (5.144)
would be a hyperplane in "k-dimensional space".

Assume now that N observed sample values of x_1 are obtained.
By the methods of least squares, the coefficients a_i would now be
selected so as to minimize the sum of the squares of the deviations
of the observed values of x_1 from the predicted values x_1'. That is,
the a_i must be such that

$$\sum (x_1 - x_1')^2 = \text{minimum} \qquad (5.145)$$

where the sum here ranges over the N sample values.

5.6.1 The Least Squares Equations

It is more convenient to work with the variables as measured
from their sample means $\bar{x}_i$ where

$$\bar{x}_i = \frac{\sum_{j=1}^{N} x_{ij}}{N} \qquad (5.146)$$

Therefore let

$$X_i = x_i - \bar{x}_i \qquad (i = 1, 2, \ldots, k) \qquad (5.147)$$

Also note that if $X_1' = x_1' - \bar{x}_1$,

then

$$x_1 - x_1' = X_1 + \bar{x}_1 - (X_1' + \bar{x}_1) = X_1 - X_1' \qquad (5.148)$$

Now the small x's can be expressed in terms of the capital X's and Eq. (5.144) may be written with appropriate coefficients b_i in place of a_i,

$$X_1' = b_0 + b_2 X_2 + b_3 X_3 + \ldots b_k X_k \qquad (5.149)$$

Equation (5.149) is known as a regression equation. Equation (5.148) indicates that minimizing $\sum (X_1 - X_1')^2$ is equivalent to minimizing $\sum (x_1 - x_1')^2$. Hence, determining the b_i to minimize the sum $\sum (X_1 - X_1')^2$ will effectively give the a_i determined by minimizing the sum $\sum (x_1 - x_1')^2$.

From Eq. (5.149) one obtains a function of the b_i as follows:

$$(X_1 - X_1')^2 = f(b_0, b_2, \ldots, b_k) = (X_1 - b_0 - b_2 X_2 - \ldots - b_k X_k)^2$$

$$(5.150)$$

Next, in order to minimize f with respect to the b_i, the partial derivatives of f are taken with respect to the b_i and set equal to zero, giving the following system of equations.

$$- 2\sum(X_1 - b_0 - b_2X_2 - \ldots - b_kX_k) = 0$$

$$- 2\sum X_2(X_1 - b_0 - b_2X_2 - \ldots - b_kX_k) = 0$$

$$\cdot \qquad \cdot \qquad \cdots \qquad \cdot \qquad \cdot$$

$$\cdot \qquad \cdot \qquad \cdots \qquad \cdot \qquad \cdot \qquad (5.151)$$

$$\cdot \qquad \cdot \qquad \cdots \qquad \cdot \qquad \cdot$$

$$- 2\sum X_k(X_1 - b_0 - b_2X_2 - \ldots - b_kX_k) = 0$$

Now these equations are multiplied by 1/2 and the first sum is transposed to the right which gives

$$Nb_0 + b_2\sum X_2 + \ldots + b_k\sum X_k = \sum X_1$$

$$b_0\sum X_2 + b_2\sum X_2^2 + \ldots + b_k\sum X_2X_k = \sum X_2X_1$$

$$\cdot \qquad \cdot \qquad \cdots \qquad \cdot \qquad \cdot$$

$$\cdot \qquad \cdot \qquad \cdots \qquad \cdot \qquad \cdot$$

$$\cdot \qquad \cdot \qquad \cdots \qquad \cdot \qquad \cdot$$

$$b_0\sum X_k + b_2\sum X_kX_2 + \ldots + b_k\sum X_k^2 = \sum X_kX_1$$

Since the sum of deviations from a mean is zero,

$$\sum X_i = \sum(x_i - \bar{x}_i) = 0 \qquad (5.153)$$

all terms vanish in the first equation of Eq. (5.152) except Nb_o, which implies $b_o = 0$. Thus the number of Eq. (5.152) to solve is reduced by one.

The quantity r_{ij} as defined by

$$r_{ij} = \frac{\sum (x_i - \bar{x}_i)(x_j - \bar{x}_j)}{Ns_i s_j} \qquad (5.154)$$

where the s_i and s_j are the sample standard deviations, is known as the sample correlation coefficient between the variables x_i and x_j. This quantity r_{ij} is a constant with numerical value between -1 and 1. Since

$$\sum (X_i X_j) = \sum (x_i - \bar{x}_i)(x_j - \bar{x}_j) = Nr_{ij}s_i s_j \qquad (5.155)$$

the set of linear Eq. (5.152) may be rewritten as

$$b_2 r_{22} s_2 + b_3 r_{23} s_3 + \cdots + b_k r_{2k} s_k = r_{21} s_1$$

$$b_2 r_{32} s_2 + b_3 r_{33} s_3 + \cdots + b_k r_{3k} s_k = r_{31} s_1$$

$$\cdot \qquad \cdot \qquad \cdots \qquad \cdot \qquad \cdot$$

$$\qquad (5.156)$$

$$\cdot \qquad \cdot \qquad \cdots \qquad \cdot \qquad \cdot$$

$$\cdot \qquad \cdot \qquad \cdots \qquad \cdot \qquad \cdot$$

$$b_2 r_{k2} s_2 + b_3 r_{k3} s_3 + \cdots + b_k r_{kk} s_k = r_{k1} s_1$$

The set of Eq. (5.152) may now be solved to obtain the coefficients b_i for the regression Eq. (5.149). Solving these, of course, effectively gives the coefficients a_i for Eq. (5.144) also. The general procedure discussed above will now be illustrated on the important simple special case for three variables.

5.6.2 Special Case for Three Variables

A. Regression Equation Coefficients

The desired equation to be obtained is

$$x_1 = a_0 + a_2 x_2 + a_3 x_3 \qquad (5.157)$$

or the equivalent expression

$$X_1 = b_2 X_2 + b_3 X_3 \qquad (5.158)$$

where the coefficients are obtained from the following set of equations:

$$b_2 \sum X_2^2 + b_3 \sum X_2 X_3 = \sum X_2 X_1$$

$$b_2 \sum X_3 X_2 + b_3 \sum X_3^2 = \sum X_3 X_1 \qquad (5.159)$$

To solve for the a_i, substitute $(x_i - \bar{x}_i)$ for X_i. Then

$$(x_1 - \bar{x}_1) = b_2(x_2 - \bar{x}_2) + b_3(x_3 - \bar{x}_3) \qquad (5.160)$$

$$x_1 = \bar{x}_1 - b_2 \bar{x}_2 - b_3 \bar{x}_3 + b_2 x_2 + b_3 x_3 \qquad (5.161)$$

This gives the coefficients a_i as

$$a_0 = \bar{x}_1 - b_2 \bar{x}_2 - b_3 \bar{x}_3$$

$$a_2 = b_2 \qquad (5.162)$$

$$a_3 = b_3$$

Equations (5.159) may be solved by many methods. A systematic method will be demonstrated that conveniently gives other quantities

to be used in testing for significance the regression coefficients and the multiple correlation coefficient (to be defined below). First compute the sample means $\bar{x}_i$ and standard deviations s_i. Note that two of the quantities $\sum x_2^2$ and $\sum x_3^2$ are obtained in the process of computing the standard deviations and must be saved. Next the terms $\sum x_2 x_3$, $\sum x_2 x_1$, and $\sum x_3 x_1$ may be computed from the following formula:

$$\sum x_i x_j = \sum (x_i - \bar{x}_i)(x_j - \bar{x}_j) = \sum x_i x_j - N\bar{x}_i\bar{x}_j \qquad (5.163)$$

To solve the system Eq. (5.159) two sets of auxiliary equations may be first formed, namely,

$$c_{22}\sum x_2^2 + c_{23}\sum x_2 x_3 = 1$$

$$c_{22}\sum x_2 x_3 - c_{23}\sum x_3^2 = 0 \qquad (5.164)$$

and

$$c_{23}\sum x_2^2 + c_{33}\sum x_2 x_3 = 0$$

$$c_{23}\sum x_2 x_3 + c_{33}\sum x_3^2 = 1 \qquad (5.165)$$

The solutions of Eq. (5.164) are

$$c_{23} = \frac{-\sum x_2 x_3}{\sum x_2^2 \sum x_3^2 - \left(\sum x_2 x_3\right)^2} \qquad (5.166)$$

$$c_{22} = \frac{\sum x_3^2}{\sum x_2^2 \sum x_3^2 - \left(\sum x_2 x_3\right)^2}$$

From Eq. (5.165)

$$c_{33} = \frac{\sum x_2^2}{\sum x_2^2 \sum x_3^2 - (\sum x_2 x_3)^2}$$ (5.167)

Now, compute the b's from

$$b_2 = c_{22} \sum x_2 x_1 + c_{23} \sum x_3 x_1$$

$$b_3 = c_{23} \sum x_2 x_1 + c_{33} \sum x_3 x_1$$ (5.168)

Finally, the a's for the original regression equation are obtained from Eq. (5.162).

B. Standard Deviations of the b's.

The standard deviation of the distribution of the b's can be shown to be:

$$s_{b_2} = s_{1.23} \sqrt{c_{22}}$$

$$s_{b_3} = s_{1.23} \sqrt{c_{33}}$$ (5.169)

where $s_{1.23}$ is the standard deviation for the plane of regression of x_1 on x_2 and x_3. The quantity $s_{1.23}^2$ is computed from (see Ref. [5]):

$$s_{1.23}^2 = \frac{\sum x_1^2 - b_2 \sum x_1 x_2 - b_3 \sum x_1 x_3}{N}$$ (5.170)

which has (N - 3) d.f.

The assumptions involved in deriving these standard deviations are:
(1) that a least squares procedure is used in obtaining the b's,
(2) every point (x_2, x_3) is duplicated in repeated experiments with x_1 being the only variable, (3) that the variance of x_1 is constant for any point (x_2, x_3), and (4) that the deviations of the observed values of x_1 from the predicted values of x_1 are normally distributed.

C. Multiple Correlation Coefficient

The multiple correlation coefficient $\eta_{1.23}$ is defined by its squared relation

$$(\eta_{1.23})^2 = 1 - \frac{\sigma_{1.23}^2}{\sigma_1^2} \qquad (5.171)$$

where σ_1 and $\sigma_{1.23}$ are the true population values.

The square of the multiple correlation coefficient, namely $(\eta_{1.23})^2$, can be interpreted as the percentage of the variance of x_1 that is accounted for by x_2 and x_3. The multiple correlation coefficient $\eta_{1.23}$ gives an indication of the improvement in the precision of estimating x_1 from the regression equation rather than from the mean $\bar{x}_1$. These statements are amplified in part (e) below.

The sample multiple correlation coefficient $R_{1.23}$ is defined as

$$(R_{1.23})^2 = 1 - \frac{s_{1.23}^2}{s_1^2} \qquad (5.172)$$

However, to make this an unbiased estimate of $(\eta_{1.23})^2$, the unbiased estimates of $\sigma_{1.23}^2$ and σ_1^2, which are $[N/(N-3)]s_{1.23}^2$ and $[N/(N-1)]s_1^2$ respectively, should be used. This gives the relation

$$(\hat{R}_{1.23})^2 = \frac{(N-1)}{(N-3)}(R_{1.23})^2 - \frac{2}{N-3} \qquad (5.173)$$

where $(\hat{R}_{1 \cdot 23})^2$ is the "best" estimate of $(\eta_{1 \cdot 23})^2$. This correction is unnecessary for large values of N.

D. Tests for Significance

The regression coefficients b_2, b_3 may be tested for significance quite easily. It has been shown (see Ref. [6]) that the statistic

$$ t = \frac{b_i - \beta_i}{\sqrt{\frac{N}{(N-3)}} \, s_{b_i}} \qquad i = 2, 3 \qquad (5.174) $$

where β_i is the hypothesized population regression coefficient, has a "t" distribution with $(N-3)$ d.f. The case of interest is for $\beta_i = 0$. That is, it is desirable to know whether or not the coefficient b_i for the variable x_i is significantly different from zero. Therefore, for a two-tailed test at the α level of significance, compute

$$ t = \frac{b_i}{s_{b_i}} \cdot \sqrt{\left(\frac{N-3}{N}\right)} \qquad (5.175) $$

Reject the hypothesis $b_i = 0$ if $t > t_{\alpha/2}$, where $t_{\alpha/2}$ is obtained from tables of the t-distribution with $(N-3)$ d.f. For the one-tailed test, reject if $t > t_\alpha$ for the α level of significance.

A confidence interval for b_i may also be computed. This interval with a $(1-\alpha)$ probability of containing the population co-efficient β_i is given by

$$ b_i \pm \left(\frac{N}{N-3}\right) s_{b_i} t_{\alpha/2} \qquad (5.176) $$

It can be shown (see Ref. [6]) that the statistic

$$ F = \frac{(\hat{R}_{1 \cdot 23})^2}{- (\hat{R}_{1 \cdot 23})^2} \cdot \frac{N-3}{2} \qquad (5.177) $$

has an F distribution with $n_1 = 2$ and $n_2 = N - 3$ d.f. Therefore the multiple correlation coefficient may be tested for being significantly different from zero at the α level of significance by comparing F with F_α obtained from the tables of the F-distribution with $(2, N - 3)$ d.f. The hypothesis of the population value being zero is rejected if $F > F_\alpha$.

This test of significance for the multiple correlation coefficient arises from a partition of the variance of the observed values of x_1 from its mean $\bar{x}_1$. One sum of squares has 2 d.f. (k - 1 in general) and the other $N - 3$ (N - k in general) leading to the F ratio given.

E. Interpretation of Results and Restrictions

The main assumption present in obtaining the preceding results is as follows. For each point (x_2, x_3) the variable to be predicted, x_1, is a random variable with a normal distribution about some mean which depends on the point (x_2, x_3), but the standard deviation is the same for all values of (x_2, x_3). Also it is assumed that if the sampling is repeated, the same set of (x_2, x_3) would be observed each time with again x_1 the random variable.

The interpretation of the multiple correlation coefficient was mentioned previously. That is, the square $\hat{R}^2_{1.23}$ gives the percentage of variation which is "explained" by the variables x_2, x_3. Stated another way, $1 - (\hat{R}_{1.23})^2$ is the percentage of variation due to other sources or left unexplained. From Eq. (5.172),

$$\frac{s_{1.23}}{s_1} = \sqrt{1 - (\hat{R}_{1.23})^2} \tag{5.178}$$

In this sense Eq. (5.178) gives an indication of the size of the ratio of the deviations of observed x_1 from the predicted values x_1' as compared to the deviations of x_1 from its mean $\bar{x}_1$. Since this ratio is less than unity, it yields a measure of the improvement in the precision of prediction when using a regression plane as compared to the mean value of x_1.

The regression coefficients b_2 and b_3 represent the effect on x_1 of a unit increase in x_2 when x_3 is held constant, and a unit increase in x_3 when x_2 is held constant, respectively. The coefficient a_0 merely accounts for the means not being zero. The b's being significantly different from zero indicates whether or not the predictor variables involved have significant effects on the variable to be predicted.

Large sample sizes are normally quite necessary to obtain significant results in multiple regression problems. That is, one prefers sample sizes of $N = 100$ or preferably even larger. Also, the full range of interest of the variables must be sampled. It is very seldom that one can extrapolate his results beyond the range of values which have been obtained. Figure 5.13 below gives an indication for the two variable case of what could happen. The range that was observed can be fitted very well by a linear equation, but clearly this approximately linear portion could be part of a quadratic relation.
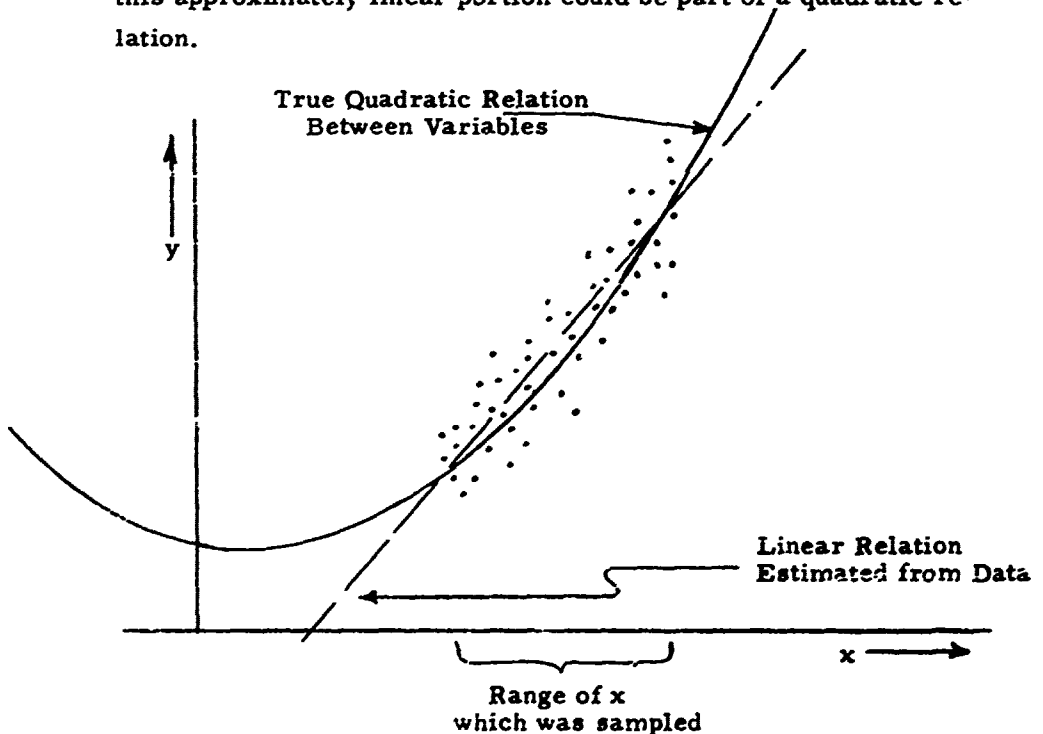
Figure 5.13 Hypothetical True Quadratic Relation and Linear Estimate Obtained from Restricted Data

5.6.3 Computational Example

The following table of hypothetical data for 3 variables will serve to illustrate application of the preceding equations.

$$k = 3 \qquad N = 25$$

x_1	x_2	x_3	$x_1 x_2$	$x_1 x_3$	$x_2 x_3$	x_1^2	x_2^2	x_3^2
1	1	1	1	1	1	1	1	1
1	1	2	1	2	2	1	1	4
2	2	1	4	2	2	4	4	1
1	2	2	2	2	4	1	4	4
3	3	2	9	6	6	9	9	4
2	3	3	6	6	9	4	9	9
3	4	3	12	9	12	9	16	9
2	1	1	2	2	1	4	1	1
0	1	1	0	0	1	0	1	1
1	0	0	0	0	0	1	0	0
0	1	0	0	0	0	0	1	0
1	1	1	1	1	1	1	1	1
4	3	3	12	12	9	16	9	9
5	3	4	15	20	12	25	9	16
4	4	3	16	12	12	16	16	9
5	6	5	30	25	30	25	36	25
4	6	5	24	20	30	16	36	25
4	3	4	12	16	12	16	9	16
5	4	4	20	20	16	25	16	16
4	5	5	20	20	25	16	25	25
6	5	5	30	30	25	36	25	25
5	5	5	25	25	25	25	25	25
6	4	5	24	30	20	36	16	25
2	2	2	4	4	4	4	4	4
2	3	2	6	4	6	4	9	4
$\sum$ 73	73	69	286	269	265	295	283	259

$$\bar{x}_1 = \frac{\sum x_1}{N} = 2.9 \qquad (\bar{x}_1)^2 - 8.41 \qquad \bar{x}_1 \bar{x}_2 = 8.41$$

$$\bar{x}_2 = \frac{\sum x_2}{N} = 2.9 \qquad (\bar{x}_2)^2 = 8.41 \qquad \bar{x}_1 \bar{x}_3 = 8.12$$

$$\bar{x}_3 = \frac{\sum x_3}{N} = 2.8 \qquad (\bar{x}_3)^2 = 7.84 \qquad \bar{x}_2 \bar{x}_3 = 8.12$$

$$\sum x_1^2 = \sum x_1^2 - N(\bar{x}_1)^2 = 295 - 2.0.25 = 84.75$$

$$\sum x_2^2 = \sum x_2^2 - N(\bar{x}_2)^2 = 283 - 210.25 = 72.75$$

$$\sum x_3^2 = \sum x_3^2 - N(\bar{x}_3)^2 = 259 - 196.00 = 63.00$$

$$\sum x_1 x_2 = \sum x_1 x_2 - N \bar{x}_1 \bar{x}_2 = 286 - 210.25 = 75.75$$

$$\sum x_1 x_3 = \sum x_1 x_3 - N \bar{x}_1 \bar{x}_3 = 269 - 203.00 = 66.00$$

$$\sum x_2 x_3 = \sum x_2 x_3 - N \bar{x}_2 \bar{x}_3 = 265 - 203.00 = 62.00$$

From the preceding data, the following computations are made: from Eq. (5.159),

$$b_2 \, 72.75 + b_3 \, 62.00 = 75.75$$

$$b_2 \, 62.00 + b_3 \, 63.00 = 66.00$$

Solution of auxiliary Eq. (5.164) and (5.165) by Eq. (5.166) and (5.167):

Let $\Delta = \sum x_2^2 \sum x_3^2 - \left(\sum x_2 x_3\right)^2 = (72.75)(63.00) - (62.00)^2 = 739.25$

$$c_{23} = \frac{-62.00}{\Delta} = \frac{-62.00}{739.25} = -.0839$$

$$c_{22} = \frac{63.00}{\Delta} = \frac{63.00}{739.25} = .0852$$

$$c_{33} = \frac{72.75}{\Delta} = \frac{72.75}{739.25} = .0984$$

Then b_2 and b_3 are obtained from Eq. (5.168)

$$b_2 = (75.75)(.0852) + (66.00)(-.0839) = .9165$$

$$b_3 = (75.75)(-.0839) = (66.00)(.984) = .1390$$

and from Eq. (5.162)

$$a_0 = 2.9 - (2.9)(.9165) - (2.8)(.1390) = -.1471$$

Next, using Eq. (5.170):

$$s_{1.23}^2 = \frac{84.75 - (.9165)(75.75) - (.1390)(66.00)}{25} = .2460$$

$$s_{1.23} = .4960$$

and with Eq. (5.169)

$$s_{b_2} = .4960 \sqrt{.0852} = .1437$$

$$s_{b_3} = .4960 \sqrt{.0984} = .1545$$

Now $c_1^2 = (84.75/25) = 3.39$ and using Eq. (5.172):

$$(R_{1.23})^2 = 1 - \frac{.2460}{3.39} = .93$$

and with Eq. (5.173)

$$(\hat{R}_{1.23})^2 = \frac{24}{22}(.93) - \frac{2}{22} = 1.01 - .09 = .92$$

$$\hat{R}_{1.23} = .96$$

In practice, it would be quite unusual to obtain a correlation coefficient this large.

To test the b's for significance compute "t" from Eq. (5.175):

$$\text{for } b_2, \quad t = \left(\frac{.9165}{.1437}\right) \cdot \left(\frac{22}{25}\right) = 5.56$$

$$\text{for } b_3, \quad t = \left(\frac{.1390}{.1545}\right) \cdot \left(\frac{22}{25}\right) = 0.82$$

From the tables of the t-distribution with 22 d.f. one finds $t_{2.5} = 2.07$. Therefore, working at the 5% level of significance b_2 is only significant while b_3 is not. This would indicate that the variable x_3 has no significant effect on x_1.

To test $\hat{R}_{1.23}$, compute F from Eq. (5.177):

$$F = \left(\frac{.92}{.08}\right) \cdot \left(\frac{22}{2}\right) = 126.5$$

In the tables of the F-distribution one finds $F_{1.0}(2, 22) = 99.46$ so even at the 1% level of significance the multiple correlation coefficient is significant.

5.6.4 General Case for k Variables

A. Regression Coefficients

The auxiliary Eq. (5.164) and (5.165) given for solution in the three variable case actually are equivalent to finding the inverse of the matrix of the given sums of cross products. That is, letting x_{ij} represent the sum $\sum X_i X_j$ where $i, j = 2, 3, \ldots, k$, the c_{ij}'s represent a matrix such that

$$\begin{vmatrix} x_{22} & x_{23} & \cdots & x_{2k} \\ x_{32} & x_{33} & \cdots & x_{3k} \\ x_{42} & x_{43} & \cdots & x_{4k} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ x_{k2} & x_{k3} & \cdots & x_{kk} \end{vmatrix} \cdot \begin{vmatrix} c_{22} & c_{23} & \cdots & c_{2k} \\ c_{32} & c_{33} & \cdots & c_{3k} \\ c_{42} & c_{43} & \cdots & c_{4k} \\ \cdot & & \cdots & \cdot \\ \cdot & & \cdots & \cdot \\ \cdot & & \cdots & \cdot \\ c_{k2} & c_{k3} & \cdots & c_{kk} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \cdot & & \cdot & \cdots & \cdot \\ \cdot & & \cdot & \cdots & \cdot \\ \cdot & & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{vmatrix} \quad (5.179)$$

After the multiplication is performed, columns can be equated to columns, and $k - 1$ sets of $k - 1$ equations, each set with $k - 1$ unknowns, are obtained.

$$x_{22} c_{22} + x_{23} c_{23} + \cdots + x_{2k} c_{2k} = 1$$

$$x_{32} c_{22} + x_{33} c_{32} + \cdots + x_{3k} c_{k2} = 0$$

$$\begin{matrix} \cdot & & \cdot & \cdots & \cdot & & \cdot \\ \cdot & & \cdot & \cdots & \cdot & & \cdot \\ \cdot & & \cdot & \cdots & \cdot & & \cdot \end{matrix} \qquad (5.180)$$

$$x_{k2} c_{22} + x_{k2} c_{32} + \cdots + x_{kk} c_{k2} = 0$$

etc. for the other $k - 2$ columns of the matrix.

Of course, the amount of computation involved as the number of variables increases becomes quite extensive and a digital computer becomes a very desirable tool. Another computational method will be given without justification in Section 5.6.6 which is suitable for programming on a digital computer.

To continue from Eq. (5.180) the b_i's are computed by performing another matrix multiplication. That is,

$$
\begin{Vmatrix} b_2 \\ b_3 \\ \cdot \\ \cdot \\ \cdot \\ b_k \end{Vmatrix} = \begin{Vmatrix} c_{22} & c_{23} & \cdots & c_{2k} \\ c_{32} & c_{33} & \cdots & c_{3k} \\ \cdot & & \cdots & \cdot \\ \cdot & & \cdots & \cdot \\ \cdot & & \cdots & \cdot \\ c_{k2} & c_{k3} & \cdots & c_{kk} \end{Vmatrix} \cdot \begin{Vmatrix} x_{21} \\ x_{31} \\ \cdot \\ \cdot \\ \cdot \\ x_{k1} \end{Vmatrix} \tag{5.181}
$$

This gives the generalization of Eq. (5.168):

$$
b_2 = c_{22} x_{21} + c_{23} x_{31} + \cdots + c_{2k} x_{k1}
$$
$$
b_3 = c_{32} x_{21} + c_{33} x_{31} + \cdots + c_{3k} x_{k1}
$$
$$
\cdot \quad \cdot \quad \cdot \quad \cdots \quad \cdot
$$
$$
\cdot \quad \cdot \quad \cdot \quad \cdots \quad \cdot
$$
$$
\cdot \quad \cdot \quad \cdot \quad \cdots \quad \cdot
$$
$$
b_k = c_{k2} x_{21} + c_{k3} x_{31} + \cdots + c_{kk} x_{k1}
$$

Let B be the $\left[(k - 1) \times 1\right]$ regression coefficient vector, X the $\left[(k - 1) \times (k - 1)\right]$ matrix of sums of cross products for the predictor variables, and Y the $\left[(k - 1) \times 1\right]$ vector of the cross product sums involving X_1, the predicted variable. Then the preceding can be concisely expressed as the matrix equation

$$
XB = Y \tag{5.183}
$$

with the solution

$$B = X^{-1} Y \tag{5.1}$$

where $C = X^{-1}$ as defined by Eq. (5.179)

B. Standard deviations of the b's.

Standard deviation of estimate $s_{1.23\ldots k}$ for the "plane" regression of x_2, x_3, ... x_k on x_1 is the square root of

$$s^2_{1.23\ldots k} = \frac{\sum x_1^2 - b_2 \sum x_1 x_2 - b_3 \sum x_1 x_3 - \ldots - b_k \sum}{N}$$

$$\tag{5.1}$$

which has $(N - k)$ d. f.   The standard deviations of the b's

$$s_{b_i} = s_{1.23\ldots k} \sqrt{c_{ii}} \qquad , \qquad i = 2, 3, \ldots, k \tag{5.1}$$

C. Multiple Correlation Coefficient

The sample multiple correlation coefficient $R_{1.23\ldots k}$ is given by the square root of

$$(R_{1.23\ldots k})^2 = 1 - \frac{s^2_{1.23\ldots k}}{s^2_1} \tag{5.1}$$

The unbiased estimate is then

$$(\hat{R}_{1.23\ldots k})^2 = \frac{(N-1)}{(N-k)} (R_{1.23\ldots k})^2 - \frac{k-1}{N-k} \tag{5.1}$$

D. Tests for Significance

The "t" test for the regression coefficient becomes

with the solution

$$B = X^{-1} Y$$
(5.184)

where $C = X^{-1}$ as defined by Eq. (5.179)

B. Standard deviations of the b's.

Standard deviation of estimate $s_{1.23\ldots k}$ for the "plane" of regression of $x_2, x_3, \ldots, x_k$ on x_1 is the square root of

$$s_{1.23\ldots k}^2 = \frac{\sum x_1^2 - b_2 \sum x_1 x_2 - b_3 \sum x_1 x_3 - \ldots - b_k \sum x_1 x_k}{N}$$
(5.185)

which has $(N - k)$ d.f. The standard deviations of the b's

$$s_{b_i} = s_{1.23\ldots k} \sqrt{c_{ii}} \qquad , \qquad i = 2, 3, \ldots, k \qquad (5.186)$$

C. Multiple Correlation Coefficient

The sample multiple correlation coefficient $R_{1.23\ldots k}$ is given by the square root of

$$(R_{1.23\ldots k})^2 = 1 - \frac{s_{1.23\ldots k}^2}{s_1^2}$$
(5.187)

The unbiased estimate is then

$$(\hat{R}_{1.23\ldots k})^2 = \frac{(N-1)}{(N-k)} (R_{1.23\ldots k})^2 - \frac{k-1}{N-k}$$
(5.188)

D. Tests for Significance

The "t" test for the regression coefficient becomes

$$t = \frac{b_i - \beta_i}{\dfrac{N}{(N - k)} \, s_{b_i}} \qquad (5.189)$$

where t has a "t" distribution with $(N - k)$ d.f.

For the multiple regression coefficient;

$$F = \frac{(\hat{R}_{1.23 \ldots k})^2}{1 - (\hat{R}_{1.23 \ldots k})^2} \cdot \frac{(N - k)}{(k - 1)} \qquad (5.190)$$

where F has an "F" distribution with $(k - 1, N - k)$ d.f.

See Ref. [5] for the above generalizations to k variables.

5.6.5 Nonlinear Regression

Powers or cross products of the predictor variables may be attached to the regression equations as additional variables. For instance, in the three variable case for linear regression one must solve Eq. (5.159). If it is suspected that an interaction exists, a term in the form of $x_2 x_3$ could be attached as a fourth variable x_4. Then instead of two equations in two unknowns, the dimension of the problem has been raised and there would now exist three equations in three unknowns. Powers such as x_2^2 and x_3^2 could be attached as variables x_5 and x_6 if so desired.

If an exponential relation is proposed such as

$$x_1 = a_1 x_2^{b_2} x_3^{b_3} \qquad (5.191)$$

then logarithms of Eq. (5.191) may be taken to linearize the relation; now,

$$\log x_1 = \log a_1 + b_2 \log x_2 + b_3 \log x_3 \qquad (5.192)$$

and this system is clearly analogous to Eq. (5.157). The least squares fit obtained to Eq. (5.192) is not precisely the same as that obtained by fitting directly to Eq. (5.191). However, the error is generally small enough so that the linearization by logarithms is useful in practice.

5.6.6 Alternative Computing Scheme

Form the following matrix of the correlation coefficients r_{ij} where r_{ij} is defined by Eq. (5.155).

$$
\begin{Vmatrix}
r_{11} & r_{12} & r_{13} & \cdots & r_{1k} \\
r_{21} & r_{22} & r_{23} & \cdots & r_{2k} \\
r_{31} & r_{32} & r_{33} & \cdots & r_{3k} \\
\cdot & \cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot \\
r_{k1} & r_{k2} & r_{k3} & \cdots & r_{kk}
\end{Vmatrix} = R \qquad\qquad (5.193)
$$

Compute the inverse

$$ A = R^{-1} $$

If a_{ij} are the elements of the matrix A, the following may be computed:

(1) The regression coefficients

$$ b_j = -\left(\frac{a_{1j}}{a_{11}}\right)\left(\frac{s_1}{s_j}\right) \qquad , \qquad j = 2, 3, \ldots, k \qquad (5.195) $$

(2) The multiple correlation coefficient

$$ R_{1.23\ldots k} = \sqrt{1 - (1/a_{11})} \qquad\qquad (5.196) $$

(3) The standard deviation of the errors of estimate

$$s_{1.23\ldots k} = \frac{s_1}{\sqrt{a_{11}}} \qquad\qquad (5.197)$$

(4) The standard deviation of the regression coefficients

$$s_{c_j} = \frac{1}{a_{11}} \sqrt{\frac{a_{11}^2 \ _j + a_{1j}^2}{n}} \quad , \quad j = 2, 3, \ldots, k \quad (5.198)$$

Table 5.1 Areas of the Normal Curve, $\text{Area} = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{0}^{z} e^{-t^2/2}\,dt$

(in terms of σ units)

z	Area	z	Area	z	Area
.00	.0000	.45	.1736	.90	.3159
.01	.0040	.46	.1772	.91	.3186
.02	.0080	.47	.1808	.92	.3212
.03	.0120	.48	.1844	.93	.3238
.04	.0160	.49	.1879	.94	.3264
.05	.0199	.50	.1915	.95	.3289
.06	.0239	.51	.1950	.96	.3315
.07	.0279	.52	.1985	.97	.3340
.08	.0319	.53	.2019	.98	.3365
.09	.0359	.54	.2054	.99	.3389
.10	.0398	.55	.2088	1.00	.3413
.11	.0438	.56	.2123	1.01	.3438
.12	.0478	.57	.2157	1.02	.3461
.13	.0517	.58	.2190	1.03	.3485
.14	.0557	.59	.2224	1.04	.3508
.15	.0596	.60	.2257	1.05	.3531
.16	.0636	.61	.2291	1.06	.3554
.17	.0675	.62	.2324	1.07	.3577
.18	.0714	.63	.2357	1.08	.3599
.19	.0753	.64	.2389	1.09	.3621
.20	.0793	.65	.2422	1.10	.3643
.21	.0832	.66	.2454	1.11	.3665
.22	.0871	.67	.2486	1.12	.3686
.23	.0910	.68	.2517	1.13	.3708
.24	.0948	.69	.2549	1.14	.3729
.25	.0987	.70	.2580	1.15	.3749
.26	.1026	.71	.2611	1.16	.3770
.27	.1064	.72	.2642	1.17	.3790
.28	.1103	.73	.2673	1.18	.3810
.29	.1141	.74	.2703	1.19	.3830
.30	.1179	.75	.2734	1.20	.3849
.31	.1217	.76	.2764	1.21	.3869
.32	.1255	.77	.2794	1.22	.3888
.33	.1293	.78	.2823	1.23	.3907
.34	.1331	.79	.2852	1.24	.3925
.35	.1368	.80	.2881	1.25	.3944
.36	.1406	.81	.2910	1.26	.3962
.37	.1443	.82	.2939	1.27	.3980
.38	.1480	.83	.2967	1.28	.3997
.39	.1517	.84	.2995	1.29	.4015
.40	.1554	.85	.3023	1.30	.4032
.41	.1591	.86	.3051	1.31	.4049
.42	.1628	.87	.3078	1.32	.4066
.43	.1664	.88	.3106	1.33	.4082
.44	.1700	.89	.3133	1.34	.4099

Table 5.1 (cont'd)

z	Area	z	Area	z	Area
1.35	.4115	1.80	.4641	2.25	.4878
1.36	.4131	1.81	.4649	2.26	.4881
1.37	.4147	1.82	.4656	2.27	.4884
1.38	.4162	1.83	.4664	2.28	.4887
1.39	.4177	1.84	.4671	2.29	.4890
1.40	.4192	1.85	.4678	2.30	.4893
1.41	.4207	1.86	.4686	2.31	.4896
1.42	.4222	1.87	.4693	2.32	.4898
1.43	.4236	1.88	.4699	2.33	.4901
1.44	.4251	1.89	.4706	2.34	.4904
1.45	.4265	1.90	.4713	2.35	.4906
1.46	.4279	1.91	.4719	2.36	.4909
1.47	.4292	1.92	.4726	2.37	.4911
1.48	.4306	1.93	.4732	2.38	.4913
1.49	.4319	1.94	.4738	2.39	.4916
1.50	.4332	1.95	.4744	2.40	.4918
1.51	.4345	1.96	.4750	2.41	.4920
1.52	.4357	1.97	.4756	2.42	.4922
1.53	.4370	1.98	.4761	2.43	.4925
1.54	.4382	1.99	.4767	2.44	.4927
1.55	.4394	2.00	.4772	2.45	.4929
1.56	.4406	2.01	.4778	2.46	.4931
1.57	.4418	2.02	.4783	2.47	.4932
1.58	.4429	2.03	.4788	2.48	.4934
1.59	.4441	2.04	.4793	2.49	.4936
1.60	.4452	2.05	.4798	2.50	.4938
1.61	.4463	2.06	.4803	2.51	.4940
1.62	.4474	2.07	.4808	2.52	.4941
1.63	.4484	2.08	.4812	2.53	.4943
1.64	.4495	2.09	.4817	2.54	.4945
1.65	.4505	2.10	.4821	2.55	.4946
1.66	.4515	2.11	.4826	2.56	.4948
1.67	.4525	2.12	.4830	2.57	.4949
1.68	.4535	2.13	.4834	2.58	.4951
1.69	.4545	2.14	.4838	2.59	.4952
1.70	.4554	2.15	.4842	2.60	.4953
1.71	.4564	2.16	.4846	2.61	.4955
1.72	.4573	2.17	.4850	2.62	.4956
1.73	.4582	2.18	.4854	2.63	.4957
1.74	.4591	2.19	.4857	2.64	.4959
1.75	.4599	2.20	.4861	2.65	.4960
1.76	.4608	2.21	.4864	2.66	.4961
1.77	.4616	2.22	.4868	2.67	.4962
1.78	.4625	2.23	.4871	2.68	.4963
1.79	.4633	2.24	.4875	2.69	.4964

Table 5. 1 (concluded)

z	Area		z	Area		z	Area
2.70	.4965		2.80	.4974		2.90	.4981
2.71	.4966		2.81	.4975		2.91	.4982
2.72	.4967		2.82	.4976		2.92	.4982
2.73	.4968		2.83	.4977		2.93	.4983
2.74	.4969		2.84	.4977		2.94	.4984
2.75	.4970		2.85	.4978		2.95	.4984
2.76	.4971		2.86	.4979		2.96	.4985
2.77	.4972		2.87	.4979		2.97	.4985
2.78	.4973		2.88	.4980		2.98	.4986
2.79	.4974		2.89	.4981		2.99	.4986
						3.00	.4987

For a more comprehensive table, see Reference [9], p. 456.

Table 5.2 p-percent Values of Normal Distribution

The probability that x differs from its mean by more than λ_p times the s.d. is equal to $(p/2)\%$. Alternatively, $\mu \pm \lambda_p \sigma$ contains $(100-p)\%$ of the area under the normal probability density function, see Reference [2], p. 274.

P	λ_p
100	0.0000
95	0.0627
90	0.1257
85	0.1891
80	0.2533
75	0.3186
70	0.3853
65	0.4538
60	0.5244
55	0.5978
50	0.6745
45	0.7544
40	0.8416
35	0.9346
30	1.0364
25	1.1503
20	1.2816
15	1.4395
10	1.6449
5	1.9600
1	2.5758
0.1	3.2905
0.01	3.8906

Table 5.3 The χ^2 Distribution

χ_p^2 as a Function of n and p

Degrees of Freedom n	p=95	80	50	30	20	10	5	2	1
1	0.004	0.064	0.455	1.074	1.642	2.706	3.841	5.412	6.635
3	0.352	1.005	2.366	3.665	4.642	6.251	7.815	9.837	11.341
5	1.145	2.343	4.351	6.064	7.289	9.236	11.070	13.388	15.086
10	3.940	6.179	9.342	11.781	13.442	15.987	18.307	21.161	23.209
15	7.261	10.307	14.339	17.322	19.311	22.307	24.996	28.259	30.578
20	10.851	14.578	19.337	22.775	25.038	28.412	31.410	35.020	37.566
25	14.611	18.940	24.337	28.172	30.675	34.382	37.652	41.566	44.314
30	18.493	23.364	29.336	33.530	36.250	40.256	43.773	47.962	50.892

The probability that χ^2 assumes a value larger than χ_p^2 is equal to p%.

For n>30, $\qquad \chi_p^2 = \frac{1}{2} \left| \sqrt{2n-1} + \lambda_{2p} \right|^2 \qquad$ where λ_{2p} is found in Table 5.2.

For a more comprehensive table, see Reference [9], p. 464.

Table 5.4 p-percent Values of "Student's t" Distribution

n	$t_{10.0}$	$t_{5.0}$	$t_{2.5}$	$t_{1.0}$	$t_{0.5}$
1	3.08	6.31	12.71	31.82	63.66
2	1.89	2.92	4.30	6.96	9.92
3	1.64	2.35	3.18	4.54	5.84
4	1.53	2.13	2.78	3.75	4.60
5	1.48	2.02	2.57	3.36	4.03
6	1.44	1.94	2.45	3.14	3.71
7	1.42	1.89	2.36	3.00	3.50
8	1.40	1.86	2.31	2.90	3.36
9	1.38	1.83	2.26	2.82	3.25
10	1.37	1.81	2.23	2.76	3.17
12	1.36	1.78	2.18	2.68	3.05
14	1.34	1.76	2.14	2.62	2.98
16	1.34	1.75	2.12	2.58	2.92
18	1.33	1.73	2.10	2.55	2.88
20	1.32	1.72	2.09	2.53	2.85
22	1.32	1.72	2.07	2.51	2.82
24	1.32	1.71	2.06	2.49	2.80
26	1.32	1.71	2.06	2.48	2.78
28	1.31	1.70	2.05	2.47	2.76
30	1.30	1.70	2.04	2.46	2.75
∞	1.28	1.65	1.96	2.32	2.58

For a more comprehensive table, see Reference [9] p. 465.

Table 5.5   F Distribution

n = d. f. for numerator   m = d. f. for denominator

(a)

$F_{5.0}$

n\m	5	10	20	30	∞
5	5.05	4.74	4.56	4.50	4.37
10	3.33	2.98	2.77	2.70	2.54
20	2.71	2.35	2.12	2.04	1.84
30	2.53	2.16	1.93	1.84	1.62
∞	2.21	1.83	1.57	1.46	1.00

(b)

$F_{2.5}$

n\m	5	10	20	30	∞
5	7.15	6.62	6.33	6.23	6.02
10	4.24	3.72	3.42	3.31	3.08
20	3.29	2.77	2.46	2.35	2.09
30	3.03	2.51	2.20	2.07	1.79
∞	2.57	2.05	1.71	1.57	1.00

(c)

$F_{1.0}$

n\m	5	10	20	30	∞
5	11.0	10.1	9.55	9.38	9.02
10	5.64	4.85	4.41	4.25	3.91
20	4.10	3.37	2.94	2.78	2.42
30	3.70	2.98	2.55	2.39	2.01
∞	3.02	2.32	1.88	1.70	1.00

(d)

$F_{0.5}$

n\m	5	10	20	30	∞
5	14.9	13.6	12.9	12.7	12.1
10	6.87	5.85	5.27	5.07	4.64
20	4.76	3.85	3.32	3.12	2.69
30	4.43	3.54	3.01	2.82	2.38
∞	3.35	2.52	2.00	1.79	1.00

$$\text{Prob}(F > F_p) = p$$

For a more comprehensive table, see Reference [5], p. 878.

Table 5.6 Tolerance Factors

Factors K such that the probability is .95 that at least a proportion P will be included between $\bar{x} \pm Ks$, where $\bar{x}$ and s are computed from a sample of size N.

P / N	.90	.95	.99	P / N	.90	.95	.99	P / N	.90	.95	.99
2	32.02	37.67	48.43	25	2.208	2.531	3.457	100	1.874	2.233	2.934
3	8.380	9.916	12.86	30	2.140	2.549	3.350	110	1.861	2.218	2.915
4	5.369	6.370	8.299	35	2.090	2.490	3.272	120	1.850	2.205	2.898
5	4.275	5.079	6.634	40	2.052	2.445	3.213	130	1.841	2.194	2.883
6	3.712	4.414	5.775	45	2.021	2.408	3.165	140	1.833	2.184	2.870
7	3.369	4.007	5.248	50	1.996	2.379	3.126	150	1.825	2.175	2.859
8	3.136	3.732	4.891	55	1.976	2.354	3.094	160	1.819	2.167	2.848
9	2.967	3.532	4.631	60	1.958	2.333	3.066	170	1.813	2.160	2.839
10	2.839	3.379	4.433	65	1.943	2.315	3.042	180	1.808	2.154	2.831
12	2.655	3.162	4.150	70	1.929	2.299	3.021	190	1.803	2.148	2.823
14	2.529	3.012	3.955	75	1.917	2.285	3.002	200	1.798	2.143	2.816
16	2.437	2.903	3.812	80	1.907	2.272	2.986	500	1.737	2.070	2.721
18	2.366	2.819	3.702	85	1.897	2.261	2.971	1000	1.709	2.036	2.676
20	2.310	2.752	3.615	90	1.889	2.251	2.958	∞	1.645	1.960	2.576

For a more comprehensive table, see Reference [6], p. 102.

Table 5.7 Values for Variance Equality Test

95th and 99th Percentile Values of $F_{max} = \left(s^2_{max} / s^2_{min}\right)$ in a Set of k Mean Squares each based on n Degrees of Freedom.   Upper Value is 95th and Lower the 99th Percentile in each cell.

n \ k	2	3	4	5	6	7	8	9	10
2	39.0 / 199.	67.5 / 448.	142. / 729.	202. / 1036.	266. / 1362.	333. / 1705.	403. / 2063.	475. / 2432.	550. / 2813.
3	15.4 / 47.5	27.8 / 85.	39.2 / 120.	50.7 / 151.	62.0 / 184.	72.9 / 216.	83.5 / 249.	93.9 / 281.	104. / 310.
4	9.60 / 23.2	15.5 / 37.	20.6 / 49.	25.2 / 59.	29.5 / 69.	33.6 / 79.	37.5 / 89.	41.1 / 97.	44.6 / 106.
5	7.15 / 14.9	10.8 / 22.	13.7 / 28.	16.3 / 33.	18.7 / 38.	20.8 / 42.	22.9 / 46.	24.7 / 50.	26.5 / 54.
8	4.43 / 7.50	6.00 / 9.9	7.18 / 11.7	8.12 / 13.2	9.03 / 14.5	9.78 / 15.8	10.5 / 16.9	11.1 / 17.9	11.7 / 18.9
10	3.72 / 5.85	4.85 / 7.4	5.67 / 8.6	6.34 / 9.6	6.92 / 10.4	7.42 / 11.1	7.87 / 11.8	8.28 / 12.4	8.66 / 12.9
15	2.86 / 4.07	3.54 / 4.9	4.01 / 5.5	4.37 / 6.0	4.68 / 6.4	4.95 / 6.7	5.19 / 7.1	5.40 / 7.3	5.59 / 7.5
20	2.46 / 3.32	2.95 / 3.8	3.29 / 4.3	3.54 / 4.6	3.76 / 4.9	3.94 / 5.1	4.10 / 5.3	4.24 / 5.5	4.37 / 5.6
30	2.07 / 2.63	2.40 / 3.0	2.61 / 3.3	2.78 / 3.4	2.91 / 3.6	3.02 / 3.7	3.12 / 3.8	3.21 / 3.9	3.29 / 4.0
60	1.67 / 1.96	1.85 / 2.2	1.96 / 2.3	2.04 / 2.4	2.11 / 2.4	2.17 / 2.5	2.22 / 2.5	2.26 / 2.6	2.30 / 2.6

For a more comprehensive table, see Reference [9], p 462.

Table 5.8 Factors for Converting Sample Range to Sample Standard Deviation

Values of d_2. The sample standard deviation $s = \bar{R}/d_2$ where $\bar{R}$ is based on the average of k sample ranges computed from samples of size N.

Number of Samples k	Sample Size N						
	2	3	4	5	6	7	8
2	1.28	1.81	2.15	2.40	2.60	2.77	2.91
3	1.23	1.77	2.12	2.38	2.58	2.75	2.89
4	1.21	1.75	2.11	2.37	2.57	2.74	2.88
5	1.19	1.74	2.10	2.36	2.56	2.73	2.87
8	1.17	1.72	2.08	2.35	2.55	2.72	2.87
10	1.16	1.72	2.08	2.34	2.55	2.72	2.86
14	1.15	1.71	2.07	2.34	2.54	2.71	2.85
∞	1.13	1.69	2.06	2.33	2.53	2.70	2.85

For a more comprehensive table, see Reference [5]. p. 374

Table 5.9 Factors for Computing Sample Size
and Number of Defects

Factors for computing N and c for a sampling plan with $\alpha = .05$ and $\beta = .10$[*].

c	P_α N	P_β N	P_β/P_α
0	0.051	2.30	45.10
1	0.355	3.89	10.96
2	0.818	5.32	6.50
3	1.366	6.68	4.89
4	1.970	7.99	4.06
5	2.613	9.28	3.55
6	3.285	10.53	3.21
7	3.981	11.77	2.96
8	4.695	12.99	2.77
9	5.425	14.21	2.62
10	6.169	15.41	2.50

[*] For a more comprehensive table, see Reference [1]. p. 832.

5.7 REFERENCES

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6. ANALYTICAL PROCEDURES FOR DETERMINING VIBRATION ENVIRONMENT

The procedures for determining the vibration environment at a single point on the structure of a modern flight vehicle may be divided logically into two parts:

Part 1. the procedure for analyzing the pertinent statistical properties of a single vibration time history record.

Part 2. the procedure for establishing the over-all vibration environment given the statistical properties of each of a collection of vibration time history records.

Section 6.1 will describe an over-all recommended procedure for Part 1 while Section 6.2 will discuss a recommended procedure for Part 2.

6.1 PROCEDURE FOR ANALYZING INDIVIDUAL VIBRATION RECORDS

The over-all recommended procedure for analyzing the pertinent statistical properties of a single vibration time history record is summarized in Fig. 6.1. Each block in Fig. 6.1 will be discussed in this section. The discussion will be in terms of analog techniques and instrumentation, but any part of the recommended procedure, or all of it, could be accomplished by digital techniques.

Actual procedures to accomplish the functions shown in Fig. 6.1 have in many cases been covered in other sections of the report. The discussions here will then often consist of simply defining the need of the function in the over-all procedure while referring the reader to another appropriate section of the report for details.

6.1.1 Vibration Transducer

The first step in the procedure obviously is to obtain an analog signal of the physical parameter of interest as noted by block A in Fig. 6.1. The physical parameter of interest will probably be some measure of vibration response at a point on the structure of the vehicle, such as acceleration, velocity, displacement, or stress. The transducer would then be an accelerometer, velocity signal generator, displacement signal generator (if such
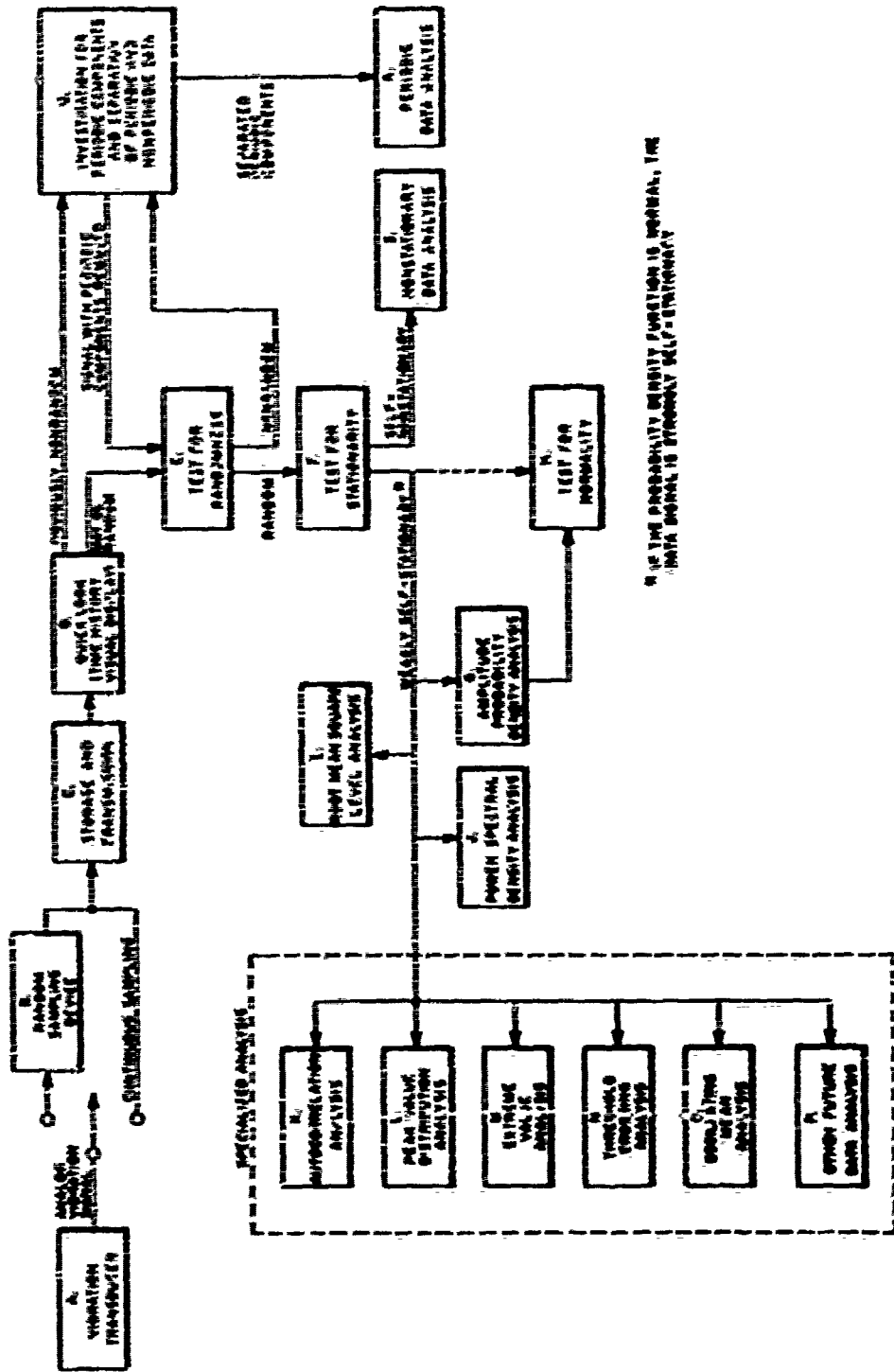
Figure 6.1. Over-all Recommended Procedure for Analyzing Individual Vibration Records

a transducer exists), or strain gage. However, the procedure is not limited to parameters of structural vibration response. The transducer could just as well be, for example, a microphone which generates a voltage signal proportional to acoustical sound pressure at some point inside the vehicle.

The physical characteristics of conventional transducers applicable to the flight vehicle vibration problem are detailed in Section 7.1. Additional discussions concerning the selection of transducers are presented in Section 3.4.

6.1.2 Vibration Data Sampling

Samples of the analog signal from the transducer must be obtained by some sampling procedure as noted by block B in Fig. 6.1. The length of each time history sample obtained is of direct importance to the statistical accuracy of the analyses to follow. The desired sample length T will be discussed for each type of analysis.

On the other hand, the number of samples obtained and the manner in which they are selected is of direct importance to establishing the overall vibration environment at a single point on the structure of the vehicle. In general, it will be recommended that samples be obtained by either one long continuous record, or by a set of short records of predetermined length each selected randomly during a flight. This entire subject, including a description of a suitable random sampling device, will be detailed in Section 6.2.

6.1.3 Storage and Transmission

After the vibration time history samples are obtained, the samples must either be analyzed directly or recorded and stored on magnetic tape (or some other suitable means of storage) for later analysis. As will be seen, the over-all analytical procedure and analysis techniques recommended here will require a repeated investigation of each time history sample. Although direct analysis of vibration time history signals was sometimes done a few years ago, this approach is no longer feasible. The recording and storage of the time history signals will definitely be required. Magnetic tape recording is recommended since it offers the most

accurate and convenient means available at the present time for storing analog signals with wide amplitude and frequency ranges. A discussion of magnetic tape recording is presented in Section 7.2.2.

The vibration time history samples might be recorded on magnetic tape inside the vehicle or transferred to the ground by telemetry and recorded there. The recovery of an in-flight magnetic tape recording is often not possible, particularly in the case of pilotless missiles. Even in smaller piloted airplanes, an airborne tape recorder may not be permitted because of size and weight. However, it is recommended that sample records be obtained directly in flight by an airborne recorder when feasible. A telemetry system is an additional source of calibration error as well as a source of test failure due to malfunctions, and should be employed only when absolutely necessary. A discussion of telemetry is presented in Section 7.2.6.

5.1.4 "Quick Look" Analysis

The over-all recommended procedure requires at this point that the sample records be divided into two categories: records which are random and records which are nonrandom. An explicit statistical procedure for establishing whether or not a record is random is proposed in the next section. However, before proceeding with detailed analysis, it would be desirable to take a "quick look" at the vibration time history to see if the record is obviously nonrandom as indicated by block D in Fig. 5.1. This may be done by displaying the record on a cathode ray oscilloscope or a high frequency galvanometer oscillograph. If strong periodic components are obviously present, the sample record may be rejected immediately as being nonrandom without wasting time on more detailed analysis.

The "quick look" analysis step may be as elaborate as desired. For example, a frequency spectrum display will reveal the presence of periodicies in a signal to the experienced vibration engineer, even when the amplitude of the periodic component is quite small relative to the random noise background. Direct visual display spectrum analyzers, which are commercially available, would be ideal for this application. Of course, more advanced detection techniques

such as autocorrelation analysis could be employed, but elaborate analysis of that type is not justified at this early stage in the over-all procedure. In any case, sample records which are obviously nonrandom should be transferred to block Q in Fig. 6.1, to be discussed later.

6.1.5 Test for Randomness

Those sample records accepted as possibly being random by the "quick look" analysis must now be investigated more rigorously as indicated by block E in Fig. 6.1. It is necessary that one have considerable confidence that the sample record is random because analysis procedures to be later recommended will involve statistical procedures based on randomness. One suggested method of testing a sample record for randomness is the Run Test which will now be described.

Suppose in tossing a coin 20 times the following sequence is observed:

$$\underset{1}{\underline{H\,H}} \;\; \underset{2}{T} \;\; \underset{3}{H} \;\; \underset{4}{\underline{T\,T}} \;\; \underset{5}{\underline{H\,H\,H}} \;\; \underset{6}{T} \;\; \underset{7}{H} \;\; \underset{8}{\underline{T\,T}} \;\; \underset{9}{H} \;\; \underset{10}{\underline{T\,T}} \;\; \underset{11}{H} \;\; \underset{12}{\underline{T\,T\,T}}$$

A run is defined as a sequence of identical symbols which are followed and preceded by different symbols or no symbols at all. In the above example, the number of runs is 12. The total number of runs in a sample of any given size gives an indication of whether or not the sample is random. If very few runs occur, a time trend or some bunching due to lack of independence is suggested. If a great many runs occur, systematic short-period cyclical fluctuations are indicated.

Let n_1 = the number of elements of one kind, and n_2 = the number of elements of another kind, the total number of elements being $n = n_1 + n_2$. In the above example, n_1 would be the number of heads ($n_1 = 9$), and n_2 the number of tails ($n_2 = 11$). Let r equal the number of runs as defined above.

For n_1 and n_2 larger than 20, it can be shown (Ref. [7] and [8]) that for random data a good approximation to the sampling distribution of r is the normal distribution with mean μ_r and variance σ_r^2 given by,

$$\mu_r = \frac{2n_1 n_2}{n_1 + n_2} + 1 = \frac{2n_1 n_2}{n} + 1 \tag{6.1}$$

$$\sigma_r^2 = \frac{2n_1 n_2 (2n_1 n_2 - n_1 - n_2)}{(n_1 + n_2)^2 (n_1 + n_2 - 1)} = \frac{2n_1 n_2 (2n_1 n_2 - n)}{n^2 (n - 1)} \tag{6.2}$$

For example, suppose $n_1 = 40$ and $n_2 = 60$. Then $\mu_r = 49$ and $\sigma_r \sim 5$. Thus, for a 95% confidence level, the value of r should fall in the interval between 39 and 59. Stated another way, if r lies outside this interval, the data may be considered nonrandom at the 5% level of significance.

For vibration data analysis, if n discrete measurements of the instantaneous amplitude value are taken, the number n_1 may be taken as the number of measurements which lie above the average value of the n amplitudes at hand, while n_2 may be taken as the number of measurements which lie below this average value. A single run would be defined as a sequence of above average (below average) measurements which are followed and preceded by below average (above average) measurements or no measurements at all. The total number of runs may be denoted by r, as before, and the sampling distribution of r described by Eq. (6.1) and (6.2), assuming n_1 and n_2 are larger than 20.

When instantaneous amplitude values of the vibration data are recorded continuously, the effective number of samples is $n = 2 BT$ where T is the sample length in seconds, and B is the frequency bandwidth in cps. The quantity n_1 would be obtained by multiplying the percentage (P_1) of time per unit time that the instantaneous amplitude values lie above their average value by n. Thus, $n_1 = P_1 n$. Similarly, $n_2 = P_2 n$ where P_2 is the percentage of time per unit time that the instantaneous amplitude values lie below their average value.

For example, suppose $T = 5$ seconds and $B = 100$ cps. Suppose also that $P_1 = P_2 = 50\%$. Then,

$$n = 2BT = 1000$$

$$n_1 = P_1 n = 500 = n_2$$

$$\mu_r = \frac{2n_1 n_2}{n} + 1 = 501$$

$$\sigma_r = \sqrt{\frac{2n_1 n_2(2n_1 n_2 - n)}{n^2(n - 1)}} \approx 15.8$$

Then, for a 95% confidence interval, if the record is random, the value
of r (the number of times that the instantaneous amplitude values pass
through their average value) should fall in the interval between 469 and
533 If the value of r falls outside this interval, the record may be
rejected as nonrandom at the 5% level of significance (that is, probability
of Type I error equals 5%).

The application of the Run Test to vibration time history records
will require only a few instruments. Because an explicitly defined signal
bandwidth B is required, the record will have to be filtered through a
low pass filter with an extremely sharp high frequency cutoff. If the
filter does not have a cutoff of, say, at least 60 db per octave, the trans-
fer function for the filter should be determined to calculate the equivalent
noise bandwidth B_N of the signal as described in Section 4.9.2. The
noise bandwidth B_N should then be used for obtaining the number of de-
grees of freedom n for the record by the formula $n = 2B_N T$.

The quantity r, which is the number of times the instantaneous
amplitude values pass through some arbitrary level in either direction,
might be determined by using a polarity device followed by a counter.
For convenience, the polarity device should be centered over the average
value of the signal record so that n_1 will equal n_2. The quantity r
could also be determined by simply obtaining a visual recording of the
instantaneous signal amplitudes on an oscillograph, for example, and
manually counting the number of average value crossings occurring in
the record length T.

It is conceded that the proposed Run Test is not a totally fool-proof
method for establishing randomness. Obviously, a specially selected
sinusoidal signal with an appropriate frequency will be accepted by the

Run Test as being random. Furthermore, it would not be surprising if the Run Test fails to perform properly when the signal record being analyzed has sharp peaks in its power spectral density function. These possible factors will have to be investigated experimentally in a controlled laboratory program, as outlined in Section 8, before the Run Test for randomness can be incorporated into a standard procedure. Nevertheless, it is believed that the Run Test when augmented by good engineering judgment and experience will permit the detection of nonrandom sample records with a high level of confidence.

Sample records which are found to be nonrandom by the Run Test are transferred to block Q in Fig. 6.1, to be discussed later.

6.1.6 Tests for Stationarity

Sample records which have been confirmed to be random must now be divided into two categories: records which are stationary and records which are nonstationary. This is necessary because analysis procedures to be recommended later will involve statistical procedures which require that the data be stationary. An explicit statistical procedure to establish whether or not a random vibration time history sample record is stationary will now be proposed, as indicated by block F in Fig. 6.1.

First, it is required to define clearly what stationarity means in terms of a single random signal record. In Section 4.4, stationarity is defined for a random process consisting of a collection or ensemble of records. However, a somewhat different concept is involved when stationarity is considered for a single record from the ensemble. These matters will now be discussed.

Suppose a single experiment is performed yielding only one output vibration record, say, the vibration amplitude as a function of time. Furthermore, suppose that it is not convenient or possible to obtain other results which would generate an ensemble of records. Thus, it is now meaningless to perform ensemble averages, or to compare time averages from this one record with any other record. Pictured below is a sketch of the single available record.

SINGLE RANDOM RECORD

What does the word "stationary" imply with reference to a single record? The answer to this question is obviously a different interpretation for "stationary" than considered previously in Section 4.4. To distinguish this case, the terms "stationary-within-itself" or "self-stationary" will be introduced because these terms convey the idea of breaking up the single record into a reasonable number of shorter sub-records and comparing their statistical properties as obtained by time averages in each subrecord. If all statistical properties from the sub-records agree with one another, then the original record will be said to be strongly self-stationary; if only the mean values, mean square values, and autocorrelation functions are in good agreement, then the original record will be called weakly self-stationary. Thus, the class of weakly self-stationary records has as a subclass the class of strongly self-stationary records. If the amplitude values from the subrecords follow a Gaussian distribution and if the record is weakly self-stationary, then the record will also be strongly self-stationary. Thus, for the Gaussian case, weakly and strongly self-stationary are synonymous. Single records which are not weakly self-stationary will be said to be self-nonstationary.

It is important for the reader to keep clear the essential distinctions between such ideas as "stationary", "ergodic", and "self-stationary". The latter term applies to a single record; the first two terms apply to a random process, that is, an ensemble of records. If the single record is also a particular member of a random process, then the random process may be stationary while the particular single record under consideration may or may not be self-stationary. For example, individual records may be self-nonstationary by themselves, but when considered in an ensemble of records, the entire ensemble may still be stationary. For this to occur, special fluctuations within one record must be balanced out by different fluctuations within other records. Thus, knowledge of the properties of a single record by itself yields no information about stationary nature of the random process. However, if the random process is ergodic, the individual records must be self-stationary since each record is representative of the original random process. Hence, if an individual member is not self-stationary, then the random process can not be ergodic. On the other hand, individual records of a random process

may be self-stationary without the process itself being ergodic because the individual records do not have to agree with one another. Of course, the random process will be stationary when all the individual records are self-stationary.

Figure 6.2 is a display of various categories for a single record and an ensemble of records, using arrows to indicate subclasses.
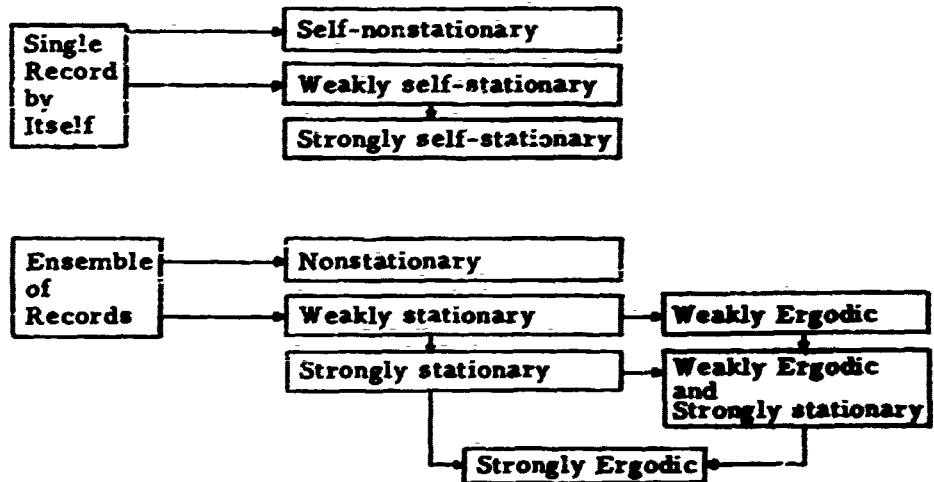
Figure 6.2. Categories for Single Records and Ensembles

How may one proceed to test whether or not a given random process is stationary (weakly or strongly) or ergodic (weakly or strongly)? How may one test whether or not a given single record is self-stationary (weakly or strongly)? What special properties should be considered if the given single record is a member of a random process?

Consider, first of all, the case where a given single record is a member of a random process, and suppose one has determined for this record that it is either self-stationary (weakly or strongly) or self-nonstationary. If the record is self-nonstationary, then the underlying random process can not be urgodic, but is otherwise not restricted. If the record is self-stationary, no conclusions can be drawn about the random process based on this information alone. For a record whose amplitude values follow a Gaussian distribution, the record is automatically

strongly self-stationary if it is found to be weakly self-stationary. If the amplitude values follow a non-Gaussian distribution, weakly self-stationary properties do not imply strongly self-stationary properties. Fortunately, one is usually satisfied to know merely whether or not the record is weakly self-stationary.

Consider, next, the case where a collection (ensemble) of records is available from a given random process. Suppose that one has determined for each record whether or not it is self-stationary of self-nonstationary. If mixed results are obtained, then the random process is not ergodic, but is otherwise not restricted. Statements about the stationarity of the random process would now depend upon performing ensemble averages of statistical quantities over the random process at different times, and checking to see whether or not these statistical results are invariant with respect to time translations. If every individual record is self-stationary, then the random process will automatically be stationary and no ensemble averaging is required. Furthermore, if the individual records also yield essentially the same self-stationary properties, then the random process will be ergodic. Thus, determination of self-stationarity of each record in a large collection of related records will yield desired information about the underlying random process.

It is now necessary to develop a quantitative procedure which tests whether or not a given single record is self-stationary. More explicitly, it is usually of practical interest to test only whether or not the given record is weakly self-stationary. Towards this goal, the record must be broken up into a number of shorter subrecords, and it is required to observe how closely the mean values, mean square values and auto-correlation functions as calculated from each subrecord agree with one another. For simplicity, as well as because of its widespread occurrence in random phenomena, it will be assumed that the mean values are essentially zero (or can be subtracted out) and that these mean values agree to within a desired confidence level. Also, it is reasonable to expect that autocorrelation functions from the various subrecords will be in close agreement if this is true of their mean square values, since autocorrelation functions are less than or equal to their associated mean square values. Hence, it is deemed sufficient to examine merely the mean square values from the various subrecords.

Before proceeding with the details of a test for weak self-stationarity, certain mathematical relationships associated with the mean square value estimation problem are required. These relationships will be developed in detail in the following subsection on "Analysis of Mean Square Measurements". Those not interested in such mathematical details are advised to bypass this subsection and go directly to the next subsection entitled "A Test for Weak Self-Stationarity".

6.1.7 Analysis of Mean Square Measurements

Consider a random record $x(t)$ of length T (seconds) to be passed through a squaring circuit and then through a perfect integrator where it is averaged for a time $T_1 < T$. Let the output be denoted by $y(t)$. This output is an estimate of the mean square value of $x(t)$. See sketch below.
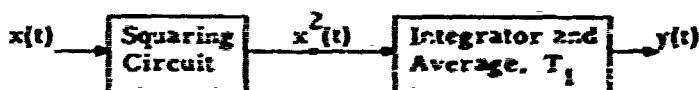
$$y(t) = \frac{1}{T_1} \int_{t-T_1}^{t} x^2(t)\, dt \quad : \quad T_1 < t < T \tag{6.3}$$

If the record length T is such that $NT_1 = T$ where N is an integer, and if the output is obtained only at discrete times T_1, $2T_1$, ..., NT_1, then

$$y(T_1) = \frac{1}{T_1} \int_{0}^{T_1} x^2(t)\, dt$$

$$y(2T_1) = \frac{1}{T_1} \int_{T_1}^{2T_1} x^2(t)\, dt$$

$$y(NT_1) = \frac{1}{T_1} \int_{(N-1)T_1}^{NT_1} x^2(t)\, dt$$

would represent a discrete set of mean square output values as computed from N different subrecords of $x(t)$. An alternative equivalent representation for the above outputs is given by

$$ y_1 = \frac{1}{T_1} \int_0^{T_1} x_1^2(t)\, dt $$

$$ y_2 = \frac{1}{T_1} \int_0^{T_1} x_2^2(t)\, dt $$

$$ y_N = \frac{1}{T_1} \int_0^{T_1} x_N^2(t)\, dt $$

where the range of integration is now the same in all cases $(0, T_1)$, and the different subrecords $x_i(t)$, $i = 1, 2, \ldots, N$, are indicated by the different subscripts. This set of outputs is a special case of Eq. (6.3) where $t = T_1$, a constant, and $y(t) = \left\{ y_i \right\}$, $i = 1, 2, \ldots, N$.

The mean value of the set of possible output records $\left\{ y(t) \right\}$, at any fixed instant of time, computed over the ensemble of possible input records $\left\{ x(t) \right\}$, is given by

$$ \mu_y = E\left[y(t) \right] = \frac{1}{T_1} \int_{t-T_1}^{t} E\left[x^2(t) \right] du = E\left[x^2(t) \right] = R_x(0) \qquad (6.4) $$

where it is assumed that $x(t)$ is weakly self-stationary. The quantity $R_x(0)$ is the stationary autocorrelation function $R_x(\tau)$ at $\tau = 0$ of $x(t)$ as defined by

$$ R_x(\tau) = E\left[x(t)\, x(t + \tau) \right] \quad ; \quad \text{independent of } t \qquad (6.5) $$

Equation (6.4) shows that $y(t)$ is an <u>unbiased</u> estimate of the mean square value of $x(t)$.

The variance in the estimate $y(t)$ is given by

$$\sigma_y^2 = E\left[y(t) - \mu_y\right]^2 = E\left[y^2(t)\right] - \mu_y^2 \tag{6.6}$$

$$= \frac{1}{T_I^2} \int_{t-T_I}^{t} \int_{t-T_I}^{t} \left[E\left[x^2(u)\, x^2(v)\right] - R_x^2(0)\right] du\, dv$$

$$= \frac{2}{T_I^2} \int_{t-T_I}^{t} \int_{t-T_I}^{t} R_x^2(v - u)\, du\, dv \tag{6.7}$$

using the stationary relation

$$E\left[x^2(u)\, x^2(v)\right] = 2R_x^2(v - u) + R_x^2(0) \tag{6.8}$$

which is valid if $x(t)$ follows a <u>Gaussian (normal) distribution</u> with zero mean value. This relation is approximated closely also by other distributions. From Eq. (6.6) letting $\gamma = v - u$, $d\gamma = dv$, one obtains

$$\sigma_y^2 = \frac{2}{T_I^2} \int_{t-T_I}^{t} \int_{t-T_I-u}^{t-u} R^2(\gamma)\, d\gamma\, du \tag{6.9}$$

Next, reverse the order of integration between γ and u. See sketch below. This yields

$$\sigma_y^2 = \frac{2}{T_I^2} \int_0^{T_I} \int_{t-T_I}^{t-\gamma} R_x^2(\gamma)\, du\, d\gamma + \frac{2}{T_I^2} \int_{-T_I}^{0} \int_{t-T_I-\gamma}^{t} R_x^2(\gamma)\, du\, d\gamma$$

$$= \frac{2}{T_I^2} \int_0^{T_I} (T_I - \gamma)\, R_x^2(\gamma)\, d\gamma + \frac{2}{T_I^2} \int_{-T_I}^{0} (T_I + \gamma)\, R_x^2(\gamma)\, d\gamma$$

$$= \frac{4}{T_I^2} \int_0^{T_I} (T_I - \gamma)\, R_x^2(\gamma)\, d\gamma \tag{6.10}$$
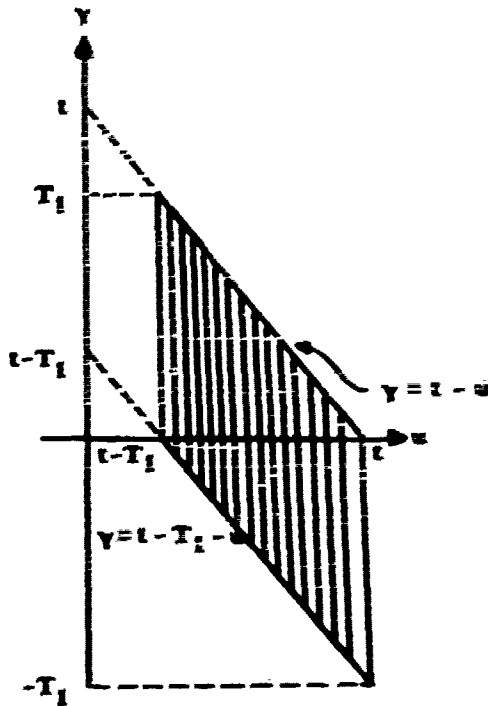
For definiteness and simplicity, as well as because of its wide-spread observance in physical phenomena, the autocorrelation function $R_x(\gamma)$ will be represented by a damped exponential function:

$$R_x(\gamma) = R_x(0)e^{-2B|\gamma|} \tag{6.11}$$

where B may be interpreted as an ideal low-pass bandwidth (cps). Substituting Eq. (6.11) into Eq. (6.10), and carrying out the simple integration,

$$\sigma_\gamma^2 \approx \frac{R^2(0)}{BT_1} \quad \text{assuming } BT_1 > 1 \tag{6.12}$$

If the integration time T_1 equals the full length T of the record, Eq. (6.12) should be replaced by

$$\sigma_\gamma^2 \approx \frac{R^2(0)}{BT} \quad \text{assuming } BT > 1 \tag{6.13}$$

Equation (6.12) shows that σ_y^2 approaches zero as T_i becomes large, proving that $y(t)$ is a <u>consistent</u> estimate of the mean square value of $x(t)$.

The result obtained in Eq. (6.12) gives the variance in estimates of $y(t)$ for times t in the range $T_i < t < T$, and as such indicates the mean square fluctuation to be expected in $y(t)$ during this time interval.

R-C Averager

Instead of using a perfect integrator and averager, it is frequently necessary to employ an R-C averaging circuit. A direct comparison will now be carried out between statistical results obtained previously for the perfect integrator-averager and those expected for the R-C circuit. To avoid confusion with the previous work, let the output of the R-C circuit be denoted by $z(t)$. See sketch below.
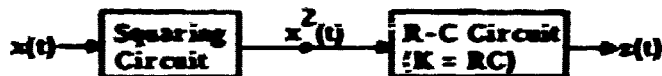
As is well known,

$$z(t) = \int_0^t x^2(u)\, h(t - u)\, du \quad ; \quad t \geq 0 \tag{6.14}$$

where

$$h(\tau) = \frac{1}{K}\, e^{-\tau/K} \quad ; \quad \tau \geq 0 \tag{6.15}$$

$$K = RC = \text{time-constant of R-C circuit}$$

Thus,

$$z(t) = \frac{e^{-t/K}}{K} \int_0^t x^2(u)\, e^{u/K}\, du \tag{6.16}$$

The mean value of the set of possible $z(t)$ is given by

$$F_z = E\left[z(t)\right] = \frac{e^{-t/K}}{K} \int_0^t E\left[x^2(u)\right] e^{u/K} du$$

$$= \frac{R_x(0) e^{-t/K}}{K} \int_0^t e^{u/K} du = R_x(0)\left(1 - e^{-t/K}\right) \qquad (6.17)$$

The table below shows how $\mu_z/R_x(0)$ varies with t.

t	K	2K	3K	4K	5K	6K	7K
$\mu_z/\ldots$	0.632	0.865	0.950	0.982	0.993	0.998	0.999

It is clear that for $t \geqslant 3K$, μ_z may be approximated by $R_x(0)$ with, at most, ± 5 percent error; for $t \geqslant 4K$, the approximation is good to within a 2 percent error. For definiteness, assume from henceforth that $t \geqslant 4K$ so as to make μ_z essentially independent of t. Thus,

$$\mu_z \approx R_x(0) \quad : \quad t \geqslant 4K \qquad (6.18)$$

and $z(t)$ for $t \geqslant 4K$ becomes an unbiased estimate of the mean square value of $x(t)$.

Analogous to Eq. (6.6), the variance in the estimate is here given by

$$\sigma_z^2 = E\left[z(t) - F_z\right]^2 = E\left[z^2(t)\right] - F_z^2 \qquad (6.19)$$

$$= \frac{e^{-2t/K}}{K^2} \int_0^t \int_0^t e^{(u+v)/K}\left\{E\left[x^2(u) x^2(v)\right] - F_z^2\right\} du\, dv$$

$$\approx \frac{2e^{-2t/K}}{K^2} \int_0^t \int_0^t e^{(u+v)/K} R_x^2(v - u)\, du\, dv \qquad (6.20)$$

provided $t \geq 5K$, assuming $\{x(t)\}$ is weakly stationary and normally distributed. Now, letting $\gamma = \tau - u$, $d\gamma = d\tau$,

$$\sigma_{\dot{r}}^2 \approx \frac{2e^{-2t/K}}{K^2} \int_0^t \int_{-u}^{t-u} e^{(2u+\gamma)/K} R_x^2(\gamma)\, d\gamma\, du \qquad (6.21)$$

Next, as in the derivation of Equation (6.18), reverse the order of integration between γ and u. See sketch below.
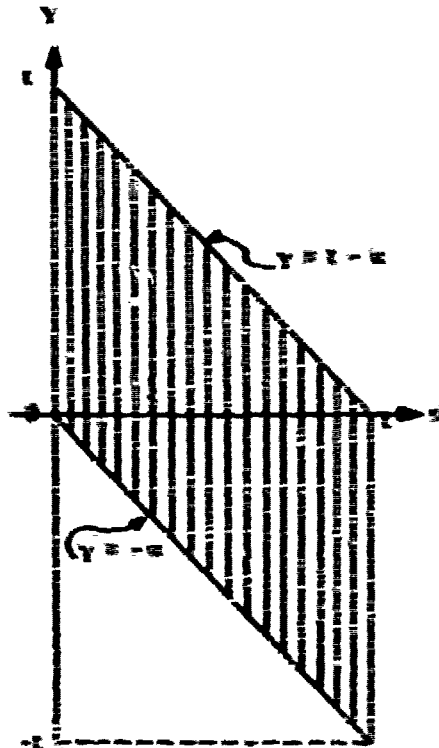
This procedure yields

$$\sigma_z^2 \approx \frac{2e^{-2t/K}}{K^2} \int_{-t}^{0} \int_{-u}^{t} R^2_{(\gamma)} \, e^{\gamma/K} \, e^{2u/K} \, du \, d\gamma$$

$$+ \frac{2e^{-2t/K}}{K^2} \int_{0}^{t} \int_{0}^{t-\gamma} R^2_{(\gamma)} \, e^{\gamma/K} \, e^{2u/K} \, du \, d\gamma$$

$$\approx \frac{e^{-2t/K}}{K} \int_{-t}^{0} R^2_{(\gamma)} \, e^{\gamma/K} \left[e^{2t/K} - e^{2\gamma/K} \right] d\gamma$$

$$+ \frac{e^{-2t/K}}{K} \int_{0}^{t} R^2_{(\gamma)} \, e^{\gamma/K} \left[e^{2(t-\gamma)/K} - 1 \right] d\gamma$$

$$\approx \frac{2}{K} \int_{0}^{t} R^2_{(\gamma)} \left[e^{-\gamma/K} - e^{-2t/K} \, e^{\gamma/K} \right] d\gamma \qquad (6.22)$$

Assume now the same autocorrelation function $R_x(\gamma)$ specified previously in Eq. (6.11), namely,

$$R_x(\gamma) = R_x(0) \, e^{-2B|\gamma|} \qquad (6.23)$$

where B is the bandwidth (cps) occupied by x(t). Substitution of Eq. (6.23) into Eq. (6.22) leads to the result

$$\sigma_z^2 \approx \frac{R^2(0)}{2\,BK} \qquad \text{assuming } t \gg 4K \text{ and } BK > 1 \qquad (6.24)$$

Direct comparison of Eq. (6.24) with Eq. (6.12) reveals that the effective integration time (T_1) is equal to twice the R-C circuit time constant (2K), that is, independent of B,

$$T_1 = 2K \qquad (6.25)$$

This important result holds in the time range $t \geq 4K$, and all formulas involving T_1 are valid for the R-C circuit in this time range merely by replacing T_1 by 2K. In particular, Eq. (6.24) indicates the mean square fluctuation to be expected in $z(t)$ when $t \geq 4K$.

6.1.8 A Test for Weak Self-Stationarity

The test for weak self-stationarity to be developed here is actually a test for whether or not a particular record is self-nonstationary at some given level of significance. The test is applied by determining whether measured mean square values fluctuate outside an allowable range more than a defined percentage of the time. Records which fail to be self-nonstationary by the test will be accepted as weakly self-stationary.

Let it be assumed that the input record is weakly self-stationary and occupies a frequency bandwidth of B cps. The normalized variance in the measurement of the mean square value s^2 using a perfect integrator with a finite integration time (equal to the subrecord sample length) of T_1 seconds is shown in Eq. (6.12) to be estimated closely by

$$\text{Variance} = \frac{\sigma^2}{s^2} = \frac{1}{BT_1} \qquad (6.26)$$

For an RC averaging circuit with time constant $K = RC$, after say four or more time constants have elapsed, the effective integration time T_1 is shown in Eq. (6.25) to be related to K by

$$T_1 = 2K \qquad (6.27)$$

Hence, after time = 4K, the normalized variance for an RC circuit is given by

$$\text{Variance} = \frac{1}{2BK} \tag{6.28}$$

The number of degrees of freedom associated with a record of length T_1 (sec) and frequency bandwidth B (cps) is $n = 2BT_1$ since the record can be reconstructed from its samples taken (1/2B) seconds apart on the time scale (Ref. [1], p. 57). Thus, T_1 divided by (1/2B) or $2BT_1$ numbers completely determine a record which is T_1 seconds long.

$$n = 2BT_1 = \frac{T_1}{(1/2B)} \tag{6.29}$$

For an RC averaging circuit, after time = 4K, the equivalent number of degrees of freedom becomes

$$n = 4BK \tag{6.30}$$

From other statistical considerations (see Section 5.3.3[c]), it is known that for samples with n degrees of freedom, the measured mean square value s^2 and the true mean square value σ^2 are such that

$$\frac{ns^2}{\sigma^2} = \chi^2 \tag{6.31}$$

follows a χ^2 distribution with (n - 1) degrees of freedom. Values of χ^2 are tabulated according to n and a desired level of significance α (in percent) as defined by

$$\text{Prob}\left[\chi^2 > \chi^2_{(\alpha/2)}\right] = \alpha/2 \tag{6.32}$$

For example, if $\alpha/2 = 5\%$ or 95% ,

$$\text{Prob}\left[\chi^2 > \chi^2_{.05}\right] = .05$$

$$\text{Prob}\left[\chi^2 > \chi^2_{.95}\right] = .95$$

Hence, a 90% confidence interval is given by

$$\text{Prob}\left[\chi^2_{.95} < \chi^2 \leqslant \chi^2_{.05}\right] = .90$$

In general, for an arbitrary confidence interval $(1 - \alpha)$, in percent,

$$\text{Prob}\left[\chi^2_{1-(\alpha/2)} < \chi^2 \leqslant \chi^2_{(\alpha/2)}\right] = 1 - \alpha \tag{6.33}$$

The quantities $\chi^2_{1-(\alpha/2)}$ and $\chi^2_{(\alpha/2)}$ yield the lower and upper limits of χ^2 in the desired confidence interval. From Eq. (6.31), equivalent expressions to Eq. (6.33) are

$$\text{Prob}\left[\chi^2_{1-(\alpha/2)} \leqslant \frac{ns^2}{\sigma^2} \leqslant \chi^2_{(\alpha/2)}\right] = 1 - \alpha \tag{6.34}$$

$$\text{Prob}\left[\sigma^2\left(\frac{\chi^2_{1-(\alpha/2)}}{n}\right) \leqslant s^2 \leqslant \sigma^2\left(\frac{\chi^2_{(\alpha/2)}}{n}\right)\right] = 1 - \alpha \tag{6.35}$$

$$\text{Prob}\left[s^2\left(\frac{n}{\chi^2_{(\alpha/2)}}\right) < \sigma^2 \leqslant s^2\left(\frac{n}{\chi^2_{1-(\alpha/2)}}\right)\right] = 1 - \alpha \tag{6.36}$$

Equations (6.34), (6.35), and (6.36) are confidence statements about the random variable ns^2/σ^2 in different forms. The confidence meaning in each of these equations is that if the experiment of measuring s^2 is repeated a large number of times, then the inequality in question will be satisfied in $(1 - \alpha)$ percent of these cases.

To interpret Eq. (6.36), one would say that the range interval constructed from a measured s^2 by calculating $s^2(n/\chi^2_{(\alpha/2)})$ and $s^2(n/\chi^2_{1-(\alpha/2)})$ will contain the true σ^2 with $(1 - \alpha)$ percent confidence, that is, in $(1 - \alpha)$ percent of the times that the test is conducted. Similarly, to interpret Eq. (6.35), assuming σ^2 to be known, one

would say that the measured values s^2 will fail in the range shown from $s^2(\chi^2_{1-(\alpha/2)}/n)$ to $s^2(\chi^2_{(\alpha/2)}/n)$ with $(1 - \alpha)$ percent confidence. Equation (6.35), rather than Eq. (6.36), is fundamental to the later analysis. First, however, Eq. (6.36) will be discussed.

Table 6.1 below, based on Eq. (6.36), contains calculated values of (n/χ^2) as a function of n and α, and thus provides confidence limits and intervals for the true σ^2 from measured mean square values of s^2. For simplicity, only 80% and 95% confidence intervals are shown, and n is taken equal only to 2, 10, 20, 40, 60, or 120. Other values are readily available from general χ^2 tables. For large values of n, the χ^2 distribution approaches a normal distribution with mean equal to n and variance approximately equal to $2n$, (see Section 5.3.2).

Table 6.1. Confidence Intervals for True Mean Square Values

No. of degrees of freedom		n = 2	n = 10	n = 20	n = 40	n = 60	n = 120
80% Confidence interval	Lower limit	0.43	0.62	0.70	0.77	0.81	0.85
	Upper limit	9.48	2.05	1.67	1.37	1.29	1.19
95% Confidence Interval	Lower limit	0.27	0.49	0.58	0.67	0.72	0.79
	Upper limit	39.20	3.07	2.08	1.63	1.48	1.31

Tabulated Limit x Measured Mean Square Value = Confidence Limit

Confidence Interval = Upper Confidence Limit - Lower Confidence Limit

To illustrate use of Table 6-1, suppose $n = 60$ for a particular record as computed from Eq. (6.29) or Eq. (6.30), and suppose the measured mean square value $s^2 = 3$ volts. Then, the 80% lower and upper chi-square confidence limits for σ^2 are equal to $(0.81)(3) = 2.43$ volts and $(1.29)(3) = 3.87$ volts, respectively, and one would state here with 80% confidence that, if the record is weakly self-stationary, then the true mean square value lies in the range 2.43 to 3.87 volts. That is, in repeated measurements of the mean square value, there is a constant probability of 80% that this statement about the location of the true

mean square value is true, and a constant probability of 20% that this statement is false.

Table 6.2 below (based on Eq. 6.35) contains calculated values of (x^2/n) as a function of n and α, and thus provides confidence limits and intervals for s^2 assuming knowledge of the true mean square value σ^2. Table 6.2 and Table 6.1 are, in a mathematical sense, inverse to one another.

Table 6.2. Confidence Intervals for Measured Mean Square Values

No. of degrees of freedom		n = 2	n = 10	n = 20	n = 40	n = 60	n = 120
80% Confidence Interval	Lower Limit	0.10	0.48	0.60	0.73	0.77	0.84
	Upper Limit	2.33	1.62	1.43	1.30	1.24	1.18
95% Confidence Interval	Lower Limit	0.03	0.32	0.48	0.51	0.67	0.76
	Upper Limit	3.71	2.04	1.73	1.50	1.39	1.27

Tabulated Limit x True Mean Square Value = Confidence Limit

Confidence interval = Upper Confidence Limit - Lower Confidence Limit

To illustrate use of Table 6.2, suppose $n = 60$ and suppose the true mean square value is known to be $\sigma^2 = 3$ volts. Then the 80% lower and upper chi-square confidence limits for s^2 are equal to $(0.77)(3) = 2.31$ volts and $(1.24)(3) = 3.72$ volts, respectively. One would assert here that the following statement is correct: namely, if the record is weakly stationary, and if the experiment is repeated over and over again, it would be expected that 80% of the measured mean square values would fall inside the range 2.31 to 3.72 volts. It follows that 20% of the measured mean square values would be expected to fall outside the range 2.31 to 3.72 volts.

In applying Table 6.2 or extensions thereof to practical problems, it is required that n and σ^2 be known. The main problem is estimation of σ^2, the true mean square value. Since the expected value over a set of measurements of s^2 gives an unbiased and consistent estimate of σ^2, this is clearly the approach to take to estimate σ^2. By definition, a set

of measurements of any quantity is said to be unbiased if, independent of the number of degrees of freedom n, the expected value is the true value. The set of measurements is said to be consistent if the mean square deviation from the true value approaches zero as n approaches infinity. Thus, one should choose for σ^2 the average value s^2 about which the different measurements on s^2 are varying. In the case of an RC filter, only the time history after four time constants have elapsed should be considered in determining this average value.

Assume now that a sample record of length T and bandwidth B is to be tested for weak self-stationarity. The record is divided into N number of subrecords of equal length $T_i = (T/N)$. Then the number of degrees of freedom for each subrecord is $n = (2BT/N)$. Estimate the mean square value s_i^2 for each of the N subrecords. Establish a $(1 - \alpha)$ chi-square confidence interval and an estimated true mean square value σ^2 for the entire record by the procedures previously discussed.

If the sample record is weakly self-stationary, it would be expected that αN of the mean square estimates s_i^2 would fall outside the $(1 - \alpha)$ chi-square confidence interval. The number αN will hereafter be called the expected number of violations. Remember that αN is only an expected number of violations and for any given test for self stationarity on a weakly self-stationary record, the actual number of violations q may be somewhat more or less than αN. In fact, the actual number of violations q will be distributed in some undefined manner about a mean of αN. If it is assumed that the undefined distribution of q is symmetrical, then the mean αN will also be the median, and q will actually be greater than αN in 50% of the time that the test is applied to self-stationary data. The remaining question is, how many violations q should be permitted before the sample record is considered self-nonstationary.

Consider the test for self-stationarity as a set of N experiments where each experiment is the estimation of a mean square value s_i^2 for the ith subrecord formed from the original record being tested. Each experiment has two possible outcomes, the value of s_i^2 will fall outside a $(1 - \alpha)$ chi-square confidence interval (a failure), or the value of s_i^2

will fall inside a $(1 - \alpha)$ chi-square confidence interval (a success). Assuming the series of N experiments are statistically independent, the test may be considered as a set of Bernoulli trials (see Section 5.5, or where the probability of a failure is α and the probability of a success is $(1 - \alpha)$. The resulting random variable for the number of failures q (violation of the $(1 - \alpha)$ confidence interval) will have a binomial probability density function as follows.

$$b(q) = \binom{N}{q} \alpha^q (1 - \alpha)^{N-q} \tag{6.37}$$

The expression $\binom{N}{q}$ is the number of different combinations of N number of experiments taken q number at one time.

$$\binom{N}{q} = \frac{N!}{q! \, (N - q)!} \tag{6.38}$$

where $N! = N(N - 1)(N - 2) \cdots (3)(2)(1)$.

Remember that the sum of all probabilities must be one.

Hence,

$$\sum_{q=1}^{N} \binom{N}{q} \alpha^q (1 - \alpha)^{N-q} = 1 \tag{6.39}$$

Also, the probability of any given number of failures k can be established as follows:

$$\sum_{q=1}^{k} \binom{N}{q} \alpha^q (1 - \alpha)^{N-q} = b(1) + b(2) + \ldots + b(k) \tag{6.40}$$

Consider the following example. Assume the sample record is divided into $N = 10$ subrecords, and the mean square value s_i^2 is estimated for each of the ten subrecords formed from the original record. Any chi-square confidence interval $(1 - \alpha)$ desired can be calculated for the estimates. Since this confidence coefficient is of no direct importance

to the present example, use a 50% chi-square confidence interval for simplicity.

Then

$$(1 - \alpha) = \alpha = 0.5, \text{ and for } N = 10,$$

$$b(q) = \binom{10}{q} 0.5^q 0.5^{(10-q)} = \binom{10}{q} (0.5)^{(10)} = \binom{10}{q} \frac{1}{1024}$$

Now,

$$b(q = 10) = 0.001 = b(q = 0)$$
$$b(q = 9) = 0.010 = b(q = 1)$$
$$b(q = 8) = 0.044 = b(q = 2)$$
$$b(q = 7) = 0.117 = b(q = 3)$$
$$b(q = 6) = 0.205 = b(q = 4)$$
$$b(q = 5) = 0.246$$

It may be said that the probability of 7 or more of the ten mean square estimates falling outside the 50% chi-square confidence interval is $b(q = 7 \text{ or } 8 \text{ or } 9 \text{ or } 10) = 0.172$, if the original record were self-stationary. Hence, if 7 or more of the ten estimates fall outside the 50% chi-square confidence interval, the record can be considered as self-nonstationary at the 17.2% level of significance (Type I error = 17.2%). Higher confidence is obtained if a larger number of estimates fall outside the confidence interval. The following higher confidence statements could be made.

A. If 8 or more of the ten estimates fall outside the 50% confidence interval, it may be said with 94.4% confidence that the record is self-nonstationary.

B. If 9 or more of the ten estimates fall outside the 50% confidence interval, it may be said with 99% confidence that the record is self-nonstationary.

C. If all 10 of the estimates fall outside the 50% confidence interval, it may be said with 99.9% confidence that the record is self-nonstationary.

If n is large and α is small, then the binomial distribution may be approximated by a Poisson distribution (see Section 5.5.4), such that

$$p(q) = \frac{\lambda^q e^{-\lambda}}{q!} \qquad \text{where} \qquad \lambda = \alpha N \qquad (6.41)$$

The expected value $E(q)$ and standard deviation $\sigma(q)$ of the number of failures q are now given by

$$E(q) = \lambda \qquad : \qquad \sigma(q) = \sqrt{\lambda} \qquad (6.42)$$

A further limiting approximation of the Poisson distribution for large N and small α leads to a normal (Gaussian) distribution with the above mean and standard deviation.

For example, suppose $N = 250$ and $\alpha = 0.10$. Then $\lambda = \alpha N = 25$ and $\sigma = \sqrt{25} = 5.0$. Suppose, to a first order of approximation that a normal distribution applies. If the record is self-stationary, then the probability that $(\lambda + 2\sigma) = 35$ or more of the 250 samples lie outside the $(1 - \alpha) = 90\%$ chi-square confidence interval is 2.5%. Hence, if this occurs, the record may be considered self-nonstationary with 97.5% confidence. The probability that $(\lambda + \sigma) = 30$ or more of the samples lie outside this same 90% chi-square confidence interval is 16%. For this case, the record may be considered nonstationary with 84% confidence. Finally the probability that $\lambda = 25$ or more of the samples lie outside the 90% chi-square confidence interval is 50%. Now, the record would be considered self-nonstationary with only 50% confidence.

The same ideas discussed for a sample record divided into discrete subrecords can be applied to continuous estimates of mean square values obtained by filtering the output of a square law rectifier with an RC filter of time constant K. For this case, attention is directed to the percentage of time per unit time (P_q) that the continuous mean square estimate lies outside a $(1 - \alpha)$ chi-square confidence interval. From Eq. (6.27), the total number of experiments will be

given by $N = (T/2K)$ where T is the total record length. Then the number of failures $q = P_q N$.

From the foregoing discussion, one might be tempted to test for self-stationarity at the lower self-nonstationary level of significance to reduce the risk of rejecting self-stationary data as being self-nonstationary (reduce Type I error). However, it should be noted that as the level of significance for the test is decreased, as the probability of Type I error is reduced), the risk of accepting a record as self-stationary when in fact it is truly self-nonstationary is increased. This risk of accepting a self-nonstationary record as self stationary is the Type II error of the test (see Section 5.1).

The Type II error associated with the test for self-stationarity is a function of both the number of statistical degrees of freedom $(n = 2BT_i)$ of the mean square estimates for each of the original sub-records, as well as the level of significance for the test (Type I error). A Type II error can (with some reservations, perhaps) be developed for this test by using conventional quality control procedure as discussed in Section 5.5. However, if the number of degrees of freedom for the mean square estimates is reasonably large (greater than, say, $n = 20$) the Type II error should not be a serious problem even when testing at a level of significance of 1%.

In conclusion, a specific analytical procedure has been proposed to establish with defined confidence whether or not a single vibration time history sample record is self-nonstationary. Those records not rejected by the test as self-nonstationary will be accepted as being weakly self-stationary. As stated for the recommended test for randomness (Section 6.1.5), the general procedures should be thoroughly investigated by laboratory experiments, as outlined in Section 6.2.2, before the test is incorporated into a standard procedure.

Sample records which are found to be self-nonstationary by the test for stationarity are transferred to block S in Figure 6.1, to be discussed later. All other records will be considered weakly self stationary until further tests for normality are conducted, also to be discussed later.

nc   ments needed to accomplish the test for stationarity
include   ·, .s filter with a very sharp high frequency cutoff, a
rms... sig   .mating device such as a true rms vacuum tube volt-
rate ...   .' ed in Section 7.3, and a voltage level recorder.

6.1.9 Ar ·it   . Probability Density Analysis

One . f the fundamental types of analysis included in the over-
all recor . endec ·rocedure is amplitude probability density analysis,
as indic: 1 by b·ck G in Figure 6.1. Probability density analysis
has not b··n a co mon data analysis procedure in the past. This has
been tru cr two ·imary reasons as follows:

(a) The est ation of probability density for the instantaneous
     signal a ·plitudes at three or four times the rms value of
     the sign . (at 3 or 4 σ) requires rather long sample
     records, is shown in Section 7.5. For example, an
     18.5 min ·e long record is required to estimate the prob-
     ability d· sity with a standard error (normalized standard
     deviation of 30% at the 4 σ point on a normal (Gaussian)
     probabili . density function.

 ·   Enginee: end to be readily receptive to the idea that all
     random r ponses in nature are probably distributed in a
     Gaussian inner and that confirmation is not required.

· · · first reaso· for avoiding probability density analysis in a
data r .c iction proce· . e has considerable practical support and poses
a real oblem. The · cond reason, however, is not justified. Even
if it c ·d be ass·m·e· at the underlying stochastic processes pro-
ducin; .o structural . ·ponse are nearly always distributed in a
Gaussi. a manner, the ·nssibility of nonlinear structural character-
istic: · .ay sometime· · oduce a response amplitude probability density
funct .. that deviates v. 'ely from a Gaussian probability density func-
tion. ·he subject of no ·.inear structural response is reviewed in
Sect:. 9.5.

. is recommend· hat amplitude probability density analysis
ec ·: ·ded early in the · ·r-all procedure for analyzing a sample
record for the followiny · asons:

(a) The estimate of the probability density function of a sample record can be used to test the record for normality. The fact that a sample record does or does not have a Gaussian probability density function renders important conclusions concerning the self-stationarity of the record and the accuracy of other types of analyses, as has been discussed in Section 6.1.8.

(b) If the amplitude probability density function of the sample record is not Gaussian, the estimated probability density function may be one of the most important statistical results of the over-all analysis procedure, particularly if the primary objective of the analysis is the prediction of structural fatigue life. For details, see the review of structural fatigue presented in Section 9.4.

The estimation of amplitude probability density functions of sample records by analog techniques is detailed in Section 7.5.

6.1.10 Test for Normality

After the amplitude probability density function of the sample record is estimated, the record should be tested for normality, as indicated by block H in Fig. 6.1. A test for normality will now be proposed.

From Section 7.5, any given confidence interval (assuming a normal distribution of estimates) can be established for estimates at any point on a probability density function curve, given the signal bandwidth B and the record length T. Let it be assumed that a sample record has a Gaussian amplitude probability density function. Calculate the upper and lower normal confidence interval limits desired for the appropriate values of B and T, as well as the appropriate amplitude window (Δx) for the specific analyzer being used. Plot a theoretical Gaussian probability density function with the confidence limits at all amplitude points as shown in Fig. 6.5.

It now remains to determine if the estimated probability density function $\hat{p}(x)$ of the sample record will properly fit a given normal confidence interval as shown in Fig. 6.5. The question really is, what does a proper fit convey?

Remember that the estimate $\hat{p}(x)$ is actually a set of estimates, each for some different amplitude window, Δx. If $\hat{p}(x)$ is estimated
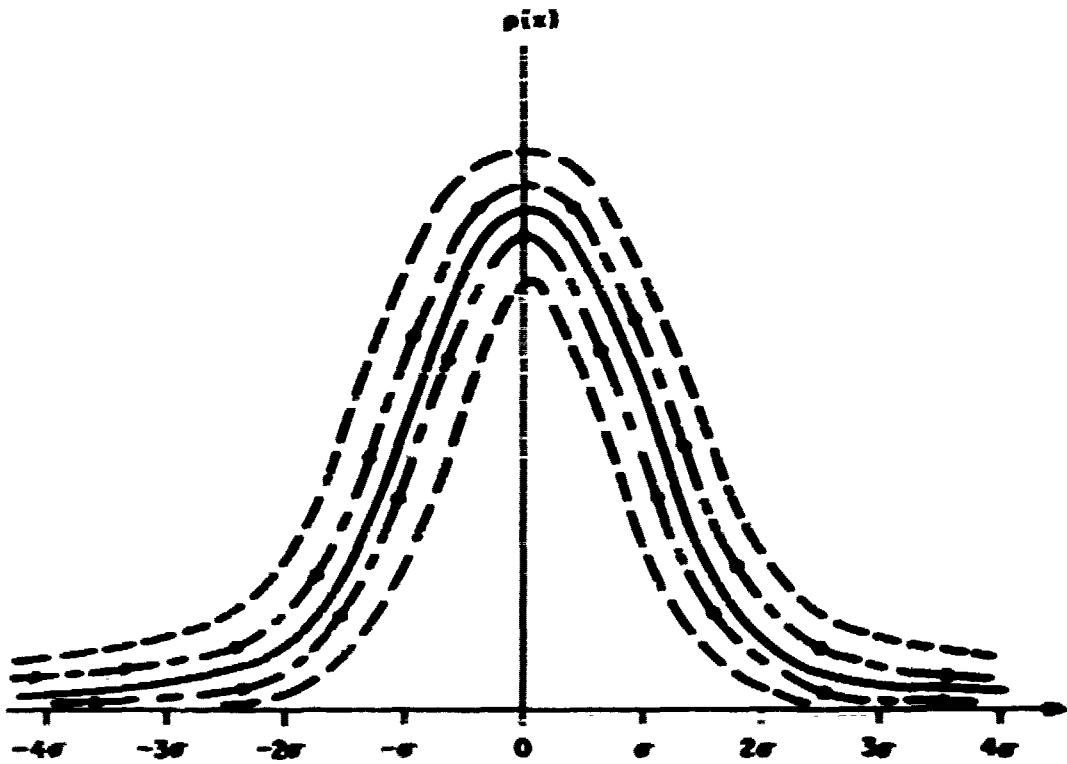
Figure 6.3. Test for Normality

over some amplitude range $(x_2 - x_1)$, the total number of experiments, N, represented by different $\hat{p}(x)_i$ will be,

$$N = \frac{(x_2 - x_1)}{\triangle x} \tag{6.43}$$

It would then be expected that αN of the individual estimates $\hat{p}(x)_i$ would fall outside a $(1 - \alpha)$ normal confidence interval. For example, if the amplitude range of $\hat{p}(x)$ is $(x_2 - x_1) = 8 \sigma$, and the amplitude window is $\triangle x = 0.1 \sigma$, then $N = 80$. For an 80% normal confidence interval $(\alpha = 0.20)$, it would be expected that $\alpha N = 16$ of the 80 estimates would fall outside the 80% confidence interval. For a continuous probability density estimate curve, it would be expected that the estimate curve would be outside the 80% confidence interval along 20% of the range $(x_2 - x_1)$.

The procedure for establishing a level of significance at which $\hat{p}(x)$ will be rejected as non-normal is precisely the same as was developed for the test for self-stationarity in Section 6.1.8. That is, assume each of the N number of individual estimates $\hat{p}(x)_i$ to be a statistically independent experiment with two possible outcomes, failure or success. The resulting random variable (failure or success) will have a binomial distribution. As before, if N is large and α is small, the binomial distribution may be approximated by a Poisson distribution or further approximated in the limit by a normal distribution. Appropriate confidence intervals may be established as detailed in Section 6.1.8 to reject a sample record as being non-normal at any desired level of significance.

Remember that the above proposed test for normality is basically a null hypothesis test as are the proposed tests for randomness and stationarity. It is hypothesized that the sample record is obtained from a vibration time history with a normal amplitude probability density function. A test is conducted to see if there is strong reason to believe that the hypothesis is not true, and if so, the sample record is rejected as non-normal at a given level of significance. The proposed test for normality is then really a test for non-normality.

Those records which are not rejected as non-normal are accepted as being normal. The confidence in the result of the test is a confidence in a record being non-normal and not a confidence in a record being normal.

As the level of significance of the test is decreased, the risk of rejecting records which are truly normal as being non-normal (Type I error) is reduced. However, as the Type I error is reduced, the risk of accepting non-normal records as being normal (Type II error) is increased. It is believed that the arguments concerning Type II error presented at the end of Section 6.1.3 also apply to the proposed test for normality. If the equivalent number of degrees of freedom, n, for the probability density function estimate $\hat{p}(x)$ is, say, at least 20, the Type II error should be incidental.

Now consider in more detail the significance of the results of the test for normality. Assume a sample record is found to be normal. Referring to Section 6.1.6, the record may now be considered as strongly self-stationary rather than only weakly self-stationary. Furthermore, the test for stationarity is based upon an assumed chi-square distribution of mean square values. Theoretically, the distribution of mean square values is known to be a true chi-square distribution when the original amplitude distribution is normal. Hence, if a sample record is found to be normal, the results of the test for stationarity can take on more technical rigor.

One may wonder why the test for normality is not conducted before the test for stationarity. The reason is that the procedure for estimating an amplitude probability density function from a sample record, needed to test for normality, requires that the sample record be self-stationary. If one wanted to be completely safe, it might be argued that neither the test for normality nor the test for stationarity is rigorously valid unless the sample record is accepted by both tests as being normal and self-stationary. However, because of the Central Limit Theorem and its practical consequences, it is not necessary to require that the original record be normally distributed before applying the test for stationarity. The concepts of stationarity and normality are quite distinct, and a stationary random record may or may not be normally distributed.

In view of the above statement, the over-all recommended procedure for analysis will not be terminated because a weakly self-stationary record is found to be non-normal. Furthermore, the above considerations should not pose a serious problem unless the amplitude density function for a sample record deviates drastically from a normal distribution, which is not considered likely. It should simply be kept in mind that the confidence in the test for stationarity will be somewhat less rigorous if the record is found to be non-normal. The same position will be taken in regard to some of the analyses procedures to come, where the statistical accuracy is developed assuming a normal probability density function for signal amplitudes.

One additional comment should be noted. The test for normality described here is applicable to an analog sample record. If the data is available in digital form, the normality of the sample record may be tested directly by the chi-square goodness of fit test as detailed in Section 5.3.2.

6.1.11 Root Mean Square Level Analysis

The next step indicated by block I in Fig. 6.1 is an rms level measurement. This step actually will not supply any new or significant information. An estimate of the rms level of the record will automatically result from the amplitude probability density analysis discussed in Section 6.1.9, since the rms level is simply the standard deviation, σ, of the probability density function. The rms level also will be yielded by the power spectrum of the signal, to be discussed in the next section. It is included as a cardinal step in the over-all procedure because, in the past, it has sometimes been the only statistic of structural response that was measured and analyzed. By itself, the rms level of a sample record will give only a "ball park" idea of the intensity of a vibration environment.

The general procedures involved in estimating the rms level of a sample record, however, are of considerable importance because they are fundamental to the power spectral density estimation problem. As a result, voltmeter estimation of rms levels of sample records

and the statistical accuracy of such estimates are discussed in detail in Sections 7. 3 and 8. 3. 1.

6. 1. 12 Power Spectral Density Analysis

Perhaps the most important single statistic of a self-stationary random vibration time history record is the power spectral density function. The power spectrum of all self-stationary sample records should be estimated as indicated by block J in Fig. 6. 1.

The power spectral density function describes the frequency composition of the vibratory response. For linear systems, the response power spectrum is equal to the input power spectrum multiplied by the square of the transfer function (magnitude response function) for the structure. Thus, power spectra estimates can yield information concerning the dynamic characteristics of the structure. The total area under the power spectrum curve is the mean square value of the response. To be more general, the mean square value of the response in any frequency range of concern is determined by the area under the power spectrum curve bounded by the limits of that frequency range. Obviously, power spectra data will be required for any analysis objective.

Physical significance of the power spectral density function of structural vibration response data is shown clearly in Section 9 and elsewhere throughout this report. A detailed discussion of the techniques for estimating the power spectrum of a vibration signal from a self-stationary sample record, and the associated accuracy of such estimates, is presented in Sections 4. 8 and 7. 4.

6. 1. 13 Autocorrelation Analysis

Other analyses remaining in the over-all recommended procedure shown by Fig. 6. 1 are considered specialized analyses. This is to say that they should be accomplished only when required by particular applications and analysis objectives.

The first such specialized analysis is autocorrelation as indicated by block K in Fig. 6. 1. The autocorrelation function of a vibration time history is the inverse Fourier transform of the power spectral

density function. Thus, the determination of the autocorrelation function will technically not yield any new information over the power spectrum. However, there are situations when the autocorrelation function will better define the information than will the power spectral density function. An example is the case of detecting periodic components in an otherwise random signal. Autocorrelation analysis will quickly identify periodicities that might be missed in the power spectrum, as discussed in Section 7.6. As a result, autocorrelation analysis would be very valuable in support of block Q in Fig. 6.1, to be discussed later.

A detailed discussion of techniques for estimating the autocorrelation function of a vibration signal from a self-stationary sample record, and the associated statistical accuracy of such estimates, is presented in Section 7.6. The importance of cross-correlation functions is shown there by an example of vibration source localization.

6.1.14 Peak Value Distribution Analysis

The next specialized analysis, indicated by block L in Fig. 6.1, is peak value distribution analysis. The distribution of peak values in the structural response time history is of prime importance if the general analysis objective is the prediction of structural fatigue life. The significance of peak value distributions in the problem of fatigue prediction is discussed in Section 9.4

Peak value distribution analysis is actually an extension of amplitude probability density analysis as discussed in Sections 4.9.3 and 6.1.7.

A practical technique for estimating peak value distributions from sample records is detailed in Section 7.5.

6.1.15 Extreme Value Analysis

Another specialized analysis that might be of interest, particularly for fatigue life predictions, is an analysis of extreme values as indicated by block M in Fig. 6.1. A brief analytical discussion of extreme value is presented in Section 4.9.6.

6.1.16 Threshold Crossing Analysis

An investigation of threshold crossings, as indicated by block N in Fig. 6.1, may be desired if the mean time between arbitrary level crossings is of interest. Such information would be useful, for example, in predicting collisions between a piece of equipment mounted on the structure and some nearby object. In a more general case, threshold crossing data could be used to design the optimum spacing between adjacent pieces of shock mounted equipment in the vehicle. Of course, the vibration time history data in this case would have to be an analog of displacement. Threshold crossing analysis might also be of some aid in fatigue life prediction.

A brief analytical discussion of threshold crossings is presented in Section 4.9.3.

6.1.17 Oscillating Mean Analysis and Other Future Data Analysis

An "oscillating mean" analysis, as indicated by block O in Fig. 6.1, is included only to note that research is currently in progress on the subject. The oscillating mean is defined as a line which will pass through the mid-point between each peak and following valley of the vibration time history record. It is believed that the distribution of this oscillating mean, combined with the distribution of instantaneous amplitudes about it, will present an improved technique for predicting structure fatigue life. Those interested in further details are referred to Ref. [9].

Block P in Fig. 6.1 provides for other future data analysis, or more specialized analysis for particular applications, which are not covered by procedures already discussed.

6.1.18 Investigation for Periodic Components and Separation of Periodic and Nonperiodic Data

Let attention now be returned to blocks D and E in Fig. 6.1. Those sample records that are found to be nonrandom in either block are transferred to block Q which will now be discussed.

The most common reason for a record being considered non-random by either visual inspection of the time history (Section 6.1.4), or by application of the Run Test (Section 6.1.5), will undoubtedly be the presence of periodic components. If periodic components are present, it may often be possible to separate them from the random components and then return the record with periodicities removed to the over-all recommended procedure for random data.

The first problem is the detection problem. The two most effective detection procedures have already been discussed. A narrow band spectrum analyzer will often reveal periodic components in a random background to the experienced engineer. The difficulty is to distinguish between a periodic component and a sharp peak in the spectrum caused by the random response of a lightly damped structural resonance. The power spectral density analyzer of block J in Fig. 6.1 could be used for this sort of investigation. A second and more powerful detection procedure is autocorrelation analysis of the record, as discussed in Section 7.6.

The next problem is the separation problem. The required separation can be accomplished with high "Q" notch filters. The so-called peak-notch filters found in many random vibration shaker systems in vibration laboratories should be acceptable for this application. Of course, when a sine wave is filtered out of the record by a notch filter, a narrow frequency band of the random component will also be removed. However, if the "Q" of the notch filter is quite high (if the bandwidth is very narrow), the random data removed will be incidental in terms of the total broadband random energy represented in the record.

After all periodic components are removed, the record should be returned to the over-all recommended procedure at block E in Fig. 6.1, the test for randomness. If the record is again rejected as nonrandom, it should be discarded, or removed for special analysis not covered by the present procedures.

6.1.19 Periodic Data Analysis

Periodic components separated from otherwise random records in block Q may be analyzed separately, as indicated by block R in

Fig. 6.1. Procedures and techniques associated with periodic data analysis are thoroughly covered in the literature. A summary of pertinent mathematical properties of periodic signals is presented in Sections 4.1 and 4.2, with additional physical material available in Sections 3.3 and 9.1.

6.1.20 Nonstationary Data Analysis

Those sample records that are found to be self-nonstationary by the test for stationarity in block F, are set aside for special consideration as indicated by block S in Fig. 6.1. The analysis of random vibration time history records which are self-nonstationary is beyond the scope of this report. Considerably more theoretical work is needed on this subject.

6.1.21 Statistical Errors and Instrument Errors

Errors associated with the analysis of a single vibration time history sample record by the over-all recommended procedure may be divided into two main categories:

   a.   statistical estimation errors
   b.   instrument errors (sometimes called calibration errors)

Statistical estimation errors associated with each recommended analysis technique in the over-all procedure have been developed for each technique. It should be remembered that the predicted statistical errors presented are over and above the conventional instrument errors to be expected. The instrument or calibration error for each analysis procedure is, of course, a function of the specific equipment employed, and must be evaluated in terms of an actual data reduction instrumentation system.

6.2 PROCEDURE FOR ANALYZING COLLECTION OF VIBRATION RECORDS

The preceding section has presented methods for analyzing each individual vibration record by itself. A procedure for gathering and

analyzing statistical properties associated with a collection of records will be presented in this section. Sections 6.2.1 through 6.2.5 to follow discuss theoretical considerations involved in selecting an appropriate sampling technique so as to reduce the amount of data to be gathered. Section 6.2.6 then displays a block diagram for carrying out this technique for general situations. The final Section 6.2.7 outlines a step-by-step procedure for statistical analysis over the collection of records which have been gathered. This procedure is related closely to material in Sections 5 and 6, and the reader will be referred to these sections for details.

6.2.1 Random Sampling Considerations

It is the purpose of this section to present one possible approach to obtain the maximum amount of information from a minimum amount of data. Such an optimization procedure can often be handled best by applying known statistical methods. To reduce the quantity of data that need be gathered, and to avoid the human influence in biasing the results, a random sampling scheme is proposed. Knowing the mean time between samples, the distribution about this mean, and the length of each sample, quantitative results can be obtained allowing the prediction of any given vibration level occurring, how often a given level might occur, how well the samples represent the entire vibration life history of a vehicle, and the minimum number of samples required for a given confidence in the results.

Another reason for a random sampling technique, as opposed to predetermined sampling, is that if samples are taken at predetermined times, the probability of recording events which occur between samples will be zero. Random sampling handles this problem by providing, within a known probability and confidence, that the samples will record high and low vibration levels in the same proportion as they occur in flight.

It is not intended that the random collection of data, which is proposed here, take the place of specific test flights made to collect specific data. Rather it is to be used where it is desired to know the over-all long range vibration life history of flight vehicles.

Since no single given mean time between samples will fit all flight vehicles, or take into account the lengths of different flight phases of any one vehicle, a breakdown is needed to help determine what type of sampling scheme would apply to a particular vehicle, and/or each flight phase of the vehicle. Prime concern is the duration of each flight phase and the frequencies most likely to be present during each phase. A summary of this information is shown in Table 3.1 of Section 3.2. It is realized that not every particular flight vehicle will fit exactly into one of the four categories shown in Table 3.1. This breakdown just serves a qualitative purpose to indicate some of the relative lengths of flights and flight phases to help decide the number of samples to be taken and their time duration.

Some of the additional factors that must be considered are:

1. End use of data (analysis objective).

2. Frequency range and dynamic range of telemetry and recording equipment available.

3. Frequency range of the vibration to be measured.

4. Length of recording time and number of channels that are available.

5. Before deciding to use a random sampling technique, it has to be determined if the savings in the amount of data and data reduction time are greater than the weight penalty and expense of equipment required to accomplish random sampling.

Some of the equipment (mentioned in item 5 above) needed to accomplish random sampling, would be a random noise source to trip a relay at random time intervals to actuate telemetering or recording equipment. If it is required to know exactly when during the flight each random record was taken, a Time-of-Occurrence-Marker should also be used.

Random noise sources, for example, are made by the General Radio Co., West Concord, Massachusetts, Automation Laboratories, Inc., Westbury, L.I., N.Y. and others, while Time-of-Occurrence-Markers are made by the Applied Science Corp., Princeton, New Jersey.

Other companies working in this area are B & K Instruments, Cleveland, Ohio, Flow Corp., Arlington, Massachusetts, Panoramic Radio Products, Inc., Mount Vernon, New York, and many others.

Moreover, everyday smaller and lighter telemetering equipment becomes available and possibly by combining several of the above items into a single random-sampling-telemetering unit, no weight penalty at all may be involved.

For reasons of simplicity both in instrumentation and in later statistical analysis, it is recommended that sample lengths be of fixed duration, and only the time interval between samples should be random. Prior knowledge of the sample length, and of the frequency range being recorded, enables one to make valid statistical estimates about the accuracy of various measurements of interest, such as power spectrum measurements or probability density measurements, as discussed in Section 6.1. On the practical side, instrumentation problems of recording such data appears easier if it is decided in advance that each sample length should be of a certain definite duration.

6.2.2 Probability of Missing Particular Events

The first problem of concern is that, if data is not recorded continuously, certain important unexpected events might be missed. Therefore, the probability of missing unexpected events should be known or estimated. This is best illustrated by an example. Let

e = length of unexpected event in seconds

T = length of each sample in seconds

and $\bar{L}$ = mean time between samples in seconds (measured from center to center)
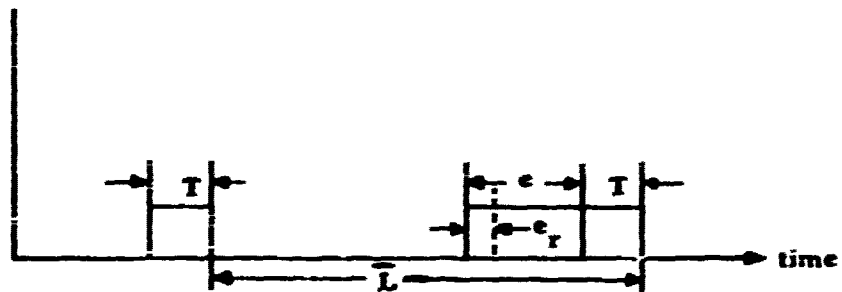
Figure 6.4. Example of Random Sampling

Figure 6-4 shows one possible sequence, given that e will occur once (at random) during time $\overline{L}$. The "average" probability (since is also an average) of missing all parts of event e is then

$$\mathbf{P}\text{ (missing } e\text{)} = \frac{\overline{L} - (T + e)}{\overline{L}} = 1 - \frac{T + e}{\overline{L}} \qquad (6.44)$$

assuming $e > 0$ and $T + e < \overline{L}$.

To illustrate Eq. (6.44), consider the limiting case as e approaches zero. For $e = 0$, from Eq. (6.44),

$$\mathbf{P}\text{ (missing } e = 0\text{)} = \frac{\overline{L} - T}{\overline{L}}$$

Now, if $\overline{L} = 2T$, one obtains

$$\mathbf{P}\text{ (missing } e = 0\text{)} = \frac{1}{2}$$

which agrees with the observation that there is an average probability of (1/2) of missing unexpected events of infinitesimal size if the mean time between samples is twice the sample length.

Equation (6.44) does not apply to situations of predetermined fixed sample lengths and fixed times between samples. For predetermined fixed sampling techniques,

$$\text{Prob (missing } e\text{)} = \begin{cases} 0 & \text{if } e \text{ occurs inside sample length} \\ 1 & \text{if } e \text{ occurs outside sample length} \end{cases}$$

Similarly, all further remarks do not apply to these fixed cases.

For random sampling techniques, from Eq. (6.44), the average probability of recording any part of the event e is

$$\mathbf{P}\text{ (recording } e\text{)} = 1 - \left(1 - \frac{T + e}{\overline{L}}\right) = \frac{T + e}{\overline{L}} \qquad (6.45)$$

For example, if $\overline{L} = 100$ seconds, $T = 5$ seconds, and $e = 10$ seconds

$$P \text{ (recording } e) = \frac{5 + 10}{100} = 0.15$$

i. e. , there is a 15 percent probability (on the average) that any part of e will be recorded if the sample length is 5 seconds, the mean time between samples is 100 seconds and the length of the random event is 10 seconds. This would appear to be quite low. However, if e does occur only once, it might not be of very great interest anyway. Yet if e would occur, for example, 10 times during a 1000 second interval, a much higher probability would be required and, in fact, exists. Using the same values for e, T, and $\overline{L}$ as above the average probability of recording any part of e at least once is now

$$P \text{ (recording } e) = 1 - \left(1 - \frac{T + e}{\overline{L}} \right)^{10} = 1 - 0.16 = 0.84$$

The relationships above apply to the probability of recording "any part of e", the length of which could be infinitesimally small. It would be more satisfactory to know "What is probability of recording at least 1 second of a 10 second event?" Let e_r be the minimum portion of e to be recorded. Equation (6.45) now becomes for $e_r \leq T$ and $e_r \leq e$

$$P \begin{pmatrix} \text{recording at} \\ \text{least } e_r \end{pmatrix} = 1 - \left[1 - \frac{T + (e - e_r)}{\overline{L}} \right] = \frac{T + (e - e_r)}{\overline{L}} \qquad (6.46)$$

subject to the assumptions that $e > 0$ and $T + e \leq \overline{L}$. In deriving Eq. (6.46), the probability of recording at least e_r is obtained from the probability of missing at most $(e - e_r)$ the sum of these two probabilities being unity. For the example values $\overline{L} = 100$ sec., $T = 5$ sec., $e = 10$ sec., and $e_r = 1$ sec., the result would become $P = 0.14$.

Nothing has been said so far about the probability density of times between samples except that some sort of random distribution is implied. If this distribution is known then a confidence level can be established for the probability of recording e. Assume the time between samples has a normal density function with a known standard deviation of σ (as determined in advance for a particular piece of recording equipment), then the probability of missing e will have a range

$$\frac{(L - \lambda\sigma) - (T + e)}{L - \lambda\sigma} < P(\text{missing } e) < \frac{(L + \lambda\sigma) - (T + e)}{L + \lambda\sigma} \qquad (6.47)$$

where λ is the number of standard deviations required for a given confidence. In particular, for 95 percent confidence with a normal distribution of times between samples, the constant $\lambda = 2$. The word "confidence" should be interpreted here by the statement that if a series of measurements are made and the probabilities determined, then 95% of these probabilities will lie in the range expressed by Eq. (6.47).

Equation (6.47) can also be written

$$1 - \frac{T + e}{L - \lambda\sigma} < P(\text{missing } e) < 1 - \frac{T + e}{L + \lambda\sigma}$$

Hence, the probability of recording any part of e will have the range

$$\frac{T + e}{L + \lambda\sigma} < P(\text{recording } e) < \frac{T + e}{L - \lambda\sigma} \qquad (6.48)$$

and the probability of recording at least an e_r time interval of an event e will have the range given by

$$\frac{T + (e - e_r)}{L + \lambda\sigma} < P\left\{\begin{matrix} \text{recording at} \\ \text{least } e_r \end{matrix}\right\} < \frac{T + (e - e_r)}{L - \lambda\sigma} \qquad (6.49)$$

For example, if $\overline{L}$ is normally distributed with $\sigma = 5$ sec., and $\overline{L} = 100$ sec., $T = 5$ sec., $\epsilon = 10$ sec., $e_r = 1$ sec., then for $\lambda = 2$, it can be said with 95 percent confidence that the probability of recording at least one second of the unexpected event e, will have the range

$$\frac{5 + (10 - 1)}{100 + 10} < P \begin{pmatrix} \text{recording at} \\ \text{least } e_r \end{pmatrix} < \frac{5 + (10 - 1)}{100 - 10}$$

$$0.127 < P \begin{pmatrix} \text{recording at} \\ \text{least } e_r \end{pmatrix} < 0.156$$

If ten such unexpected events should occur during a period of 1000 seconds, then the probability of recording at least 1 second of one of the events, with the other parameters the same as above, will be

$$1 - \left[1 - \frac{5 + (10 - 1)}{100 + 10}\right]^{10} < P \begin{pmatrix} \text{recording at} \\ \text{least } e_r \text{ once} \end{pmatrix} < 1 - \left[1 - \frac{5 + (10 - 1)}{100 - 10}\right]^{10}$$

$$0.742 < P \begin{pmatrix} \text{recording at} \\ \text{least } e_r \text{ once} \end{pmatrix} < 0.813$$

with 95 percent confidence.

From the equations above, the sample length and mean time between samples can now be determined for a given confidence in a known risk of missing unexpected severe vibration levels of arbitrary lengths e. Of course a qualitative evaluation still has to be made as to what constitutes a "risk," i.e., how high should the vibration level be and how long should it last before it is considered of importance to know the probability of missing it.

6.2.3 Probability of Including Range of Events

Another criteria that determines sample length and mean time between samples is the total number of samples required for a given

confidence to predict the total vibration life history from these samples, namely, the total range of possible events. This can best be illustrated by approaching the problem in reverse, i.e., given a number of samples, what can be said of their accuracy in representing the continuous process from which they were taken.

For example, suppose 50 samples, each 5-seconds long, are available which have been taken at random times. Many parameters may be calculated from these samples involving amplitude or frequency properties. For purposes of illustration here, consider only one such parameter. To be definite, suppose the rms acceleration for each 5 second sample is calculated. A single number, g's rms, now represents each of these samples and can be plotted vs. time as shown in Fig. 6.5.
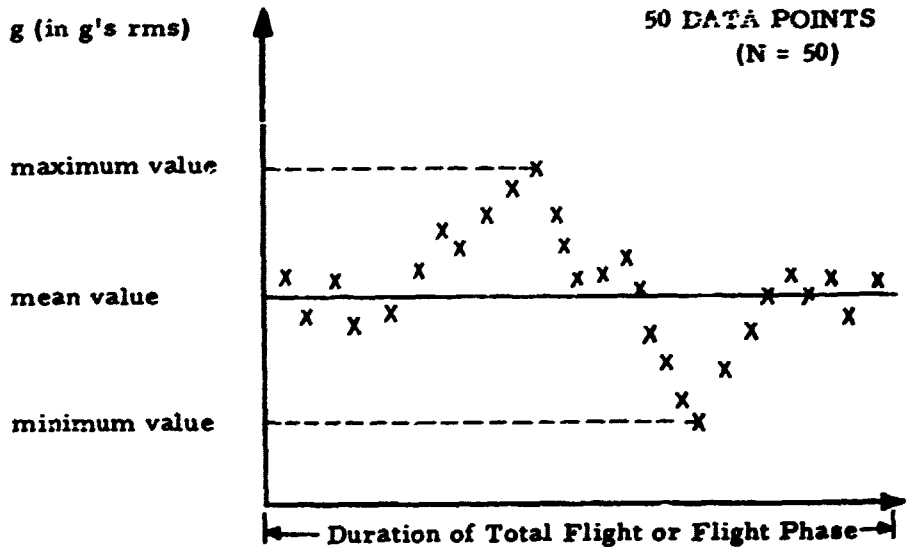
Figure 6.5. Example of Range of Events

(It should be noted that the division of the time axis may be arbitrary. It is assumed here that the real time relationship is to be maintained since it may be of interest to know which particular sample point is taken during any given flight maneuver.)

The minimum and maximum values can be noted from Fig. 6.5, and the mean value and standard deviation can be calculated without knowing anything about the distribution of these points. Let m be the mean value and s the standard deviation from the mean, then

$$m = \frac{1}{N} \sum_{i=1}^{N} g_i \qquad (6.50)$$

where N is the sample size, and

$$s = \sqrt{\overline{g^2} - m^2} \qquad (6.51)$$

where

$$\overline{g^2} = \frac{1}{N} \sum_{i=1}^{N} (g_i)^2 \qquad (6.52)$$

Several conclusions can now be drawn from Fig. 6.5 if the various values are statistically independent. This assumption appears reasonable in nearly all physical applications, provided that the autocorrelation function of the process that is being sampled is zero for time delays equal to the mean time $\overline{L}$ between samples.

It has been shown (Ref. [1], pp. 201-204) that a stationary exponential-cosine autocorrelation function $R_g(\tau)$ (see Fig. 6.7) applies to a common power spectrum as shown in Fig. 6.6. As long as $S_g(f) \to 0$ as $f \to \infty$ where f is frequency, then $R_g(\tau) \to 0$ as $\tau \to \infty$. (For the vibration environment in missiles and aircraft the spectral density is approximately zero above 10,000 cps and often is considered negligibly small above 2000 or 3000 cps.)
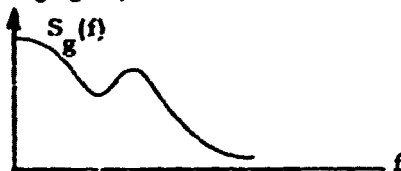
Figure 6.6. Common Power Spectrum
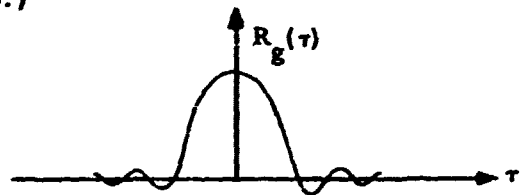
Figure 6.7. Exponential-Cosine Autocorrelation Function

For the particular case of a flat (bandwidth-limited) power spectrum (Fig. 6.8) the Sampling Theorem (Ref. [1], pp. 55-60) shows that
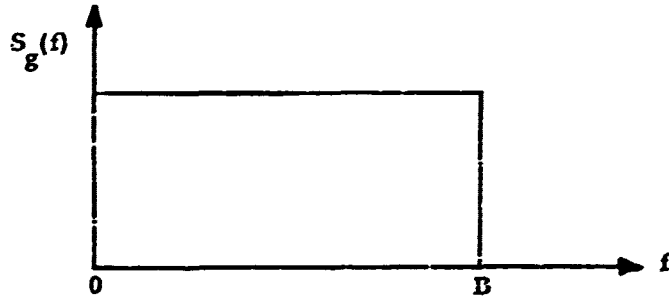
Figure 6.8. Flat Power Spectrum

the first zero crossing of $R_g(\tau)$ occurs at $\tau = (1/2B)$, where B is the bandwidth of $S_g(f)$. It can now be stated, that as long as $\bar{L} > (1/2B)$, the rms accelerations in Fig. 6.5 are approximately independent, with the approximation rapidly becoming better as $\bar{L} \rightarrow \infty$. It can be seen this is really no limitation at all on $\bar{L}$ since for B = 2000 cps, $L >$ $(1/4000) = 0.00025$ seconds.

A more stringent limitation is that $\bar{L}$ should be at least as long as the period of the lowest frequency of interest. If the lowest frequency of interest is 5 cps, then $\bar{L}$ should be at least 0.2 seconds long to assume statistical independence. Again, this criteria can usually be met but should be kept in mind as one of the limitations on $\bar{L}$. The lowest frequency of interest is also a consideration when determining the sample length T.

Another assumption that is usually made is that the statistical parameters of the random process (i.e., vibrations) do not change with time. This is another way of saying that the process should be stationary.

This assumption will usually not be valid for an entire flight and therefore the flight will have to be broken up into several periods, each of which may be reasonably assumed stationary. Probably few random processes of interest in the physical world are truly stationary. However, the assumption still provides a useful and simple model which can give results in good agreement with experiments.

The ergodic property is also quite often of concern. If it is only required to know how well the samples represent the one single flight from which they were taken, ergodicity does not enter the picture at all. It only becomes a problem if it is required to know how well these samples represent flights to be made in the future, or other flights from which no samples are available.

For example, samples taken during the cruise phase may or may not represent the environment of another aircraft during cruise. This will only be true if the cruise phase was made under "similar" conditions (i.e., weather, throttle setting, pay-load, etc.). The word "similar" also implies that there are certain random variations which cannot be controlled. Therefore, more than one flight should be made so that the random variations can be taken into account.

1. <u>One-Sided Test.</u> As mentioned above, it is assumed that the distribution of these rms values is not known. The proportion of the population (the number of rms accelerations for all 5-second intervals possible during the flight, i.e., for a 1 hour flight there are 720 five-second intervals) occurring below the maximum sample value can be calculated with a known confidence from the formula (Ref. [2], pp. 162-163),

$$(P)^N = \alpha \tag{6.53}$$

where

$$P = \text{proportion of population}$$
$$1 - \alpha = \text{confidence coefficient}$$
$$N = \text{sample size.}$$

For example from Fig. 6.5, N = 50; therefore it can be said with $(1 - \alpha)$ percent confidence that 100 P percent of the population will be less than the maximum value shown in Fig. 6.5. If the maximum value is for example, 9 g's, and 92 percent confidence is desired then

$$(P)^{50} = 0.08$$
$$P = 0.95$$

and it can be said with 92 percent confidence that 95 percent of all rms accelerations (of 5-second duration) for the entire flight will be less than 9 g's if 50 samples are taken. See Fig. 6.9.

2. __Two-Sided Test (Maximum and Minimum Value).__ If it is required to know what proportion of the population will be between the maximum and minimum sample values recorded, the relationship becomes (Ref. [2], pp. 162-163),

$$NP^{N-1} - (N - 1) P^N = \alpha \tag{6.54}$$

where N, P, and α are defined as before. If for example the maximum value in Fig. 6.5 is 9 g's and the minimum 1 g then for $\alpha = 0.28$ and N = 50

$$50(P)^{49} - 49(P)^{50} = 0.28$$

and the solution for P is

$$P = 0.95$$

Thus, it can be said here with 72 percent confidence that 95 percent of all rms accelerations (of 5-second duration) for the entire flight or flight phase under consideration will be between 1 and 9 g's. To consider another example, for $P = 0.90$, N = 50 it can be said with 96.6 percent confidence that 90 percent of all rms accelerations will be between 1 and 9 g's, since α turns out to be equal to 0.034.

For Eq. (6.53) and (6.54), α and $(1 - \alpha)$ have been plotted against N for various values of P in Fig. 6.9 and 6.10 respectively. These curves clearly show that very little additional confidence is gained by taking a continuous record, no matter how long, rather than approximately 50 samples, if $P = 0.9$.

The assumption is made here that the population (all possible 5-second samples during a flight or flight phase, i.e., a continuous record) is infinite. This assumption is generally considered valid if

Figure 6.9. One-Sided Tent

CONFIDENCE $(1-\alpha)$ VS SAMPLE SIZE N FOR EQUATION (6.54)

$$N(P)^{N-1} - (N-1) P^N = a$$

P=0.95

P=0.90

P=0.85

SAMPLE SIZE: N

CONFIDENCE $(1-\alpha)$
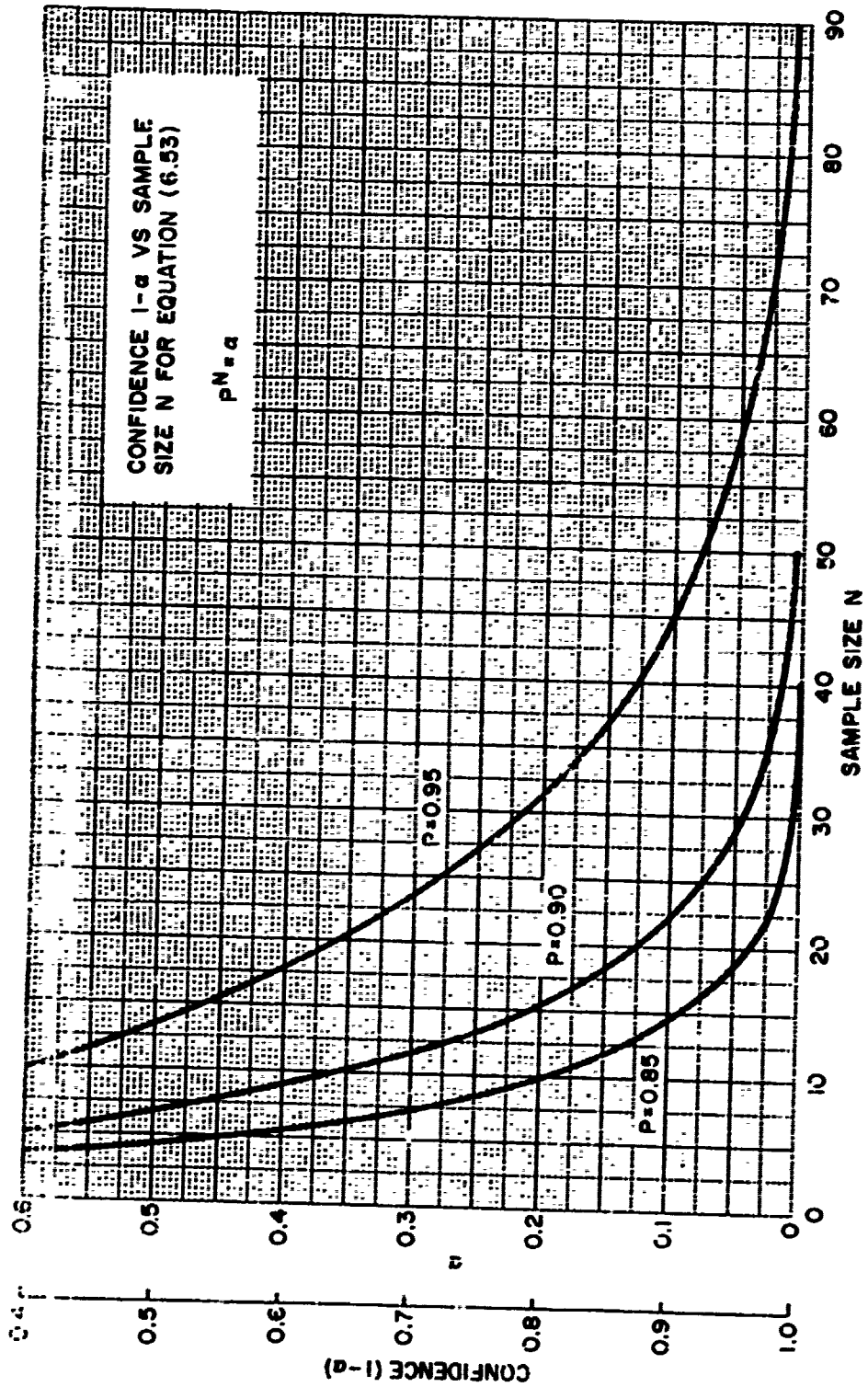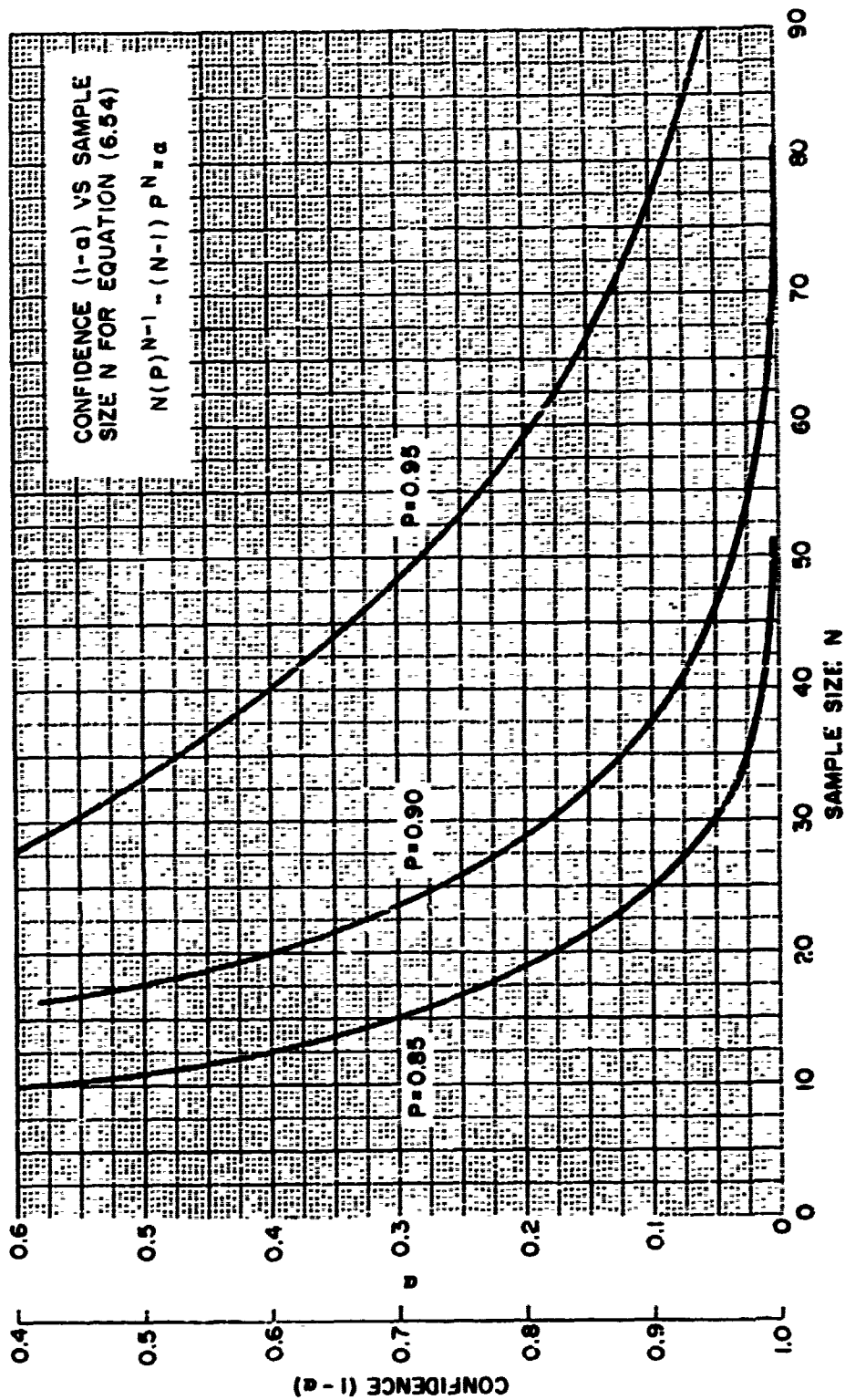
the population is at least ten times as large as the number of samples taken. Yet, for instance, five 5-second samples may have been taken during a flight phase which is only one minute long. In this case the population is only twelve and Eq. (6.53) and (6.54) do not apply. In this case Eq. (6.44) through (6.49) should be used to determine the probability of missing unexpectedly high vibration levels. This is also illustrated in Section 6.2.4, where numerical examples are considered.

3. Two-Sided Test (Arbitrary Values). It may also be of interest to know the probability of the population occurring between $m - \lambda s$ and $m + \lambda s$, i.e., if the mean of the sample in Fig. 6.5 is 5 g's and the standard deviation $s = 1$ g, what is the probability of the population occurring between 3 and 7 g's ($\lambda = 2$)? For $\lambda > 1$, the Tchebycheff inequality (Ref. [1], p. 102 or Ref. [10], pp. 176-177) can be applied, and

$$P\left[\mu - \lambda\sigma < g < \mu + \lambda\sigma\right] > 1 - \frac{1}{\lambda^2} \qquad (6.55)$$

where μ is the true population mean, and σ the true population variance.

Before using the inequality (6.55), μ has to be evaluated. If $E(m)$ is the expected value of the sample mean then

$$E(m) = \mu \qquad (6.56)$$

This cannot be evaluated directly, but the distribution of m can be found and therefore the range of $E(m)$ will be known since the distribution of m, from the Central Limit Theorem (see Section 4.3.3(b)), will approach the normal distribution as the number of samples, N, approach infinity. It would appear that a large number of samples would be required. However it has been shown that even with $N = 10$ or 12, a very good approximation to normality is obtained (Ref. [8], p. 138 and Ref. [10], pp. 180-183). The variance of the distribution of the sample mean is given by (see Eq. (5.33)),

$$\sigma_m^2 = \frac{\sigma^2}{N} \qquad (6.57)$$

Now the problem arises of determining σ^2, the variance of the population. About all that can be said, is that the expected value of the sample variance, $E(s^2)$ is equal to $\left[(N - 1)/N\right]\sigma^2$. Therefore for samples larger than 10, $E(s^2) \sim \sigma^2$. To find the distribution of s^2, or even the higher moments such as the standard deviation, is very difficult and is discussed in Ref. [11], pp. 113-138. However, since the population is not infinite, it can be stated intuitively that the standard deviation of s^2 will decrease with an increase in sample size. For purposes of simplicity it will now be assumed that $\sigma^2 = s^2$, but it should be kept in mind that s^2 is a random variable and not precisely known.

From Eq. (6.57) with $\sigma = s$, and letting $N = 50$ as in Fig. 6.5,

$$\sigma_m = \frac{s}{\sqrt{N}} = \frac{1}{\sqrt{50}} = 0.14$$

Therefore, assuming normality, if the sample mean value $m = 5$ g's, the probability of the true population mean μ lying between 4.72 and 5.28 g's ($\pm 2\sigma_m$) is 0.95.

Again the above estimates of the population mean and population variance are only valid for an infinite population. As previously stated this assumption is considered valid if the population is at least ten times the sample size.

If the population is smaller than this, the estimate of the population variance becomes (see Ref. [4], pp. 253-254),

$$\hat{\sigma}^2 = \left(\frac{M - 1}{M}\right)\left(\frac{N}{N - 1}\right) s^2 \tag{6.58}$$

where M is the size of the population and N the number of samples. The standard deviation of the distribution of the sample mean becomes

$$\sigma_m = s \sqrt{\frac{M - 1}{M(N - 1)}} \tag{6.59}$$

Now the problem arises of determining σ^2, the variance of the population. About all that can be said, is that the expected value of the sample variance, $E(s^2)$ is equal to $\left[(N - 1)/N\right]\sigma^2$. Therefore for samples larger than 10, $E(s^2) \sim \sigma^2$. To find the distribution of s^2, or even the higher moments such as the standard deviation, is very difficult and is discussed in Ref. [11], pp. 113-138. However, since the population is not infinite, it can be stated intuitively that the standard deviation of s^2 will decrease with an increase in sample size. For purposes of simplicity it will now be assumed that $\sigma^2 = s^2$, but it should be kept in mind that s^2 is a random variable and not precisely known.

From Eq. (6.57) with $\sigma = s$, and letting $N = 50$ as in Fig. 6.5,

$$\sigma_m = \frac{s}{\sqrt{N}} = \frac{1}{\sqrt{50}} \approx 0.14$$

Therefore, assuming normality, if the sample mean value m = 5 g's, the probability of the true population mean μ lying between 4.72 and 5.28 g's ($\pm 2\sigma_m$) is 0.95.

Again the above estimates of the population mean and population variance are only valid for an infinite population. As previously stated this assumption is considered valid if the population is at least ten times the sample size.

If the population is smaller than this, the estimate of the population variance becomes (see Ref. [4], pp. 253-254),

$$\hat{\sigma}^2 = \left(\frac{M - 1}{M}\right)\left(\frac{N}{N - 1}\right) s^2 \tag{6.58}$$

where M is the size of the population and N the number of samples. The standard deviation of the distribution of the sample mean becomes

$$\sigma_m = s\sqrt{\frac{M - 1}{M(N - 1)}} \tag{6.59}$$

It can be seen that for large M and N, with $N \ll M$, Eq. (6.59) approaches Eq. (6.57).

Another complication arises now, which is, that the sample mean is not normally distributed. However, the assumption of normality will only introduce a small error since the Tchebycheff inequality is a very conservative estimate regardless of the nature of the distribution.

Applying these results to Eq. (6.55) it can be stated with a given confidence that the probability of rms acceleration values (g) occurring in a given range is by letting $\sigma = s$, $\mu = m \pm \lambda' \sigma_m$ (or $\sigma = \sigma_{est.}$ and $\mu = m \pm \lambda' \sigma_m'$ depending on population size) and $\lambda > 1$

$$P\left[(m - \lambda'\sigma_m) - \lambda s < g < (m + \lambda'\sigma_m) + \lambda s\right] > 1 - \frac{1}{\lambda^2} \qquad (6.60)$$

where λ' is the number of σ_m's required for a given confidence.

For example if $m = 5$, $s = 1$, $\sigma_m = 0.14$, $\lambda = 2$, and $\lambda' = 2$

$$P\left[2.72 < \bar{g} < 7.28\right] > 1 - \frac{1}{4} = 0.75$$

i. e., the probability of an rms acceleration (of 5-second duration) occurring between 2.72 and 7.28 g's during the entire flight is greater than 0.75 or the probability of the population being greater than 7.28 g's or less than 2.72 g's is less than 0.25, with a 95 percent confidence.

This probability can be improved considerably if certain assumptions can be made about the distribution of g, namely if

a. The probability density function f(g) of the statistic g is unimodal with the modal value coinciding with the mean; and

b. The function f(g) is monotonic on either side of the modal value.

Then the Camp-Meidell inequality applies (see Ref. [3]), which is

$$P\left[(\mu - \lambda\sigma) < g < (\mu + \lambda\sigma)\right] > 1 - \frac{1}{2.25\lambda^2} \qquad (6.61)$$

If the modal value g_0 does not coincide with the mean μ, the right hand side of Eq. (6.71) becomes (see Ref. [4], pp. 133-184),

$$1 - \frac{1 + c^2}{2.25(\lambda - |c|)^2}$$

where c is a measure of the skewness defined as

$$c = \frac{\mu - g_0}{\sigma}$$

Making the substitutions for μ and σ as in Eq. (6.60), Eq. (6.61) becomes

$$P\left[(m - \lambda'\sigma_m) - \lambda s < g < (m + \lambda'\sigma_m) + \lambda s\right] > 1 - \frac{1}{2.25\lambda^2} \qquad (6.72)$$

and for the same values used in the previous problem:

$$P(2.72 < g < 7.28) > 0.89$$

The assumptions made here are quite important. If, for instance, the sample points had been distributed as shown in Fig. 6.11, the probability density function $f(g)$ would not be unimodal (see Fig. 6.12) and Eq. (6.71) does not apply. It also should be noted that the assumption of stationarity would require special investigation. Such a bimodal distribution might result from a change in flight phase, such as changing from taxi to run-up.

An additional increase of this probability can be obtained if the assumption is made that g is distributed normally. Of course this distribution can never be truly normal since $g > 0$ and normality requires $-\infty < g < \infty$. However, this assumption may be quite good in the region of interest (i.e., $m \pm 3s$). Two tests to determine the error of an assumption of normality are available and are discussed in Sections 5.3.2 and 6.1.10 of this report.
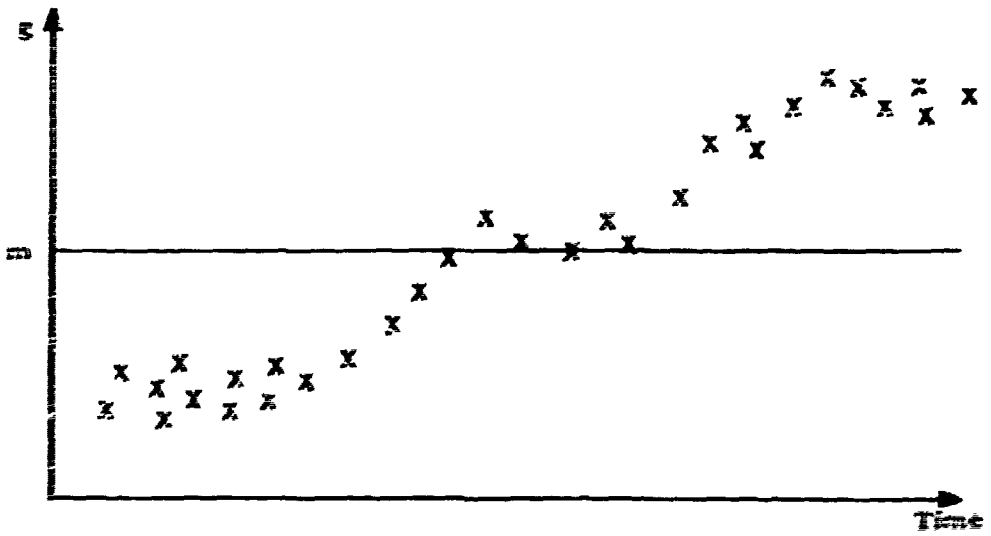
Figure 6.11. Bimodal Sample Set
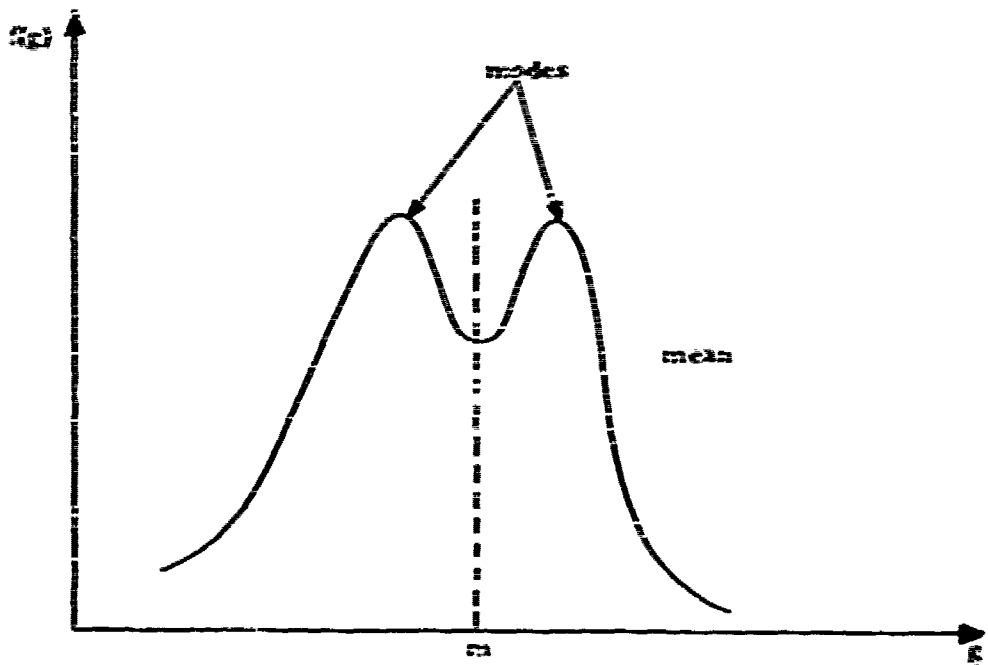
Figure 6.12. Bimodal Probability Density Function

If the region of interest is $m \pm 3s$ the first requirement that should be met for a normality assumption is

$$m - 3s > 0 \qquad (6.63)$$

If subsequently it has been shown that the approximation of normality is good the following relationship applies (Ref. [1], pp. 93-96), see Section 4.3.3(b),

$$P\left[g > \mu + \lambda\sigma\right] = \int_{\mu+\lambda\sigma}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \, e^{-\frac{(g-\mu)^2}{2\sigma^2}} \, dg \qquad (6.64)$$

$$= \int_{\lambda\sigma}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \, e^{-\frac{x^2}{2\sigma^2}} \, dx$$

by letting $x = g - \mu$, $dx = dg$. Similarly,

$$P\left[g < \mu - \lambda\sigma\right] = \int_{-\infty}^{-\lambda\sigma} \frac{1}{\sigma\sqrt{2\pi}} \, e^{-\frac{x^2}{2\sigma^2}} \, dx \qquad (6.65)$$

and therefore

$$P\left[\mu - \lambda\sigma < g < \mu + \lambda\sigma\right] = \int_{-\lambda\sigma}^{\lambda\sigma} \frac{1}{\sigma\sqrt{2\pi}} \, e^{-\frac{x^2}{2\sigma^2}} \, dx \qquad (6.66)$$

is the general equation desired.

Making the previous substitution $\sigma = s$ and $\mu = m \pm \lambda'\sigma_m$ (or $\sigma = \hat{\sigma}$ and $\mu = m \pm \lambda'\sigma_m'$ depending on population size) Eq. (6.65) becomes

$$P\left[(m - \lambda'\sigma_m) - \lambda s < g < (m + \lambda'\sigma_m) + \lambda s\right] = \int_{-\lambda s}^{\lambda s} \frac{1}{s\sqrt{2\pi}} e^{-\frac{x^2}{2s^2}} dx \quad (6.67)$$

The probability integral in Eq. (6.67) is tabulated in various forms in most handbooks on probability theory and statistics. (See Tables 5.1 and 5.2.) For several values of λ the value of this integral is shown in Table 6.3 below and compared to the values obtained when the Tchebycheff and Camp-Meidell inequalities are used (see Ref. [5], pp. 239-240).

Table 6.3. Comparison of Normal, Camp-Meidell, and Tchebycheff Results

	Deviate	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2.0$	$\lambda = 2.5$	$\lambda = 3.0$
P>	Normal	0.6827	0.8664	0.9545	0.9876	0.9973
	Camp-Meidell	0.5556	0.8025	0.8889	0.9289	0.9506
	Tchebycheff	0	0.5556	0.7500	0.8400	0.8889

From Eq. (6.67), for the same values assumed previously,

$$P(2.72 < g < 7.28) > 0.95 \quad \text{(with 95 percent confidence)}$$

6.2.4 Numerical Examples of Random Sampling Technique

When applying the relationships developed in the preceding sections, a considerable number of engineering judgments have to be made. Therefore, the development of a step-by-step procedure is best accomplished by considering and solving a hypothetical example to illustrate in general how this sampling method can be effective.

Problem 1. Jet Aircraft. A sensitive piece of electronic equipment is to be installed in a jet aircraft. The vibration environment in the region where this equipment is to be mounted is not known. One or

more test flights are to be made to establish the vibration levels present in this region. The length of the mission of the aircraft is eight hours, from engine start to shutdown.

Question: How should the data be taken for maximum accuracy and a minimum amount of data?

Step No. 1

Prepare a chart similar to that shown in Table 3.1, Section 3.2, of this report. Category 4 can be used as a guide. A hypothetical chart for the aircraft under consideration is shown in Table 6.4 below.

Step No. 2

From Table 6.4 it is apparent that a single mean time between samples will not adequately cover all flight phases. Therefore, a "long" mean time will be chosen for phases e, f and g (the three longest phases), and a "short" mean time for each of the other phases.

If desired, one may pool together phases of short duration whose lengths are not clearly defined and base calculations on their combined longer time interval. This latter procedure, of course, will destroy individual information about the various short phases, and so should not be followed if individual information is sought.

Step No. 3

A numerical value now has to be determined for the two mean times between samples. The mean time between samples is a function of:

a)   Length of each sample

b)   Number of samples required for a given confidence of not exceeding the maximum value (which in turn depends on length of flight and each flight phase)

c)   Total recording time available for the flight and each flight phase

d)   Probability of recording parts of an unexpected event.

Some engineering judgment is required to reconcile these different considerations.

Table 6.4 Flight Characteristics for Aircraft XXXX

Flight Phase	Approximate Time Duration	Vibration Sources	Relative Severity	Remarks
a. Warmup	1 - 7 Minutes	Jet exhaust	M	M = 1.0 to 10.0g's rms
b. Taxi	5 Minutes	Runway roughness, Jet exhaust	M	
c. Runup	2 - 6 Minutes	Jet exhaust	H	H = 10.0g's rms or higher
d. Takeoff	1 Minute	Runway roughness, Jet exhaust, Atmospheric turbulence	H	
e. Climb	30 Minutes	Jet exhaust, Atmospheric Turbulence. Boundary layer turbulence.	M-H	
f. Cruise and Flight Maneuvers etc.	6.5 Hours	Jet exhaust, Atmospheric turbulence. Boundary layer turbulence buffet	M-H	Time duration may be varied as appropriate for different applications
g. Descent	30 Minutes	Atmospheric turbulence. Boundary layer turbulence	M	
h. Landing gear down. Flaps down	5 Minutes	Flap buffet, Atmospheric turbulence	M	
i. Landing	1 Minute	Runway roughness	M	
j. Taxi	5 Minutes	Runway roughness, Jet exhaust	M	

TRIAL A

a) **Sample Length.** In Section 6.2.3, it is noted that each sample should be at least as long as the period of the lowest frequency of interest. To be on the safe side it will be required here that

$$T \geqslant (2/f_T) \qquad (6.68)$$

where f_T is the lowest frequency of interest in cycles per second. If f_T is 5 cps then

$$T \geqslant (2/5) = 0.4 \text{ seconds}$$

However, there are also other considerations that have to be taken into account. These are:

    1) **Adequate length of tape to form a loop.**

This should preferably be no less than 15 inches. If tape recorder speed is 30 inches per second, then the minimum sampling time should be 0.5 seconds.

    2) **Statistical accuracy in the analysis of each sample.**

If n = number of degrees of freedom, B = filter bandwidth, and T the sample length then, see Section 4.8.5,

$$n = 2BT \qquad (6.69)$$

For reasonable accuracy and confidence in results, the minimum number of degrees of freedom should be at least 20, preferably larger.

Consider the case where the primary analysis will be a power spectral density analysis using a filter bandwidth B. If B = 5 cps, then from Eq. (6.69)

$$T = \frac{n}{2B} = \frac{20}{10} = 2 \text{ seconds}$$

.

It should be noted here that for n to be large, B should be large, yet for good resolution B should be small. A detailed discussion of this matter is presented in Section 7.4.

From the above considerations a minimum sample length of 2 seconds appears here to be a good choice.

b) Number of samples required to include maximum values. When establishing the vibration environment for electronic equipment it is generally most important to know the highest acceleration that might occur during flight. Therefore the one-sided relationship (Eq. (6.53)) will be used to determine the number N of samples required. A decision now must be made to determine a satisfactory probability that the vibration environment will not be higher than the values recorded. It will be assumed that a 95 percent confidence is desired in the claim that 95 percent of the vibrations during any one flight phase will be less than the maximum value recorded. Therefore from Eq. (6.53)

$$(0.95)^{N} < 0.05$$

and from Fig. 6.9,

$$N \geqslant 58$$

For $N < 58$, the above claim and confidence must be reduced. At this point it is obvious that 58 samples, each 2 seconds long, cannot be taken during phases d and i of the flight in question (see Table 6.4), since these phases last only for 1 minute each. Also, no consideration has been given here to a mean time between samples which requires longer phase durations.

c) Short flight phases and probability of recording unexpected events. The first alternative is to continuously record during these two flight phases. But then flight phases a, b, c, h, and j should also be recorded continuously to avoid 6 changes in the sampling procedure. This would require approximately 30 minutes of recording, or 4,500 ft

of tape for a recording speed of 30 inches per second, with only 6 percent of the total flight covered.

The other alternative is to take fewer samples. As a first try it will be decided to take 10 samples, each 2 seconds long. Since now the population (total number of 2-second samples possible in one minute) is only three times as large as the sample size, Eq. (6.53) and (6.54) do not apply. Instead Eq. (6.44), (6.45) or (6.46) can be used to calculate the probability of missing unexpected events. If, for instance, during 5 percent of the time of the flight phase (3 seconds out of 60 seconds) some severe vibration level would occur, it is only necessary that 5 percent of the sample records show this severe vibration level in order to have the same ratio of length of severe vibration to length of normal vibration. In this example the total recording time is 20 seconds. Therefore approximately 1 second of the 3-second event should be recorded. Using Eq. (6.46) with $T = 2$, $e = 3$, $e_r = 1$ and $\overline{L} = 6$, the average probability

$$\mathbf{P} \text{ (recording at least } e_r) = \frac{2 \div (3 - 1)}{6} = \frac{2}{3}$$

Thus, on the average, this result may be satisfactory.

For some applications it might be important to detect the unexpectedly severe vibration levels even if they last, say, only 2 seconds. If they occur 10 percent of the time, there now would be three such unexpected events, each 2 seconds long. To record 2 seconds from these 3 events, 0.67 seconds have to be recorded from each event. The probability of recording at least 0.67 seconds of one event occurring 3 times is for $T = 2$, $e = 2$, $e_r = 0.67$ and $\overline{L} = 6$

$$\mathbf{P} \text{ (recording at least } e_r) = 1 - 1 - \frac{2 + (2 - .67)^3}{6} = 0.91$$

and the probability of recording e_r all of the 3 times is

$$\mathbf{P} \text{ (recording at least } e_r \text{ 3 times)} = (0.91)^3 \approx 0.76$$

These calculations show that engineering judgment is required on how long an unexpected event should be before it is considered important not to miss it, and if the resulting probabilities of detection are high enough.

For the purpose of this example the two second sample length and a mean time between samples of six seconds will suffice. Also, since the mean time between samples is quite short, the samples do not have to be taken at random and a fixed time between samples can be used, if so desired. Using this same sample length and time between samples for the other short flight phases results in a recording time of 10 minutes (or a tape length of 1500 ft for a 30 in/sec recording speed).

d) **Long flight phases and probability of recording unexpected events.** Using a two-second sample length for phases e, f and g, Eq. (6.53) can be applied since the total number of possible two-second samples for the shortest of these three phases is 900.

For 58 samples, each two seconds long, the mean time between samples for phases e and g is

$$\bar{L} (e, g) = \frac{1800}{58} \cong 31 \text{ seconds}$$

These samples will now be taken at random. Assuming $\bar{L}$ to be normally distributed, a standard deviation of times between samples of 5 seconds will be chosen. This will result in a spread of $\bar{L} \pm 15$ seconds with 99.7 percent probability. Of course, the assumption of normality is only approximate since the tails of the distribution of $\bar{L}$ cannot go to $\pm \infty$ (i. e. the probability of a succeeding sample being taken before the preceding one is zero).

Now the probability of missing certain events can be calculated (as was done for the short flight phases) with a given confidence, once it has been decided how long an unexpected event should be before it is of importance not to miss it, by use of Eq. (6.47), (6.48) and (6.49).

For example, if $\bar{L}$ is normally distributed with mean $\bar{L} = 31$ seconds and standard deviation $\sigma = 5$ seconds, then for $T = 2$ sec, $e = 6$ sec, and $e_r = 1$ sec, from Eq. (6.49) with $\lambda = 2$, it can be said with 95 percent confidence that the probability of recording at least one second of the unexpected event e will have the range

$$0.17 < P \text{ (recording at least } e_r) < 0.33$$

For ten such unexpected events the probability of recording at least one second of e will have the range

$$0.84 < P \text{ (recording at least } e_r) < 0.98$$

e) <u>Total recording time.</u> For a mean time between samples of 31 seconds for phase f, the number of samples for phase f will be

$$N(f) = \frac{6.5 \times 3600}{31} \approx 756$$

Then the total number of samples for phases e, f, and g is

$$N(e, f, g) = 2(58) + 756 = 872$$

This results in a recording time of

$$\frac{2 (872)}{60} = 29.1 \text{ minutes}$$

The total recording time for the entire flight is now $29.1 + 10 = 39.1$ minutes.

b) <u>Number of flights.</u> If no conflicts have occurred, the above sampling plan can now be used. The question that next arises, is, what is the number of flights that should be made. This again will depend on the

problem under consideration. Is the electronic equipment to be in-
stalled in more than one airplane? How representative is the one
test flight going to be of other flights? How representative is the
airplane of other airplanes? The answers to these questions fall in
the general category "Statistical Results from Repeated Experiments"
and are discussed in Section 5.4 of this report. It should be noted
here that if more than one flight can be made, fewer samples may
be required for each of the flights than were determined in Steps 2
and 3 above.

Problem 2. Ballistic Missile. A piece of electronic equip-
ment is to be installed in an air-launched ballistics missile (ALBM).
The vibration environment for this equipment is to be determined.
The ALBM will be carried by a jet-aircraft for approximately 4 hours
and then launched. Total powered flight for the ALBM is 2 minutes.

TRIAL A

Step No. 1

Prepare a chart similar to that shown in Table 3.1, Section 5.2.
A hypothetical chart for the ALBM is shown in Table 6.5 below.

Steps No. 2 and 3

For phase (a) of the ALBM shown in Table 6.5, a similar
argument can be made as for phases a-f in Problem #1. It will be
assumed that the same mean times between samples are applicable
here. The short mean time between samples should be resumed
in phase (b), Table 6.5, which calls for firing of the ALBM. How-
ever if data is telemetered throughout the mission of the ALBM one
additional improvement can now be made. Rather than taking no
records at all during the period between samples, a commutator can
be used to transmit the vibration levels from two (or more) additional
points on the structure. This will result in a considerable savings

Table 6.9 Flight Characteristics for ALBM XXXX

Flight Phase	Approximate Time Duration	Vibration Sources	Relative Severity	Remarks
a. Transportation (Airborne only)	See Table 6.4 Phases a - f	See Table 6.4 Phases a - f	See Table 6.4 (a - f)	
b. Takeoff	5 - 30 Seconds	Rocket exhaust, Boundary layer turbulence, Buffet, Atmospheric turbulence	H	Includes flight maneuver to release missile and ignition H = 10.0g's rms or higher
c. Subsonic Flight	15 - 30 Seconds	Rocket exhaust, Boundary layer turbulence, Wind shear and gusts.	H	
d. Transonic Flight	2 - 6 Seconds	Oscillating shocks, Buffet, Boundary layer turbulence	H	(.8M = 1.2M)
e. Supersonic Flight	2 - 4 Minutes	Boundary layer turbulence	M	M= 1.0 to 10.0g's rms
f. Ballistic Trajectory	16 Minutes	None	L	L= 0 to 1.0g's rms
g. Re-entry	3 - 10 Minutes	Boundary layer turbulence Base pressure fluctuations.	M	

of telemeter channels without any loss of statistical accuracy. For a random sampling plan, the commutator can be instructed to change channels at random to retain the statistical significance of the results.

6.2.5 Further Remarks on Random Sampling

As shown in preceding Sections 6.2.1 through 6.2.4, the decision as to how much data should be gathered cannot be reduced entirely to an objective process. However the discussion does present several ways in which quantitative evaluation and prediction of results can be obtained for many of the problems confronting the vibration engineer. These techniques can eliminate, within the present state of the art, certain gross intuitive decisions which often lead to erroneous conclusions, and they show promise of considerable savings in cost.

This approach attaches a numerical confidence to the question of how representative a certain number of samples are from different flight phases of a flight vehicle. It also provides a numerical value for the risk of missing certain events if a continuous record is not taken.

The only parameter under consideration up to now, has been the rms acceleration of vibration samples of some arbitrary length. Nothing has been said so far about the frequency content of these samples and it has been assumed (by implication) that there were no periodicities superimposed on these samples, since sample values were considered to be independent.

It will now be shown qualitatively how the general statistical methods shown above can also be applied to other parameters of the samples taken during flight.

1. If any periodic components are present in the samples, they should be filtered out as discussed in Section 6.1.18. If desired, the samples can be filtered to determine an rms acceleration for any given bandwidth. For example, if the samples covered the frequency range from 5-2000 cps, a plot similar to that shown in Fig. 6.5 can now be made for each of twenty frequency bands, say 5-100 cps,

100-200 cps, ..., 1900-2000 cps. Or, if desired, a narrower band-width could be used. Each of the resulting plots can now be analyzed in the same way as Fig. 6.5.

2. The power spectrum could be obtained for a number of related samples. For each frequency, or small range of frequencies over which the spectrum is of interest, an average power spectrum value can be calculated using all samples of a flight or flight phase, and the standard deviation from this average can be determined and these values analyzed as shown.

3. The amplitude probability density could be handled in a similar manner as that used for the power spectrum by computing the amplitude probability density for a number of related samples. For particular amplitude values of interest, an average probability density can be calculated as well as the standard deviation from this average.

4. Arbitrary level (threshold) crossings might also be of interest. From each sample it can be determined how often a given level of acceleration is exceeded. This parameter also can then be analyzed similar to the method shown.

For each of the parameters listed above (and others) the procedure may have to be modified somewhat. It is beyond the scope of this report to do this quantitatively, but it is recommended that such analysis be done in the future.

6.2.6 Block Diagram for Selection of Sampling Scheme

The preceding examples have illustrated analytical methods involved in the selection of a random sampling scheme for collecting vibration records from flight vehicles. Figure 6.13 presents a block diagram to outline this procedure. A short discussion will follow explaining each block in the diagram. It should be noted that the diagram presents procedures applicable mainly to data collection in a laboratory program. Modifications necessary when concerned with actual flight data will be indicated where appropriate.

```
┌──────────────┐        ┌──────────────────┐        ┌──────────────┐
│     A.       │        │       B.         │        │      C.      │
│   Flight     │───────▶│  Sample Length   │───────▶│  Number of   │
│Specification │        │  Determination,  │        │ Samples for  │
│              │        │   Mean Time      │        │  Unexpected  │
│              │        │ Between Samples  │        │    Events    │
└──────────────┘        └──────────────────┘        └──────────────┘
                                                            │
                                                            ▼
                                                    ┌──────────────┐
                                                    │      D.      │
                                                    │  Number of   │
                                                    │  Samples to  │
                                                    │Contain Fraction│
                                                    │of Population │
                                                    └──────────────┘
                                                            │
                                                            ▼
                                                    ┌──────────────┐
                                                    │      E.      │
                                                    │Select Over-all│
                                                    │Sampling Scheme│
                                                    └──────────────┘
                                                            │
                                                            ▼
                                                    ┌──────────────┐
                                                    │      F.      │
                                                    │Implement Scheme│
                                                    │ Collect Data │
                                                    └──────────────┘
                                                            │
                                                            ▼
                                                    ┌──────────────┐
                                                    │      G.      │
                                                    │ Analyze Data:│
                                                    │1. Unexpected │
                                                    │  Events Recorded?│
                                                    │2. Range of   │
                                                    │  Values Sampled?│
                                                    └──────────────┘
```
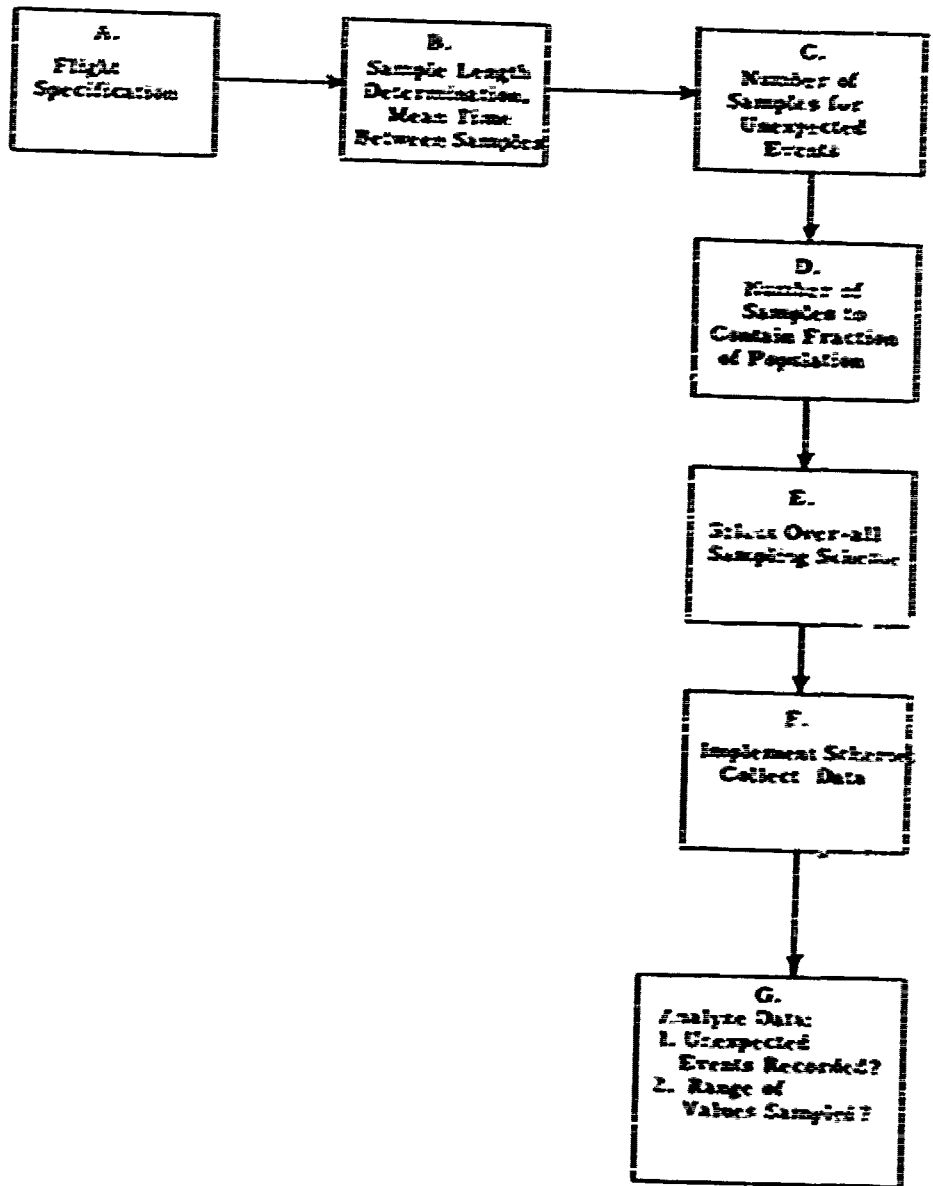
Figure 6.13 Over-all Recommended Procedure for Selection of
Sampling Scheme

Block A. Flight Specifications

Step No. 1 of the numerical examples (Problems No. 1 and 2),
in the preceding Section 6.2.4, explain this procedure and also give
charts for a hypothetical aircraft and missile.

Block B. Sample Length Determination, Mean Time Between Samples

Steps No. 2 and 3 of Problem No. 1 present considerations in-
volved in the selection of the mean time between samples and the sample
length. These are shown to be interrelated matters and, also, may
have to be modified when considering the number of samples required
as a result of other statistical demands.

Block C. Number of Samples for Unexpected Events

Engineering judgment is, of course, necessary here in deciding
on what probability one wants to maintain of detecting an unexpected
event. Analytical details of this important consideration are presented
in Section 6.2.2, and discussed also in Problem No. 1, Steps 3c and 3d.

Block D. Number of Samples to Contain Fraction of Population

As pointed out in Problem No. 1, Step 3b, if it is desired to
know certain maximum values that might be attained, the sample size
will have to be increased to include larger portions of the population.
This topic is discussed in Section 6.2.3, and illustrated in Fig. 6.9
and 6.10.

Block E. Select Over-all Sampling Scheme

After the above steps have been accomplished, an over-all sam-
pling scheme may now be devised. This again will require some judg-
ment in adjusting the sample length, mean time between samples, and
number of samples to fit together as a meaningful whole. The material
presented in Sections 6.2.1 through 6.2.5 is of considerable assistance
to this end.

Block F. Implement Scheme; Collect Data

Here, there is a great difference as to whether the data is to be collected in a laboratory or from actual flight vehicles. Many considerations enter, such as instrumentation and flight details, which are covered in various sections throughout this entire report.

Block G. Analyze Data

If the data is the result of laboratory experimentation, final calculated results may be compared with the a priori statistics. There are two main tests:

1. In the laboratory where the parameters are controlled, one can now check whether or not any "built in" unusual events were recorded as often as expected. Section 6. 2. 2 presents details on these calculations.

2. The range of values allowed is known in advance and therefore comparisons can be made with the calculated results. See Section 6. 2. 3 for details on these calculations.

If the data is recorded from actual flights, these comparisons cannot be made. Instead, one now would have certain information of interest about the flight. The following section will present further procedures to follow in analyzing a collection of flight records, with tests designed first for a laboratory program.

6. 2. 7 Block Diagram for Analysis of a Collection of Records

As in the previous section, the block diagram in Fig. 6. 14 presents procedures which involve laboratory experimentation. Again, the areas that do not apply for the analysis of actual flight vehicle data will be indicated as appropriate. The experimental program in Section 8. 4 is mainly an extension of the present section, while basic statistical matters appear in Section 5. 4.
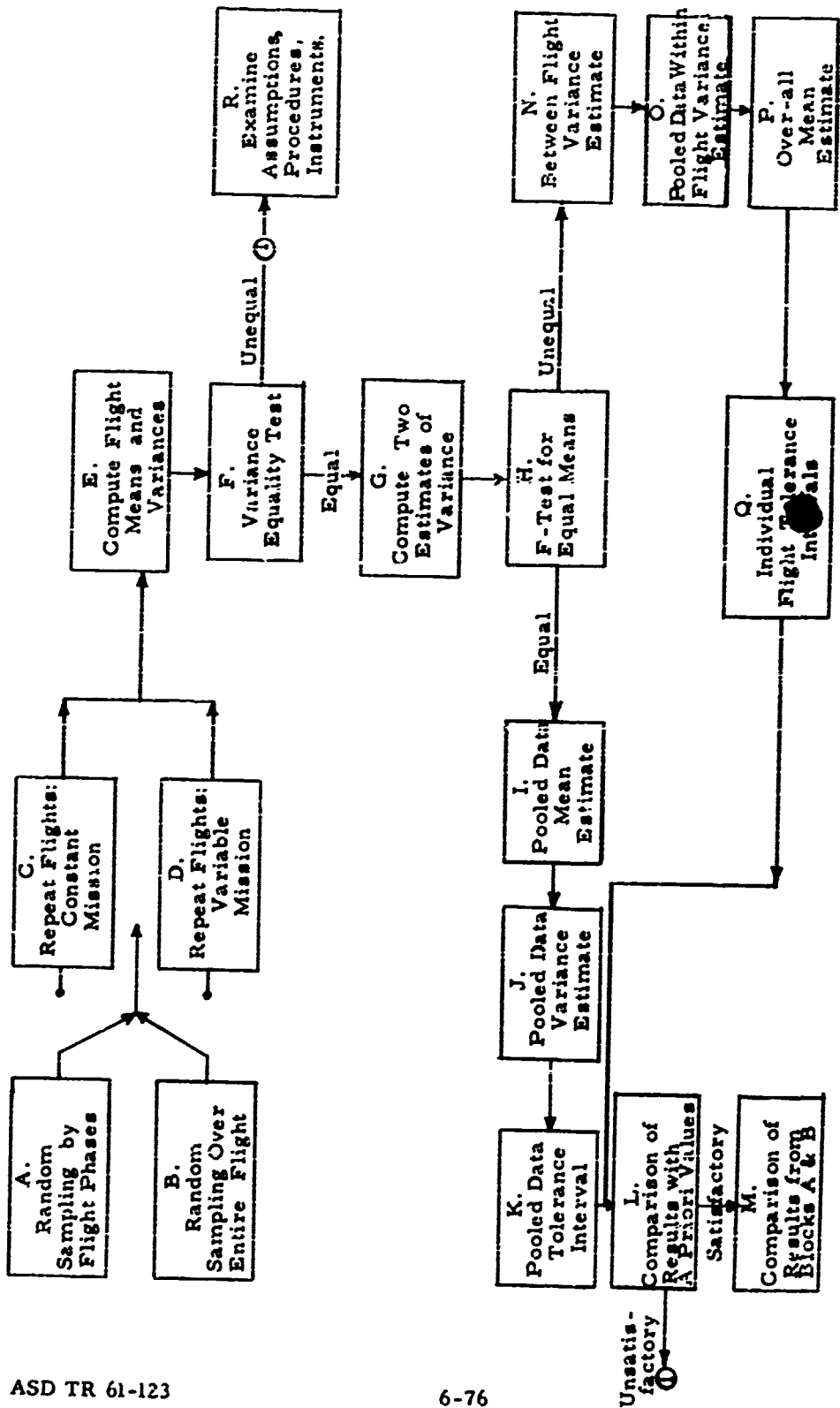
Figure 6.14 Over-all Recommended Procedure for Analyzing Collection of Vibration Records.

Blocks A and B. Random Sampling

Section 6.2 outlines procedures involved in selecting a random sampling scheme. In Section 8.4 of this report, there is further discussion concerning the selection of sample sizes and the number of flights for repeating flights. This suggests a test for trying two variations in the laboratory, and then making the final decision for actual flights based on the laboratory results. Namely, it is suggested that in one case the flight phases should be considered separately. In the other case, ignore the flight phases and consider the flight as a whole. A comparison of results may now be made which could possibly indicate that the consideration of flight phases is unnecessary in some situations.

Blocks C and D. Repeat Flights

Again, the reader is referred to Sections 5.4 and 8.4 for analytical details. In the laboratory, the "mission" from flight-to-flight can be controlled precisely. Then, upon analysis of the final results, it can be determined if the variations are detected as predicted. Two procedures are suggested: (1) that of repeating the same simulated flight (constant mission); and (2) that of repeating a set of different flights (variable mission), with the variation between flights being controlled so that statistical procedures may be verified. In actual flight test, of course, these options do not exist.

Block E. Compute Flight Means and Variances

The means and variances of whatever parameter is being considered should now be computed. In this situation, as contrasted with that of the individual record analysis, each record will be a single observation and the sample size is therefore the number of records. Complete details for these calculations appear in Section 5.4.

Block F. Variance Equality Test

The procedure for application of this test is given in Section 5.4.2. Before the analysis can continue further, the assumption of equal variances, necessary for later calculations, must be verified. This, apparently, is not an unreasonable assumption to make in actual practice.

Block G. Compute Two Estimates of the Variance

Section 5.4.2 presents complete details of this desired calculation and its theory. Basically, two separate estimates of the over-all population variance are computed and lead to the test for equal flight means.

Block H. F-Test for Equal Means

This test is given in Section 5.4.2. The verification of equal means allows the pooling of the data from all the flights. When this pooling is justified, this is the most important result of the repeated flight analysis. The data may then be considered as one large sample with the associated increased confidence for the estimates.

Blocks I, J and K. Pooled Data Estimates

These calculations are covered in detail in Sections 5.4.2 and 5.4.4. It is here that precise estimates are obtained about parameters of interest for future predictions.

Block L. Comparison of Results with A Priori Values

At this point, if the flights have been simulated in a laboratory, comparisons may be made with the predetermined values. Any discrepancies here, such as an indication of equal means when the mission was varied, or an indication of different means when the mission was constant, require some further consideration as indicated in Block R. Supplementary discussion on these matters is presented in Section 8.4.3. Clearly, this block does not apply for actual flight test data.

Block M. Comparison of Results from Different Sampling Schemes

Again, this block does not apply for an actual flight test. However, results obtained from the laboratory may now be applied to future flight test procedures. See Section 8.4 for further discussion.

Block N. Between-Flight Variance Estimate

If the means from flight-to-flight are different, an estimate of the variance of this distribution is desired. See Section 5.4.2 for required analytical details.

Block O. Within-Flight Variance Estimate

The data may still be pooled for an estimate of the variance of the distribution of values within a given flight. See Section 5.4.2 for required analytical details.

Block P. Over-all Mean Estimate

An estimate of the mean of the over-all distribution of flight means may be obtained here. This involves the same computations as for Block I.

Block Q. Individual Flight Tolerance Intervals

In this case, the pooling of the data from all the flights is not allowed, and therefore, the intervals will not be as precise as that obtained in Block K. See Section 5.4.1 for the calculations.

Block R. Examine Assumptions, Procedures, Instruments

This block applies mainly to laboratory data, but may also be pertinent for flight test data. If at any point in a laboratory program, the results of the analysis do not agree with the controlled parameter, some examination of the procedures involved will have to be made. Perfect agreement cannot be expected. However, any extreme deviations must result in further critical examinations of the underlying assumptions, procedures and instruments.

6.3 REFERENCES

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7. INSTRUMENTATION TO MEASURE
VIBRATION CHARACTERISTICS

Previous sections of this report have discussed many mathematical, statistical, and physical aspects of acquiring and evaluating vibration phenomena in a flight vehicle. The present section is concerned with problems of actual hardware instrumentation generally available in laboratories or required for experimental verification of analytical material contained in the report. The section is divided into various parts which attempt to cover a large number of the more important instrumentation problems. For illustration purposes only, specific equipment and manufacturers are named in some instances, and are not intended to reflect on the merits of other available equipment which is not discussed.

7.1 TRANSDUCER CONSIDERATIONS

Many different types of transducers have been developed over the years for the purpose of converting mechanical motions into equivalent electrical signals. These include strain gage, piezoelectric, variable inductance, electrokinetic, magnetostrictive, potentiometer, variable capacitance, and permanent magnet self-generating instruments. The most commonly used of these elements for structural response measurements in flight vehicles are the strain gage and piezoelectric crystal. Both elements are employed to generate analog acceleration signals. Bonded strain gages are also widely used to obtain direct analog strain (stress) signals.

7.1.1 Characteristics of Piezoelectric Crystal Accelerometers

The small size, light weight, and high frequency response characteristics of the piezoelectric crystal accelerometer have resulted in its wide application for structural vibration response measurements in flight vehicles. Crystal elements have been designed with natural frequencies as high as 100KC permitting reasonably accurate acceleration measurements up to 20 KC.

A summary of the general characteristics of commercially available piezoelectric crystal accelerometers is presented below. For more

specific information, one should refer to the literature of accelerometer manufacturers, some of which is given in references at the end of this section.

Dynamic Range	0.01 g to 10,000 g
Frequency Response	2 cps to 20 KC
Temperature Range	-100°F to $+525^\circ$F
Linearity	$\pm 1\%$ and $\pm 2\%$
Sensitivity	.075 MV/g to 75 MV/g
Lowest Resonant Frequency	1.6 to 60 KC
Transverse Sensitivity	$\pm 5\%$ and $\pm 10\%$
Acoustic Response	Less than 0.5g at 140 db

7.1.2 Characteristics of Strain Gage Accelerometers

A flat frequency response down to 0 cps (DC) is the primary advantage of the strain gage accelerometer. Unfortunately, the high frequency response is limited to a maximum of 600 cps. A general summary of the characteristics of commercially available strain gage accelerometers is presented below.

Dynamic Range	0g to 200g
Frequency Response	0 cps to 600 cps
Temperature Range	-40°F to $+200^\circ$F
	or -65°F to $+125^\circ$F
Linearity	$\pm 1\%$
Sensitivity	0.4 MV/g to 35 MV/g
Lowest Resonant Frequency	21 cps to 850 cps
Transverse Sensitivity	$\pm 2\%$
Acoustic Response	Unknown

7.1.3 Characteristics of Bonded Strain Gages

In contrast to the acceleration transducers discussed above, the strain gage, as its name implies, has an output that is proportional to strain, rather than acceleration. The restrictions on strain gage data, such as frequency response, dynamic range, etc., are usually imposed by the associated instrumentation and not by the gage. The subject of

strain measurements is well known throughout the industry and is covered in great detail in many publications. Therefore, it will not be discussed any further here.

7.1.4 Transducer Applications

Appropriate applications for the transducers mentioned above fall into the following broad categories:

1. Piezoelectric and strain-gage accelerometers are generally used to determine the vibration levels of primary structure, and the vibration environment for equipment and human comfort considerations.

2. Bonded strain gages are generally used to obtain panel response data which may be interpreted in terms of stress levels or displacement amplitudes, either of which is valuable for fatigue life prediction.

Accelerometers have been employed at times for fatigue studies, but then it is important to know something about the non-linear characteristics of the structure, and one should measure phase information.

7.2 TRANSMISSION AND RECORDING

Signals from dynamic transducers are transmitted to magnetic tape recorders either directly or through telemetry systems. The use of telemetry is obviously required in missile applications. Additional instrumentation is provided by amplifiers designed to match transducer impedances and to amplify the signal.

7.2.1 Telemetering Systems

Various telemetry systems are available for the transmission of flight data. Because of the diverse instrumentation requirements of aircraft, missiles, and space probes, no single telemetering package with fixed capabilities will satisfy even a majority of applications. However, the FM-FM (frequency modulation) system has been the method which has been most widely employed to compile dynamic data.

In the FM-FM system, the transducer signal modulates a subcarrier frequency which is mixed with other subcarriers to create a composite signal that, in turn, frequency modulates the main carrier

frequency (usually from 216 to 235 megacycles/sec.) A study of FM-FM telemetering characteristics requires an examination of the main units that comprise the FM-FM system. Major units include subcarrier oscillators, radio-frequency transmitters, band-pass filters, and sub-carrier discriminators. FM-FM channels are standardized according to Inter-Range Instrumentation Group (IRIG) specifications. Eighteen subcarrier frequencies are available for FM modulation, with center frequencies ranging from 400 cps to 70,000 cps. The maximum frequency response available is DC to 2100 cps.

7.2.2 Magnetic Tape Recorders

Magnetic tape recorders fall into three main classes: (1) direct amplitude modulated (AM) recorders, (2) digital recorders, and (3) frequency modulated (FM) recorders. The direct recorders do not have the ability to reproduce low-frequency signals but do have an excellent high-frequency response and consequently, are useful in recording telemetry signals. The digital recorders have fast stop and start features, and are utilized to record pulse data wherein the magnetic tape is either saturated or non-saturated. Pulse code modulation signals are recorded on the digital tape units. The FM tape units are ideal for recording vibration and acoustic signals. Most FM tape units provide frequency responses from dc to 10,000 cycles per second at tape speeds of 60 inches per second. FM recorders are presently available with responses from dc to 20,000 cycles per second. The AM recorders which are used to record composite telemetry signals are, generally, ground-based units. Airborne AM units are available, but are not employed to record dynamic data because of their low-frequency limitations. This leaves the FM tape units to record airborne vibration and acoustic signals both in airborne and ground installations. For further information, the reader is referred to the literature and the references at the end of this section.

7.2.3 Calibration of Transducer-Telemetry-Recorder Systems

Perhaps the greatest source of instrument (calibration) error in any data gathering and processing procedure is the combination trans-ducer, telemetry, and recorder system errors. Each of these items

will undoubtedly be calibrated individually at regular intervals or before each test. Nevertheless, complete system calibrations are a definite necessity to produce valid dynamic information. This calls for end-to-end system calibrations wherein a full range of frequencies and amplitudes are injected into the transducer end of the instrumentation system, transmitted through each unit in the chain, and recorded on magnetic tape for final data reduction and analysis.

The material to follow will now consider properties of available instrumentation equipment to perform various desired measurements on the vibration data which has been gathered.

7.3 VOLTMETER MEASUREMENTS OF RANDOM DATA

As discussed in the previous two sections, random vibration response measurements in modern flight vehicles are obtained using transducers which produce electrical voltage signals proportional to displacement, velocity, acceleration, or stress. The analog voltage signals may be processed in many ways but ultimately must be read out in terms of some amplitude level such as the mean square or root mean square level for a given bandwidth. A voltage level measuring device of some sort is then a necessary instrument for random vibration response measurements.

Electrical engineers have always been concerned primarily with the power aspects of electrical systems, particularly the association between voltage and power. For direct current systems, the relationship is quite simple: power is proportional to the square of the voltage. When alternating current came into the picture, the parameter of root mean square (rms) was adopted to characterize an alternating voltage because it permitted the same simple association with power: power is proportional to the square of the rms voltage. The true rms value for any periodic voltage, v(t), is as follows:

$$v_{rms} = \left[\frac{1}{2\pi} \int_0^{2\pi} v^2(t) dt \right]^{1/2} \qquad (7.1)$$

Specifically for a sinusoidal voltage, $v(t) = V_{(max)} \sin \omega t$;

$$V_{rms} = 0.707 V_{(max)}. \qquad (7.2)$$

Early AC voltmeters were designed to actually measure and indicate the true rms voltage of a signal. The two most popular rms measuring voltmeters were the dynamometer type and the thermocouple type, as discussed in Section 7.3.7. The limited frequency response of the dynamometer and the fragility of the thermocouple voltmeter forced these instruments out of popular use when the more rugged broad frequency response vacuum tube and dry disk rectifier type voltmeters were developed.

7.3.1 AC Rectifier Type Voltmeters

At the present time, the vast majority of commercial AC voltage measuring instruments are rectifier type voltmeters. These voltmeters normally have scales calibrated to read the rms voltage of sinusoidal signals, but do not actually measure the rms voltage of the signal. The two most common rectifier circuits are as follows:

1. An arithmetic average value rectifier which schematically consists of a D'Arsonval meter in series with a diode (half-wave average value rectifier) or a D'Arsonval meter across a bridge of four diodes (full wave average value rectifier).

2. A peak value rectifier which schematically consists of a D'Arsonval meter in series with a diode plus a shunt capacitor (half-wave peak value rectifier) or a D'Arsonval meter across a bridge of four diodes with a shunt capacitor (full wave peak value rectifier).

Full wave bridges are employed in preference to half-wave circuits in the more expensive instruments, but the result is the same; an average value rectifier voltmeter measures the rectified average voltage of the signal and the peak value rectifier voltmeter measures the peak voltage of the signal. Of the two types of rectifier circuits, the average value circuit (or something similar to it) is far more prevalent in commercial

instruments. Due to the non-linear characteristics of many types of diodes in the lower region of their usable voltage range, some AC rectifier type voltmeters actually measure something near the rms voltage at the low end of the scale and something near the average voltage at the high end of the scale. In any case, the meter scales are calibrated to read the rms voltage of a sinusoidal signal.

The conventional vacuum tube and dry disk rectifier type AC voltmeters just discussed are completely satisfactory for electrical power work since alternators produce sinusoidal voltage signals. However, these voltmeters will give erroneous readings when used to measure signals with wave forms that are not sinusoidal. The magnitude of errors that may result are well illustrated for the case of a conventional average value rectifier type AC voltmeter in the following table taken from Ref. [20].

	Sine Wave	Square Wave	Rectangular Peaks Occupy-ing 1% of Time Axis
True rms Voltage	1.00	1.00	1.00
Reading of a Conventional AC Voltmeter	1.00	1.11	0.35

Erroneous voltage readings will also result when the conventional AC voltmeter is used to measure random voltage signals. For example, if an ideal average value rectifier type voltmeter were used to measure the rms voltage of a random signal with a Gaussian probability density function, the rms voltage of the signal would be $2/\sqrt{\pi}$ or approximately 1.3 times the rms voltage reading indicated by the voltmeter (Ref. [21]). If a peak value rectifier type voltmeter were used for the same measurement, a completely nonsensical reading would obviously result.

The increasing amount of random vibration data being gathered in modern high speed flight vehicles has produced a new need for instruments which will measure the rms voltage of any aperiodic signal. There are of course the many types of true rms voltage measuring instruments that have been developed over the years, as summarized in Section 7.3.7. All of these instruments, however, possessed at least one major dis-advantage for general application in vibration analysis. As a result, several instrument companies have developed vacuum tube rectifier type

voltmeters designed to actually measure the true rms voltage of a signal.
These instruments are usually called true rms vacuum tube voltmeters
(true rms VTVM) to distinguish th· n from the conventional AC VTVM's.
They actually measure the rms voltage of non-sinusoidal signals with
the advantages of conventional VTVM's; i.e., broad frequency response,
rugged dependability, high input impedance, high sensitivity, etc.

7.3.2 Vacuum Tube (true rms) Voltmeters

The electronic circuit used in true rms VTVM's varies with manu-
facturer, but in general it is a rectifier circuit which approximates a
parabolic transfer characteristic so that instantaneous output voltage is
proportional to the square of instantaneous input voltage. Such a transfer
characteristic can be obtained in several ways. As stated before, many
types of diodes have non-linear transfer characteristics particularly in
the lower region of the usable voltage range. The transfer characteristic
is often quite close to the desired square law over a properly limited
voltage range. Another approach is to use a circuit of biased diodes
designed so that each diode contributes a straight line segment to a
polygon transfer characteristic which approximates a parabola. In any
case, the squaring circuit must be followed by an averaging circuit to
obtain the mean square of the signal.

The ideal averaging circuit would integrate the input signal over
some averaging time, T, and divide the result by T. An actual integrating
circuit such as is used in analog computers (an operational amplifier with
a feed back condenser) could be employed. A true integrater type averag-
ing circuit would give a one number average value at the end of the
integrating time, T, for the input voltage occurring during that time.
However, a simpler and cheaper way of averaging is to use a resistance-
capacitance (RC) circuit. In this case, the output is a continuous signal
representing at any instant the approximate average of the entire past
history of the input voltage weighted by an amount which depends upon
how long ago the voltage was applied. Commercial true rms VTVM's
employ RC circuits to obtain the mean square of voltage signals. The
square root of the resulting mean square measurement is obtained by
proper calibration of the meter scale. The specifications including

instrument error for one of the commercially available true rms VTVM's (Ballantine Laboratories Model 320) is presented in Table 7.1. This instrument is chosen merely for convenience and should not be regarded as superior or inferior to other available similar equipment. As one might expect, the major disadvantage of a true rms VTVM is cost.

Table 7.1 Specifications For Ballantine Laboratories Model 320 True RMS Voltmeter

Voltage Range: 100 microvoit to 320 volts RMS in 13 ranges, in steps of 10 db.

Frequency Range: 5 to 500,000 cps.

Crest Factor Range: 1 to 4.5 (0 to 13 db) for full scale readings; 1 to 15 (0 to 23 db) for bottom scale readings.

Accuracy - Sine Waves: 3 percent from 15 to 150,000 cps; 5 percent from 5 to 15 cps and from 150,000 to 500,000 cps at any point on the scale.

Accuracy - Non-Sinusoidal Waves: 3 percent if all component frequencies lie in the range from 15 to 150,000 cps; 5 percent for all other conditions within the allowable frequency and crest factor range. All accuracy figures apply to any point on the meter scale and for all ranges.

Stability of Calibrating Source: 0.5 percent for line voltage variations of 105-125 volts or of 210-250 or of 210-250 volts and for long term usage.

Input Impedance: 10 megohms shunted by approximately 25 MMf up to 10 millivolts, and by approximately 8 MMf above 10 millivolts.

7.3.3 Statistical Accuracy of Measurements

Before proceeding with the discussion of specific true rms voltmeters, consider the statistical accuracy associated with the use of these meters to measure the mean square level of a random signal. Mean square rather than rms measurements will be discussed since the mean square voltage is the parameter actually measured by true rms VTVM

circuits. As mentioned before, the square root of the measurement or the rms voltage is obtained by proper meter calibration.

Assume an ideal voltage measuring instrument with a perfect squaring circuit followed by an averaging circuit which integrates for a mean square voltage over some averaging time, T. Such a voltmeter would give an exact mean square measurement for any steady state periodic input signal. Consider the case where the input signal is a random signal which is stationary in time with a frequency bandwidth of B. The measurement obtained is now only an <u>estimate</u> of the mean square voltage of the signal, since the measurement constitutes a sample of the signal over a finite period of time. The statistical accuracy or quality of the measurement is a function of the averaging time (sample length) T, the signal bandwidth B, and the power spectral density function $S(f)$. For the present, assume the signal has a uniform power spectrum, $S(f) = $ constant, and an ideally defined bandwidth, B, with infinitely sharp cutoff frequencies. The more general case will be discussed later in Section 7.4.2. Now the quality may be expressed in terms of a normalized variance of the measurement as follows:

$$\epsilon^2 = \frac{1}{BT} \tag{7.3}$$

Another measure of quality is the equivalent number of statistical degrees of freedom, n, for the measurement, as follows (Ref. [7]):

$$n = 2BT \tag{7.4}$$

Consider a signal with a Gaussian probability density function and a true mean square voltage of σ^2. For a mean square voltage measurement of s^2, the following statistical relationship exists:

$$\frac{s^2}{\sigma^2} = \frac{\chi^2}{n} \tag{7.5}$$

where the value of χ^2 (chi-square) for various confidence intervals may be obtained from standard statistical tables (such as page 309 of Ref. [9]).

Now, if a mean square voltage of s^2 is measured for a sample of length T from a stationary random signal, the actual mean square voltage, σ^2, for the random signal is between the limits defined for a given confidence interval by the equation:

$$\sigma^2 = \left(\frac{n}{\chi^2}\right) s^2 \quad \text{where} \quad n = 2BT \tag{7.6}$$

A limited table of confidence intervals from mean square measurements as a function of the number of degrees of freedom for a measurement of unity is presented in Table 7.2.

Table 7.2 Confidence Limits From Mean Square Measurements
As Function of Number of Degrees of Freedom

No. of Degrees of Freedom		n = 2	n = 10	n = 20	n = 40	n = 60	n = 120
80% Confidence	Lower limit	.43	.62	.70	.77	.81	.85
	Upper limit	9.48	2.05	1.67	1.37	1.29	1.19
95% Confidence	Lower limit	.27	.49	.58	.67	.72	.79
	Upper limit	39.20	3.07	2.08	1.63	1.48	1.31

For example, assume an ideal instrument with an averaging time of one second is used to obtain a mean square measurement of s^2 equal to one volt for a random signal with a bandwidth of 60 cps. The equivalent number of degrees of freedom for the measurement is 120. From Table 7.3, it may be said with 80 percent confidence that the actual mean square voltage of the random signal is between 0.85 volts and 1.19 volts.

7.3.4 Accuracy Using RC Filters

The quality of a mean square voltage measurement obtained using an ideal instrument has been established. However, as previously noted, commercial true rms VTVM's do not average ideally by integration over the sample length but rather by use of R-C filters. The output voltage time history of a simple R-C filter for a constant DC voltage input is shown as follows.
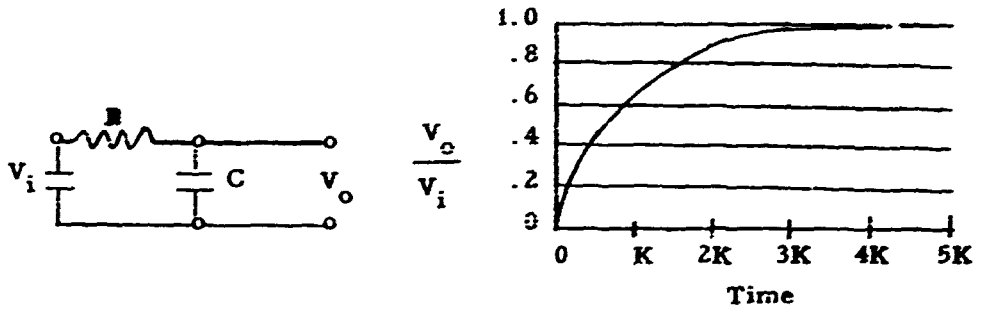
Time Constant $K = RC$

$$\frac{V_o}{V_i} = 1 - e^{-(t/K)}$$ (7.7)

As seen from the above graph, the output of the simple RC filter will reach a voltage nearly equal to the input after the signal has been applied for three or four time constants ($V_o = 0.98\ V_i$ at four time constants). Thus for a steady state periodic input signal to a true rms VTVM, the RC filtering of the square law rectifier output will yield a reasonably accurate and steady meter indication of mean square voltage after a few time constants have elapsed. This assumes of course that the filter time constant is long compared to the period of the fundamental frequency of the periodic signal being measured. Now consider the case of a stationary random input signal. After a few time constants have elapsed, the meter will indicate a continuous estimate of the mean square voltage which at any instant is the result of all signals that have gone before. The quality of the measurement at any time is once again a function of the frequency bandwidth and the averaging time, which in turn is dependent upon the time constant of the voltmeter. As the voltmeter time constant and/or bandwidth of the input signal become smaller, the fluctuations of the meter reading become larger (the quality of the measurement at any instant becomes poorer).

The relationship between the voltmeter time constant and the averaging time T needed to establish the statistical quality of the meter reading

at any instant of time is not precise until the signal has been applied for several time constants. After, say, four or more time constants have elapsed, the averaging time T in Eqs. (7.3) and (7.4) may be replaced by 2K where K = RC is the filter time constant. This important relationship is derived in an earlier section of this report. Now, the normalized variance of the mean square measurement at any instant after time = 4K becomes,

$$e^2 = \frac{1}{2 \, BK} \tag{7.8}$$

and the number of equivalent degrees of freedom becomes,

$$n = 4 \, BK \tag{7.9}$$

For example, if B = 50 cps and K = 0.1 seconds, then n = 20. From Table 7.2, it follows that for an 80 percent confidence interval, the actual mean square voltage of a random signal is between 0.70 and 1.67 of the mean square voltage measured at any instant of time after, say, 4K or 0.4 seconds have elapsed. This assumes that the sample length of the signal is at least 0.4 seconds long. It should be noted that the time constant of commercial true rms VTVM's is fixed, and thus for a given bandwidth, the quality of an instantaneous measurement is fixed no matter how long the sample length of the signal is. It would be desirable if the time constant of the meter could be adjusted so that K could always be 1/4 of any sample length being measured. The measurement would then be the meter indication at the end of the sample and would be the highest quality measurement attainable for this type of voltmeter.

It should be pointed out that a true rms VTVM with a fixed time constant often is used to measure the voltage of a stationary random signal where the sample length is many times longer than the time constant of the voltmeter. In this case, the observer does not obtain a measurement by reading the voltmeter indication at one particular instant, but rather by mentally averaging the continuous voltmeter indication over the entire sample length. The quality of a measurement obtained by averaging the continuous mean square voltage indications over a long sample is better than the quality of one instantaneous observation.

7.3.5 Physical Example

The time constant of true rms VTVM's is usually between 0.1 and 1.0 seconds. Time constants of less than 0.1 second will seriously limit the instruments' capability to measure low frequency signals. Time constants of longer than a second will result in an unduly long wait to obtain a measurement, although the resulting meter indication would be more steady (the quality of the measurement would be higher). A physical example might help give a better feel for the association between meter indication, time constant, and input signal bandwidth for true rms VTVM voltage measurements of random noise. Such an example is presented in Fig. 7.1.A true rms vacuum tube voltmeter was used to measure the mean square voltage of the output of a random noise generator limited in bandwidth by a controllable filter.. The time constant of the voltmeter was measured and found to be about 0.1 seconds. The mean square measurement signal was monitored and recorded just prior to the voltmeter meter. The mean square voltage time history for a 3-second long sample is presented for various bandwidths. Bandwidth rather than time constant was varied because the time constant of the voltmeter is fixed. The random noise generator was permitted to warm up for several hours to assure a stationary output signal.

From Fig. 7.1, note that the continuous mean square measurement (and the rms indication of the meter) for a bandwidth of 20,000 cps was quite steady over the 3-second sample. In this case, the quality of the measurement at any instant is so high (4 BK = 8000 degrees of freedom) that the estimation error is negligible as compared to the instrument error. Now consider the measurement for a bandwidth of 5 cps, and note the large variations in the measurement over the 3-second long sample. Here the measurement at any instant of time (after four or more time constants have elapsed) has a quality of 2 degrees of freedom. If the indicated mean square voltage at any instant is s^2, one could say with 80 percent confidence that the true mean square voltage of the signal measured was between $0.43 s^2$ and $9.48 s^2$. This experiment shows that true rms VTVM's may be used to obtain random signal voltage measurements with high confidence for short samples of broad band input signals, even when the time constant is as short as 0.1 seconds. However, when
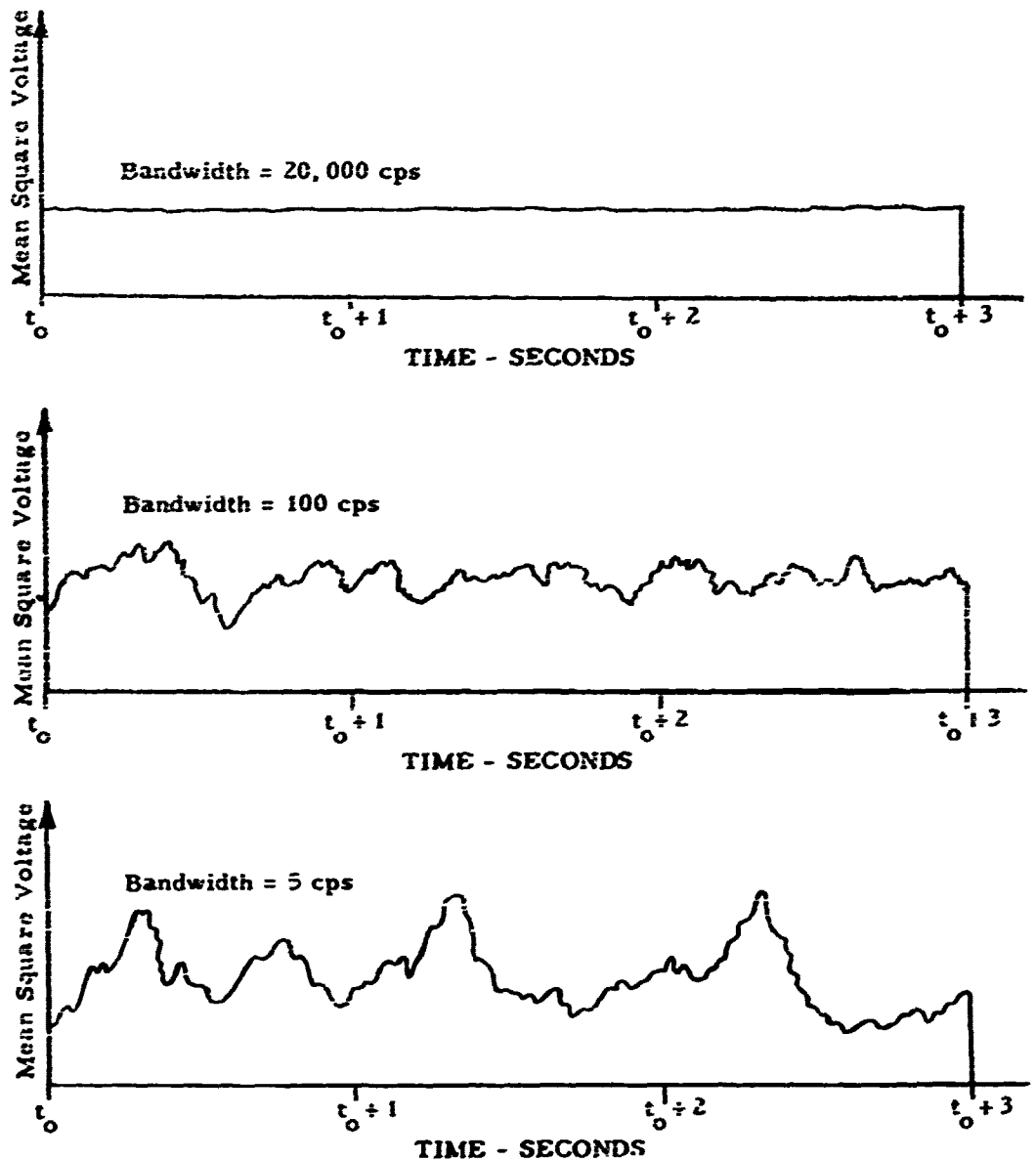
Figure 7.1   Mean Square Output vs. Time
True rms Voltmeter

Time constant of equivalent RC filter is approximately 0.1 seconds. t_o is several time constants after stationary random (Gaussian) signal was applied.

such instruments are used to measure short samples of narrow band random signals, the confidence in the resulting measurment is rather poor and must receive consideration. This point is extremely important because spectral analysis of random signals is normally accomplished by measuring the mean square voltage output of a narrow band wave analyzer. The subject will be covered more fully in the next Section 7.4 on power spectral density measurements.

7.3.6 Further Remarks on Voltmeter Measurements

The rectifier type true rms VTVM has been discussed in some detail here because it appears to be the most practical voltage measuring instrument for random vibration analysis that is readily available commercially. This does not mean that other types of voltmeters are not available. Thermocouple type voltmeters incorporating electronic circuits have been developed which provide the high sensitivity and high input impedance of rectifier type voltmeters. Furthermore, these instruments do not appear to present the problem of damage susceptibility that has been characteristic of thermocouple type voltmeters in the past. Thermocouple voltmeters with equivalent RC time constants of as long as 16 seconds are commercially available. If long enough random signal samples are obtainable, fluctuations of the measurement indication will be quite small even for narrow frequency bandwidths.

One additional point concerning true rms voltage measurements of random signals should be mentioned. Any true rms voltage measuring instrument must be designed to electrically deal with instantaneous input voltages over a very wide dynamic range. In the case of conventional AC voltmeters used to measure sine wave signals, the voltmeter circuit will not see instantaneous voltage levels above 1.414 times the rms voltage of the signal being measured (the peak of a sine wave is 1.414 rms). However, non-sinusoidal periodic signals may have instantaneous voltage peaks which are many times larger than the rms voltage of the signal. The ratio of peak voltage to rms voltage is called the crest factor of the signal (sometimes called peak factor), and the ability of a voltmeter to accept high crest factors is defined by the crest factor limit or range of

the voltmeter. The crest factor of a sinusoid is 1.414, and a voltmeter with a crest factor limit of 5 (range of 1 to 5) will measure without clipping any signal with a crest factor of 5 or less. For the case of random signals, the crest factor is undefined (it is extremely large), and some clipping must occur since no voltmeter can be designed with an infinite crest factor limit. However, if the crest factor limit of the voltmeter is sufficiently high, the clipping may be negligible. Whether or not the crest factor limit of a given voltmeter is sufficient depends upon the probability density function of the random signal being measured. For a random signal with a Gaussian amplitude density, a crest factor limit of 5 would result in clipping only 0.000057 percent of the time, and even a crest factor limit as low as 3 would result in clipping only 0.27 percent of the time. The crest factor limit for commercial true rms VTVM's is normally between 3 and 5, although voltmeters are available with crest factor limits as high as 10. A detailed discussion of the effect of clipping on random signals appears in Section 5.5 of this report.

In summary, rms voltage measurements of random signals in association with analog vibration data analysis should be obtained using true rms voltage measuring instruments. It should be remembered that most conventional AC voltmeters in use today are average value rectifier type voltmeters which do not actually measure the rms voltage of a signal. It should also be remembered that no matter how small the instrument error of a voltmeter is, the rms voltage measured for a random signal is only an estimate of the true rms voltage of the signal. A confidence interval for the estimate can be established if the signal frequency band width and instrument averaging time or sample length are known. It would be desirable to measure the rms voltage of a signal by actual integration over the sample length, but unfortunately, commercial instruments of this type are not readily available. Most commercial voltmeters average the instantaneous voltage square signal from the square low rectifier by use of an RC type filter to give a continuous mean square measurement (made rms by proper scale calibration) over the sample length. After the signal has been applied for four or more filter time constants, a confidence interval for the meter indication at any instant can be established by considering the averaging time for the measurement to be equal to two time constants.

7. 3. 7 Summary of True rms Voltage Measuring Instruments

(Compiled from References [28], [11], [30], and [22])

1. Electrodynamic type voltmeter (dynamometer).

This instrument employs two coils connected in series, one fixed and one moving, with an indicating needle attached to the moving coil. The instrument is then like a D'Arsonval meter except the permanent magnetic is replaced by a fixed coil. The resulting needle movement is proportional to the square of the instantaneous applied voltage. The inertia and damping of the moving element integrated the torque variations to give a mean square indication. Proper scale calibration permits direct rms readings.

Primary advantage: high accuracy.

Primary disadvantage: limited frequency response (DC through power frequencies).

2. Moving-iron type voltmeter.

This instrument utilizes the reaction between a moveable soft-iron vane and a magnetizing field coil. The vane movement is a function of the rms voltage to the coil.

Primary advantage: high accuracy.

Primary disadvantage: limited frequency response (DC through power frequencies).

3. Thermocouple type voltmeter.

This instrument consists of a heater, a thermocouple, and a D'Arsonval meter. The voltage signal is applied to the heater which is in physical contact with the thermocouple. The thermocouple temperature as read by the D'Arsonval meter is proportional to the average power dissipated in the heater, which in turn is proportional to the mean square voltage of the applied signal. Proper scale calibration permits direct rms readings.

Primary advantages: high accuracy and broad frequency response (DC to one megacycle or higher).

Primary disadvantage: susceptibility to damage due to overloading unless the crest factor range is sharply limited.

4. <u>Computer type voltmeter.</u>

This instrument employs analog computing circuits to square the input voltage signal and integrate over any desired time interval to obtain a mean square measurement for that interval.

Primary advantages: great flexibility; yields a one number mean square measurement for a specific sample length.

Primary disadvantage: cost.

5. <u>Electrostatic type voltmeter.</u>

This instrument utilizes the electrostatic reaction between the fixed plates and moveable plates of a variable condenser. The position of the moveable plates as indicated by an attached needle can be made proportional to the rms voltage of the applied signal.

Primary advantages: high input impedance at low frequencies.

Primary disadvantage: limited sensitivity (high voltages only).

6. <u>True rms rectifier type voltmeter.</u>

This instrument is discussed in the text.

Primary advantages: high input impedance (up to 10 megohms); high sensitivity (down to 100 microvolts); broad frequency response (5 cps to 200 kilocycles or higher).

Primary disadvantages: cost and only moderate accuracy (about 3 percent).

7.4 POWER SPECTRAL DENSITY MEASUREMENTS OF RANDOM DATA

One of the useful statistics for characterizing a random signal is the power spectral density of the signal. The power spectral density or power spectrum of a stationary random signal is a measure of the relative power per cps versus frequency and may be presented in any units which are proportional to power, i.e., g^2/cps, psi^2/cps, etc. Random vibration measurements in modern flight vehicles are usually obtained using acceleration transducers. As a result, most power spectrum data currently appearing in the literature are in the form of acceleration power spectral densities with the units of g^2/cps.

Analytically, the power spectrum of a stationary random signal is the Fourier transform of the autocorrelation function of the signal. Hence one can obtain a power spectrum either directly by filtering in the frequency domain, or indirectly by filtering in the time domain (autocorrelation) and determining the Fourier transform of the results. This section will deal only with the measurement of power spectrum by filtering in the frequency domain. Autocorrelation measurements will be the subject of a later Section 7.6.

In general, the determination of the power spectrum for a random signal consists of measuring the mean square value of the signal in each of many narrow frequency bands which together cover the frequency range of concern, and dividing each mean square value by its associated bandwidth. Ideally, one would like to measure the limit of the mean square value divided by bandwidth, as the bandwidth approaches zero, but this is beyond the practical capabilities of physical instruments. Filtering and mean square measurement are then the fundamental physical processes involved in power spectral density determination. However one cannot actually measure the true mean square value of a random signal, since a measurement constitutes a sample of the signal over some finite period of time. A mean square measurement of a sample record from a random signal is then only an estimate of the true mean square level of the signal. When one speaks of power spectral density measurements, one is actually talking about a statistical estimate. The accuracy of power spectrum estimates is considered in detail in this report.

7.4.1 General Techniques for Obtaining Power Spectra Estimates

Consider now the actual techniques that might be employed for power spectral density analysis. The discussion will be in terms of analog devices, but the general procedures could just as well be accomplished digitally. One possible technique, in terms of minimum computation time, would be to pass the input random signal through a set of very narrow band filters which cover the frequency range under consideration and simultaneously measure the mean square output of each filter by squaring and averaging the output of each filter over the entire available

record length. Then, by dividing each mean square measurement by its associated filter bandwidth and simultaneously recording the outputs on an appropriate frequency scale, an estimate of the power spectral density function for the random signal is obtained, as shown in Figure 7.2 below.
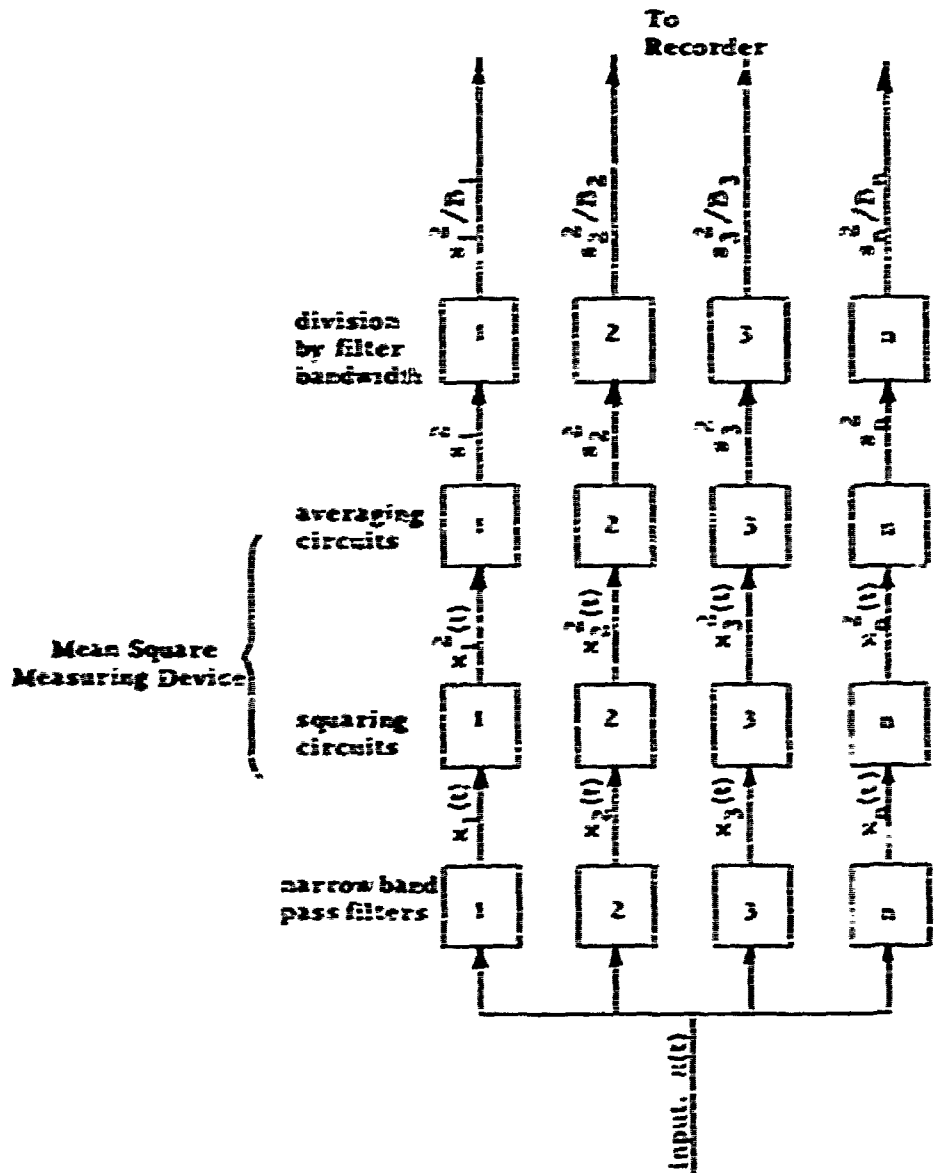
Figure 7.2 (Parallel) Filter Set Type Analyzer

The procedure shown in Figure 7.2 would require a large number of filters and mean square measuring devices, which are expensive. A great deal of money is saved at the expense of increased analysis time by using only one mean square measuring device, and determining the mean square out-put of each filter of the set individually, as shown in Figure 7.3.
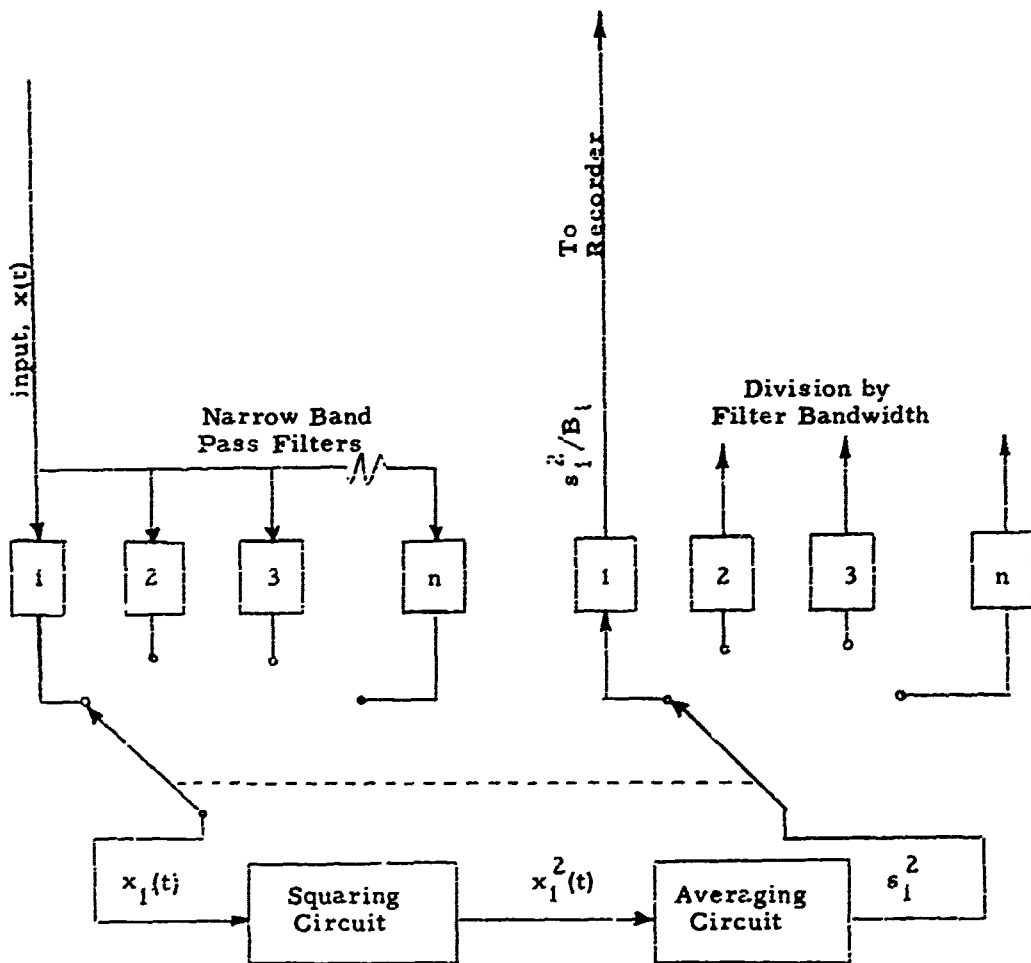
Figure 7.3 (Sequential) Filter Set Type Analyzer

The procedure of Figure 7.3 will be called a (sequential) filter set type analyzer. The filter sets of Figures 7.2 and 7.3 might be, and often are replaced by one filter with an effective variable center frequency that may be moved continuously through the frequency range under consideration. This arrangement, which will be called a variable center frequency filter type analyzer, is shown in Figure 7.4
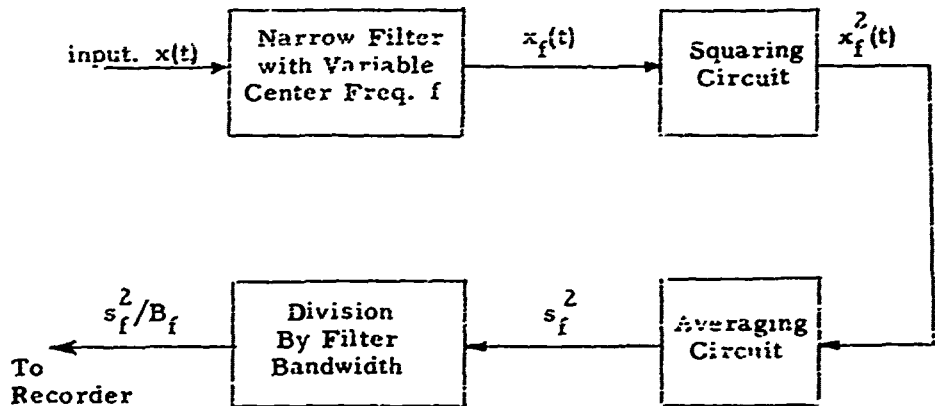
Figure 7.4 Variable Center Frequency Filter Type Analyzer

A fourth procedure involves the application of the heterodyne principle. Rather than moving a variable center frequency filter through the frequency range of the signal as in Figure 7.4, a heterodyne analyzer moves the frequency range of the signal past a single high frequency fixed filter. Frequency transposition of the input signal is accomplished by combining the input signal in a modulator with a signal from a variable frequency oscillator. As the oscillator frequency is varied, the modulator generated sidebands are moved in frequency past the fixed filter for analysis, as shown in Figure 7.5 below.
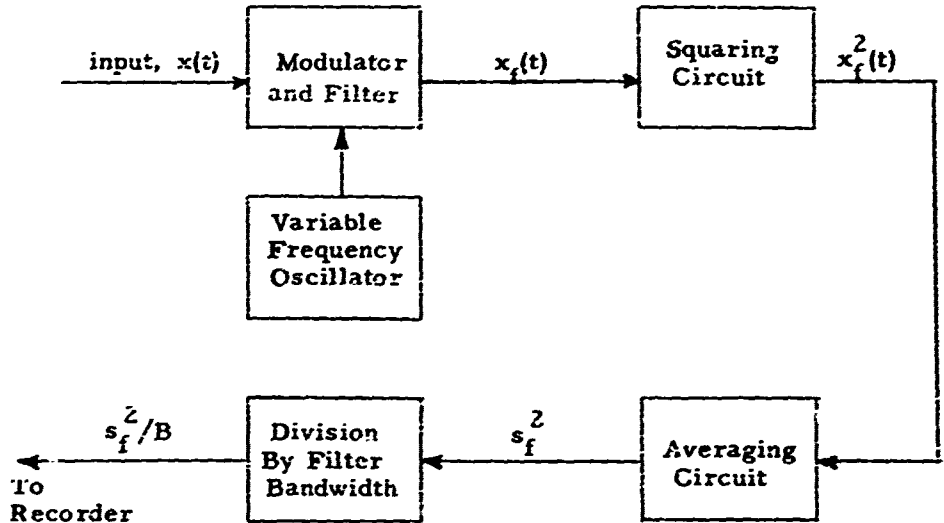
Figure 7.5 Heterodyne Type Analyzer

The heterodyne type analyzer is inherently a constant bandwidth instrument. Its primary advantage is that the single fixed filter can be carefully designed and controlled to obtain high selectivity and stability. Quartz crystal filters with a Q of 30,000 or more at a center frequency around 100 kc are used.

It has been pointed out that the heterodyne type power spectrum analyzer in Figure 7.5 filters over the frequency range under consideration with a constant bandwidth, but nothing has been said about the bandwidth of the other analyzer techniques discussed. The filter set type analyzer in Figure 7.3 may be one of two types. The first is a set of filters having equal bandwidths (constant bandwidth analysis) and the second is a set of filters having equal C (constant percentage analysis). The constant percentage filter set is far more prevalent in practice. The variable center frequency filter type analyzer in Figure 7.4 may also be either a constant bandwidth or constant percentage type filter.

There has been considerable discussion in the past on the subject of constant bandwidth versus constant percentage filtering techniques for power spectral density analysis. A good summary of pro and con

arguments is presented in Ref. [19]. In general, constant percentage analysis has certain advantages associated with a constant resolution of the power spectrum estimate, as will be discussed later. The primary argument for constant bandwidth analysis is simply the practical consideration of filter design. The sharp selectivity of the fixed crystal filter in the heterodyne type analyzer is not obtainable in the constant percentage type analyzers.

In summary, the vast majority of commercially available power spectral density analyzers employ one of the following three filtering techniques (Refs. [7]. [28], and [24]).

1. Constant percentage filter set type analyzer - Fig. 7.3.

2. Variable center frequency filter type analyzer (either constant bandwidth or constant percentage) - Fig. 7.4.

3. Constant bandwidth heterodyne type analyzer - Fig. 7.5.

Of the three types of analyzers, the heterodyne type instrument is the most widely used for detailed power spectral density analysis. Commercial heterodyne analyzers usually are equipped with several different filter bandwidth selections ranging from 2 cps to 50 cps. The frequency range of heterodyne analyzers is usually from zero to 25 kc, which is ample to cover the frequency range of interest for most vibration and acoustical data analysis.

7.4.2 Statistical Accuracy of Power Spectra Estimates

The basic measurement required for power spectral density analysis of stationary random signals is a mean square value determination. Specifically, the mean square value of the random signal is estimated by measuring the mean square value of a sample record using a squaring circuit followed by an averaging circuit. The details of actual analog mean square voltage measuring devices are presented in the previous Section 7.3 Assume for the moment that one has a mathematically precise mean square measuring device consisting of a perfect square low rectifier followed by a perfect averager which integrates the squared amplitudes over the entire sample record length and divides by the record length.

How well a mean square measurement of a sample record from a stationary random signal represents the true mean square level of the signal may be expressed in terms of the statistical quality of the measurement. The quality of a mean square measurement is a function of the record length T, the record frequency bandwidth B, and the power spectrum S(f). One expression for statistical quality is the normalized variance of the measurement given as follows, see Eq. (4.168) of Section 4.8.8,

$$\epsilon^2 = \frac{1}{BT} + \frac{1}{576} \left[\frac{B}{\lambda(f)} \right]^4 \tag{7.10}$$

where $\lambda(f) = \left| S(f)/S''(f) \right|^{1/2}$ is called the "spectral band width" of the random signal under consideration. If there are no sharp variations of the power spectrum within the bandwidth B being considered, the value of $\lambda(f)$ becomes very large and the second term of Eq. (7.10) becomes negligible. For the case of power spectral density analysis where B is the bandwidth of a narrow band pass filter, the second term may normally be neglected (although not always), and Eq. (7.10) becomes simply,

$$\epsilon^2 = \frac{1}{BT} \tag{7.11}$$

Another measure of statistical quality which is more amiable to standard statistical tables is the equivalent number of degrees of freedom of the measurement, denoted by n, as follows:

$$n = 2BT \tag{7.12}$$

The same argument now applies as developed earlier in Section 7.3.3, and confidence limits for the true mean square value from measured mean square values as a function of n are displayed in Table 7.2.

For example, assume a mean square value of s^2 is measured for a sample record of one second length from a random signal with a bandwidth of 30 cps. The equivalent number of degrees of freedom for the measurement is 60. From Table 7.2, it may be said that the actual mean square value for the random signal is between $0.72 s^2$ and $1.48 s^2$ for a confidence coefficient of 95 percent.

It should be noted that Eqs. (7.11) and (7.12) may have slightly different constant coefficients depending upon the actual configuration of the band pass filter employed, Refs. [3] and [4], but for the purposes of this discussion, the filter will be considered ideal with sharp cut off edges. Eqs. (7.11) and (7.12) are good approximations for an ideal filter.

7.4.3 Resolution of Power Spectra Estimates

Examine in more detail the effect of filter bandwidth on the quality of power spectral density estimates. It is seen from Eqs. (7.11) and (7.12) that the statistical quality of a mean square measurement (in terms of the equivalent number of degrees of freedom) is directly proportional to the bandwidth of the signal. One might then conclude for power spectral analysis that improved measurement quality can easily be obtained by simply increasing the bandwidth of the analyzer filter. However, increasing the filter bandwidth reduces the "resolution" of the analysis; i.e., it reduces the ability of the analysis to properly define sharp peaks in the power spectrum. The selection of the analyzer filter bandwidth is always a compromise between measurement quality and measurement resolution. It should be further noted that if there are sharp peaks in the filter bandwidth, the second term of Eq. (7.10) may become quite significant and the quality of the measurement will not be as good as would be predicted by Eqs. (7.11) and (7.12). The emphasis then should be placed upon selecting an analyzer bandwidth which will afford proper resolution of the spectrum being analyzed. A reasonable criteria for proper resolution might be a filter bandwidth that is one fourth the bandwidth (between half power points) of the narrowest peak in the power spectrum to be analyzed. This points out the rather annoying fact that power spectra measurements with known statistical confidence cannot be obtained

without some prior knowledge of at least the general nature of the power spectrum to be analyzed.

Power spectrum resolution is sometimes defined as the bandwidth of the analyzer filter divided by the center frequency of the filter in percentage. For a power spectral density analysis with a constant 5 percent resolution, the analyzer filter bandwidth would be 5 cps at a center frequency of 100 cps and 50 cps at a center frequency of 1000 cps. Using this definition of resolution, a constant percentage type analyzer (using constant Q filters) will yield power spectra with constant resolution while constant bandwidth type analyzers will yield power spectra with a resolution that increases in direct proportion to frequency.

7.4.4 Constant Bandwidth Power Spectra Estimates - Maximum Filter Scan Rates

Consider in detail the statistical accuracy of power spectral density estimates obtained using constant bandwidth type analyzers, such as a heterodyne type analyzer, which scan the frequency range under consideration with a continuous sweep of a single filter. This type of analyzer is being considered first and in most detail because it is the most widely used. Assume a sample record of length T seconds was obtained from a band limited stationary random signal with a frequency range of F cps. The power spectrum of the signal is to be measured (estimated) over the bandwidth of the signal by analysis of the sample record. Since the analysis will be done by an analog instrument, the record should be on magnetic tape or some other form of storage suitable for electrical playback. The record can then be made into a loop for continuous playback into the analyzer. Let the bandwidth of the analyzer filter be determined by resolution considerations to some value B. The maximum statistical quality attainable in the power spectrum measurement has now been fixed by Eq. (7.12). For example, if the record length is T = 6 seconds, and the analyzer filter bandwidth is B = 10 cps, the maximum quality of the power spectrum measurement is 120 equivalent degrees of freedom. Analysis techniques employed from this point on cannot increase the statistical accuracy of the measurement beyond that accuracy defined by 120 degrees of freedom.

The next question is how fast may the analyzer filter scan the frequency range F of the record without unnecessarily reducing the quality of the measurement. If the frequency range F were scanned in discrete steps, the filter would have to be stepped through (F/B) different center frequencies and remain at each center frequency for at least T seconds if maximum statistical quality is to be obtained. For example, if F = 2000 cps, B = 10 cps, and T = 6 seconds, the filter would have to be moved through 200 positions and remain at each for 6 seconds to obtain the maximum available quality of 120 equivalent degrees of freedom. The total analysis time would be at least 1200 seconds. If the frequency range F is scanned by a continuous sweep of the filter (as is the case in actual practice), the same general limitation applies. The scan rate should not be faster than one bandwidth per record length or B/T. For the previous example, the maximum scan rate would be S.R. = 1.67 cps/sec. for maximum quality. A slower scan rate would not increase the quality of the measurement, but a faster scan rate would reduce the quality of the measurement. On a basis of measurement quality, the maximum analyzer filter scan rate (S.R.) has thus been established to be,

$$S.R. \leq \frac{B}{T} \qquad (7.13)$$

However, there are other problems which must be considered. The transient response of the analyzer filter is one such problem. The filter must be swept slow enough to permit proper response to abrupt changes in the power spectrum being analyzed. The response of an ideal rectangular narrow band filter to a suddenly applied sinusoid is detailed in Chapter XI of Ref. [15]. It is shown that the time required for the filter output to rise from zero to 100% of the input, which shall be called "build up time", is inversely proportional to the bandwidth of the filter. It can be shown that this inverse proportionality of transient response to bandwidth also applies for simple single tuned and double tuned filters (Ref. [19], 1002-Q-1, and Ref. [3]). For the case of the ideal rectangular narrow band filter, the build up time to a suddenly applied input signal (or any abrupt change in the input signal level) is

approximately equal to (1/B). However, the filter output does not rise to the input level and stop there after (1/B) seconds. It overshoots the input level and then goes into a decayed oscillation about it. These oscillations will decay to less than 2% in about (8/B) seconds after the input signal is applied. Thus a criteria for proper filter response should be that each frequency increment of the spectrum be viewed by the analyzer filter for at least (8/B) seconds or 8 times the build up time for the analyzer filter.

It should be noted that there is another important reason to support such a criteria. Assume that each frequency increment is viewed for an interval of only (1/B) seconds. The full output of the filter will occur only for an instant at the end of the interval. For a random signal input where a mean square estimate of the filter output is obtained by averaging all squared amplitudes over the entire interval, it is obvious that the resulting mean square measurement would be too low. By limiting the minimum interval that each frequency increment is viewed by the filter to (8/B), we assure ourselves that the resulting mean square measurement of the filter output will be biased by no more than (1/8) of the interval. Then, on a basis of proper filter response, the scan rate should be limited to $\dfrac{B}{(8/B)}$ or,

$$S.R. \leq \frac{B^2}{8} \tag{7.14}$$

It should be noted that the constant factor of (1/8) in Eq. (7.14) would be somewhat different for other filter characteristics (single tuned, double tuned, etc). However, Ref. [19] (1002-Q-1) indicates the build up time for an ideal rectangular filter is longer than for other conventional filter characteristics, and Eq. (7.14) should be conservative when applied to any real filter.

Another factor to be considered in the maximum scan rate that should be used for constant bandwidth spectral analysis is the characteristics of the read-out or mean square measurement circuit of the analyzer. If the read-out is accomplished by a true integration of the output of a square law rectifier over the record length T, no further

problems arise. However, the read-out is often accomplished by averaging the output of a square law rectifier by an equivalent RC filter (not to be confused with the scanning filter of the analyzer). As discussed in Section 6.1.7 of this report, when a random signal is passed through an RC filter, the output of the filter is not a true linear time integration over a specific time interval as is required mathematically to obtain a true average of the input signal. The output of the RC filter is a continuous signal which at any instant of time is a time weighted integration of all signal amplitudes that have gone before. However, it is shown in Section 6.1.7 that after the signal has been applied to the RC filter for 3 or 4 time constants the output of the RC filter at any instant approximately corresponds to a true averaging process with an effective integration time equal to twice the time constant of the RC filter. Then when the output of a square law rectifier is passed through an equivalent RC filter, the output of the RC filter at any instant (after 3 or 4 time constants have elapsed) is equivalent to a mean square measurement with a statistical quality given by,

$$\text{Number of degrees of freedom, } n = 4 \, BK \qquad (7.15)$$
$$\text{where } K = RC \text{ time constant}$$

Consider now the significance of Eq. (7.15) in the constant bandwidth analysis of a record length T with a scanning filter of a bandwidth B. As previously stated, the maximum quality (degrees of freedom) of the measurement is fixed by $n = 2 \, BT$. Then if this maximum quality is to be maintained in the mean square read-out, the time constant K of the equivalent RC filter associated with the read-out should be $K = (T/2)$. A larger time constant cannot improve the quality of the measurement, but a shorter time constant will reduce the quality of the measurement.

Consider next the effect of the read-out time constant on the scan rate. If the scanning filter were stepped from one center frequency to another frequency B cps away, several time constants must elapse to permit the equivalent RC filter of the read-out to respond to the new information. Specifically, let 4K be the criteria for proper response (the output of an RC filter rises to 98% of the input at 4 time constants).

For proper read-out by an RC averaging network, the scan rate of the analyzer filter is limited to one filter bandwidth every 4 RC filter time constants, or,

$$S. R. \leq \frac{B}{4K} \qquad ; \qquad K = RC \qquad (7.16)$$

Remembering that for maximum measurement quality, $K = (T/2)$, it follows that,

$$S. R. \leq \frac{B}{2T} \qquad (7.17)$$

Note that Eq. (7.17) limits the maximum scan rate for the analyzer filter to (1/2) the scan rate permitted by Eq. (7.13). This leads to the important conclusion that if power spectrum read-out (mean square measurement divided by bandwidth) is accomplished by averaging the squared amplitudes with an equivalent RC filter rather than by true integration, the maximum scan rate for the analyzer filter must be reduced by (1/2) to obtain the same statistical quality in the measurement.

Before proceeding, briefly review the discussion of power spectral density measurements by scanning with a constant bandwidth filter. Given a stationary random signal record of length T seconds, the maximum statistical quality of a power spectral density estimate at any frequency in terms of the equivalent number of degrees of freedom, n, for the measurement is,

$$n = 2BT \qquad (7.18)$$

where B is the bandwidth of the analyzer filter. The bandwidth B selected for analysis should be as large as possible while still maintaining proper spectral resolution in the measurement. In general, B should be no greater than (1/4) the bandwidth of the narrowest peak to be expected in the power spectrum. After B is selected, the maximum attainable quality of the measurement is fixed. In order to maintain this quality, the scan rate, S. R., of the analyzer should be as follows:

(1) For power spectrum read-out by averaging squared amplitudes using a true integrating circuit.

$$S. R. \leqslant \frac{B}{T} \quad \text{or } S. R. \leqslant \frac{B^2}{8} \quad \text{whichever is smaller} \qquad (7.19)$$

$$\text{Total analysis time} = \frac{F}{S. R.} \qquad (7.20)$$

S. R. = scan rate in cps/sec

B = bandwidth of scanning filter in cps

T = record length in seconds

F = frequency range to be analyzed in cps

(2) For power spectrum read-out by averaging squared amplitudes using an equivalent RC filter circuit.

$$S. R. \leqslant \frac{B}{4K} \quad \text{or } S. R. \leqslant \frac{B^2}{8} \quad \text{whichever is smaller} \qquad (7.21)$$

$$\text{Optimum value of K} = (T/2) \qquad (7.22)$$

$$\text{Total Analysis Time} = \frac{F}{S. R.} \qquad (7.23)$$

S. R. = scan rate in cps/sec

B = bandwidth in cps

T = record length in seconds

K = time constant of equivalent RC filter in seconds

F = frequency range to be analyzed in cps

Commercial heterodyne type power spectral density analyzers are equipped with several different fixed filter bandwidths ranging from 2 cps to 50 cps. In order to obtain proper resolution in the lower frequencies, it might be necessary to select a bandwidth of say 5 cps or perhaps even as narrow as 2 cps. However, at higher frequencies a filter bandwidth of 50 cps may provide completely acceptable resolution. It is good practice to divide the frequency range to be analyzed into two or three ranges and use a different filter bandwidth in each range. The higher frequency ranges can be scanned with wider filters. The result is a faster scan rate (reduced analysis time) with adequate statistical accuracy.

Example: Consider the problem of estimating the power spectrum of a stationary random signal by constant bandwidth analysis of a 6-second long sample record over the frequency range from 10 to 2000 cps. By a brief preliminary frequency scan, it is determined that proper spectral resolution will be obtained using filter bandwidths as follows:

(1) 10 - 100 cps: 2 cps
(2) 100 - 500 cps: 10 cps
(3) 500 - 2000 cps: 50 cps

The maximum measurement quality in each of the three frequency ranges will be as follows:

(1) 24 degrees of freedom
(2) 120 degrees of freedom
(3) 600 degrees of freedom

Assume power spectral density read-out is accomplished by averaging the output of a square law rectifier with an equivalent RC filter and dividing by bandwidth. The time constant for the equivalent RC filter should then be adjusted to 3 secs and the scan rates in each of the three frequency ranges should be limited as follows:

(1) less than 0.167 cps/sec
(2) less than 0.833 cps/sec
(3) less than 4.17 cps/sec

Note that the scan rates are not further limited by the response characteristics of the analyzer filter. The total analysis time will be at least 1360 seconds or 23 minutes.

The statistical confidence limits (for an 80% confidence coefficient) which may be placed on the resulting power spectral density estimate in each of the three frequency ranges are as follows (obtained from page 309 of Ref. [9]):

(1) One may be 80% confident that the true power spectral density of the signal at any frequency between 10 and 100 cps is between $0.72 \hat{S}(f)$ and $1.53 \hat{S}(f)$ where $\hat{S}(f)$ is the measured power spectral density at that frequency.

(2) One may be 80% confident that the true power spectral density of the signal at any frequency between 100 and 500 cps is between $0.85 \hat{S}(f)$ and $1.19 \hat{S}(f)$.

(3) One may be 80% confident that the true power spectral density of the signal at any frequency between 500 and 2000 cps is between $0.86 \hat{S}(f)$ and $1.14 \hat{S}(f)$.

7.4.5 Constant Percentage Power Spectra Estimates - Maximum Filter Scan Rates

The general relationships developed for constant bandwidth power spectral density analysis also apply to constant percentage power spectral density analysis. The only difference is that for constant percentage analysis, the analyzer filter bandwidth increases in direct proportion to the center frequency of the filter. Here $B = Pf$ where P is resolution (some constant fraction less than one) and f is center frequency. For constant percentage analysis, Eqs. (7.12), (7.13), (7.14), and (7.17) become, respectively,

$$n = 2(Pf) T \qquad (7.24)$$

$$S.R. \leq \frac{(Pf)}{T} \quad \text{(read-out averaging by true integration)} \qquad (7.25)$$

$$S.R. \leq \frac{(Pf)^2}{8} \qquad\qquad (7.26)$$

$$S.R. \leq \frac{(Pf)}{2T} \quad \text{(read-out averaging by equivalent RC filter} \\ \text{with } K = \frac{1}{2}T) \qquad\qquad (7.27)$$

Note from Equation (7.24) that the quality of the measurement increases with frequency. Also, from Equations (7.25) and (7.27), the maximum scan rate increases with frequency. Specifically, for Equation (7.25),

$$\text{max. S.R.} = \frac{df}{dt} = \frac{Pf}{T} \quad \text{or} \quad \frac{df}{f} = \frac{Pdt}{T}$$

Then

$$\int_{f_1}^{f_2} \frac{df}{f} = \frac{P}{T} \int_0^t dt$$

Hence

$$\ln(f_2/f_1) = \frac{Pt}{T} \qquad \text{of} \qquad f_2 = f_1\, e^{(Ft/T)} \qquad\qquad (7.28)$$

where f_2 is the upper limit and f_1 the lower limit of the frequency range under consideration, and t is analysis time.

If the maximum scan rate is to be maintained, the above formulation shows that the frequency range under consideration must be scanned by a logarithmic sweep. Furthermore, the total analysis time $t = \frac{T}{P} \ln \frac{f_2}{f_1}$, multiplied by two for read-out averaging by an RC filter. These conclusions are of course subjected to the possible limitations imposed by Eq. (7.26).

Example: Consider the problem of estimating the power spectrum of a stationary random signal by constant percentage analysis of a 6-second long sample record over a frequency range of 10 cps to 2000 cps. By a brief preliminary frequency scan, it is determined that proper spectral resolution will be obtained using a 10% resolution filter (P = 0.1).

The maximum measurement quality will range from 12 equivalent degrees of freedom at 10 cps to 2400 equivalent degrees of freedom at 2000 cps, increasing linearly with frequency.

Assume power spectral density read-out is accomplished by averaging the output of a square law rectifier with an equivalent RC filter and dividing by bandwidth. The time constant for the equivalent RC filter should then be adjusted to 3 seconds and the frequency range should be scanned with a logarithmic sweep of no shorter than 636 seconds (10.6 minutes) which is the minimum analysis time. Note that the scan rate is not further limited by the response characteristics of the analyzer filter.

The statistical confidence intervals which may be placed on the resulting power spectral density estimate are different at every frequency. The widest confidence interval (poorest estimate) will be in the lowest frequency analyzed (the narrowest analyzer filter). At 10 cps, we may be 30% confident that the true power spectral density of the signal is between $0.64 \hat{S}(f)$ and $1.90 \hat{S}(f)$ where $\hat{S}(f)$ is the measured power spectral density. At 100 cps, we may be 80% confident that the true power spectral density of the signal is between $0.85 \hat{S}(f)$ and $1.19 \hat{S}(f)$.

7.4.6 Power Spectra Estimates for Nonstationary Random Data

The statistical accuracy of power spectral density estimates for stationary random data has been reviewed in this section. It is important to remember that all discussions have been limited to the analysis of a sample signal record taken from a stationary random process (the statistical properties of the process are invariant with time translations). If the signal is non-stationary, it is obvious that no statistical property of the signal over all time, such as the power spectral density function, can be estimated from a sample record of finite time. It is quite important that one has confidence that a sample record used for power spectral density analysis was obtained from a stationary random signal before that analysis is used to estimate the power spectrum of the random signal. Such confidence can be obtained by testing the record to be analyzed for self stationarity by the procedures presented in detail in Section 6.1.8.

Assume a long sample record of the random acceleration time history at a point on the structure of a modern flight vehicle is to be analyzed to estimate the power spectral density. Also assume a test of the record for stationarity fails to establish confidence that the record represents a stationary random signal. Such records are common for vibration measurements obtained during missile flights where the vibration environment may continually change with time. The engineer is often interested in obtaining some definition of the environment represented by the record. Procedures for analyzing non-stationary records have been proposed from time to time, such as in Ref. [20]. Nearly all such proposals involve the same general approach, which may be summarized as follows:

(1) The long record of a non-stationary vibration signal is divided into many short sub-records.

(2) Each of the short sub-records is assumed to be representative of a stationary random signal.

(3) An estimate of the power spectral density of the assumed stationary signal is obtained for each of the sub-records.

(4) The results are presented as a variation of power spectral density versus time.

The argument for the procedure is that the resulting vibration data which is sometimes referred to as a "power spectral density time history", may be correlated with flight events that are also changing with time. Time lags between instantaneous flight events and the response power spectrum are usually ignored as being very small.

There is no doubt that worthwhile qualitative engineering information may often be obtained by such an analysis procedure. However, the statistical significance of the resulting data is at best rather questionable. If one could not confidently accept the original record as being stationary, there is certainly no quantitative justification for arbitrarily accepting all sub-records formed from the original record as being stationary. It follows then that confidence statements about the statistical accuracy of the power spectra estimates obtained from the sub-records would not

be justified. All that can be said is that the equivalent number of degrees of freedom of the resulting measurements must be less than $n = 2BT$, or less than the statistical quality that would have been attained if the measurement had been an estimate of the power spectrum of a stationary signal. These matters for non-stationary data are worthy of further theoretical investigation.

7.4.7 Conclusions

To summarize the preceding discussion, general procedures have been described for obtaining power spectral density estimates by analog techniques. Emphasis has been placed upon the statistical accuracy of the estimates as related to sample record length, filter bandwidth, and filter scan rates. Both constant bandwidth filtering and constant percentage filtering techniques have been explained, and statistical confidence intervals for the resulting estimates have been reviewed assuming ideal filter characteristics.

An analog instrument used for power spectral analysis must accomplish four functions.

(1) Filter over the frequency range under consideration.

(2) Square the output of the filter.

(3) Average the output of the squaring device.

(4) Divide the output of the averaging device by the associated filter bandwidth.

Several commercial companies produce complete power spectral density analyzing systems, for sale at prices today ranging from $5,000 to $20,000, or more, depending upon the amount of support equipment desired. The more elaborate systems are equipped with both an integrator type averaging circuit and an equivalent RC filter type averaging circuit to give a choice in the technique desired for mean square level estimation. The cycle length for the integrator averager and the time constant for the RC filter averager are both variable from one second or less to 30 seconds or longer. The great flexibility of such commercial

power spectral density analyzers permits maximum quality estimates of power spectra from record lengths of less than one second to longer than 30 seconds.

Many laboratories cannot afford the expense of a complete commercial power spectral density analyzer system. However, with the sacrifice of convenience and flexibility, power spectra estimates can often be obtained using common laboratory equipment that may be on hand for other functions. Many commercial wave analyzers are available, and often found in vibration laboratories, which are not actually power spectral density analyzers because their read-out is accomplished using a conventional average value rectifier type voltmeter circuit. These instruments will accomplish only the first function stated above. However, with the assistance of some additional laboratory instruments, power spectral density estimates can be obtained using such wave analyzers. The laboratory may have available a true rms vacuum tube voltmeter, as described in detail in . on 7.5 of this report, which will effectively accomplish functions 2 and 3. A voltmeter such as the one specified in Table 7.1 squares and averages (by an equivalent RC filter) the input signals and has a mean square signal output jack on the panel. A number of common laboratory chart recorders can be operated from this mean square output signal. The output of the wave analyzer could be intercepted just before its own read-out circuit and monitored with the true rms voltmeter. The mean square output of the voltmeter could be recorded as a function of time and correlated with the center frequency of the analyzer filter. If the analyzer scans with a constant bandwidth filter, function No. 4 is easily accomplished by dividing the resulting spectrum plot by a constant.

Such a makeshift set up will, of course, present problems. For one thing, the time constant of the final read-out would be a fixed value. Without a variable RC time constant, the set up would not have sufficient flexibility to obtain the maximum quality in measurement of the available data. However, a specific statistical quality can be associated with the power spectral density measurements that are obtained.

7.5 PROBABILITY DENSITY MEASUREMENTS OF RANDOM DATA

The amplitude time history of any steady state periodic signal is defined by an explicit mathematical equation, and the exact amplitude of the signal may be determined at any future instant of time. On the other hand, the amplitude time history of a random signal cannot be characterized by an explicit mathematical function and statements concerning the exact amplitude of a random signal of any future instant of time are not possible. However, if the random signal is stationary in time (See Sections 4 and 6 of this report), statistics can be employed to establish the probability of certain amplitudes occurring at any future instant of time. The amplitude probability density function of a stationary random signal (sometimes called the amplitude probability distribution although this term is also reserved for the integral of the probability density function) describes the probability of given amplitude occurrences and is an important statistical property of random signals.

Usually only two parameters of a random signal which may be calculated from the probability density function are of interest. These are the mean, μ, of the random signal and the variance, σ^2, the variance being the mean square value about the mean. For analog vibration data, the mean (defined by a DC voltage level) is normally zero, and interest is limited to the variance. The variance for a random signal (with mean of zero) is the mean square value of the signal, and is the parameter that one actually estimates with a true rms voltmeter as described in Section 7.3 The rms (root mean square) value of a random signal is simply the positive square root of the variance, and is called the standard deviation, σ.

In some applications, the mean square value of a random signal may not be a sufficient description for the engineer. Consider the case of an engineer analyzing an analog signal of the acceleration response at a point on the structure of a modern flight vehicle. His interest may be in structural fatigue, in which case he would like to know the probability of accelerations occurring at, say, 2σ or 3σ, and not just the mean square acceleration. In other words, he would like to know the over-all acceleration probability density function for the response. The estimation of such over-all probability density functions of random data by analog techniques is the subject of this section.

7.5.1 Analog Instrumentation

As discussed and illustrated in Section 4.9.2 of this report, an estimate of the probability density function is provided by,

$$\hat{p}(x) = \frac{1}{T}\left(\frac{\Delta T}{\Delta x}\right) \tag{7.29}$$

The terms of Equation (7.29) define the operations which an analog instrument must accomplish:

(1) Measurement of the total time, ΔT, that the signal amplitude falls within a narrow amplitude window, Δx.

(2) Division by the window Δx.

(3) Division by the total sampling time T.

Of course, the center amplitude of the window, Δx, would have to be variable to cover the entire range of signal amplitudes under consideration. In terms of $\hat{p}(x)$,

$$\hat{P}(x,\ x + \Delta x) \approx \hat{p}(x)\Delta x \tag{7.30}$$

where $\hat{P}(x,\ x + \Delta x)$ is an estimate of the probability that a random record $x(t)$ assumes amplitude values between x and $x + \Delta x$ in a sampling time T.

The above operations for measuring $\hat{p}(x)$ can be physically accomplished in many ways. In general, the factor ΔT could be determined by using a voltage gate (narrow band voltage discriminator) to actuate a clock. The width of the voltage gate would correspond to some narrow range of signal amplitudes. When the input voltage from a sample record falls within the gate, the clock operates. For all other input voltages, the clock does not operate. The division by the record length T would be accomplished by a second clock which operates over the total record length. If the width of the voltage gate is constant, the required division by Δx can be obtained by proper read-out calibration.

Statistical estimates of the amplitude probability density function in accordance with Eq. (7.29) would result. See Figure 7.6 below.
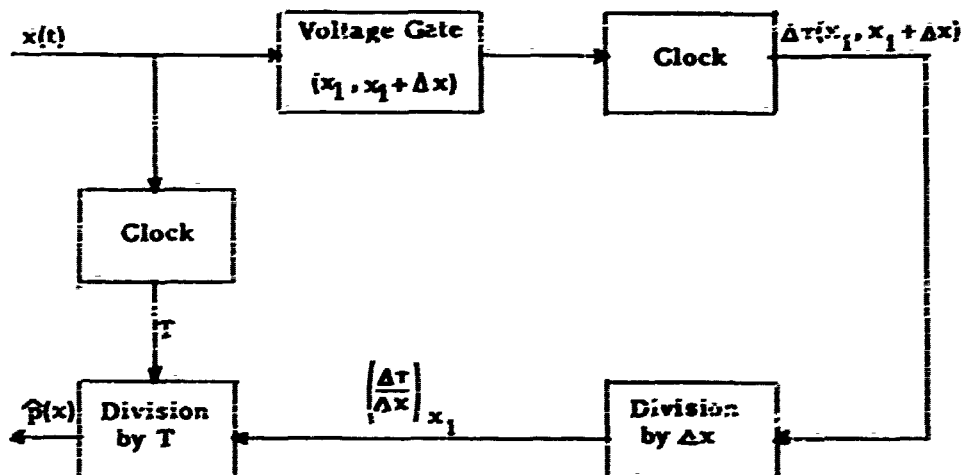
Figure 7.6 Probability Density Analyzer

Analog probability density analyzers have been manufactured by several commercial companies. Experiments conducted on one of these laboratory analyzers which is commercially available at the present time is discussed in Section 7.5.4.

7.5.2 Distribution Functions of Instantaneous Values and Peak Values

By definition, the (cumulative) probability distribution function $P(x)$ is defined as the probability (i.e., the fraction of time, on the average) that $x(t)$ assumes particular amplitude values between $-\infty$ and x. In terms of the probability density function $p(x)$

$$P(x) = \text{Prob}\left[x(t) \leq x\right] = \int_{-\infty}^{x} p(x)\, dx \quad ; \quad \frac{dP(x)}{dx} = p(x) \qquad (7.31)$$

These different concepts of (cumulative) probability distribution function $P(x)$ versus probability density function $p(x)$ are confused occasionally in engineering literature. The above relationship shows that knowledge of either one determines the other by appropriate integration or differentiation.

Probability distribution functions of instantaneous values of a signal $x(t)$, and of peak values of $x(t)$, are of considerable importance in vibration analysis. Analog instrumentation to measure these two probability distribution functions directly will now be described.

(1) Instantaneous Values

To determine the probability distribution of instantaneous values of a signal $x(t)$, the sum of time increments τ_i $(i = 1, 2, 3, \cdots;$ that $x(t)$ exceeds any preselected value of x is measured and divided by that total time (record length) T. This yields

$$\text{Prob}\left[x(t) \geq x\right] = \frac{\sum_{i=1}^{k} \tau_i}{T} \qquad (7.32)$$

Then

$$P(x) = \text{Prob}\left[x(t) \leq x\right] = 1 - \text{Prob}\left[x(t) \geq x\right] = 1 - \frac{\sum_{i=1}^{k} \tau_i}{T} \qquad (7.33)$$

See sketch below.
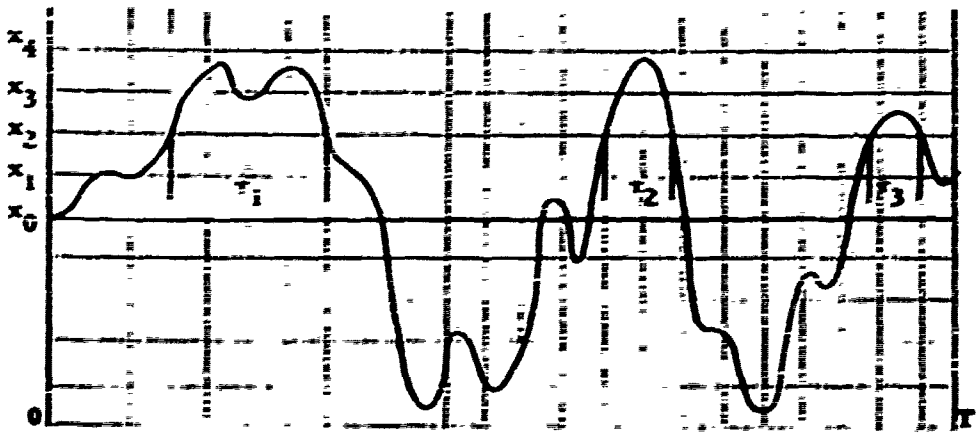
For example,

$$\text{Prob}\left[x(t) \gtrless x_2\right] = \frac{\tau_1 + \tau_2 + \tau_3}{T}$$

$$P(x_2) = \text{Prob}\left[x(t) \lessgtr x_2\right] = 1 - \left(\frac{\tau_1 + \tau_2 + \tau_3}{T}\right)$$

Required equipment to perform the above operations are (1) an optional polarity device if it is desired that positive and negative values of $x(t)$ should be analyzed separately, (2) a bank of discriminators which determine whether the signal is greater or less than a preselected value, (3) a clock to record the time interval that a particular level is exceeded, (4) a separate clock to record the time interval between these events. At the end of the record length T, the separate clock reading is divided by T to give the value of the probability distribution function at the given level. If the values at adjacent levels are subtracted from one another, an estimate would be obtained of the average probability density function between these two levels.

(2) Peak Values

To determine the probability distribution function $P_p(x)$ of peak values in $x(t)$, a slightly different procedure may be employed. Let x_0 be the level such that there is a 50% probability that $x(t)$ exceeds x_0. For a signal with a symmetrical probability density function about its mean value μ, the level $x_0 = \mu$. Let $x_p(t)$ represent peak values of $x(t)$. Then the probability that $x_p(t)$ is greater than a preselected peak value x_p equals the number of times that $x(t)$ exceeds x_p divided by twice the total number of times that $x(t)$ exceeds x_0. In equation form,

$$\text{Prob}\left[x_p(t) \geq x_p\right] = \frac{N\left[x(t) \geq x_p\right]}{2N\left[x(t) \geq x_0\right]} \tag{7.34}$$

Then

$$P_p(x) = \text{Prob}\left[x_p(t) \leq x_p\right] = 1 - \frac{N\left[x(t) \geq x_p\right]}{2N\left[x(t) \geq x_0\right]} \tag{7.35}$$

where

$$N\left[x(t) \geq x_p\right] = \text{number of times that } x(t) \text{ exceeds } x_p$$

$$N\left[x(t) \geq x_0\right] = \text{number of times that } x(t) \text{ exceeds } x_0$$
$$\text{(called the zero level count)}$$

Required equipment to obtain the above probability distribution function of peak values are (1) a bank of discriminators, (2) a set of associated counters to register one count each time the level of the particular discriminator is exceeded. At the end of the record length T, the various counter readings are recorded, and divided by the zero level count. As before, a polarity device may be employed to analyze separately the positive and negative values of $x(t)$.

7.5.3 Statistical Relationships and Accuracy

The statistical accuracy of probability density estimates corresponding to Eq. (7.29) has been studied previously in Section 4.9.2. A final result, contained in Eq. (4.205), gives for the standard error ϵ (normalized standard deviation) of the estimate

$$\epsilon = \left[\frac{1}{2 \, f_0 \, T \, \hat{p}(x)(\Delta x)} \right]^{1/2} \tag{7.36}$$

when band limited white noise is passed through an idelized low pass filter with a sharp cut off at frequency f_0 cps.

Another different approach to defining the standard error of probability density estimates may be based upon work done at Bolt Beranek and Newman, Inc. The following development originates from correspondence with Mr. Herbert L. Fox of that company.

The normalized variance of an amplitude density estimate obtained using an amplitude window of width Δx over a sample record of length T may be given by,

$$\epsilon = \frac{1}{n} = \frac{1}{(\Delta x)\,\bar{\gamma}T} \tag{7.37}$$

where $\bar{\gamma}$ is the number of times x passes through the interval $(x_0, x_0 + \Delta x)$ per unit time. The denominator $n = (\Delta x)\bar{\gamma}T$ equals the number of statistical degrees of freedom (the effective number of observations), as given by the product of the number of observations per event (Δx), multiplied by the number of events per unit time $\bar{\gamma}$, multiplied by the sampling time interval T.

If Δx is small enough, it may be assumed that the probability of x entering Δx and changing direction in Δx without passing through is very small. Therefore, $\bar{\gamma}$ is approximately twice the number of times that x exceeds some x_0 per unit time, since on the average there would occur the same number of upward and downward crossings.

If it is assumed that the amplitude density function for the signal is Gaussian, then it can be shown (based upon the work of S. O. Rice discussed in Section 4.9.3 of this report) that,

$$\bar{Y} = 8\pi \ p(x) \left[\frac{\int_0^\infty f^2 S(f) \, df}{\int_0^\infty S(f) \, df} \right]^{1/2} \tag{7.38}$$

where $S(f)$ is the power spectral density function of the signal being analyzed. Once again, a power spectrum for the signal must be known or assumed to obtain a solution to Eq. (7.38).

Assume, as before, that the signal is band limited white noise cut off sharply at some frequency f_0. Then,

$$\bar{Y} = \sqrt{\frac{8\pi}{3}} \ p(x) \ f_0 \tag{7.39}$$

Substituting into Eq. (15),

$$\epsilon^2 = \frac{\sqrt{\frac{8\pi}{3}}}{f_0 \ T \ p(x)(\Delta x)} \tag{7.40}$$

Then the standard error of the probability density estimate becomes,

$$\epsilon = \left[\frac{1}{2.89 \ f_0 \ T \ p(x)(\Delta x)} \right]^{1/2} \tag{7.41}$$

Note that Eq. (7.41) is identical in form to the corresponding result of the first development, Eq. (7.36). However, the constant coefficients of the two equations are different by a factor of about 20%; 0.707 for Eq. (7.36) as compared to 0.588 for Eq. (7.41). In view of the totally different approaches used to arrive at these equations, the agreement is considered quite good. One comparison with experimental tests now follows.

7.5.4 Experimental Tests

Bolt Beranek and Newman have conducted unpublished experimental investigations of the statistical accuracy of the Bruel and Kjaer Model 160 Probability Density Analyzer when employed to estimate the probability density function of random noise cut off sharply at several different frequencies f_0. The results of these experiments produced the following empirical equation for ϵ.

$$\epsilon = \left[\frac{2}{\pi^3 f_0 T \, \hat{p}(x)(\Delta x)} \right]^{1/2} \tag{7.42}$$

The coefficient of Eq. (7.42) is 0.254. Thus, the results of experiments with an actual piece of hardware indicate the standard deviation of the probability density estimates to be less than half the theoretically predicted value for ϵ. It is believed that this discrepancy could be due in part to the possible failure of the experimental conditions to comply fully with the assumptions employed in the statistical developments. Remember that both developments assume the random signal is band limited white noise with an infinitely sharp cut off at f_0, and the Bolt Beranek and Newman theoretical development further assumes the random signal has a Gaussian probability density function.

7.5.5 Physical Example

The discussion will be solidified by considering a detailed physical example of probability density estimation by an analog instrument. Assume there is given a 10 second long sample record of a stationary random signal with a flat power spectrum cut off sharply at $f_0 = 500$ cps by an ideal low pass filter. Also assume one has available a probability density analyzer which scans the amplitude range of the signal with an amplitude window having a width of (1/10)th the rms voltage of the signal. Then, if the rms level of the input signal (one σ) is one volt, the window, Δx, is 0.1 volts wide. The window is now set at some center voltage x_1, and the time interval $\Delta \tau$, that the signal occupies the window over the record length, $T = 10$ seconds, is measured and divided by T and Δx_1 to obtain $\hat{p}(x_1)$. For this example, $\hat{p}(x_1) = \Delta \tau_1$.

The center voltage of the window is now moved 0.1 volts, and the proba-
bility density, $\hat{p}(x_2) = {}_{\Delta}T_2$, is measured. The procedure is repeated
until the amplitude range of interest is covered.

If the amplitude range of interest were say $\pm 4\sigma$ (± 4 volts), 80
measurements would be required and the minimum analysis time would
be 800 seconds or 13.33 minutes. Note that the amplitude range could
be scanned either in discrete steps as described above or continuously.
If a continuous window scan is used, the scan rate (S.R.) should be
slow enough to permit each increment of amplitude to be viewed by the
window over the length of the record. Thus,

$$S.R. \leq \frac{\Delta x}{T} \qquad (7.43)$$

The maximum scan rate for this example would be 0.1/10 = 0.01 volts/sec.

Now consider the standard error of this estimate. Eq. (7.36) will
be used since it is the most conservative estimate of standard error.
From Eq. (7.36), for this example,

$$\epsilon = \left[\frac{1}{2(500)(10)(0.1)\hat{p}(x)} \right]^{1/2} = \sqrt{\frac{0.001}{\hat{p}(x)}}$$

The standard error at any voltage amplitude x_i is a function of the
probability density measured at that voltage. For simplicity, assume
the probability density function measured in the example followed a
Gaussian probability density function. Suppose the measurements at
principal points and their associated standard errors were as follows:

Center Voltage of window	Value for p(x)	ε
0 volts	0.40	0.050
$\pm$ 1	0.24	0.064
$\pm$ 2	0.054	0.14
$\pm$ 3	0.0044	0.48
$\pm$ 4	0.0001	3.2

From the estimate at zero volts, for a 68 percent confidence interval, the true probability density, p(x), at zero volts is within the range of $\hat{p}(x) \pm \epsilon \hat{p}(x)$ or between 0.38 and 0.42. At one volt, one can be 68 percent confident that p(x) is between 0.22 and 0.26. At two volts, for a 68 percent confidence interval, p(x) is between 0.046 and 0.062. At three and four volts, the standard error has become too large to permit an assumption that the distribution of $\hat{p}(x)$ is normal. Note that the standard error at four volts indicates that p(x) would be between minus 0.0002 and 0.0004 with 68 percent confidence. This is of course meaningless since p(x) cannot take on negative values. Thus, one does not know the statistical accuracy of these measurements at three and four volts except it is undoubtedly much better than indicated by ϵ.

It is of interest to reverse the problem and solve for the required record length T to obtain an appropriate value of ϵ for the estimate of probability density at three and four volts. Specifically, assume one is analyzing a signal with a true Gaussian amplitude density function and the estimate at three and four volts is desired to have a standard error of $\epsilon = 0.3$. At three volts, p(x) = 0.0044. Then,

$$0.3 = \left[\frac{1}{2(500)(0.0044)(0.1) \ T} \right]^{1/2}$$

Solving for T yields,

$$T = 25.2 \text{ seconds} \quad (3 \, \sigma \text{ point})$$

At four volts, p(x) = 0.0001. Then,

$$0.3 = \left[\frac{1}{2(500)(0.0001)(0.1) \ T} \right]^{1/2}$$

Solving for T yields,

$$T = 1111 \text{ seconds}$$

$$= 18.5 \text{ minutes} \quad (4 \, \sigma \text{ point})$$

The above example illustrates clearly the real problems associated with probability density estimations. The primary problem is that long samples are necessary to obtain estimates of very low probability densities with known statistical confidence coefficients. This is not to imply that probability density analysis is not practical. Often, an estimate of the probability density of a random signal out to $\pm 3 \sigma$ is a satisfactory definition, and can be accurately obtained from samples of less than 30 seconds. It should be added that more theoretical work is needed on the problem of error analysis for small probability density measurements. It appears that non-parametric techniques might be applicable to this particular estimation problem.

7.6 CORRELATION MEASUREMENTS OF RANDOM DATA

In the past, vibration engineers have generally limited themselves to power spectra analysis when dealing with random vibration. This may account for the limited number of commercially available correlation analyzers covering the frequency range from 5 to 2000 cps. In recent years, however, engineers have become aware of the importance to determine if one set of measurements is linearly dependent upon another set of measurements and several companies (see Refs. [13], [14], and [23]) have built analog correlation analyzers for their own problems. Usually both the autocorrelation and the cross-correlation functions are desired.

The autocorrelation function provides information about the dependence of two measurements taken from the same random process. Under certain conditions the autocorrelation function (except for a constant factor) is the Fourier transform of the power spectral density and therefore contains the same information, except that the autocorrelation function is obtained by manipulating in the time domain while the power spectral density is a function in the frequency domain. In particular, the autocorrelation function will help to bring out any hidden periodicities contained in the random process. This information cannot be obtained as easily from the power spectral density function since a sharp peak in the power spectrum only indicates a large amount of "power" at the particular frequency, but the amplitude at that frequency may still be randomly distributed.

The cross-correlation function gives information about the dependence between two different random processes. For instance, the vibration level occurring at a point on a structure can be separated into components according to (1) the sources from which it originated; (2) the transit time from the source to the point in question; and (3) the frequency, all from a knowledge of the cross-correlation functions involved.

This section first discusses the underlying theory of correlation functions, then describes the basic features of two privately manufactured analog correlators, and concludes with a practical example indicating the importance of correlation analysis to vibration analysis.

7. 6. 1 Correlation Coefficient

As pointed out in the preceding remarks, it often is desirable to know something about the dependence of one set of measurements on another set of measurements. Assume that one wants to know the dependence of a random variable x on a random variable y. A way of determining this would be to plot the outcomes of a particular experiment as points in the (x, y) plane and observe the resultant figure. Such a scatter plot might look like that shown below.

Scatter Diagram
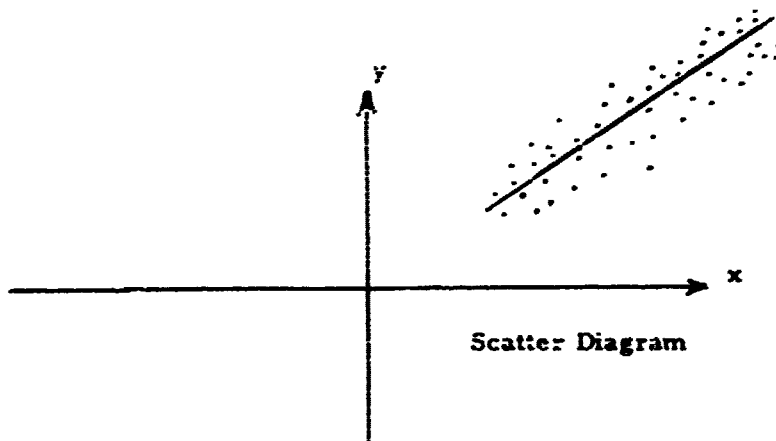
If x and y are not dependent upon one another the sample points would be more or less scattered throughout the plane. On the other hand, if x and y are strongly dependent upon one another, the sample points would then be clustered around some curve describing their

functional dependence. The simplest type of dependence would be linear dependence in which case the curve would be a straight line. If there is an indication of linear dependence, it would then be of interest to determine which straight line

$$y_p = a + bx \tag{7.44}$$

gives the "best" predicted value y_p of the random variable y based upon a particular value of the random variable x. A generally accepted definition of "best" prediction is in terms of minimizing the mean square deviation σ^2 between the true sample value of y and its predicted value:

$$\sigma^2 = E\left[(y - y_p)\right]^2 \tag{7.45}$$

where E indicates the Expected Value. This is known also as a "least squares" fit.

It can be shown (see Ref. 7), that by using this criteria for goodness, Eq. (7.44) becomes

$$y_p = \mu_y + \frac{\rho_{xy}}{\sigma_x^2} \ (x - \mu_x) \tag{7.46}$$

where $\mu_y = E\{y\}$ is the mean of y, $\mu_x = E(x)$ is the mean of x, $\sigma_x^2 = E\left[(x - \mu_x)^2\right]$ is the variance of x, and ρ_{xy} is the covariance of x and y defined by

$$\rho_{xy} = E\left[(x - \mu_x)(y - \mu_y)\right] \tag{7.47}$$

The correlation coefficient Γ_{xy} is defined by

$$\Gamma_{xy} = \frac{E\left[(x - \mu_x)(y - \mu_y)\right]}{\sigma_x \sigma_y} = \frac{\rho_{xy}}{\sigma_x \sigma_y} \tag{7.48}$$

These terms were introduced previously in Section 4. 3. 2 of this report.

In terms of Γ, one may prove from Eq. (7. 45) that the minimum mean square deviation σ^2 becomes

$$\sigma^2 = \sigma_y^2 \, (1 - \Gamma^2) \qquad\qquad (7.49)$$

Note that if the covariance $\rho_{xy} = 0$, then $\Gamma = 0$ and $\sigma^2 = \sigma_y^2$. When this case occurs, the "best" predicted value for y_p is $y_p = \mu_y$, the mean value of the y's. On the other hand, if $y = x$, the $\rho_{xy} = \sigma_x \sigma_y$ and $\Gamma = 1$. When this case occurs, the minimum mean square deviation $\sigma^2 = 0$. Since it can be shown that Γ is always between -1 and $+1$, the minimum mean square deviation σ^2 will vary between 0 and σ_y^2. The correlation coefficient thus determines the dependence of y to x and indicates how much improvement is possible using Eq. (7. 46) as opposed to merely choosing $y_p = \mu_y$.

7. 6. 2 Correlation Functions

Let x_1 and x_2 be the random variables representing possible values of sample functions $\begin{bmatrix} ^k x(t) \end{bmatrix}$, $(k = 1, 2, 3, \dots)$, from a random process at times t_1 and t_2. See sketch below.

Random Process
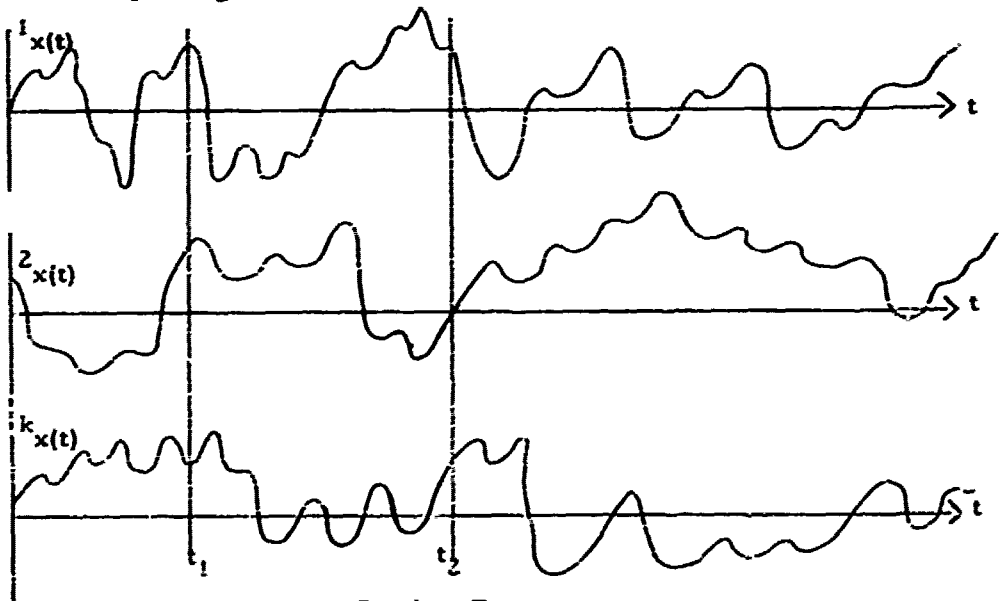
Since the joint probability distribution of x_1 and x_2 may change as t_1 and t_2 change, the statistical average $E[x_1 x_2]$ over $\{^k x(t)\}$ could be a function of both instants of time. This statistical average is called the "autocorrelation function" of the random process and, as in Section 4.4.1, denoted by

$$R_x(t_1, t_2) = E[x_1 x_2] \tag{7.50}$$

Now the corresponding correlation coefficient is also a function of t_1 and t_2. It is called the "normalized autocorrelation function" and denoted by $\Gamma_x(t_1, t_2)$, namely,

$$\Gamma_x(t_1, t_2) = \frac{E[(x_1 - \mu_1)(x_2 - \mu_2)]}{\sigma_1 \sigma_2} \tag{7.51}$$

where $\mu_1 = E(x_1)$ and $\mu_2 = E(x_2)$. Eq. (7.51) should not be confused with Eq. (7.48) where x and y are not functions of time.

If the random process is stationary, the autocorrelation is a function only of the time difference $\tau = t_1 - t_2$ and one can write

$$R_x(t, t - \tau) = R_x(\tau) = E[x_t \, x_{t-\tau}] \tag{7.52}$$

Since now $\mu_x = \mu_1 = \mu_2$ and $\sigma_x = \sigma_1 = \sigma_2$, the normalized autocorrelation function becomes

$$\Gamma_x(\tau) = \frac{R_x(\tau) - \mu_x^2}{\sigma_x^2} \tag{7.53}$$

Consider next two random processes $\{^k x(t)\}$ and $\{^k y(t)\}$, each of which has an autocorrelation function $R_x(t_1, t_2)$ and $R_y(t_1, t_2)$ respectively. For these two processes, two cross-correlation functions are defined by,

$$R_{xy}(t_1, t_2) = E[x_1 y_2] \tag{7.54}$$

$$R_{yx}(t_1, t_2) = E[y_1 x_2] \tag{7.55}$$

where the expected values are obtained by statistical averages over the random processes. All statistical correlations between $\left\{{}^k x(t)\right\}$ and $\left\{{}^k y(t)\right\}$ can then be specified by a correlation matrix

$$\overline{\Gamma} = \begin{bmatrix} R_x(t_1, t_2) & R_{xy}(t_1, t_2) \\ R_{yx}(t_1, t_2) & R_y(t_1, t_2) \end{bmatrix} \tag{7.56}$$

The time autocorrelation function $R_x(\tau)$ associated with a particular record $x_p(t)$ may be defined by a time average

$$R_{x_p}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x_p(t) x_p(t + \tau)\ dt \tag{7.57}$$

If the random process is ergodic (see Section 4.4.3), independent of the choice of $x_p(t)$,

$$R_{x_p}(\tau) = R_x(\tau) \tag{7.58}$$

where $R_x(\tau)$ is the previous statistical average shown in Eq. (7.52).

Similarly, the time cross-correlation function for a particular paid of records $x_p(t)$ and $y_p(t)$ may be defined as

$$R_{x_p y_p}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x_p(t) y_p(t + \tau)\ dt \tag{7.59}$$

If the processes are jointly ergodic, independent of choice of $x_p(t)$ and $y_p(t)$,

$$R_{x_p y_p}(\tau) = R_{xy}(\tau) \tag{7.60}$$

All of the above relationships require manipulation in the time domain. Sometimes it is easier to perform calculations in the frequency domain, and for a stationary random process one can obtain in this manner the power spectral density function $S_x(f)$. Simple relationships exist between this power spectral density function and the time auto-correlation function, one form of which is, (see Eqs. (4.86) and (4.87),

$$S_x(f) = 2 \int_0^\infty R_x(\tau) \cos 2\pi f \tau \, d\tau \qquad (7.61)$$

$$R_x(\tau) = 2 \int_0^\infty S_x(f) \cos 2\pi f \tau \, df \qquad (7.62)$$

Some general properties of stationary correlation functions are reviewed below without proof. A more detailed discussion on this subject can be found in Section 4.4.1.

(a) The autocorrelation function of a stationary random process is an even function of its argument, hence

$$R_x(\tau) = R_x(-\tau)$$

This property does not apply to cross-correlation functions.

(b) For a stationary random process the autocorrelation function has its maximum value at $\tau = 0$. Therefore, for all τ,

$$\left| R_x(\tau) \right| \leq R_x(0) \qquad (7.63)$$

An upper bound relationship for a stationary cross-correlation function is

$$\left| R_{xy}(\tau) \right| \leq \sqrt{R_x(0)} \sqrt{R_y(0)} \qquad (7.63a)$$

(c)
$$R_x(0) = E[x^2]$$

In words, the autocorrelation function evaluated at zero is equal to the mean square value of the stationary random process $\left\{ {}^k x(t) \right\}$. For zero means this is equal to the variance of $\left\{ {}^k x(t) \right\}$.

(d) Note that for ergodic random processes $R_x(\tau)$ can be replaced by $R_{x_p}(\tau)$, where x_p is an arbitrary particular record of the random process.

(e) Hidden periodicities in the sample function of a stationary random process can be uncovered from a knowledge of the autocorrelation function. For physically generated random processes $R_x(\tau)$ will approach zero as $|\tau|$ becomes large, since widely separated values will not be correlated. If hidden periodicities are present $R_x(\tau)$ will not become zero for large $|\tau|$ since $R_x(\tau)$ will now also be periodic.

(f) In general, "uncorrelated" (i.e., $R_x(\tau_0) = 0$ for some τ_0) does not imply statistical independence of two values at t_1 and $t_2 = t_1 + \tau_0$ from a stationary random process. It will, however, if the random process has a joint Gaussian probability density function at the two times. It is important to remember this fact since many statistical tests that might be applied to a set of various parameters (such as rms acceleration) of a random process require statistical independence of these parameters. Proving that the autocorrelation function is zero is therefore not sufficient, in general, unless the process also has a Gaussian distribution.

7.6.3 Methods for Correlation Measurements

Vibration data in flight vehicles is obtained usually by recording displacement, velocity or acceleration as a function of time. Eqs (7.57) and (7.59) therefore indicate the operations to be performed on such records to obtain autocorrelation and cross-correlation functions respectively. A block diagram for determining a cross-correlation function is shown below. By letting $y(t) = x(t)$, an autocorrelation function can be obtained.
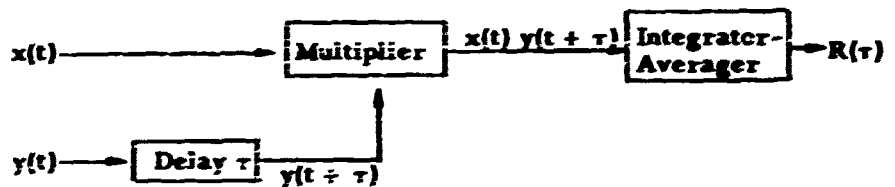
In practice, records of infinite length are not available. In addition, if a very long record has been obtained (for example, 1000 seconds), it would be desirable if only a short portion (say 10 seconds) would have to be used to obtain information about the whole record. Generally, different 10 second portions of the 1000 second record would be expected to give different results for the autocorrelation function. If the record is stationary, $R(\tau)$ would still not be the same for different 10 second samples, but now the results would be expected to be within a certain range, so that $R(\tau)$ for one single 10 second sample would provide an estimate of $R(\tau)$ for the whole record. One would also expect the estimate to be better as the sample length covers a greater portion of the total record. To indicate this dependence on sample length, measured values of correlation functions will be indicated by $R(\tau, T)$. In addition, since $R(\tau, T)$ is an even function for a self-stationary record, see Section 6, only positive values of τ have to be measured and Eqs. (7.57) and (7.59) become

$$R_x(\tau, T) = \frac{1}{T} \int_0^T x(t) \, x(t + \tau) \, dt \qquad (7.64)$$

$$\tilde{R}_{xy}(\tau, T) = \frac{1}{T} \int_0^T x(t) \, y(t + \tau) \, dt \qquad (7.65)$$

It should be noted that nonstationary random processes are not considered, since knowledge of the correlation function for a sample will provide no information about the correlation function of the entire record.

Two correlators will be discussed in this section. The principal differences between them are the frequency response and the method for obtaining the time shift. Other commercial correlators are available.

Type A Correlation Computer

This computer was built by the University of Texas and is discussed in detail by C. E. McCullough (Ref. [23]). A block diagram for this correlation computer is shown in Fig. 7.7.

The time functions are recorded on magnetic tape using a frequency modulation system to extend the low frequency response to zero. To obtain the time shift (τ), a continuous loop of two-channel magnetic tape is passed through two separate recorders. One of the recorders drives the loop at 30 inches per second. Between the two heads, the tape passes over a pulley which is driven up or down by a motor-driven lead screw. As the lead screw moves the pulley in one direction, the length of tape between heads decreases causing $y(t)$ to lag $x(t)$, and as it moves in the other direction $y(t)$ will lead $x(t)$.

Since this shift mechanism runs continuously, some error is introduced because the multiplication and averaging over the sample length T is not made for various fixed values of τ. This error can be reduced by slowing down the shift mechanism, but then the computing time would be increased. The range of τ used for this computer is variable from 0 to 1 second.

A frequency response curve for the computer is shown in Fig. 7.8. This indicates that for random vibration analysis the frequency range of this device would have to be extended considerably since most of the vibration data of interest today ranges from 5 cps to 2000 cps, with some data as high as 3000 cps.

Type B Correlation Computer

This computer was built by the Acoustics Laboratory, Massachusetts Institute of Technology, and is described in detail by Kenneth W. Goff (Ref. [14]).

The frequency range for this correlator is from 100 cps to 10,000 cps. It employs a magnetic drum instead of tape recorders and τ can be varied
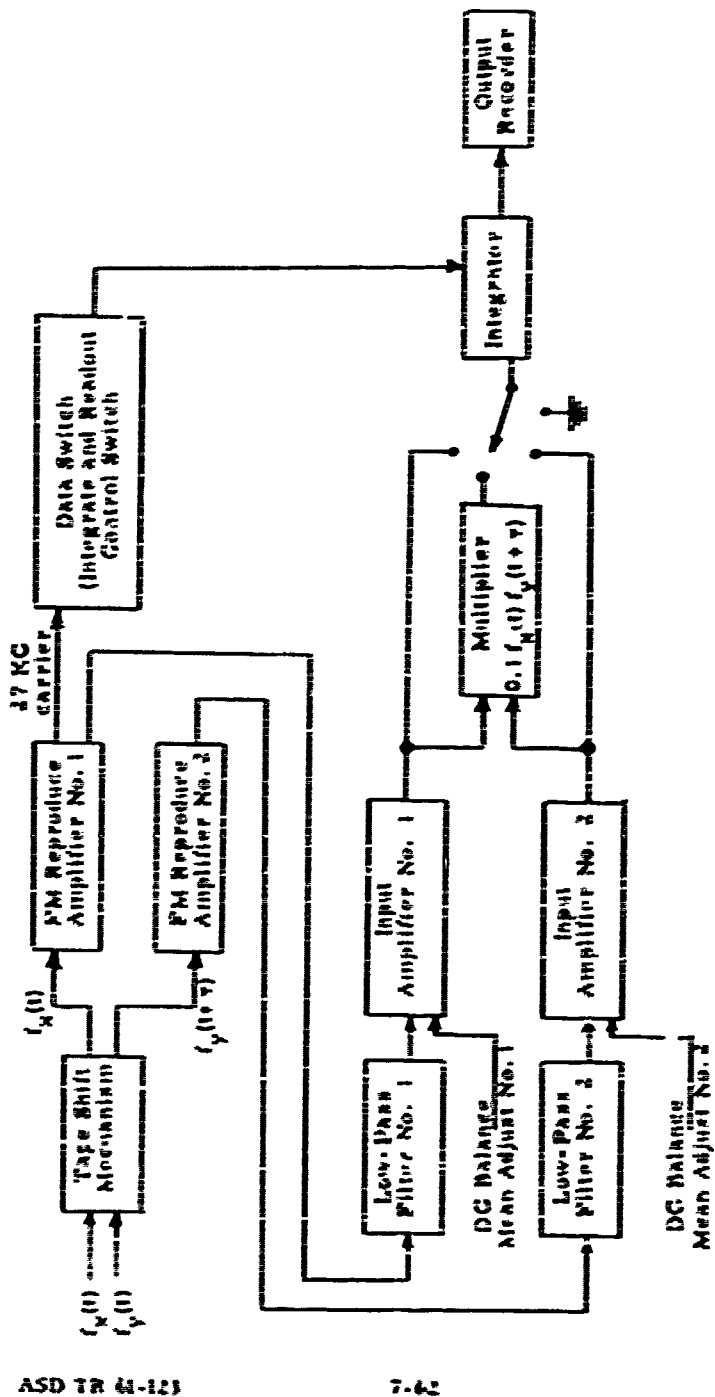
Figure 7.7 Block Diagram for Type A Correlation Computer (Reference [22], p. 24)
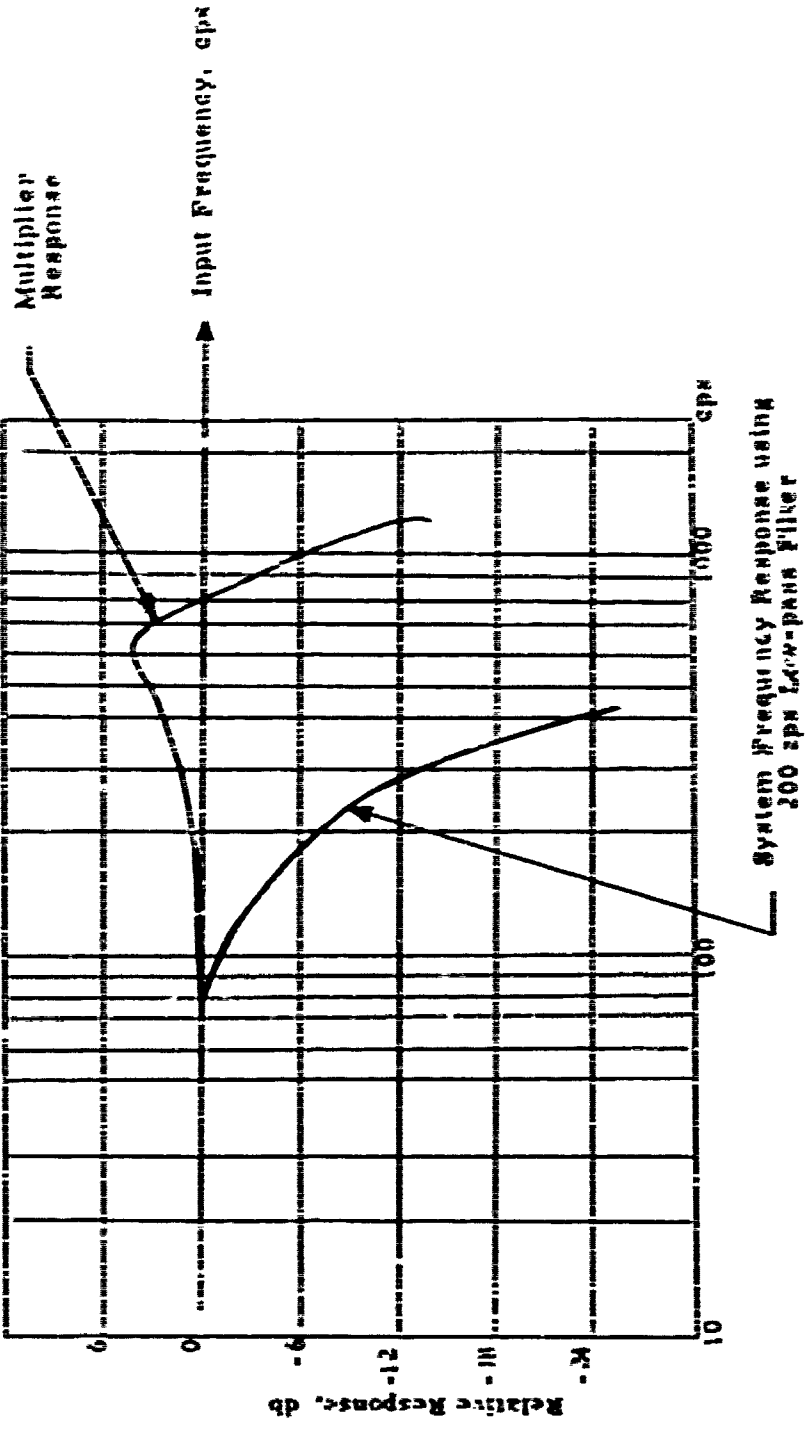
Figure 7.8 Frequency Response Curve for Type A Correlation Computer
(Reference [2], p. 30)

either in steps (one step for each revolution of the drum) or continuously.
The range for τ is from -15 milliseconds to +100 milliseconds. A
block diagram for the computer is shown in Fig. 7.9. It should be noted
that the sample length T is now fixed and equal to the circumference
of the drum. For a 100 in/sec surface speed this would allow a sample
length of 0.25 second since the circumference of the drum is 25 inches.

A qualitative comparison of Types A and B is given below.

	Type A	Type B
Frequency Range:	0 - 300 cps	100 - 10,000 cps
Range of time-delay τ:	0 - 1 second	-15 to + 100 milli-seconds
Error in τ:	?	Less than 5%
Change in τ:	Continuously	Stepped or Continuous
Signal Source:	Tape Recorders	Magnetic Drum
Type of Integrator:	Operational Amplifier	RC-Type

7.6.4 Errors in Correlation Measurements

The errors associated with correlation measurement can be
divided into two types:

1. Computer errors due to

(a) method employed to obtain τ

(b) type of integrators

(c) inherent phase-shift

(d) drum or tape-speeds

(e) accuracy of other electronic parts and recorder

2. Statistical errors in estimating the correlation function from
a short sample.
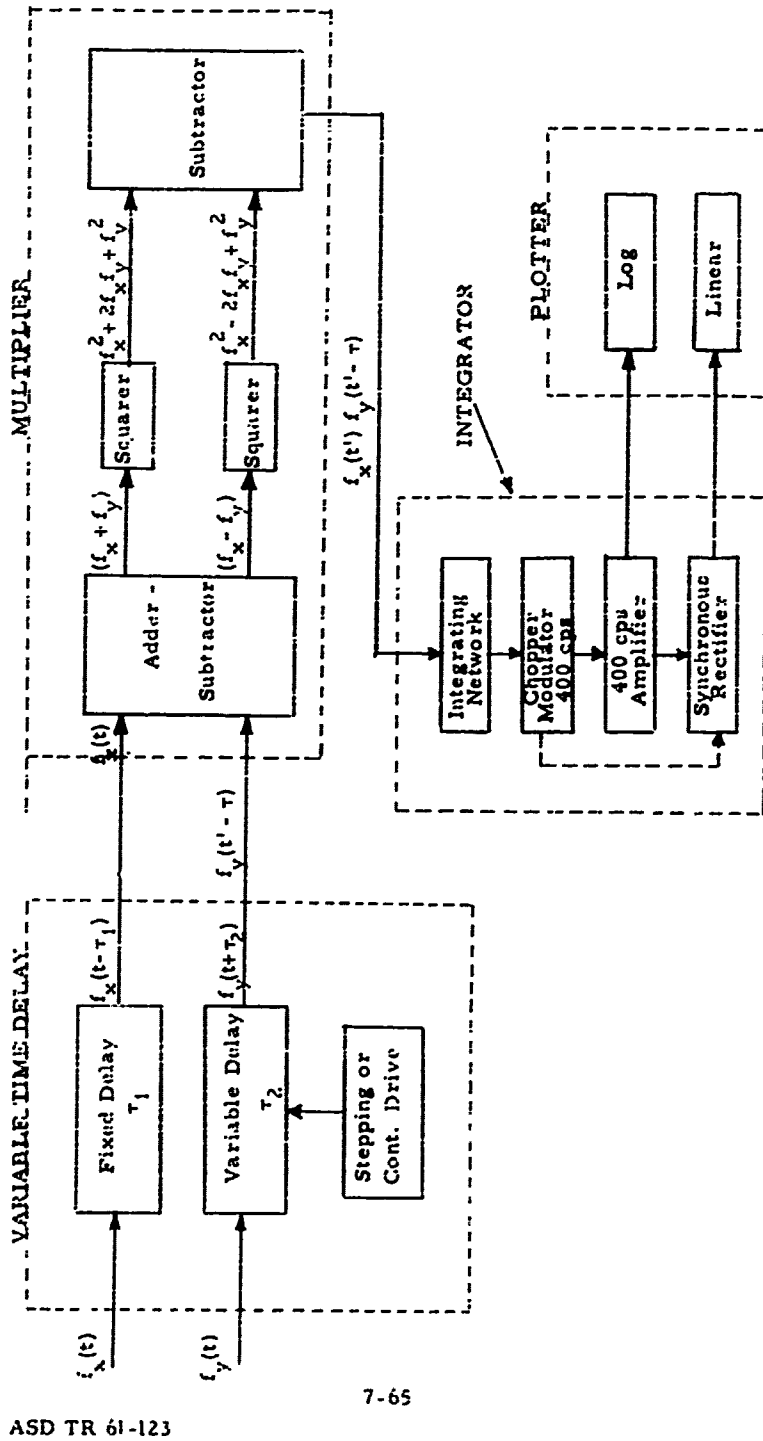
Figure 7.9 Block Diagram for Type B Correlation Computer
(Reference [14], p.224)

The effect of these errors in the time delay τ and the correlation function $R(\tau, T)$ will now be discussed in a quantitative manner wherever possible. Detailed proofs will not be given and the reader is referred to discussions by Bendat (Ref. [3]), Blackman and Tukey (Ref. [4]), Goff (Ref. [14]), and McCullough (Ref. [23]).

The error in the time delay τ is generally due to three causes:

1. Fluctuations in the tape or drum speed

2. Inherent phase shift in the system

3. Using a continuously changing time-shift mechanism

The exact magnitude of the total error in τ cannot be determined analytically. The calculations have to be supplemented by experimental data obtained from various components of the computer. The approach to this problem is discussed by McCullough (Ref. [23]) and Goff (Ref. [14]), each obtaining an answer for their particular computer. McCullough does not give any value for the error in τ but states that it can be determined by analyzing a known function (such as band-limited white noise) and comparing the experimental autocorrelation function with the theoretical values. This was also done by Goff who obtained an over-all error of less than 5 percent (Ref. [14], page 236).

The error in the correlation function $R(\tau, T)$ depends on:

1. Type of integrators

2. Drum or tape speed

3. Associated instrumentation and recorder accuracy

McCullough (Ref. [23]) uses an operational amplifier in the integrator with an associated error of less than one percent. Goff (Ref. [14]), however, uses an RC integrator. A comparison of integration times for these two types are covered in detail in Section 6.1.7 of this report.

The drum or tape speed is governed by the desired frequency resolution, which in turn is a function of the incremental shift in time delay ($\Delta\tau$) between adjacent points on the correlation curve. As $\Delta\tau$ is decreased, the highest frequency that can be detected increases. The highest frequency which may be detected is one-half the sampling frequency (Ref. [23]). For the computation of correlation functions,

the effective sampling frequency is $(1/\Delta\tau)$ cycles per second. Therefore, any frequencies above $(1/2\Delta\tau)$ cps cannot be detected since at least two cycles should be available for good resolution. If the original time function contains any significant components above $(1/2\Delta\tau)$ cps they should be removed by filtering or they will appear as low frequency components. This is generally referred to as "Aliasing" of the data (Ref. [3], pages 51-52).

What is usually desired is an estimate for $R(\tau)$, the stationary correlation function for a relatively long record. This estimate is $R(\tau, T)$, and now the question arises as to how well this estimate represents $R(\tau)$.

The general expression for the mean-square error (variance) in measuring the cross-correlation function is developed in Eq. (4.128) and is given by

$$\sigma^2(\tau, T) \approx \frac{1}{T} \int_{-\infty}^{\infty} \left[R_x(\gamma)R_y(\gamma) + R_{xy}(\gamma + \tau) R_{yx}(\gamma - \tau) \right] d\gamma \qquad (7.66)$$

The autocorrelation function estimate is obtained by replacing $y(t)$ by $x(t)$.

For the particular case of white noise passed through a rounded low-pass filter, the normalized variance in the estimate of the auto-correlation function reduces to

$$\epsilon^2(\tau, T) = \frac{\sigma^2(\tau, T)}{R^2(\tau)} \approx \frac{k}{BT} \quad ; \quad 1 < k < 2 \qquad (7.67)$$

where k is a constant of proportionality, T is the sample length, and B is the bandwidth of the filter (see Ref. [3], pages 271-272). Eq. (7.67) indicates that the mean square error is inversely proportional to the BT product, and that the uncertainty in the measurement can be reduced by increasing the correlator bandwidth, the length of the sample, or both.

For cross-correlation measurements, the mean square error becomes (Ref. [3], pages 274-275), letting $\beta = \pi BT$,

$$\sigma_{xy}^2(\tau, T) = S^2 \sigma_x^2(\tau, T) + (SN_1 + SN_2 + N_1 N_2) \left[\frac{2\beta - 1 - e^{-2(\beta)^2}}{2(\beta)^2} \right]; \quad K \ll 2$$

(7. 68)

where N_1 and N_2 represent the $(rms)^2$ (power noise) values in x(t) and y(t) respectively and S represents the $(rms)^2$ value of the signal. For a normalized signal (i. e., S = 1), and no noise in either record, Eq. (7. 68) reduces to Eq. (7. 67).

If the correlation point moves continuously during the measurement, an additional statistical error is introduced. If $\Delta\tau$ is the smallest increment in τ to be distinguished and τ varies from τ_0 to $\tau_0 + \Delta\tau$ during the correlation time T, τ becomes a function of time and is given by (Ref. [3], pages 278-282)

$$\tau(t) = \tau_0 + \lambda t \tag{7.69}$$

where

$$\lambda = \frac{\Delta\tau}{T} \tag{7.70}$$

For large BT, (say $BT > 10^3$), it has been shown by Bendat (Ref. [3]) that the additional normalized variance along the autocorrelation curve for the autocorrelation function of white noise passed through a low-pass filter is approximated by

$$e_{xy}^2(\lambda, T) < \lambda^2 BT \tag{7.71}$$

It should be noted that this term is directly proportional to BT, whereas the fundamental variance, Eq. (7. 67), is inversely proportional to BT.

7.6.5 Physical Example of Vibration Source Localization

Assume that the vibration environment at a point on a structure is to be measured, and that the vibration level is due to three sources located at various distances from the point in question. See sketch below.

y(t) → Tape Recorder

Vibration pick-up

Source No. 1 — $x_1(t)$

$x_2(t)$ — Source No. 2

$x_1(t)$

$x_3(t)$
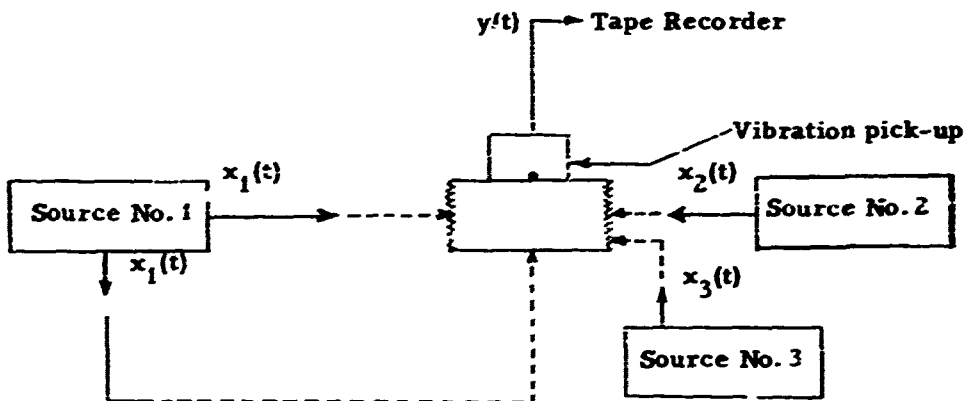
Source No. 3

Block Diagram Set-up for Localizing Vibration Sources

It is desired to know what portion of the composite record y(t) is due to each of the three sources $x_1(t)$, $x_2(t)$, and $x_3(t)$. For this problem, suppose also that the contribution of Source No. 1 is transmitted to the point in question along two different structural paths.

This problem can be solved quite simply by recording $x_1(t)$, $x_2(t)$, and $x_3(t)$ at their respective sources and then finding the cross-correlation functions $R_{x_1 y}(\tau)$, $R_{x_2 y}(\tau)$, and $R_{x_3 y}(\tau)$.

Hypothetical results are pictured in Displays A, B, and C. This analysis not only provides information about the sources, but also determines the time it takes for the vibrations to travel from the source to the point of interest.

$R_{x_1y}(\tau)$

τ_{11} τ_{12} τ
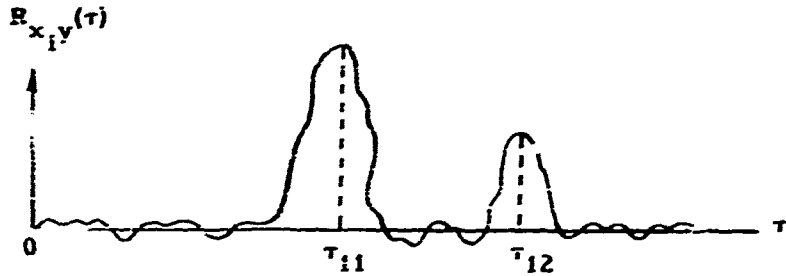
Display A.   Cross-Correlation vs. Time Delay from Source No. 1

$R_{x_2y}(\tau)$

τ
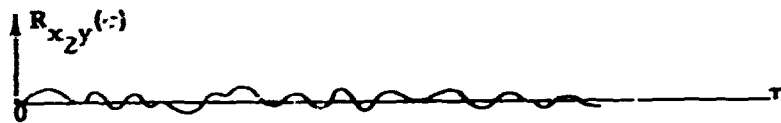
Display B.   Cross-Correlation vs. Time Delay from Source No. 2

$R_{x_3y}(\tau)$

τ_{31} τ

Display C.   Cross-Correlation vs. Time Delay from Source No. 3

Display A shows that source No. 1 travels along two different paths and gives the time it takes to travel along each of the paths before arriving at the point in question.   The relative severity of their contributions to the output is shown by the height of the correlation peaks. Display B indicates no correlation between source No. 2 and the accelerometer.   A conclusion would be made here that the vibratory energy from source No. 2 was dissipated somewhere along the way. Display C shows one point of correlation between source No. 3 and the point of interest.   The time delay τ_{31} is different from both τ_{11} and τ_{12} indicating a different length of path travelled.   The height of the correlation peak indicates a contribution something between the two paths of the first source.   This concludes the example.

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8. EXPERIMENTAL PROGRAM TO VERIFY
ANALYTICAL PROCEDURES

An experimental program will be proposed to verify the important statistical procedures developed and discussed in this report. The proposed experimental program is divided into two parts:

Part 1 - Laboratory Test Program
Part 2 - Flight Test Program

Controlled and well defined conditions available in the laboratory make laboratory experiments best suited for investigation of basic statistical techniques. As a result, emphasis will be placed on detailing comprehensive laboratory experiments. The proposed flight test program must be approached in a more general manner since the explicit details of a flight test experiment will be heavily dependent upon the results of the proposed laboratory experiments.

The material to follow begins by discussing in some detail laboratory test procedures for verifying tests of fundamental assumptions and statistical accuracy of desired measurements. This is followed by a broad treatment on statistical considerations for repeated experiments (flights) and random sampling techniques. The final portion of this section contains material relevant to conducting a flight test program.

8.1 LABORATORY TEST PROGRAM

The proposed laboratory test program will include three phases as follows:

Phase 1 - Verification of fundamental assumptions.
Phase 2 - Verification of statistical accuracy of measurements.
Phase 3 - Verification of statistical procedures for repeated experiments and random sampling.

8.1.1 Basic Laboratory Instruments

The laboratory experiments to be described will involve repeated use of several basic common laboratory instruments. Three instruments that will be continually required are as follows.

a) random noise generator

b) low pass filter

c) read-out recorder

The random noise generator should ideally provide a stationary random signal with a Gaussian probability density function and a uniform power spectral density function (band-limited white noise). A number of random noise generators are commercially available which provide a signal that acceptably approximates the above desired characteristics.

The low pass filter should ideally have an infinitely sharp high frequency cut off with a variable cut off frequency available. If the low pass filter employed does not have a relatively sharp cut off, say at least 60 db per octave, it will be necessary to determine the frequency response function for the filter and calculate the equivalent noise bandwidth B_N as described in Section 4.9.2. A sharply cut off signal bandwidth or an accurately defined equivalent noise bandwidth is essential for successful laboratory experiments.

The read-out recorder may be any conventional laboratory direct voltage recorder. Direct writing stylus type recorders can be employed, but the galvanometer type oscillograph would be more desirable because of the higher frequency response available. An upper frequency response limit of 60 cps, characteristic of direct writing stylus type recorders, will not permit as much flexibility in the experiments as the higher frequency response of the galvanometer oscillograph.

8.2 VERIFICATION OF FUNDAMENTAL ASSUMPTIONS

As noted in Section 6.1, the analysis of a single vibration signal record and the interpretation of the analysis is greatly facilitated if the record is random and self-stationary, and displays a Gaussian distribution of instantaneous amplitudes. Straightforward tests are proposed to confirm the hypotheses of randomness, self-stationarity, and Gaussian amplitude distribution. This section details laboratory experiments to substantiate the validity of the tests outlined in Section 6.1.

6.2.1 Test for Randomness

A run test for randomness of a vibration signal record is detailed in Section 6.1.5. An experimental verification of this test will involve two approaches. The first is to substantiate that the test will accept a signal record that is random. The second is to substantiate that the test will reject a signal record that is not random.

a) Verification that the run test will accept a signal record that is random.

The following test set up will be required. See Figure 5.1

Figure 5.1. Equipment for Randomness Test

The random noise generator, low pass filter, and read-out recorder should be as described in Section 8.1.1.

After the random noise generator has warmed up long enough to assure a stationary output signal, adjust the low pass filter cut off to some frequency f_0 to give a signal bandwidth of B. Remember that f_0 must be less than the upper frequency response limit of the read-out recorder.

Proceed by obtaining a record of the band-limited random signal with a record length of at least $\frac{1250}{B}$ seconds. Then if the frequency band-width of the signal is say 100 cps, the record length should be at least 12.5 seconds. Now divide the long record into N number of subrecords (N ≥ 25), each with n = 100 statistical degrees of freedom (n = 2BT). Then if B were 100 cps, the length T of each subrecord would be 0.5 seconds. Count the number of times r that the signal level crosses the zero level for each of the N subrecords.

From Equation (6.1) and (6.2), the expected number of zero crossings r is given by,

$$\mu_r = \frac{n}{2} \tag{6.1}$$

with a variance given by,

$$\sigma_r^2 = \frac{n(n-2)}{4(n-1)} \qquad (8.2)$$

Equations (8.1) and (8.2) are written in a form which assumes that the times spent by the signal above and below the zero level are about equal. This assumption is quite valid if the zero level is correctly established and the value of n is large.

For n = 100 statistical degrees of freedom, μ_r = 50 and σ_r^2 = 24.75. Then if the number of zero crossings r is measured for the N different sample records, each with n = 100 statistical degrees of freedom, it would be expected that the measured values of r would be normally distributed about a mean of μ_r = 50 with a standard deviation of σ_r ≈5.

It now remains to compare the experimentally determined values of r from the N subrecords to a normal distribution with a mean of 50 and a standard deviation of 5. Specifically, the following relationships should be closely approximated.

95% of the values of r should exceed 42
73% of the values of r should exceed 47
50% of the values of r should exceed 50
27% of the values of r should exceed 53
5% of the values of r should exceed 58

The above general procedure should be repeated using a random signal which does not have a uniform power spectral density function. A non-uniform power spectrum can be obtained by complex filtering of the random noise generator output. Extensive experiments of this type will be necessary to establish any limitation that may be imposed on the test for randomness by the power spectrum characteristics of a sample record.

b) Verification that the run test will reject a signal record that is not random.

The following test set up will be required. See Figure 8.2.
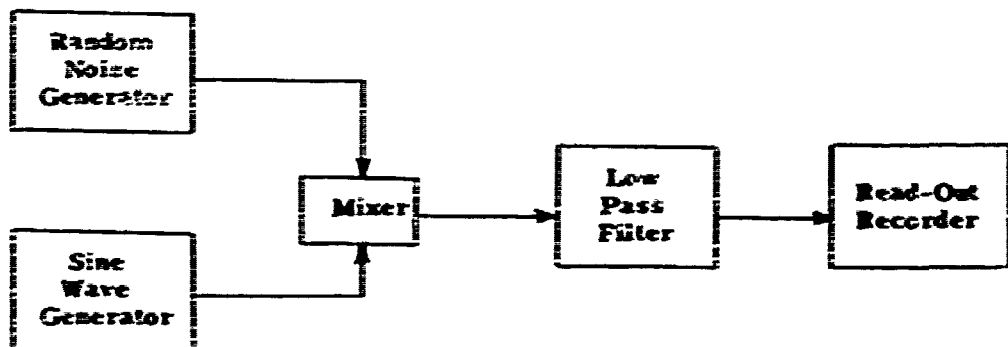
Figure 8.2 Equipment for Non-Randomness Test

The test set up is the same as required in Part (a) except provisions are made to superimpose a sinusoid on the random signal from the noise generator.

Adjust the rms voltage level of the sinusoid to be about equal to the rms voltage level of the band-limited random noise. Set the frequency of the sinusoid to be just below the high frequency cut off of the low pass filter.

Proceed by obtaining a record of the band-limited mixed sine-random signal with a length required to give 100 statistical degrees of freedom. Then if the frequency bandwidth of the signal is say 100 cps. the record should be 0.5 seconds long. Count the number of zero level crossings r and apply the run test at a 95% level of confidence.

For example assume B = 100 cps and T = 0.5 seconds. Then n = 100 and the value of r for a random signal would fall between 40 and 60 at the 95% confidence level. If r is less than 40 or greater than 60, the record may be rejected as non random at the 95% confidence level. The mixed sine-random signal record should be rejected by the test as non random, since the sine wave alone would produce nearly 100 crossings.

The above test should be repeated for various ratios of sine to random rms levels and various sinusoidal frequencies. It is quite likely

that certain combinations of rms levels and sinusoidal frequency will produce records that will be accepted by the run test as being random. These combinations represent error risks associated with the test.

8.2.2 Test for Stationarity

A test for self-stationarity of a random vibration signal record is detailed in Section 6.1.8. An experimental verification of this test will involve two approaches similar to the previous test for randomness. The first is to substantiate that the test will accept a random signal record that is self-stationary. The second is to substantiate that the test will reject a random signal record that is not self-stationary.

a) Verification that the test will accept a random signal record that is self-stationary.

The following test set up will be required. See Figure 8.3.
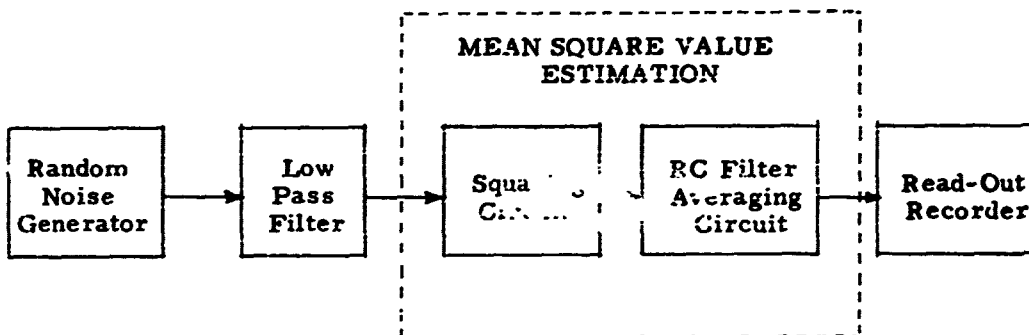
Figure 8.3 Equipment for Stationary Test

The random noise generator, low pass filter, and read-out recorder should be as described in Section 8.1.1. The squaring circuit and RC filter averaging circuit are components of many true rms vacuum tube voltmeters and power spectral density analyzers as described in Sections 7.3 and 7.4.

After the random noise generator has warmed up long enough to assure a stationary output signal, adjust the low pass filter cut off to

some frequency f_0 to give a signal bandwidth of B. The actual bandwidth B to be employed in the experiment will be a function of the RC filter time constant K. The number of statistical degrees of freedom n for the resulting mean square estimations should be rather small, say about 40, to obtain a good spread in the estimates. Then, from Eq. (6.30), the product of BK should equal about 10.

Proceed by obtaining a record of the continuous mean square estimation with a record length of at least 500K seconds. Then for n = 40 degrees of freedom, if B = 100 cps, K = 0.10 seconds, and the total record length would be at least 50 seconds. The resulting record may be considered to be a group of N numbers of subrecords, each of length T = 2K constituting a mean square estimate. Then N will be greater than 250.

Repeat the above experiment at least 25 times under the same conditions to obtain N' number of records (N' $\geq$ 25) of identical length and statistical quality. Finally, obtain one additional mean square estimate of very high quality for the same input noise level and bandwidth by increasing the time constant of the RC filter to the highest value available. A time constant K of at least 300/B to give at least 1200 statistical degrees of freedom would be desirable. The resulting estimate may be assumed to be the true mean square value σ^2 of the random signal. If the mean square estimating device does not have a variable time constant, an average of the continuous mean square estimates of all experiment records may be used for σ^2.

A chi-square confidence interval of (1 - α) for n degrees of freedom may now be established for each of the N' experiment records. Measure the actual percent of the time $\hat{\alpha}$ that the continuous mean square estimates fall outside the (1 - α) confidence interval for each of the N' records. From Section 6.1.8, if the equivalent number of mean square estimates N for each record is large, say greater than 250, and if α is small, say less than 0.10, then the measured values of $\hat{\alpha}$ multiplied by N will be distributed in an approximately normal manner with a mean value and standard deviation as follows:

$$\mu_{N\hat{\alpha}} = \lambda = N\alpha \tag{8.3}$$

$$\sigma_{N\hat{\alpha}} = \sqrt{\lambda} = \sqrt{N\alpha} \tag{8.4}$$

It now remains to compare the N' experimentally determined values for $\hat{N\alpha}$ to a normal distribution with a mean of $N\alpha$ and a standard deviation of $\sqrt{N\alpha}$.

Consider the following example. Assume 25 records of continuous mean square estimates are obtained for a random signal with a bandwidth of 100 cps using an RC filter time constant of 0.1 seconds. Then the number of statistical degrees of freedom n = 40. Assume each record is 50 seconds long. Then the effective number of estimates per record is N = 250. Consider a chi-square confidence interval of $(1 - \alpha) = 0.95$. From Table 4.2, the upper and lower confidence limits will be $1.50\,\sigma^2$ and $0.61\,\sigma^2$, respectively. Measure the percentage of time $\hat{\alpha}$ that the mean square estimate is outside the above limits and multiply by N = 250 for each of the 25 records. The 25 values for $\hat{N\alpha}$ ($\hat{N\alpha}_1$, $\hat{N\alpha}_2$, etc.) should be normally distributed with a mean of $N\alpha = 12.5$ and a standard deviation of $\sqrt{N\alpha} = 3.54$. Specifically, the following relationships should be closely approximated.

> 24 of the values of $N\alpha$ should exceed 6.3
> 20 of the values of $N\alpha$ should exceed 9.5
> 15 of the values of $N\alpha$ should exceed 11.6
> 10 of the values of $N\alpha$ should exceed 13.4
> 5 of the values of $N\alpha$ should exceed 15.5

The above general procedure should be repeated using a random signal which does not have a uniform power spectral density function. A non-uniform power spectrum can be obtained by complex filtering of the random noise generator output. Extensive experiments of this type will be necessary to establish any limitations that may be imposed on the test for self-stationarity by the power spectrum characteristics of a sample record.

b) Verification that the test will reject a random signal record that is self-nonstationary

The required test set up is identical to that shown in Figure 8.3. Use the same signal bandwidth and RC filter time constant as Part (a). A self-nonstationary random signal will be obtained by arbitrarily varying the output level of the random noise generator.

Proceed by obtaining a record of the continuous mean square estimation with a record length of at least 500K seconds, while arbitrarily varying the output level of the random noise generator. Apply the test for self-stationarity as described in Section 6.1.8 to the resulting record. The test should reject the record as being self-nonstationary.

The power of the test for self-stationarity can be experimentally investigated by repeating the above procedure for variations of the random noise generator output within several different specified limits. For example, obtain a record of the continuous mean square estimate while the noise generator output voltage is varied from 50% above to 50% below some average value. Such a record should be rejected as self-nonstationary even at a 99% confidence level. Repeat the test over and over reducing the range of noise output level variations each time. As the variations in the noise output level become limited to a very narrow range, the test will start accepting records as being self-stationary. Such a set of experiments would give at least a qualitative feel for the Type II error associated with the test for self-stationarity of random signals.

8.2.3 Test for Normality

A test for normality of instantaneous amplitudes of a random vibration signal record is detailed in Section 6.1.10. The test is accomplished by estimating the probability density function of the random signal from a record of length T and bandwidth B (using an instrument as described in Section 7.5), and comparing the estimate to a theoretical Gaussian density function with an appropriate $(1 - \alpha)$ normal confidence interval.

The validity of the test for normality rests primarily in the accuracy of predicting any $(1 - \alpha)$ normal confidence interval for a given probability density estimate, $\hat{p}(x)$. A verification of the test for normality thus involves little more than a verification of the statistical confidence for probability density estimates. This subject is discussed more fully later in Section 8.3.3.

8.3 VERIFICATION OF STATISTICAL ACCURACY OF MEASUREMENTS

In Section 7, the statistical accuracy associated with analog measurements of four important statistical properties of a random signal are developed. The four statistical properties discussed are (1) the root mean square value; (2) the power spectral density function; (3) the prbability density function, and (4) the autocorrelation (and cross-correlation) function. This section details laboratory experiments to substantiate the expected statistical accuracy in measuring (estimating) these properties.

8.3.1 Root Mean Square Value Estimates

Procedures for estimating the root mean square (rms) value of random signals and the associated statistical accuracy of the estimates are detailed in Section 7.3. Actually, mean square value estimation is the essential result of power spectral density estimation. Since the rms value of a random signal is simply the positive square root of the mean square value, the verification of the statistical accuracy of rms estimates will result directly from the verification of the statistical accuracy of power spectra estimates. This subject is discussed next in Section 8.3.2.

For some simple applications, it may be desired to confirm the statistical accuracy of rms value estimates using a commercial true rms voltmeter as described in Section 7.3. This may be done with the aid of a low pass filter (or a narrow band filter), and by applying the statistical methods discussed in Section 8.3.2.

8.3.2 Power Spectral Density Estimates

Procedures for estimating power spectra of random signals and the associated statistical accuracy of the estimates are detailed in Section 7.4. It is shown that the estimation of power spectra effectively consists of measuring the mean square value of the random signal through many adjacent narrow band filters, and dividing the mean square measurements by their associated bandwidth. An experimental verification of the statistical accuracy of power spectra estimates then involves simply the verification of the accuracy of a mean square value measurement of a random signal with a given bandwidth B. See Figure 8.4.

POWER SPECTRAL DENSITY ANALYZER

Random Noise Generator → Narrow Band Pass Filter → Squaring Circuit → Averaging Circuit (Integrator) → Read-Out Recorder
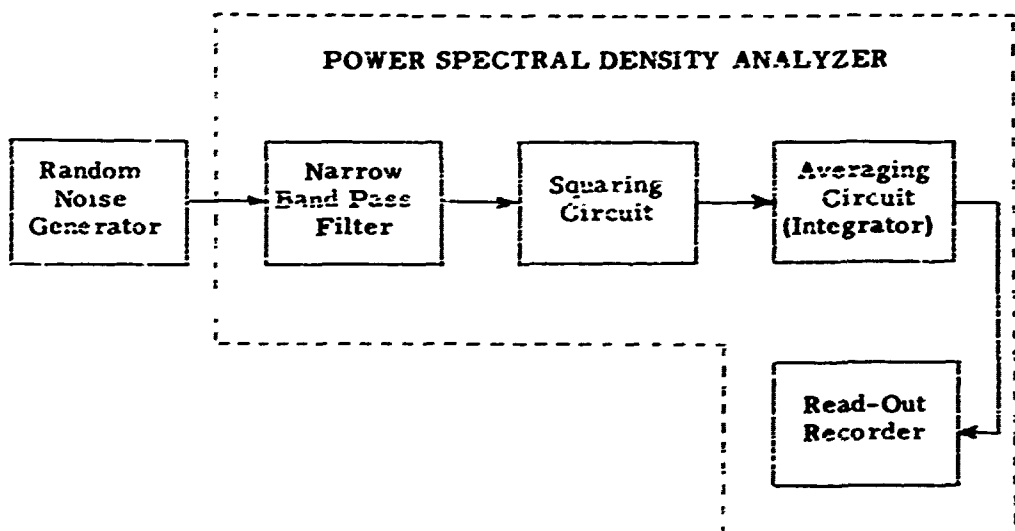
Figure 8.4 Equipment for Testing Power Spectra Measurement Accuracy

The random noise generator should be as described in Section 8.1.1. The other instruments required are integral components of any commercial power spectral density analyzer system. An analyzer with a heterodyne type crystal filter would be the most desirable since it closely approximates an ideal rectangular filter characteristic.

From Section 7.4, the equivalent number of degrees of freedom of a mean square estimate, s^2, is given by,

$$n = 2BT \tag{8.5}$$

Equation (8.5) assumes a flat power spectrum over the frequency range in a bandwidth B with infinitely sharp cut offs at the upper and lower frequency limits. After the random noise generator has warmed up long enough to assure a stationary output signal, adjust the narrow band pass filter to cover some frequency range where the power spectrum of the signal is reasonably flat. The bandwidth B should be about 10 cps

Proceed by recording a large number of mean square estimates. The number of estimates N should be at least 25. The number of statistical degrees of freedom n for each estimate should be small, say about 20, to obtain a good spread in the estimates. Then for a 10 cps

bandwidth. the record length T for each estimate (integrating time) should be about one second. Finally, measure one additional estimate of very high quality for the same input noise level and bandwidth by increasing the integrating time to obtain the longest possible record length. A record length of at least 30 seconds giving an estimate of over 600 degrees of freedom will be required. The resulting estimate may be assumed to be the true mean square value σ^2 for the signal.

If each of the N number of sample mean square estimates s_i^2 (i = 1, 2, ..., N) is an estimate of n = 2BT degrees of freedom, then

$$\frac{n\, s_i^2}{\sigma^2} = \chi^2 \tag{8.6}$$

where the distribution of χ^2 (chi-square) may be obtained from standard statistical tables. Compute the left hand side of Eq. (8.6) for each of the N estimates of s_i^2 and compare them to a χ^2 distribution of n = 2BT degrees of freedom.

For example, if a bandwidth of B = 10 cps and a record length of T = 1 second is used, then n = 20 and the following relationships should be closely approximate.

95% of the values of $\dfrac{n\, s_i^2}{\sigma^2}$ should exceed	3.94
90% " " " " " " "	4.87
75% " " " " " " "	6.74
50% " " " " " " "	9.34
25% " " " " " " "	12.5
10% " " " " " " "	16.0
5% " " " " " " "	18.3

The experimental procedure discussed here may be repeated for several different frequency bandwidths B and record lengths T to verify the general form of Equations (8.5) and (8.6).

8.3.3 Probability Density Estimates

Procedures for estimating probability density functions of random signals and the associated statistical accuracy of the estimates are detailed in Section 7.5. An experimental verification of the statistical accuracy developed for probability density estimates will require the following test set up. See Figure 8.5.

```
┌───────────┐     ┌──────────┐     ┌──────────────────┐     ┌───────────┐
│  Random   │     │   Low    │     │   Probability    │     │           │
│  Noise    │ ──▶ │   Pass   │ ──▶ │ Density Analyzer │ ──▶ │ Read-Out  │
│ Generator │     │  Filter  │     │  (Section 7.5)   │     │ Recorder  │
└───────────┘     └──────────┘     └──────────────────┘     └───────────┘
```

Figure 8.5 Equipment for Testing Probability Density Measurement Accuracy

The random noise generator and low pass filter should be as described in Section 8.1.1. The read-out recorder will probably be an integral component of the probability density analyzer.

From Section 7.5, the normalized standard deviation ϵ, also called the standard error, of an amplitude probability density estimate $\hat{p}(x)$ for any value of amplitude x is given by,

$$\epsilon = \frac{\sigma}{\hat{p}(x)} = \frac{k}{\sqrt{(\Delta x)\ TB\ \hat{p}(x)}} \qquad (8.7)$$

where k is some constant with a theoretical value of about 0.7. One purpose of the experiment will then be to establish a value of k for the actual equipment used.

After the random noise generator has warmed up long enough to assure a stationary output signal, adjust the high frequency cut off of the low pass filter to some frequency f_0 to obtain a known signal band-width B. Set the amplitude window Δx of the probability density analyzer over zero, the mean value of the signal.

Proceed by obtaining a large number of probability density estimates $\hat{p}_i(x)$ ($i = 1$, 2, 3, etc.), using records of known equal length T. The number of estimates N should be at least 25. The expected normalized standard deviation ϵ of the estimates should be rather large, say about 0.1, to obtain a good spread in the estimates and minimize the effect of instrument and observation errors.

Now compute the standard deviation estimator, s, given by,

$$s = \sqrt{\frac{\sum \left[\hat{p}_i(x) - \bar{p}(x)\right]^2}{N}} \tag{8.8}$$

where $\bar{p}(x)$ is given by,

$$\bar{p}(x) = \frac{1}{N} \sum p_i(x) \tag{8.9}$$

Substituting s for σ and $\bar{p}(x)$ for $\hat{p}(x)$ in Eq. (8.7), a value for k associated with the specific equipment used will be obtained. The value for k hopefully will be about 0.7.

The above experiment should be repeated many times for different values of bandwidth B and record length T. The value of s computed from observations for any product BT should yield approximately the same value for k when s and $\bar{p}(x)$ are substituted for σ and $\hat{p}(x)$ in Eq. (8.7). Finally, the experiment should be repeated with the amplitude window Δx set over several different amplitude points and, if the equipment permits, with several different amplitude window widths. Once again, the values for k produced by the observations should be approximately the same as before.

One may be interested in confirming that the estimates $\hat{p}_i(x)$ will be normally distributed about p(x). For any given set of parameters, B, T, Δx, and $\bar{p}(x)$, the resulting set of N number of estimates $p_i(x)$ may be tested for normality by direct application of the chi-square goodness of fit test as detailed in Section 5.3.2.

8.3.4 Autocorrelation Function Estimates

Procedures for estimating autocorrelation functions of random signals and the associated statistical accuracy of the estimates are detailed in Section 7.6. An experimental verification of the statistical accuracy developed for autocorrelation estimates will require the following test set up. See Figure 8.6.
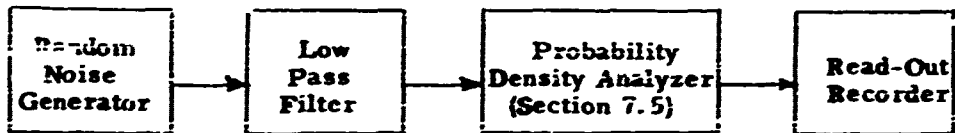


```
┌───────────┐      ┌────────┐      ┌──────────────────┐      ┌───────────┐
│ Random    │      │ Low    │      │ Correlation      │      │ Read-Out  │
│ Noise     │ ───▶ │ Pass   │ ───▶ │ Function Analyzer│ ───▶ │ Recorder  │
│ Generator │      │ Filter │      │ (Section 7.6)    │      │           │
└───────────┘      └────────┘      └──────────────────┘      └───────────┘
```

Figure 8.6 Equipment for Testing Autocorrelation Function
Measurement Accuracy

The random noise generator and low pass filter should be as described in Section 8.1.1. The read-out recorder will probably be an integral component of the correlation function analyzer.

From Section 7.6, the normalized standard deviation of an auto-correlation estimate $\hat{R}(\tau, T)$ (to be noted simply as $\hat{R}$) for any value of time delay τ is bounded by,

$$\epsilon(\tau, T) = \frac{\sigma}{\hat{R}} \leq \frac{k}{\sqrt{BT}} \qquad (8.10)$$

The equality sign applies when $\tau = 0$. The parameter k is some constant with a theoretical value between 0.5 and 1.5 u the bandwidth B is determined as described in Section 8.1.1. One purpose of the experiment will then be to establish a value of k for the actual equipment used.

After the random noise generator has warmed up long enough to assure a stationary output signal, adjust the high frequency cut off of the low pass filter to some frequency f_0 to obtain a known signal bandwidth B. Set the time delay τ of the autocorrelator to zero.

Proceed by obtaining a large number of autocorrelation function estimates, $\hat{R}_i$ ($i = 1, 2, 3$, etc.), using records of equal length T. The number of estimates N should be at least 25. The expected standard deviation of the estimates should be rather large, say about 0.1 R, to obtain a good spread in the estimates and minimize the effect of instrument and observation errors.

Now compute the standard deviation estimator, s, given by,

$$s = \sqrt{\frac{\sum_i (\hat{R}_i - \bar{R})^2}{N}} \qquad (8.11)$$

where $\bar{R}$ is given by,

$$\bar{R} = \frac{1}{N} \sum_i \hat{R}_i \qquad (8.12)$$

Substituting s for $\sigma \cdot \bar{R}$ for $\hat{R}$, in Eq. (8.10), a value for k associated with the specific equipment used will be obtained. The value for k hopefully will be between 0.5 and 1.5.

The above experiment should be repeated many times for different values of bandwidth B and record length T. The value of s computed from observation for any product BT should yield approximately the same value for k when s is substituted for $\sigma_{\hat{R}}$ in Eq. (8.10). Finally, the experiment should be repeated with the autocorrelator time delay set at several different values of $\tau \neq 0$. New values for k should be produced by the observations which are smaller than the values found earlier when $\tau = 0$.

One may be interested in confirming that the estimates $\hat{R}_i$ will be normally distributed about R. For any given product BT and time delay τ, the resulting set of N number of estimates, $\hat{R}_i$, may be tested for normality by direct application of the chi-square goodness of fit test as detailed in Section 5.3.2.

8.4 STATISTICAL CONSIDERATIONS FOR REPEATED EXPERIMENTS AND RANDOM SAMPLING

It is desirable to verify by experimental methods some of the statistical procedures described in the repeated experiments portion of Sections 5 and 6.2.7 and the random sampling techniques in Section 6.2.6. To do this, simulated "flights" may be run in the laboratory, and appropriate data collected and analyzed. If real differences are caught as predicted, and parameters estimated correctly within the predicted confidence limits, then the procedures may be assumed to be valid. If not, other methods must be devised, or underlying assumptions must be more critically examined.

Two different approaches can be carried out in an experimental program to indicate procedures that might be tried later in actual flight tests. It would be desirable, for reasons of simplicity, to be able to ignore various separate flight phases, and only to consider a flight vibration history as a whole. This is clearly impossible for many types of flight vehicles but not necessarily for all. The small number of flights (8 to 9) and samples sizes (6 to 12) indicated in the next Section 8.4.1 (Table 8.1) are probably quite suitable when considering the flight phases separately. Reasonable amounts of data would then be collected for the entire flight by combining information from the separate phases.

However, when samples are randomly selected over the total flight without regard to flight phases, more than 12 observations are clearly necessary to obtain reasonable estimates of the parameters of interest. For example, if the assumption of a normal distribution is to be verified using the χ^2 test, sample sizes of the order of 200 are desirable.

Many of the flight test problems being mentioned are actually not pertinent to laboratory experimentation due to the controlled conditions available in the laboratory. The validity of statistical procedures for small sample sizes is easily verified in a laboratory program. However, it is difficult to implement them in an optimum manner in actual flight tests.

8.4.3 Selection of Sample Size and Number of Flights

In the laboratory, the parameters of interest may be preset. Knowing these in advance, one can then theoretically optimize the experiments. That is, the number of flights k and the sample size N for each flight may be calculated so as to minimize Nk for a given probability of not detecting, for example, real differences in (rms or double-amplitude) vibration levels. In practice, of course, the real differences will not be known; however, reasonable apriori estimates may often be made. In the laboratory the real differences in the vibration levels will be known and therefore the statistical procedures can be tested more precisely.

In Section 5.4 of the report, methods for selection of the number of flights k and the sample size N are mentioned. For the laboratory experiment design, application will be made of these procedures and they will be considered in greater detail.

Referring to the above mentioned Section 5.4, one finds that the population variance estimate computed from the between group (flight-to-flight) variation consists of two components; σ^2, the population variance, and $N\sigma_{\mu}^2$, the variance due to any real differences in means multiplied by the sample size N. The second quantity, σ_{μ}^2, is the one hypothesized to be zero, and represents the quantity to be detected if it is not zero. In the laboratory experiments, a sample of "flights" may be selected such as to have mean vibration levels from a population having a variance σ_{μ}^2. With this quantity known in advance (along with σ^2), values of N and k can be selected so as to minimize Nk and give a certain probability β (the Type II error) of not detecting this component of variance. Also, there will be a certain probability α (the level of significance or the probability of Type I error) that the test will indicate $\sigma_{\mu}^2 \neq 0$ when in fact it is zero. It is desirable to have α and β very small, but these are made small at the expense of increased sample sizes.

First, values for α and β must be selected. For the purposes of this discussion, $\alpha = \beta = 0.05$ will be chosen. The ratio

$$\Delta^2 = \frac{\sigma_{\mu}^2}{\sigma^2}$$ (8.13)

will be used as a measure of deviation from the hypothesized value $\sigma_F^2 = 0$.
The values $\Delta^2 = 1$ and $\Delta^2 = 0.5$ will be selected as critical values for
illustrating computations. That is, if the flight-to-flight variation is as
great as the within flight variation in one case, or half as large in the
other. N and k will be selected so as to give a probability of at least
$(1-\beta) = 0.95$ of detecting this component of variation σ_F^2.

Application will now be made of Table 8.3 in Reference [2]. This
gives values of a function ϕ which will determine N and k as a function of
α and β. The relation to Δ^2 is as follows:

$$\phi[\alpha, \beta, k-1, k(N-1)] = N\Delta^2 + 1 \qquad (8.14)$$

Substituting the selected values of Δ^2, α, and β, one finds:

$$\phi[0.05, 0.05, k-1, k(N-1)] = N + 1 \qquad (8.15)$$

and

$$\phi[0.05, 0.05, k-1, k(N-1)] = 0.5N + 1 \qquad (8.16)$$

Since k is the primary number of interest, values of k will first be
selected. Then by inspection of Table 8.3 in Ref. [2], values of ϕ and
N will be chosen to approximately fit the desired relation of Eq. (8.15)
or (8.16). Table 8.1 below gives a tabulation of values obtained by this
procedure. The tabulated values in Table 8.1 are not precise for several
reasons. Only integral values of N make sense so a value of ϕ can not
be selected in general to maintain the relations in Equations (8.15) and
(8.16). Also interpolation is required in the table referenced which
results in inaccuracies in the computed values of Δ^2.

The values in Table 8.1 indicate that k = 8 and N = 12 are optimum
values to detect $\Delta^2 = 0.50$ (Case 1) while k = 9 and N = 6 are optimum
for $\Delta^2 = 1.00$ (Case 2). However, k = 7 and N = 8 in Case 2 do not change
the total number of observations significantly while reducing the number
of flights by two

Table 8.1 Selection of Sample Size and Number of Flights

	Case 1							
k	4	5	6	7	8	9	10	11
N	41	26	18	15	12	11	10	9
Nk	164	130	105	105	96	99	100	99
Δ^2	.53	.49	.51	.48	.49	.47	.45	.45

N and k for $\Delta^2 = 0.50$

	Case 2							
k	4	5	6	7	8	9	10	11
N	22	13	10	8	7	6	6	5
Nk	88	65	60	56	56	54	60	55
Δ^2	1.01	1.03	.96	.94	.93	.94	.88	.90

N and k for $\Delta^2 = 1.00$

It becomes apparent at this point that the selection of the sample sizes and number of flights requires the exercise of considerable engineering judgment. Sample sizes on the order of 8, 10, or 12 are undoubtedly not sufficiently large to obtain general information of interest for an entire long flight. Also, after data collection instrumentation for a flight vehicle has been accomplished it will probably be comparatively easy to collect more observations in many situations.

8.4.2 Data Collection Procedures

At this point, the detailed procedures for a random sampling scheme described in Section 6 may be applied. Flight phases and reasonable vibration levels for these phases may be simulated. Two data collection procedures might now be employed:

1. Disregard the flight phases and randomly sample throughout the entire "flight"

2. Sample within each flight phase so that each phase may be considered separately in the later analysis.

Also, two procedures for repeating the experiments should now be followed:

1. Repeat the "flight" to simulate in effect a constant "mission" from flight to flight. That is, there is no component of variance due to flight-to-flight variations.

2. Try different "flights" to simulate variable missions. The flight-to-flight vibration will be varied according to a predetermined amount.

8.4.3 Verification of Statistical Estimates

Referring to Sections 5 and 6, calculation of estimates for statistical parameters of interest may now be performed. There exist four different combinations of repeated flights which should be analyzed, and comparisons should be made between the various estimates.

- (i) Flight phases considered
  - a. Constant mission
  - b. Variable mission

- (ii) Flight phases ignored
  - a. Constant mission
  - b. Variable mission

Several checks should be made:

1. In the case of a constant mission from flight-to-flight, no significant difference of flight-to-flight mean vibration levels should be found.

2. In the case of a variable mission, the component of variance, σ_μ^2, should be detected.

3. For all cases, the mean and variance estimates along with confidence intervals can be compared with the actual known values.

4. Estimates obtained by ignoring flight phases can be compared with the estimates from the cases where flight phases are considered separately.

Perfect agreement cannot be expected, of course. In all cases, the underlying assumptions will not have been precisely fulfilled but only approximated. Here again, theory must be tempered with judgment.

8.4.4 Suggested Experimental Plan

1. Select a hypothetical "flight" plan (see for example Table 6.4 in random sampling portion of Section 6.2.4).

2. Decide upon representative (rms or double-amplitude) vibration levels for the flight phases.

3. Determine a size for the component of variance, σ_μ^2, relative to the known variance, σ^2, of the vibration levels within a flight (see Section 8.4.1 above).

4. Compute values for N and k as in Section 8.4.1 for use when flight phases are considered.

For an alternative method, application may be made of tabulated values of Eq. (8.14) for $\alpha = 0.05$ and $\beta = 0.50$ available in Ref. [1]. A smaller value for Δ^2 would then be selected to correspond to $\beta = 0.50$. Operating characteristic curves (as described in Section 5.5.7) may then be computed for various pairs of N and k so determined. Then an N and k can be selected based upon the O.C. curve giving the smallest β for the σ_μ^2 originally selected.

5. Determine a random sampling scheme for the case when flight phases are ignored. See Section 6.2 for details as to sample length, etc For this case, k may be made slightly smaller and N increased to obtain a sample size more consistent with the considerations involved in the sampling scheme in Section 6.2. One must not decrease k too much or it will be impossible to obtain good estimates of the flight-to-flight variation.

6. For the case when flight phases are considered, select samples of size N for each flight phase. Each observation will be taken randomly with respect to time in the flight phase. Again see Section 6.2 for sample length considerations, etc.

7. Determine a random selection of different rms vibration levels from a "normal" population of a predetermined mean μ and variance σ_μ^2. This must be done for each flight phase using the vibration levels selected in step (2) as the mean for each phase. These values will then determine how to run the repeated "flights" for the case when the mission is varied from flight-to-flight.

8. Perform the repeated flights and collect the data for the four cases.

9. Perform calculations as detailed in the repeated experiments section of Section 5.4 and the random sampling scheme of Section 6.2.

10. Analyze results to determine if:

    i. Flight-to-flight variations were detected as predicted.

    ii. Confidence limits for statistical parameter estimates include the actual preset known values.

    iii. Results from not considering flight phases are significantly different from those obtained when the flight phases were considered.

11. An additional procedure would be to run a "flight" introducing unusual occurrences and determine if the random sampling scheme detects these as predicted.

8.5 FLIGHT TEST PROGRAM

8.5.1 General Remarks

The exact method to be employed for obtaining information about the over-all vibration levels of flight vehicles from selected samples will to a great extent depend on the results of the Laboratory Test Program discussed in previous sections. However, there are many considerations that do not depend on the particular sampling scheme used and these will be discussed below.

In Section 3, flight vehicles are divided into four main categories, and gathering of vibration data is discussed with reference to suitable sampling methods. Once a sampling scheme has been decided upon, the method for permanently recording the vibration data and the location of the vibration sensing devices are of primary importance.

Even though the flight test program under discussion here is proposed for the main purpose of verifying certain techniques applying to a single point on a structure, other related problems cannot be entirely ignored. Vibrations are only a part of the total dynamic environment, and the high cost of a flight test program makes it mandatory to obtain the maximum amount of information from each flight. Therefore, the areas of acoustic noise and shock measurements should also be considered as part of a vibration survey. It should also be kept in mind that the data should be useful for determining the dependence of one set of measurements on another even if this is not required to satisfy the main goal of the test flight. In addition, flight recordings can be supplemented with mechanical impedance measurements on the ground, to further enhance their value for determining correlation effects.

8.5.2 Preparation for Flight Test

Each of the four flight vehicle categories can utilize two main methods for obtaining vibration data. These are by 1) direct recording, or 2) telemetering. Section 7.2 discusses in detail some of the instrumentation available for each of these two methods and their advantages and disadvantages. Generally, telemetering of data imposes greater restrictions on the frequency range, dynamic range, and the total number of transducers, than the method of direct recording.

The next consideration is the end use of the data. This includes such areas as:

    a. Establish dynamic environment for equipment
    b. Fatigue studies
    c. Human comfort considerations
    d. Verification of predictions
    e. Any combination of two or more of the above.

Each of these points has an effect on i) the total number of transducers, 2) type of transducers, and 3) the location of the transducers.

To bring these remarks into better focus, two specific examples will now be discussed. The first considers a hypothetical test flight utilizing direct recording methods, the second considers the telemetering of data.

8.5.3 Direct Recording of Vibration Data

This discussion covers a possible practical preparation of a flight test program when a direct recording method can be used.

a. Type of Vehicle: Large, manned, jet-powered aircraft.

b. Purpose of Test: Verification of the sampling scheme determined from Laboratory Program to estimate entire vibration-life-history from selected samples for several points on the structure.

c. Type of Recorders: Magnetic tape. See Section 7 for discussion.

d. Type of Transducers: Self-calibrating, piezoelectric accelerometers. See Section 7 for discussion.

e. Frequency Range: 10 to 2000 cps.

f. Acceleration Range: 0.2 to 50.0 g vector.

g. Number of Transducers: Fifteen.

h. Location of Transducers: See Figure 8.1. Each location uses three transducers, each transducer sensing along one of the three mutually perpendicular axes.

i. Number of Flights: To be determined from desired confidence limits.

Two tape recorders will be used. One will continuously record the vibration levels while the other will only record the samples whose length and method of selection were determined in the laboratory. The self-generating accelerometers contain a dual seismic system of which one element is the driver and the other the sensor. By correlating the simulated acceleration response of the sensing element with the driving system an actual calibration can be made. Since both elements can serve as sensors simultaneously during flight, the accelerometer can be used as a dual output device to measure the same acceleration in a specific area.

Figure 8.7

Location of Transducers for Flight Test Program
(Using Direct Recording Method)
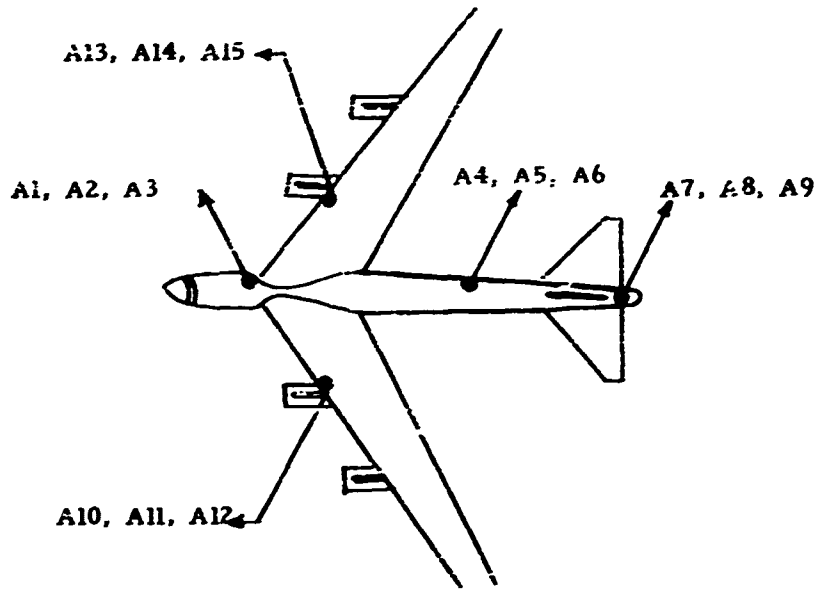
	Accelerometers		
Vertical	Longitudinal	Lateral	Location
A1	A2	A3	Attachment of Main Fwd. Spar to Fuselage (Approx. Sta. 500)
A4	A5	A6	Mid-Section of Fuselage (Approx. Sta. 1100)
A7	A8	A9	Tail-Section of Fuselage (Approx. Sta. 1800)
A10	A11	A12	Attachment of No. 2 Engine to Wing (Outboard of Wing Station 300)
A13	A14	A15	Attachment of No. 3 Engine to Wing (Outboard of Wing Station 300)

One of the sensors will then be connected to the continuously running recorder while the other sensor is sampled in accordance with the proposed sampling scheme. From these two recordings the accuracy of the sampling method can then be verified. It is also important that the accelerometers are located on primary structure with a minimum of bracketry to avoid recording local resonances which may give a distorted picture of the severity of the over-all vibration environment. If equipment is to be mounted on some secondary "soft" structure, a record of the vibration level at primary structural points, coupled with a knowledge of the transfer function from these points to the secondary structure, allows the engineer to calculate the environment for the equipment at a later time.

8.5.4 Telemetering of Vibration Data

For this example, a hypothetical four-stage Space Probe will be considered. For such a vehicle, telemeter channels are usually at a premium and it is desired to obtain the maximum amount of information from a minimum amount of data. Depending on the sampling method involved, one telemeter channel could be used for several transducers, if the time between samples for each transducer is long enough. The number of flights are also limited, sometimes only being one. It will therefore be proposed to collect acoustical data in parallel with the vibration data. This will extend the usefulness of the flight by allowing several end-results such as determining the correlation of the acoustical and vibration environments. It may also be desired to correlate the measurements with theoretical predictions made prior to the flight. The scope of these dynamic field measurements will assist the rank ordering of the magnitudes and characteristics of the environments which will occur in various space vehicles. At the same time, such a study would improve the designer's opportunity to foresee more realistic structural and equipment environments, thus fostering more confidence in design and earlier initiation of environmental testing for reliability of equipment and structure.

The vibrational environment at any point in a space vehicle is given by a complex product of external and internal exciting forces and the reception and transmission of these forces by and through the

vehicle to the point. For example, during the launch of a conventional missile, the noise generated in the rocket motors exhaust stream is propagated through the air, to, and along the outer skin of the vehicle. Because this engulfing rocket noise is of a random nature, containing all frequencies over a range of amplitudes, it excites both major and local structural and skin vibration resonances. The magnitude of this excitation depends upon the frequency spectrum, amplitude, and the space correlation of the noise, and upon the mechanical impedance of the vehicle configuration. The resulting vibrational energy is transferred throughout the missile to sub structure and equipment, and a portion is re-radiated from that as acoustic energy into the vehicle's compartments.

There are several sources of vibratory energy which, either singly or in combination, create vibration within the space vehicle. The noise generated by a rocket motor results primarily from turbulence in the subsonic mixing portion of its jet stream. The amplitude and spectrum of this noise are dependent upon various jet stream parameters, distance from the jet stream, and on the direction relative to the exhaust stream direction. All data indicate that the rocket noise reaches a maximum during launch; its exact level depending somewhat on the launch pad configuration. Rocket noise reflected from the ground plane dominates the environment for the first 1-2 seconds of flight until the nozzles reach an altitude on the order of 50 nozzles diameters. By this time, the missile is at an altitude wherein the rocket exhaust flow is unobstructed. Where upon, at this point, the effect of forward vehicle motion reduces the noise environment surrounding the missile. As the missile velocity increases, rocket noise forward of the nozzle decreases approximately with the square of the missile's velocity until Mach 0.5 is reached after which it decreases more rapidly, approaching zero at Mach one.

The spectra and levels in the rockets near field are dependent upon additional factors which include shock wave formation and stability, and the formation and stability of afterburning in the jet stream. The noise, has, in general, a continuous random level spectrum, but may contain some discrete frequencies. It is proposed that acoustic measurements be made during eight test vehicle launchings (or less, depending on the confidence desired).

In addition to the acoustical energy generated by the rocket motor exhaust, vibratory forces from the rocket motor casing are mechanically transmitted through the vehicle structure. This energy includes random forces resulting from thrust variation, combustion instability, etc., together with possibly large transient excitation during ignition. The internal vibratory energy resulting from the rocket can occur either in space or in the atmosphere, whenever a rocket is used for propulsion or steering. It is therefore proposed for this hypothetical example to install fifteen Self-Calibrating accelerometers in strategic locations of the hypothetical four stage rocket test vehicle.

An important external source is the aerodynamic noise which results from turbulence in the boundary layer. Its magnitude and spectra appear to correlate with local flow parameters in a manner which allows mathematical spectra density treatment and reasonably accurate prediction, depending upon the aerodynamic prediction of flow conditions. It is therefore highly desirable to measure this aerodynamic noise during flight.

The following list of forcing functions presents some of the vibro-acoustic sources that must be considered, measured, and analyzed to adequately define the dynamic environment of the test vehicle.

1. Acoustic Source
   a. Aerodynamic noise
   b. Rocket noise
   c. Auxiliary equipment
   d. Secondary acoustic source
   e. Interaction of shock wave and boundary layer

2. Vibration Sources
   a. Main power plant oscillations
   b. Auxiliary equipment
   c. Aerodynamic gusts
   d. Separation of stages
   e. Coning vibrations
   f. Guidance system reactions
   g. Spin rocket excitations

As stated earlier, for purposes of illustration fifteen Self-Calibrating accelerometers will be mounted within the test vehicle. These piezoelectric accelerometers contain a dual seismic system of which one element is the driver and the other the sensor. By correlating the simulated acceleration response of the sensing element with the driving system, an actual calibration can be made. Since both elements can serve as sensors simultaneously, the accelerometer can be used as a dual output device to measure the same acceleration in a specific area using different full scale outputs, thereby increasing the reliability and the dynamic range of the system. The four terminal theory, involving the reciprocity technique, can be utilized and since either element can be driven, the data obtained from the sensing element would be valid in both directions. The preferred locations for these accelerometers are listed in Table 8.2 and are displayed in Figure 8.2. Accelerometers have been located in the critical guidance bay, the nose payload, and at the aft end of the second, third and fourth stages. Following are some of the specifications applicable to these small measuring devices:

Acceleration Range:	0.2 to 500 g (vector)
Resonant Frequency:	15 KC minimum
Frequency Response:	3 to 5000 cps
Sensitivity (with a 4 ft cable):	7.5 mv/g minimum
Transverse Sensitivity:	Less than ± 5% of maximum sensitivity
Linearity:	± 2% from best straight line
Temperature Range	-65° F to +250° F
Driving Sensitivity:	1 volt drive induces a minimum output equivalent to 0.5 g acceleration on the seismic system
Driving Linearity:	± 5% up to 20 volts input
Housing:	Titanium
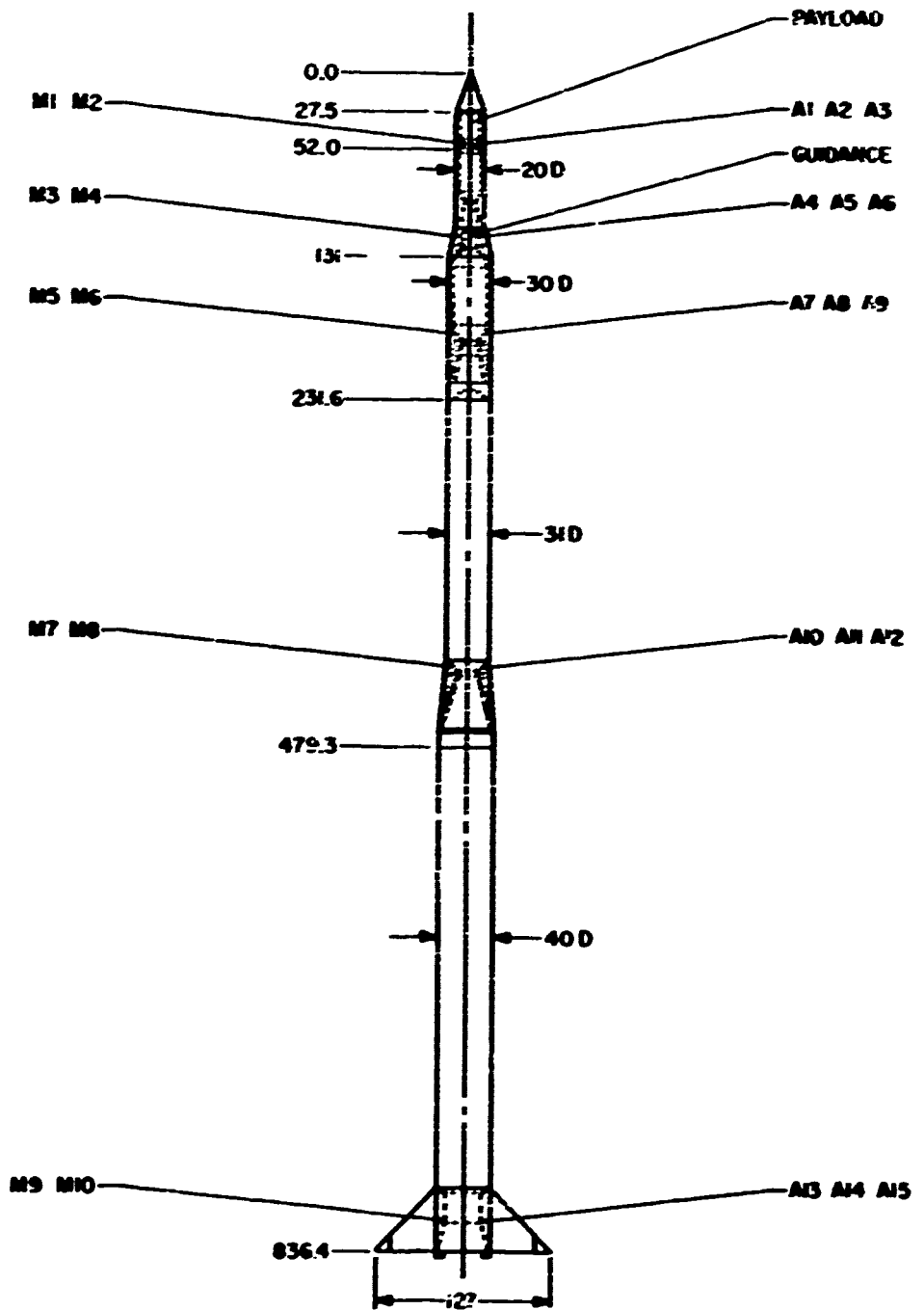
Figure 8. 8. Hypothetical Space Probe

Table 8. 2. Vibro-Acoustic Instrumentation

| Microphones | | Accelerometers | | | |
Internal	External	Vertical	Longitudinal	Lateral	Location
M1	M2	A1	A2	A3	FS50 - near guidance bay
M3	M4	A4	A5	A6	FS125 - near guidance bay
M5	M6	A7	A8	A9	FS220 - between 2nd and 3rd stages
M7	M8	A10	A11	A12	FS460 - between 3rd and 4th stages
M9	M10	A13	A14	A15	FS810 - In area of fins

Mounting	1/4 - 28 stud
Dimensions:	1. 13 L x 0.88 D Overall max.
Weight:	21 grams

The accelerometers should be installed directly on the basic struc-
ture with a minimum amount of bracketing. The brackets used (if any)
should be relatively rigid in order to minimize erroneous amplifications
near bracket natural frequencies. Wherever possible, the accelerometers
should be located on rigid structural members to avoid unwanted local
resonances. It is advisable that the accelerometer instrumentation be
calibrated to 5000 cps. Electrical voltages should be inserted to the
drivers during pre-flight and the resultant sensor response recorded.
A recording should be made also of the system electrical zero. These
calibrations should be accomplished as close to the actual flight time as
is practical. Calibrations and data runs may be identified by suitable voice
announcements on tape recordings.

The internal and external sound pressure levels of the test vehicle
may be measured by ten high-intensity microphones. The microphones
should be calibrated for frequency response in an anechoic chamber to
20,000 cps. An electrical insertion method of calibration should be used
prior to launch. Electrical voltages equivalent to selected sound pressure
levels at various frequencies should be inserted into the instrumentation
system during pre-flight. Suggested microphone locations are listed in
Table 8. 2 and are displayed in Figure 8. 2. Microphone measurements
should be made in the critical guidance bay, near the nose payload, and
at the aft end of the second, third and fourth stages.

8.5.3 Flight Conditions and Number of Flights

In Sections 5.4 and 8.4, statistical methods are developed and applied to determine the exact number of flights required for a given confidence in the hypothesis that the ratio of σ_μ^2, the between flight variance, to σ^2, the within flight variance, of vibration levels, is some number. Therefore, the smaller the ratio $\Delta^2 = (\sigma_\mu^2/\sigma^2)$, the fewer flights will be needed for a given confidence to accept or reject the hypothesis. It is apparent, that some judgment will be required to specify an estimated value for Δ^2. To aid in this estimation several possible combinations of flight conditions for a manned aircraft are given below in order of increasing ratios of Δ^2.

a. The identical aircraft flies identical missions with the same pilot under similar weather conditions.

b. The identical aircraft flies identical missions with different pilots under similar weather conditions.

c. The identical aircraft flies identical missions with different pilots under varying weather conditions.

d. Different aircraft of the same model fly identical missions with different pilots under varying weather conditions.

e. Different aircraft of the same model fly different missions with different pilots under varying weather conditions.

f. Different aircraft, using different models of the same category (such as interceptors) fly different missions with different pilots under varying weather conditions.

It should be noted that each succeeding group of flights has one change added. It is of course possible to add several additional subgroups of other combinations of changes in flight conditions which can be tested if it is felt that a significant change in Δ^2 would occur.

The above discussion shows that the exact number of flights cannot be determined at this time and will depend to a great extent on the flight vehicle under consideration and the purpose for which the data

is to be taken, i.e., is the average vibration level of a class of aircraft to be established, or is it desired to solve a vibration problem on one particular aircraft.

As a general recommendation for verifying certain parts of this report, it would be desirable to hold the variation of flight conditions to a minimum until some confidence has been established in the procedure.

There is one additional problem that should be discussed. For larger missiles, such as space-probes, sometimes only one single test flight is available. It should be realized that it is not theoretically possible then to state a numerical measure of the mathematical confidence in a prediction of the vibration environment for a second flight. In practice, of course, certain knowledge exists as to the similarity of the second flight compared to the first. A qualitative prediction can therefore be made, but a numerical confidence cannot be established.

8.6 REFERENCES

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9. APPLICATIONS TO RESPONSE OF STRUCTURES

Many physical applications to structures problems involving statistical information about random vibration phenomena have been carried out by various authors (see Ref. [4], including its Bibliography). It is the purpose of this section to review some of that material, and other references, as well as to develop some new physical insight about certain matters not previously investigated elsewhere. In particular, a survey appears here on questions of vibration induced structural fatigue, and on effects of non-linearities on response statistics. New theories are proposed for learning about the response of continuous structures to correlated random forces, and for studying modification of responses due to loading. The discussion to follow will begin by reviewing well-known material on the response of linear structures to random excitation.

9.1. RESPONSE OF LINEAR STRUCTURES TO RANDOM EXCITATION

The problem of determining the response of a linear structure to a single frequency excitation, a discrete frequency spectrum, and a continuous frequency spectrum, will be developed for a single degree of freedom system. The response of a continuous linear system will then be explained for free vibration and forced vibration.

9.1.1 Single-Degree-of-Freedom System

The linear spring mass system with viscous damping and fixed base is the simplest oscillatory system to analyze (see Section 3.3). When excited by a force $f(t)$, its equation of motion is
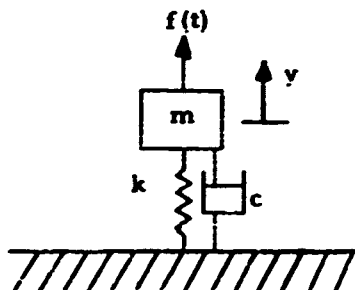
$$m\ddot{y} + c\dot{y} + ky = f(t) \qquad (9.1)$$

where the response $y = y(t)$ is a function only of a time coordinate t.

Introduce the following quantities:

$$\omega_n = \sqrt{\frac{k}{m}} = \text{undamped natural frequency} \tag{9.2}$$

$$\zeta = \frac{c}{c_{cr}} = \text{damping factor} \tag{9.3}$$

$$c_{cr} = 2\sqrt{km} = \text{critical damping coefficient} \tag{9.4}$$

Equation (9.1) can be expressed in the following form

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = \frac{1}{m} f(t) \tag{9.5}$$

When the excitation is in the form of a motion $x(t)$ of the base, an equation similar to Equation (9.5) is obtained as follows.
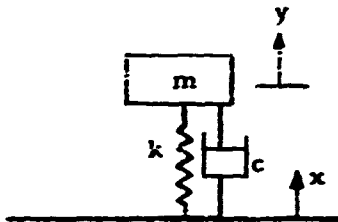
$$\ddot{z} + 2\zeta\omega_n\dot{z} + \omega_n^2 z = -\ddot{x}(t) \tag{9.6}$$

where $z = (y - x) = $ relative motion between the mass and base.
$\ddot{x}(t) = $ acceleration of the base.

Thus the problem of base motion can be analyzed by replacing $f(t)$ by $-m\ddot{x}(t)$ and y by $z = (y - x)$.

The general solution to Equation (9.5) consists of the homogeneous solution which depends on the initial conditions $y(0)$ and $\dot{y}(0)$, and which damps down in a short time, and the particular integral depending on the excitation $f(t)$.

Of particular interest here is the steady state solution to a harmonic excitation $f(t) = \mathcal{R}\left[F_o e^{i\omega t}\right]$ where $\mathcal{R}$ denotes the real part of the complex quantity in brackets. The response y and the excitation $f(t) = F_o \cos \omega t$ are representable as vectors of fixed magnitude differing by a phase ϕ and rotating together with common angular speed ω according to the equation

$$y = \mathcal{R} \frac{F_o e^{i(\omega t - \varphi)}}{m\omega_n^2 \sqrt{\left[\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta \frac{\omega}{\omega_n}\right]^2\right]}} = \mathcal{R}\left[Y e^{i\omega t}\right] \qquad (9.7)$$

where $\qquad Y = \dfrac{F_o e^{-i\varphi}}{|Z(\omega)|} = \dfrac{F_o}{Z(\omega)} \qquad\qquad (9.8)$

and

$$Z(\omega) = m\omega_n^2 \sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta \frac{\omega}{\omega_n}\right]^2}\, e^{i\varphi} \qquad (9.9)$$

The quantity $Z(\omega)$ is referred to as the __impedance__ of the system and its reciprocal $1/Z(\omega)$ is called the __frequency response function__. In these equations it is understood that the applied force and response are both real quantities, indicated by $\mathcal{R}$, so that another expression for y is

$$y = \frac{1}{2}(Y e^{i\omega t} + Y^* e^{-i\omega t}) \qquad (9.10)$$

where Y^* denotes the complex conjugate of Y.

The mean square value of y is of interest in problems which are to follow. The mean square response is found by integrating over one cycle.

$$\overline{y^2} = \frac{1}{T}\int_0^T y^2\, dt = \frac{1}{T}\int_0^T \frac{1}{4}(Y^2 e^{i2\omega t} + 2YY^* + Y^{*2} e^{-i2\omega t})\, dt$$

$$= \frac{1}{2} YY^* = \frac{F_o^2}{2Z(\omega)Z^*(\omega)} = \frac{F_o^2}{2|Z(\omega)|^2} \qquad (9.11)$$

The exponential terms integrated over one cycle are zero in the above integral and Eq. (9.11) is accurate even when integrated over a time interval not a multiple of one period provided T is large compared to the period. Since

$$\frac{F_o^2}{2}$$

is the mean square value of the harmonic excitation, the mean square response is the mean square excitation divided by the square of the impedance $|Z(\omega)|^2$.

If the excitation contains more than one harmonic component, the steady state solution becomes

$$y = \mathcal{R}\left\{\frac{F_1 e^{i\omega_1 t}}{Z_1(\omega_1)} + \frac{F_2 e^{i\omega_2 t}}{Z_2(\omega_2)} + \ldots\right\}$$

$$= \mathcal{R} \sum_j Y_j e^{i\omega_j t} = \frac{1}{2}\left[\sum_j Y_j e^{i\omega_j t} + \sum_j Y_j^* e^{-i\omega_j t}\right] \qquad (9.12)$$

The mean square response now becomes

$$\overline{y^2} = \frac{1}{T}\int_0^T \frac{1}{4}\left[\sum_j Y_j e^{i\omega_j t} + \sum_j Y_j^* e^{-i\omega_j t}\right]^2 dt = \sum_j \frac{1}{2} Y_j Y_j^* \qquad (9.13)$$

with terms like the single frequency case which must be summed over the frequency numbers j.

Consider next the problem where the excitation contains many closely spaced harmonic components approaching a continuous spectrum. For this, consider f(t) and y(t) so that the Fourier transforms exist, namely,

$$F(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t) \, dt \qquad (9.14)$$

$$Y(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} \, y(t) \, dt \qquad (9.15)$$

and determine the Fourier transform of the original differential Eq. (9.5) to obtain

$$Y(\omega) = \frac{F(\omega)}{Z(\omega)} \qquad (9.16)$$

Associated with Eq. (9.15) one has the inverse transform, and since y(t) is real, it may be written in two forms as follows.

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \, Y(\omega) \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} \, Y^*(\omega) \, d\omega \qquad (9.17)$$

The mean square value $\overline{y^2(t)} = \overline{y^2}$, can now be obtained by multiplying the two forms of Eq. (9.17) and integrating over a long time interval 2T as follows.

$$\overline{y^2} = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{\infty}^{\infty} Y(\omega) Y^*(\omega') \frac{1}{2T} \int_{-T}^{T} e^{-i(\omega-\omega')t} \, dt \, d\omega \, d\omega'$$

$$= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{T} Y(\omega) Y^*(\omega') \frac{\sin(\omega - \omega')T}{(\omega - \omega')} \, d\omega \, d\omega' \qquad (9.18)$$

Note here that the integral

$$\int_{-\infty}^{\infty} \frac{\sin(\omega - \omega')T}{(\omega - \omega')}$$

for large T is zero except when $\omega = \omega'$, in which case its value is π. This occurs because for large T the integrand above behaves like $\pi\delta(\omega - \omega')$ where $\delta(\omega - \omega')$ is a delta function peaked at $\omega = \omega'$. The mean square response then reduces to

$$\overline{y^2} = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{1}{T} Y(\omega) Y^*(\omega) \, d\omega$$

$$= \frac{1}{2\pi} \int_{0}^{\infty} \frac{1}{T} Y(\omega) Y^*(\omega) \, d\omega \tag{9.19}$$

where the prime on ω is now discarded. Equation (9.19) is a generalization of the previous case of the discrete spectrum to the continuous spectrum, and the finite summation has now been replaced by an integration over frequency. It is possible to view the integrand of this equation as the power spectral density function of the response, namely,

$$S_y(\omega) = \frac{1}{2\pi T} Y(\omega) Y^*(\omega) = \frac{|Y(\omega)|^2}{2\pi T} \tag{9.20}$$

from which the mean square response becomes

$$\overline{y^2} = \int_{0}^{\infty} S_y(\omega) \, d\omega \tag{9.21}$$

showing how $\overline{y^2}$ is distributed in the frequency range.

Similarly the power spectral density function of the input $f(t)$ is given by

$$S_f(\omega) = \frac{1}{2\pi T} F(\omega) F^*(\omega) \tag{9.22}$$

such that

$$\overline{f^2(t)} = \frac{1}{T} \int_0^T f^2(t) \, dt = \int_0^\infty S_f(\omega) \, d\omega \qquad (9.23)$$

Substituting Eq. (9.16) into (9.20) yields the following important relationship between the power spectral densities of the output and input,

$$S_y(\omega) = \frac{1}{2\pi T} \frac{F(\omega)}{Z(\omega)} \frac{F^*(\omega)}{Z^*(\omega)} = \frac{1}{|Z(\omega)|^2} S_f(\omega) \qquad (9.24)$$

Thus the power spectral density of the output is equal to the power spectral density of the input multiplied by the square of the absolute value of the frequency response function.

It should be pointed out here that in actual practical measurements, the power spectral density function should not be calculated by Eq. (9.20) or (9.22) since these definitions lead to inconsistent estimates (see other statistical analyses of power spectral density functions in Section 4.8 for a full discussion of proper methods of measurement).

In summary, the mean square response of a single-degree-of-freedom system for the three cases considered is given by:

(1) Single frequency excitation

$$\overline{y^2} = \frac{1}{2} Y Y^* \qquad (9.25)$$

where $Y = \dfrac{F_o}{Z(\omega)}$

(2) Discrete frequency spectrum:

$$\overline{y^2} = \sum_j \frac{1}{2} Y_j Y_j^*$$ (9.26)

where $Y_j = \dfrac{F_j}{Z_j(\omega_j)}$

(3) Continuous frequency spectrum

$$\overline{y^2} = \frac{1}{2\pi} \int_0^\infty \frac{1}{T} Y(\omega) Y^*(\omega) \, d\omega$$ (9.27)

where $Y(\omega) = \dfrac{F(\omega)}{Z(\omega)}$

$$F(\omega) = \int_{-\infty}^\infty e^{-i\omega t} f(t) \, dt$$

In addition the power spectral density functions for the output y(t) and input f(t), each of large finite length T, have been defined by

$$S_y(\omega) = \frac{1}{2\pi T} Y(\omega) Y^*(\omega)$$ (9.28)

$$S_f(\omega) = \frac{1}{2\pi T} F(\omega) F^*(\omega)$$ (9.29)

and it has been shown that they are related by the equation

$$S_y(\omega) = \frac{1}{|Z(\omega)|^2} S_f(\omega)$$ (9.30)

9.1.2 Continuous System

A continuous elastic body is a system of infinite number of degrees of freedom, and its equation of motion is a partial differential equation in the time and space coordinates. Its solution for the displacement is now a function of the space coordinate x and the time t and will be designated by $y(x,t)$.

(1) Free Vibration

Instead of solving the partial differential equation, which is an eigen-value problem, it is possible to view the problem from a simpler point of view of normal modes. Normal modes are <u>free vibrations</u> of the system in the absence of all external forces and damping, and hence can be considered as dynamical properties of the system. Such normal modes depend only on the geometric configuration of the body and its mass and stiffness distributions.

If a body is distorted into one of its normal mode shapes $\varphi_n(x)$ and released, it will continue to vibrate harmonically in this mode at frequency ω_n. This harmonic vibration in the time domain may be represented by $\sin \omega_n t$, $\cos \omega_n t$, or $e^{i\omega_n t}$. Thus, $\varphi_n(x)$ and ω_n completely characterize the normal mode vibration of order n. The mode shape and its natural frequency are independent of the amplitude, and the scale of $\varphi_n(x)$ is arbitrary.

The mode shapes, $\varphi_n(x)$, possess certain mathematical properties which are called orthogonal. That is, if one measures the mode shapes of the ith and the jth modes and examines the integral

$$\int_0^L \varphi_i(x)\, \varphi_j(x)\, m(x)\, dx$$

where $m(x)$ is the mass density, and the integration is over the entire body (assumed above to be of length L), the above integral will be zero when $i \neq j$, and is a finite number M_i depending on the scale of $\varphi_n(x)$ when $i = j$. It is convenient to choose the scale of $\varphi_n(x)$ such that this integral will equal one number, equal to the total mass M of the body.

There results

$$\int_0^L \phi_i(x) \phi_j(y) m(x) \, dx = \begin{cases} 0 & \text{when } i \neq j \\ M & \text{when } i = j \end{cases} \tag{9.31}$$

and one says that $\phi_n(x)$ is normalized to the total mass M.

Each normal mode can persist independently with a cyclic inter-change of kinetic and potential energies. The equation of motion is

$$D\phi_n(x) = \omega_n^2 m(x) \phi_n(y) \tag{9.32}$$

where D is a special differential operator depending on the type of structure. For instance, for the flexural vibration of a beam

$$D = \frac{d^2}{dx^2} \left(EI \frac{d^2}{dx^2} \right) \tag{9.33}$$

and for the longitudinal oscillation of a slender rod

$$D = \frac{-d}{dx} \left(AE \frac{d}{dx} \right) \tag{9.34}$$

where E, I and A represent Young's modulus, the moment of inertia, and the cross sectional area respectively.

Example:

For the longitudinal oscillation of a slender rod of uniform cross section, the differential equation for the free vibration $y(x, t)$ must satisfy

$$m \frac{\partial^2 y}{\partial t^2} - AE \frac{\partial^2 y}{\partial x^2} = 0$$

For a uniform rod of length L with free ends and fixed center, the normal modes from Eq. (9.32) are found to be

$$\phi_1(x) = \cos \frac{\pi x}{L}$$

$$\phi_3(x) = \cos \frac{3\pi x}{L}$$

$$\phi_5(x) = \cos \frac{5\pi x}{L} \qquad \text{etc.}$$

At any mode n, (n = 1, 3, 5, ...) the displacement can be written as

$$y(x, t) = q_n(t)\,\phi_n(x)$$

which substituted into the differential equation above becomes

$$(\ddot{q}_n + \omega_n^2 q)\,\phi_n(x) = 0$$

where
$$\omega_n^2 = \left(\frac{n\pi}{L}\right)^2 \frac{AE}{m} \qquad (\text{n odd})$$

Thus, the solution $q(t) = \cos \omega_n t$ is harmonic in the time domain with frequency ω_n, and its displacement is given by

$$y(x, t) = \phi_n(x) \cos \omega_n t$$

It is possible to have a free vibration which is not the simple single frequency motion indicated by $\phi_n(x)$ and ω_n. For instance, the body might be deformed into some configuration $\Phi(x)$ different from any of the normal modes and then released from this position. However, note that $\Phi(x)$ can always be represented by the sum of the normal modes with proper amplitudes as follows

$$\ddot{y}(x) = A_1 \dot{\phi}_1(x) + A_2 \dot{\phi}_2(x) + \dots \tag{9.35}$$

and since each normal mode can persist independently, the resulting motion will be

$$y = \sum_n A_n \phi_n(x) \cos \omega_n t \tag{9.36}$$

(2) Forced Vibration

When an elastic body is excited by a distributed force $f(x,t)$ including viscous damping, its differential equation of motion is

$$m(x)\ddot{y}(x,t) + c(x)\dot{y}(x,t) + Dy(x,t) = f(x,t) \tag{9.37}$$

Assume now that the displacement $y(x,t)$ can be represented by the sum of the normal modes $\phi_n(x)$ multiplied by some time function $q_n(t)$,

$$y(x,t) = \sum_n q_n(t)\phi_n(x) \tag{9.38}$$

where the $q_n(t)$ are to be determined.

Substituting Eq. (9.38) into (9.37).

$$\sum_n \ddot{q}_n(t) m(x)\phi_n(x) + \sum_n \dot{q}_n(t) c(x)\phi_n(x) + \sum_n q_n(t) D\phi_n(x) = f(x,t)$$
$$\tag{9.39}$$

Next, replace $D\phi_n(x) = \omega_n^2 m(x)\phi_n(x)$ from Eq. (9.32), multiply each term by $\phi_i(x)\,dx$ and integrate over the structure. Due to the orthogonality relation, Eq. (9.31), the first and third terms of Eq. (9.39) reduce to a single term for $n = i$. For the damping term to also vanish for $n \neq i$ it is necessary for $c(x)$ to be proportional to $m(x)$. If one lets $c(x) = 2\zeta_n \omega_n m(x)$ the summation again disappears for $n \neq i$ and one arrives at the uncoupled second-order differential equation for $q_i(t)$.

$$\ddot{q}_i(t) + 2\zeta_i\omega_iq_i(t) + \omega_i^2q_i(t) = \frac{1}{M}\int_0^L f(x, t)\phi_i(x)\ dx \qquad (9.40)$$

Since the integral on the right side is not a function of x, but depends only on t and the mode number i, let

$$f_i(t) = \int_0^L f(x, t)\phi_i(x)\ dx \qquad (9.41)$$

and write Eq. (9.40) as

$$\ddot{q}_i + 2\zeta_i\omega_i\dot{q}_i + \omega_i^2q_i = \frac{1}{M}f_i(t) \qquad (9.42)$$

Equation (9.42) is identical in form to Eq. (9.5) for the single degree of freedom system. Thus, its Fourier transform

$$Q_i(\omega) = \frac{F_i(\omega)}{Z_i(\omega)} \qquad (9.43)$$

correspond to that of Eq. (9.16), where:

$$Z_i(\omega) = M\omega_i^2\sqrt{\left[1 - \left(\frac{\omega}{\omega_i}\right)^2\right]^2 + \left[2\zeta_i\frac{\omega}{\omega_i}\right]^2}\ e^{i\phi_i} \qquad (9.44)$$

$$F_i(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t}f_i(t)\ dt$$

$$\qquad (9.45)$$

$$= \int_0^L F(x, \omega)\phi_i(x)\ dx$$

$$F(x,\omega) - \int_{-\infty}^{\infty} e^{-i\omega t} f(x, t) \, dt \qquad (9.46)$$

$$Q_i(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} q_i(t) \, dt \qquad (9.47)$$

i.e. $Q_i(\omega)$ and $F(x,\omega)$ are the Fourier transforms of $q_i(t)$ and $f(x, t)$, restricted time-wise so that the transforms exist.

For simplicity, consider first the case where all modes except one are insignificant so that in place of the summation

$$y(x, t) = q_i(t) \phi_i(x)$$

and $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (9.48)$

$$Y(x,\omega) = \frac{F_i(\omega)}{Z_i(\omega)} \phi_i(x)$$

Equation (9.19) for the mean square response now becomes a function of the space coordinate x, denoted by

$$\overline{y^2(x)} = \frac{\phi_i^2(x)}{2\pi} \int_0^{\infty} \frac{1}{T} \frac{F_i(\omega)}{Z_i(\omega)} \frac{F_i^*(\omega)}{Z_i^*(\omega)} \, d\omega$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (9.49)$$

$$= \frac{\phi_i^2(x)}{2\pi} \int_0^{\infty} \frac{1}{|Z_i(\omega)|^2} \frac{1}{T} \int_0^L \int_0^L F(u, \omega) F^*(u', \omega) \phi_i(u) \phi_i(u') \, du \, du' \, d\omega$$

One thus finds a new factor

$$\overset{\wedge\wedge\wedge}{F_{ii}}(\omega) = \frac{1}{T} \int_0^L \int_0^L F(u', \omega) F^*(u', \omega) \phi_i(u) \phi_i(u') \, du \, du' \qquad (9.50)$$

which will be referred to as the generalized power spectrum of the excitation, entering into the response equation. In terms of the notation $F_{ii}(\omega)$, one can rewrite Eq. (9.49) as

$$\overline{y^2(x)} = \frac{\phi_i^2(x)}{2\pi} \int_0^\infty \frac{\overbrace{F_{ii}(\omega)}}{|Z_i(\omega)|^2} \, d\omega \qquad (9.51)$$

Observe that the generalized power spectrum $F_{ii}(\omega)$ involves the normal modes $\phi_i(u)$ and $\phi_i(u')$, and is obtained by averaging over the space coordinates u and u' in the functions $F(u,\omega)$ and $F(u',\omega)$. These functions are Fourier transforms of the exciting functions $f(u,t)$ and $f(u',t)$, respectively, the parameters u and u' being used in place of x here to avoid confusion with the $\phi_i^2(x)$ term appearing outside the integral.

Finally, consider the most general case where Eq. (9.48) must be replaced by a summation

$$Y(x,\omega) = \sum_i \frac{F_i(\omega)\phi_i(x)}{Z_i(\omega)} = \sum_i \frac{\phi_i(x)}{Z_i(\omega)} \int_0^L F(x,\omega)\phi_i(x) \, dx$$

$$(9.52)$$

The mean square response will now be

$$\overline{y^2(x)} = \frac{1}{2\pi} \int_0^\infty \frac{1}{T} \sum_i \frac{F_i(\omega)\phi_i(x)}{Z_i(\omega)} \sum_j \frac{F_j^*(\omega)\phi_j(x)}{Z_j^*(\omega)} \, d\omega$$

$$(9.53)$$

$$= \frac{1}{2\pi} \sum_i \sum_j \int_0^\infty \frac{\phi_i(x)\phi_j(x)}{Z_i(\omega)Z_j^*(\omega)} \frac{1}{T} \int_0^L \int_0^L F(u,\omega) F^*(u',\omega) \phi_i(u) \phi_j(u') \, du \, du' \, d\omega$$

The generalized power spectrum now is composed of the influence of the ith and the jth mode and can be distinguished by the notation

$$\overset{\wedge\wedge\wedge}{F_{ij}(\omega)} = \frac{1}{T} \int_0^L \int_0^L F(u,\omega) F^*(u',\omega) \phi_i(u) \phi_j(u') \, du \, du' \qquad (9.54)$$

One can then write for the mean square response at position x,

$$\overline{y^2(x)} = \frac{1}{2\pi} \sum_i \sum_j \phi_i(x) \phi_j(x) \int_0^\infty \frac{\overset{\wedge\wedge\wedge}{F_{ij}(\omega)}}{Z_i(\omega) Z_j^*(\omega)} \, d\omega \qquad (9.55)$$

Equation (9.55) reduces to Eq. (9.51) when only one mode is involved. This completes the derivation.

9.2 CONTINUOUS STRUCTURES EXCITED BY CORRELATED RANDOM FORCES

In the previous section the equation for the mean square response at any point x of a continuous structure excited by a spacewise distributed random force was developed. Such distributed random forces are encountered in the aerodynamic flight of missiles and aircraft, whereas jet engines are an example of multisource concentrated excitation for the continuous structures. It is the purpose of this section to relate the statistical response of the continuous structure to the correlation of the excitation forces at differing positions on the structure.

9.2.1 Statistical Response and Cross-Correlation Function

From Eq. (9.52) of Section 9.1.2, the F.T. of the response of continuous system excited by a distributed random force is

$$Y(x,\omega) = \sum_i \frac{\phi_i(x)}{Z_i(\omega)} \int_0^L F(x,\omega) \phi_i(x) \, dx \qquad (9.52)$$

When the random excitation phenomenon is considered to be stationary with ergodic property, the correlation between measured quantities at two positions x and x' at different times t and t + τ, depend only

on x, x', and τ. Ensemble averages may then be replaced by time averages and the statistical properties to be deduced are independent of the origin of t chosen.

Define first the cross-correlation function of the response at two points x and x' differing in time by τ to be

$$\overline{y(x,t)\,y(x',t+\tau)} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} y(x,t)\,y(x',t+\tau)\,dt \qquad (9.56)$$

The two records y(x, t) and y(x', t) aligned in time are shown in the sketch below, and Eq. (9. 56) implies that their instantaneous values at t and t + τ are

y(x, t)

y(x', t)
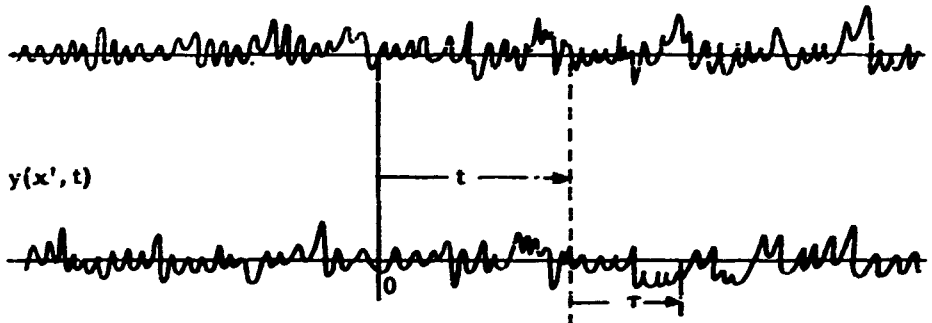
multiplied and averaged, the result being independent of where 0 is chosen.

Next, truncate these infinite records so that outside the time interval -T to T, where T is taken to be a large time, their values are zero. The equation for the cross correlation function then becomes

$$\overline{y(x,t)\,y(x',t+\tau)} = \frac{1}{2T} \int_{-\infty}^{\infty} y(x,t)\,y(x',t+\tau)\,dt \qquad (9.57)$$

where the limits of integration are extended to infinity without altering the result.

The cross power spectrum is the F. T. of the cross correlation function.

$$S_y(x, x', \omega) = \int_{-\infty}^{\infty} e^{-i\omega\tau} \overline{y(x, t)\, y(x', t + \tau)}\ d\tau \qquad (9.58)$$

Substituting Eq. (9.57) into Eq. (9.58) and introducing $e^{i\omega t}\, e^{-i\omega t} = 1$,

$$S_y(x, x', \omega) = \int_{-\infty}^{\infty} e^{-i\omega\tau}\, \frac{1}{2T} \int_{-\infty}^{\infty} e^{-i\omega t}\, e^{i\omega t}\, y(x, t)\, y(x', t + \tau)\ dt\ d\tau$$

If one interchanges the order of integration, holding t constant and varying τ, one can rearrange the above equation to

$$S_y(x, x', \omega) = \frac{1}{2T} \int_{-\infty}^{\infty} e^{-i\omega(t+\tau)}\, y(x', t+\tau) d\tau \int_{-\infty}^{\infty} e^{i\omega t}\, y(x, t)\ dt$$

and by letting $t + \tau = \xi$ in the first integral, $d\tau = d\xi$

$$S_y(x, x'\omega) = \frac{1}{2T} \int_{-\infty}^{\infty} e^{-i\omega\xi}\, y(x', \xi)\ d\xi \int_{-\infty}^{\infty} e^{i\omega t}\, y(x, t)\ dt$$

$$= \frac{1}{2T}\, Y(x', \omega)\, Y^*(x, \omega) \qquad (9.59)$$

where Y and Y^* denote the F. T. and its conjugate, and T is a large number approaching infinity.

The F. T. of Eq. (9.58) is related to its inverse by the equation

$$\overline{y(x, t)\, y(x', t + \tau)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega\tau}\, S_y(x, x', \omega)\ d\omega \qquad (9.60)$$

The substitution of Eq. (9.59) into Eq. (9.60) results in

$$\overline{y(x,t)\,y(x',t+\tau)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega\tau}\, \frac{1}{2T}\, Y(x',\omega)\, Y^*(x,\omega)\, d\omega \qquad (9.61)$$

The Y and Y^* can now be replaced from Eq. (9.52) which enables Eq. (9.61) to be written as

$$\overline{y(x,t)\,y(x',t+\tau)}$$

$$= \frac{1}{2\pi} \sum_i \sum_j \phi_i(x')\,\phi_j(x) \int_{-\infty}^{\infty} \frac{e^{i\omega\tau}}{Z_i(\omega)\,Z_j^*(\omega)} \int_0^L \int_0^L \frac{1}{2T}\, F(x',\omega)\bar{F}^*(x,\omega)\phi_i(x')\phi_j(x)\, dx'\, dx\, d\omega$$

$$(9.62)$$

$$= \frac{1}{2\pi} \sum_i \sum_j \phi_i(x')\phi_j(x) \int_{-\infty}^{\infty} \frac{e^{i\omega\tau}}{Z_i(\omega)Z_j^*(\omega)} \int_0^L \int_0^L S_f(x,x',\omega)\phi_i(x')\phi_j(x)\, dx'\, dx\, d\omega$$

In Eq. (9.62), the term

$$S_f(x,x',\omega) = \frac{1}{2T}\, F(x',\omega)\bar{F}^*(x,\omega) \qquad (9.63)$$

is the cross power spectrum of the excitation at x and x'. Equation (9.62) includes the special case of the mean square response which becomes, setting $\tau = 0$ and $x' = x$, except inside the integral sign,

$$\overline{y^2(x,t)} = \frac{1}{2\pi} \sum_i \sum_j \phi_i(x)\phi_j(x) \int_{-\infty}^{\infty} \frac{1}{Z_i(\omega)Z_j^*(\omega)} \int_0^L \int_0^L S_f(x,x',\omega)\phi_i(x')\phi_j(x)\, dx'\, dx\, d\omega$$

$$(9.64)$$

Note the cross power spectrum of the generalized force

$$\int_0^L \int_0^L S_f(x,x',\omega)\,\phi_i(x')\,\phi_j(x)\, dx'\, dx \qquad (9.65)$$

remains unchanged in Eq. (9.62) and (9.64).

The use of Eq. (9.62) or (9.64) is straightforward. However, evaluation of Eq. (9.65) for the cross power spectrum of the generalized force requires some explanation. The following examples will clarify this point.

9.2.2 Selected Physical Examples

Example 1. A general structure is loaded by two concentrated forces $P_1(t)$ and $P_2(t)$ at positions x_1 and x_2. If the correlation function between $P_1(t)$ and $P_2(t)$ is known, determine the mean square response at any point x.

The force per unit length $f(x,t)$ can be expressed as follows:

$$f(x,t) = P_1(t)\delta(x - x_1) + P_2(t)\delta(x - x_2) \qquad (9.66)$$

where $\delta(x - x_i)$ is a delta function which is zero everywhere except at x_i. The F.T. of $f(x,t)$ is

$$F(x,\omega) = P_1(\omega)\delta(x - x_1) + P_2(\omega)\delta(x - x_2)$$

and the excitation cross power spectrum from Eq. (9.63) becomes

$$S_f(x,x',\omega) = \frac{1}{2T}\left[P_1(\omega)P_1^*(\omega)\delta(x - x_1)\delta(x' - x_1) + P_2(\omega)P_2^*(\omega)\delta(x - x_2)\delta(x' - x_2) \right.$$

$$\left. + P_1(\omega)P_2^*(\omega)\delta(x - x_2)\delta(x' - x_1) + P_1^*(\omega)P_2(\omega)\delta(x' - x_2)\delta(x - x_1) \right]$$

The excitation cross power spectrum has now been resolved in terms of the power spectra

$$S_{f_{11}}(\omega) = \frac{1}{2T} P_1(\omega)P_1^*(\omega) = \frac{1}{2T}\left| P_1(\omega) \right|^2$$

$$S_{f_{22}}(\omega) = \frac{1}{2T} P_2(\omega)P_2^*(\omega) = \frac{1}{2T}\left| P_2(\omega) \right|^2$$

and the cross power spectra

$$S_{f_{12}}(\omega) = \frac{1}{2T} P_1(\omega) P_2^*(\omega)$$

$$S_{f_{21}}(\omega) = \frac{1}{2T} P_2(\omega) P_1^*(\omega)$$

However the cross power spectra $G_{f_{ij}}(\omega)$ are the F.T. of the cross correlation function between $p_1(t)$ and $p_2(t)$, whereas the power spectra $G_{f_{ii}}(\omega)$ are the F.T. of the autocorrelation functions of $P_1(t)$ and $P_2(t)$.

Equation (9.65) can now be evaluated, the result being

$$\int_0^L \int_0^L S_f(x,x',\omega)\phi_i(x')\phi_j(x)\,dx'\,dx = S_{f_{11}}(\omega)\phi_i(x_1)\phi_j(x_1) + S_{f_{22}}(\omega)\phi_i(x_2)\phi_j(x_2)$$

$$+ S_{f_{12}}(\omega)\phi_i(x_1)\phi_j(x_2) + S_{f_{21}}(\omega)\phi_i(x_2)\phi_j(x_1)$$

Thus if $p_1(t)$ and $p_2(t)$ are completely uncorrelated, i.e. $\overline{p_1(t)p_2(t)} = 0$, $S_{f_{12}}(\omega)$ and $S_{f_{21}}(\omega)$ will be zero and the terms involving the normal modes at two different stations will drop out. For $\overline{p_1(t)p_2(t)} \neq 0$, all four terms must be retained.

The mean square response at x can then be written from Eq. (9.64) as

$$\overline{y^2(x,t)} = \frac{1}{2\pi}\sum_i \sum_j \phi_i(x)\phi_j(x) \qquad \text{times}$$

$$\int_{-\infty}^{\infty} \frac{S_{f_{11}}(\omega)\phi_i(x_1)\phi_j(x_1) + S_{f_{22}}(\omega)\phi_i(x_2)\phi_j(x_2) + S_{f_{12}}(\omega)\phi_i(x_1)\phi_j(x_2) + S_{f_{21}}(\omega)\phi_i(x_2)\phi_j(x_1)}{Z_i(\omega)Z_j^*(\omega)}\,d\omega \qquad (9.67)$$

where the last two terms of the integral becomes zero for the uncorrelated excitation.

<u>Example 2.</u> A continuous structure has a random load
which is distributed over its length. Determine its mean square
response at any point x.

   This problem can be considered to be an extension of problem 1.
Assume the distributed load $f(x,t)$ to be represented by a series of
concentrated forces as follows

$$f(x,t) = p_1(t)\delta(x - x_1) + p_2(t)\delta(x - x_2) + p_3(t)\delta(x - x_3) + \ldots \qquad (9.68)$$

Its F. T. and its conjugate are

$$f(x',\omega) = P_1(\omega)\delta(x' - x_1) + P_2(\omega)\delta(x' - x_2) + P_3(\omega)\delta(x' - x_3) + \ldots$$

$$f^*(x,\omega) = P_1^*(\omega)\delta(x - x_1) + P_2^*(\omega)\delta(x - x_2) = P_3^*(\omega)\delta(x - x_3) + \ldots$$

and the cross power spectrum becomes

$$\frac{1}{2T}\left[P_1(\omega)P_1^*(\omega)\delta(x'-x_1)\delta(x-x_1)+P_1(\omega)P_2^*(\omega)\delta(x'-x_1)\delta(x-x_2)+P_1(\omega)P_3^*(\omega)\delta(x'-x_1)\delta(x-x_3)+\cdots\right.$$

$$+P_2(\omega)P_1^*(\omega)\delta(x'-x_2)\delta(x-x_1)+P_2(\omega)P_2^*(\omega)\delta(x'-x_2)\delta(x-x_2)+P_2(\omega)P_3^*(\omega)\delta(x'-x_2)\delta(x-x_3)+\cdots$$

$$\left.+P_3(\omega)P_1^*(\omega)\delta(x'-x_3)\delta(x-x_1)+P_3(\omega)P_2^*(\omega)\delta(x'-x_3)\delta(x-x_2)+P_3(\omega)P_3^*(\omega)\delta(x'-x_3)\delta(x-x_3)+\cdots\right.$$

The cross power spectrum of the generalized force, from Eq. (9.65)
is then represented by the following array of terms

$$S_{11}(\omega)\phi_i(x_1)\phi_j(x_1) \qquad S_{12}(\omega)\phi_i(x_1)\phi_j(x_2) \qquad S_{13}(\omega)\phi_i(x_1)\phi_j(x_3) \ldots$$

$$S_{21}(\omega)\phi_i(x_2)\phi_j(x_1) \qquad S_{22}(\omega)\phi_i(x_2)\phi_j(x_2) \qquad S_{23}(\omega)\phi_i(x_2)\phi_j(x_3) \ldots$$

$$S_{31}(\omega)\phi_i(x_3)\phi_j(x_1) \qquad S_{32}(\omega)\phi_i(x_3)\phi_j(x_2) \qquad S_{33}(\omega)\phi_i(x_3)\phi_j(x_3) \ldots$$

Thus it is seen that the problem requires the evaluation of the cross
correlation of the excitation $\overline{P_m(t)P_n(t)}$ between every pair of points
chosen as station on the structure: i.e.,

$$S_{mn}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} \, \overline{P_m(t) P_n(t)} \, dt$$

For uncorrelated loads, i.e., $\overline{P_m(t) P_n(t)} = 0$ for $n \neq m$, only the diagonal terms of the above array are retained and the amount of computation required is reduced considerably. It should be noted however that the diagonal terms of the form $S_{nn}(\omega) \phi_i(x_n) \phi_j(x_n)$ involve the products of the ith and j^{th} modes which remain to be summed.

Example 3. A continuous structure has a convective load expressed by the equation

$$f(x, t) = p \cos k (x - ct) \tag{9.69}$$

where p is the amplitude, c the propagation speed of the load and k the wave number. To determine the cross power spectrum of the load, first find the Fourier Transform $F(x', \omega)$ as follows. From Eq. (9.40), replacing x by x' and using a limiting operation,

$$F(x', \omega) = \lim_{T \to \infty} p \int_{-T}^{T} e^{-i\omega t} \cos k(x' - ct) \, dt$$

$$= \lim_{T \to \infty} p T e^{-ikx'} \qquad \text{when } \omega = kc$$

$$= \lim_{T \to \infty} p T e^{ikx'} \qquad \text{when } \omega = -kc$$

$$= 0 \qquad \text{when } \omega \neq \pm kc$$

$$F^*(x, \omega) = \lim_{T \to \infty} p T e^{ikx} \qquad \text{when } \omega = kc$$

$$= \lim_{T \to \infty} p T e^{-ikx} \qquad \text{when } \omega = -kc$$

$$= 0 \qquad \text{when } \omega \neq \pm kc$$

In the limit as $T \to \infty$, $F(x, \omega)$ and $F^*(x, \omega)$ become infinite, which is to be expected since the F.T. or spectrum of a harmonic function e^{ikct} is a delta function $\delta(\omega - kc)$ multiplied by the factor 2π.

From Eq. (9.63) the cross power spectrum becomes

$$S_f(x, x', \omega) = \lim_{T \to \infty} \frac{p^2 T}{2} e^{-ik(x'-x)} \qquad \text{when } \omega = kc$$

$$= \lim_{T \to \infty} \frac{p^2 T}{2} e^{ik(x'-x)} \qquad \text{when } \omega = -kc$$

$$= 0 \qquad \text{when } \omega \neq \pm kc$$

Now, replace the limiting value of T by the delta functions $(2\pi)\delta(\omega - kc)$ and $(2\pi)\delta(\omega + kc)$ at $\omega = kc$ and $\omega = -kc$, respectively, and rewrite the above equation as

$$S_f(x, x', \omega) = \pi p^2 \left[e^{-ik(x'-x)} \delta(\omega - kc) + e^{ik(x'-x)} \delta(\omega + kc) \right]$$

The substitution of $S_f(x, x', \omega)$ into Eq. (9.64) involves the following integrals

$$\pi p^2 \int_{-\infty}^{\infty} \frac{\delta(\omega - kc)}{Z_i(\omega) Z_j^*(\omega)} \int_0^L \int_0^L e^{-ik(x'-x)} \phi_i(x') \phi_j(x) \, dx' \, dx \, d\omega$$

$$+ \pi p^2 \int_{-\infty}^{\infty} \frac{\delta(\omega + kc)}{Z_i(\omega) Z_j^*(\omega)} \int_0^L \int_0^L e^{ik(x'-x)} \phi_i(x') \phi_j(x) \, dx' \, dx \, d\omega$$

$$= \frac{2 \pi p^2}{Z_i(kc) Z_j^*(kc)} \int_0^L \int_0^L \cos k(x' - x) \phi_i(x') \phi_j(x) \, dx' \, dx$$

since $Z(kc) = Z(-kc)$, this quantity being an even function of its argument. The final expression for $\overline{y^2(x,t)}$ is then

$$\overline{y^2(x,t)} = p^2 \sum_i \sum_j \frac{\phi_i(x)\phi_j(x)}{Z_i(kc)Z_j^*(kc)} \int_0^L \int_0^L \cos k(x' - x)\phi_i(x')\phi_j(x)\,dx'\,dx$$

(9.70)

This concludes Example 3.

In summary, material in this section has outlined a procedure for the determination of the cross-correlation or mean square of the response of a continuous structure excited by a distributed random load. The task resolves into one of evaluating the cross-correlation function of the excitation between every pair of points chosen as stations on the structure. Since in most cases the cross-correlation function between two stations diminish with the distance between them, the numerical values of the cross power spectrum indicated by the array in Example (2) will decrease for elements as they deviate from the main diagonal. For the uncorrelated excitation only the diagonal terms of the array will have values other than zero; however cross products of the normal modes at each station cannot be avoided in the calculation. If however the power spectrum $S_f(x, x', \omega)$ of the excitation varies along x in proportion to the mass distribution, then Eq. (9.65) indicates that for the uncorrelated excitation the terms involving the cross products of the normal modes will be zero. A uniform beam or plate with constant power spectrum distribution in x would satisfy such a requirement.

9.3 MODIFICATION OF RESPONSE DUE TO LOADING

In the problem of establishing the response of a missile component attached to the missile structure, the local response of the missile structure itself, at the point where the component is to be attached, is often specified as the environment or excitation under which the component must operate. This assumption is valid when the mass of the component is small but unrealistic for sizable masses due to its loading effect. It is evident that a large mass attached to a vibrating structure will tend to force a node at the point of attachment. It is also known that a spring mass system attached to a vibrating body acts as a vibration absorber at its natural frequency which inhibits the motion of the vibrating body. This loading effect depends on the ratio of the mass m of the component to the local mass M of the structure which is affected, and the purpose of this section is to establish this effect quantitatively.

9.3.1 Harmonic Excitation — No Damping

For the analysis of the problem, it is desirable to consider the over-all system to be divided into two parts; the primary system which excludes the missile component in question, and the secondary system consisting of the component and its mounting or isolator, as illustrated in Figure 9.1. The primary system is represented as a distributed system of mass M, whereas the secondary system is represented by the attached spring mass oscillator of mass m and spring stiffness k. Damping will be left out in this first section and the excitation will be assumed to be harmonic at frequency ω. Later sections will consider damping and random excitation.

Letting $\phi_n(x)$ be the normal modes of the unloaded primary system, the response of the loaded primary system can be represented in terms of $\phi_n(x)$ as, see Eq. (9.38).

$$y(x, t) = \sum_n q_n(t) \, \phi_n(x) \tag{9.71}$$
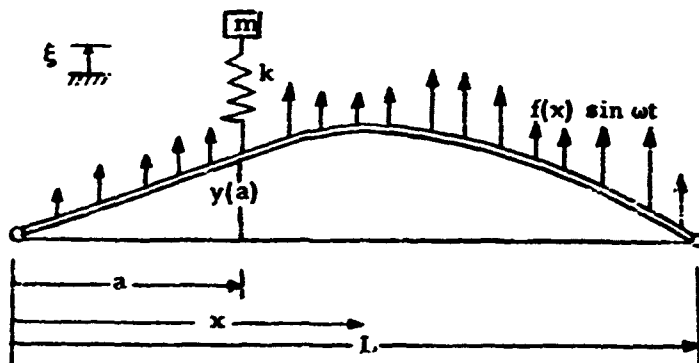
Figure 9.1. Schematic Diagram of the Primary Structure
and the Attached Component

where $q_n(t)$ is the generalized coordinate describing the time variation
of the nth mode of the loaded structure. With the component attached
at $x = a$, the generalized force due to the secondary system is $-m\ddot{\xi}\phi_n(a)$.
Assuming the exciting force to be $f(x) \sin \omega t$, the equation for the generalized coordinate of $q_n = q_n(t)$ becomes

$$\ddot{q}_n + \omega_n^2 q_n = \frac{\sin \omega t}{M} \int_0^L f(x)\, \varphi_n(x)\, dx - \frac{m}{M}\ddot{\xi}\, \phi_n(a) \qquad (9.72)$$

where:

$$M = \int_0^L \phi_n^2(x)\, dM = \text{generalized mass of the primary system}$$
which is normalized to the total mass M of the primary system.

Letting

$$K_n = \int_0^L f(x)\, \phi_n(x)\, dx \qquad (9.73)$$

and since $\ddot{\xi} = -\omega^2\xi$, and $\ddot{q}_n = -\omega^2 q_n$, the solution to the above equation is

$$q_n = \frac{K_n \sin \omega t}{M\omega_n^2 \left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]} + \frac{m}{M} \frac{\left(\frac{\omega}{\omega_n}\right)^2 \xi \, \phi_n(a)}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]} \tag{9.74}$$

The equation of motion for the secondary system is

$$m\ddot{\xi} + k\left[\xi - y(a, t)\right] = 0 \tag{9.75}$$

Thus, by letting $\omega_o^2 = \frac{k}{m}$, there results the relationship

$$\xi = \frac{y(a, t)}{1 - \left(\frac{\omega}{\omega_o}\right)^2} \tag{9.76}$$

Now, substitute Eq. (9.76) into (9.74) and Eq. (9.74) into (9.71) as follows,

$$y(a, t) = \sum_n \left\{ \frac{K_n \phi_n(a) \sin \omega t}{M\omega_n^2 \left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]} + \frac{m}{M} \frac{\left(\frac{\omega}{\omega_n}\right)^2 \phi_n^2(a) \, y(a, t)}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]\left[1 - \left(\frac{\omega}{\omega_o}\right)^2\right]} \right\} \tag{9.77}$$

Solving for $y(a, t)$ yields the result,

$$y(a, t) = \frac{\sin \omega t \sum_n \dfrac{K_n \phi_n(a)}{M\omega_n^2 \left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]}}{1 - \dfrac{\left(\frac{m}{M}\right)}{\left[1 - \left(\frac{\omega}{\omega_o}\right)^2\right]} \sum_n \dfrac{\left(\frac{\omega}{\omega_n}\right)^2 \phi_n^2(a)}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]}} \tag{9.78}$$

Note here that the response at $x = a$ with no secondary system attached (i. e., $m = 0$) is

$$y(a, t) = \sin \omega t \sum_n \frac{K_n \phi_n(a)}{M \omega_n^2 \left[1 - \left(\frac{\omega}{\omega_n} \right)^2 \right]} \qquad (9.79)$$

Thus the ratio of the response of the loaded to the unloaded system at $x = a$ is

$$R(a) = \frac{1}{1 - \frac{\left(\frac{m}{M} \right)}{\left[1 - \left(\frac{\omega}{\omega_o} \right)^2 \right]} \sum_n \frac{\left(\frac{\omega}{\omega_n} \right)^2 \phi_n^2(a)}{\left[1 - \left(\frac{\omega}{\omega_n} \right)^2 \right]}} \qquad (9.80)$$

Next, examine the amplitude ξ of the missile component, which from Eq. (9.76) and (9.78) is

$$\xi = \frac{1}{1 - \left(\frac{\omega}{\omega_o} \right)^2} \left\{ \frac{\sin \omega t \sum_n \frac{K_n \phi_n(a)}{M \omega_n^2 \left[1 - \left(\frac{\omega}{\omega_n} \right)^2 \right]}}{1 - \frac{\left(\frac{m}{M} \right)}{\left[1 - \left(\frac{\omega}{\omega_o} \right)^2 \right]} \sum_n \frac{\left(\frac{\omega}{\omega_n} \right)^2 \phi_n^2(a)}{\left[1 - \left(\frac{\omega}{\omega_n} \right)^2 \right]}} \right\} \qquad (9.81)$$

Here again one can compare the response ξ for $(m/M) \neq 0$ as compared to $(m/M) = 0$. Since $\left(\frac{\omega}{\omega_o} \right) = 0$ for $m = 0$, this ratio becomes equal to

$$\frac{R(a)}{1 - \left(\frac{\omega}{\omega_o} \right)^2}$$

where $R(a)$ is given by Equation (9.80).

9.3.2 Special Case of Rigid Attachment

If the secondary mass m is attached directly to the primary system, the case corresponds to $k = \infty$ or $\omega_o = \infty$. Equations (9.78) and (9.81) are then equal and become

$$\xi = \frac{\sin \omega t \sum_n \frac{K_n \phi_n(a)}{M \omega_n^2 \left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]}}{1 - \frac{m}{M} \sum_n \frac{\left(\frac{\omega}{\omega_n}\right)^2 \phi_n^2(a)}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]}} \tag{9.82}$$

To determine the resonant frequencies, the numerator and denominator of Eq. (9.82) are plotted in the top half of Figure 9.2, where the dotted curves are the numerator and the solid curves the denominator. It is evident that the resonant frequencies occur at points $\textcircled{1}$, $\textcircled{2}$, $\textcircled{3}$, etc. where the denominator goes to zero. Since the second branch $(n=2)$ of the denominator curve contributes very little to the first branch $(n=1)$ near $\textcircled{1}$, the first resonant frequency is given by

$$\left(\frac{\omega}{\omega_1}\right)^2 = \frac{1}{1 + \frac{m}{M} \phi_1^2(a)} \tag{9.83}$$

and the addition of the mass m to the structure lowers the quantity $(\omega/\omega_1)^2$ by the amount

$$1 - \left(\frac{\omega}{\omega_1}\right)^2 = \frac{\frac{m}{M} \phi_1^2(a)}{1 + \frac{m}{M} \phi_1^2(a)} \tag{9.84}$$

For the determination of the second resonance, it is noted that the contribution of the third branch is negligible and, since $\omega \gg \omega_1$, the denominator is approximately

Plot of Equation 9.82 ----branches of numerator
                        ——branches of denominator
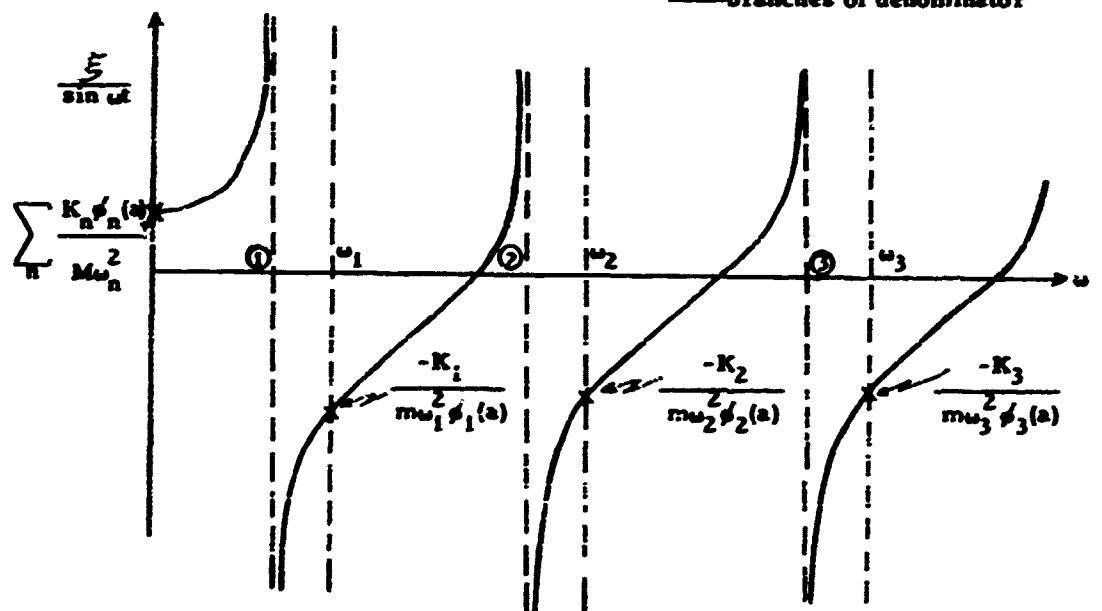
Resultant Plot of Equation 9.82

Figure 9.2 Resonance of Structure with Rigidly Attached Mass

$$1 + \frac{m}{M} \phi_1^2(a) - \frac{m}{M} \frac{\left(\frac{\omega}{\omega_2}\right)^2 \phi_2^2(a)}{1 - \left(\frac{\omega}{\omega_2}\right)^2} = 0 \qquad (9.85)$$
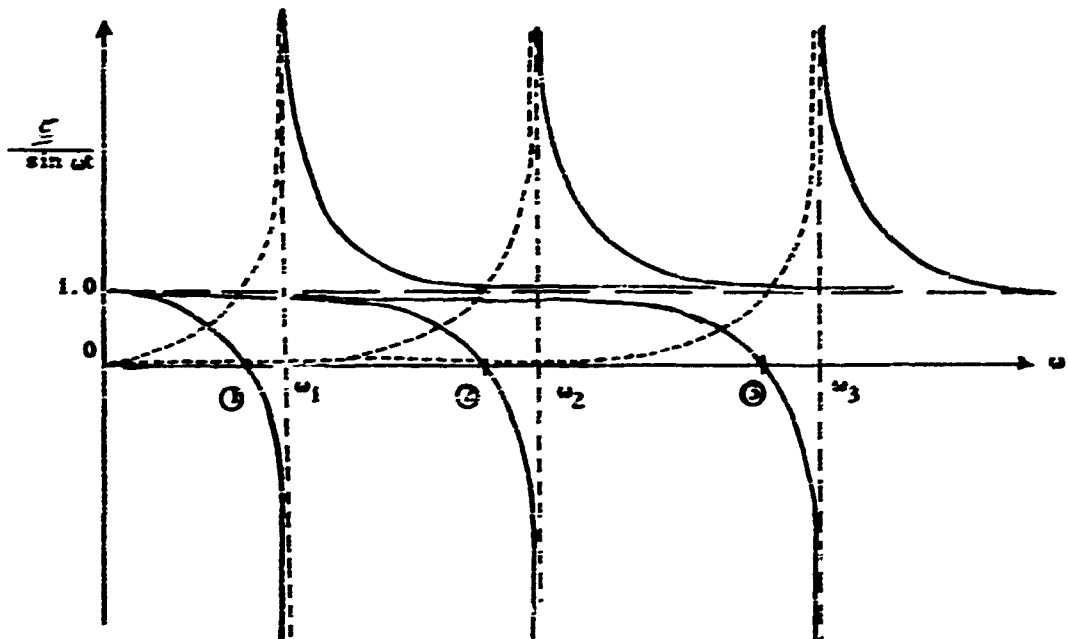
or

$$\left(\frac{\omega}{\omega_2}\right)^2 = \frac{1 + \frac{m}{M} \phi_1^2(a)}{1 + \frac{m}{M} \phi_1^2(a) + \frac{m}{M} \phi_2^2(a)} \qquad (9.86)$$

it follows that the quantity $(\omega/\omega_2)^2$ is lowered by

$$1 - \left(\frac{\omega}{\omega_2}\right)^2 = \frac{\frac{m}{M} \phi_2^2(a)}{1 + \frac{m}{M} \phi_1^2(a) + \frac{m}{M} \phi_2^2(a)} \qquad (9.87)$$

By induction, the nth resonance is given by

$$\left(\frac{\omega}{\omega_n}\right)^2 = \frac{1 + \frac{m}{M} \sum_{n=1}^{n-1} \phi_n^2(a)}{1 + \frac{m}{M} \sum_{n=1}^{n} \phi_n^2(a)} \qquad (9.88)$$

and the quantity $(\omega/\omega_n)^2$ is lowered by the amount

$$1 - \left(\frac{\omega}{\omega_n}\right)^2 = \frac{\frac{m}{M} \phi_n^2(a)}{1 + \frac{m}{M} \sum_{n=1}^{n} \phi_n^2(a)} \qquad (9.89)$$

At the frequencies ω_1, ω_2, ω_3 ... corresponding to the natural modes of the primary structure, both the numerator and denominator are infinite but their ratio is finite. It is only necessary to consider the two terms approaching infinity in the numerator and denominator which shows that the amplitudes at these frequencies are

$$\xi_{\omega=\omega_1} = \frac{-K_1 \sin \omega t}{m \omega_1^2 \phi_1 (a)} \left.\begin{array}{c}\\[4em]\\\end{array}\right\} \quad (9.100)$$

$$\xi_{\omega=\omega_2} = \frac{-K_2 \sin \omega t}{m \omega_2^2 \phi_2 (a)} \quad , \quad \text{etc.}$$

It is now possible to plot the results of the findings here as shown in the bottom half of Figure 9.2. Each normal mode frequency of the primary system is lowered due to the added mass, but the degrees of freedom remains unchanged with the same number of resonant frequencies.

9.3.3 Effect of Damping

In the first section, 9.3.1, damping has been neglected since its effect on the resonant frequencies of the system is of second order and therefore negligible. Damping is of importance, however, when peak response at resonance is of concern, and its effect will be considered in this section.

With a viscous damper between the primary and secondary system, Eq. (9.75) is changed to

$$m\ddot{\xi} + c_0 \left[\dot{\xi} - \dot{y}(a, t) \right] + k\left[\xi - y(a, t) \right] = 0 \qquad (9.101)$$

and in place of Eq. (9.76), by letting $\dot{\xi} = i\omega\xi$ and $\dot{y} = i\omega y$, one obtains

$$\xi = \frac{\left[1 + i 2\zeta_0 \dfrac{\omega}{\omega_0} \right]}{1 - \left(\dfrac{\omega}{\omega_0}\right)^2 + i 2\zeta_0 \left(\dfrac{\omega}{\omega_0}\right)} \; y(a, t) \qquad (9.102)$$

where $\zeta_c = \dfrac{c_0}{2\sqrt{km}}$ = fraction of critical damping.

The left side of Eq. (9.72) must now have an additional damping term $2\zeta_n \omega_n \dot{q}_n$ where ζ_n is the fraction of critical damping for mode n, so that the equation for q_n becomes

$$q_n = \frac{\dfrac{K_n \sin \omega t}{M \omega_n^2} + \dfrac{m}{M}\left(\dfrac{\omega}{\omega_n}\right)^2 \phi_n(a)\,\xi}{1 - \left(\dfrac{\omega}{\omega_n}\right)^2 + i2\zeta_n\left(\dfrac{\omega}{\omega_n}\right)} \tag{9.103}$$

Combining Equations (9.102), (9.103), and (9.71), and solving for y(a, t), Eq. (9.78) is replaced by

$$y(a,t) = \frac{\sin \omega t \displaystyle\sum_n \frac{K_n \phi_n(a)}{M\omega_n^2\left[1 - \left(\frac{\omega}{\omega_n}\right)^2 + i2\zeta_n\frac{\omega}{\omega_n}\right]}}{1 - \dfrac{\dfrac{m}{M}\left[1 + i2\zeta_0\left(\frac{\omega}{\omega_0}\right)\right]}{\left[1 - \left(\frac{\omega}{\omega_0}\right)^2 + i2\zeta_0\left(\frac{\omega}{\omega_0}\right)\right]}\displaystyle\sum_n \frac{\left(\frac{\omega}{\omega_n}\right)^2 \phi_n^2(a)}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2 + i2\zeta_n\left(\frac{\omega}{\omega_n}\right)\right]}} \tag{9.104}$$

and Eq. (9.81) by

$$\xi = \frac{1 + i2\zeta_0\left(\dfrac{\omega}{\omega_0}\right)}{1 - \left(\dfrac{\omega}{\omega_0}\right)^2 + i2\zeta_0\left(\dfrac{\omega}{\omega_0}\right)} \qquad \text{times Eq. (9.104)} \tag{9.105}$$

These equations are much too complicated to work with. However, one should take note of the fact that a single damper in the system will prevent infinite amplitudes. Thus one can attach a viscous damper c_0 to the isolator spring of the component and consider the primary system to be undamped.

9.3.4 Undamped Primary Structure with Damped Secondary Spring Mass

The component attached to the undamped primary structure is assumed to have a damping factor ζ_0 which will be held constant as m is varied. Of interest is the variation of the amplitude ξ in the neighborhood of resonance due to variiation of (m/M).

The equation for ξ from Equations (9.104) and (9.105), assuming $\zeta_n = 0$ for $n \neq 0$, is

$$\frac{\xi}{\sin \omega t} = \left\{ \frac{1 + i 2 \zeta_0 (\frac{\omega}{\omega_0})}{1 - (\frac{\omega}{\omega_0})^2 + i 2 \zeta_0 (\frac{\omega}{\omega_0})} \right\} \quad \text{times} \tag{9.106}$$

$$\left\{ \frac{\dfrac{K_1 \phi_1 (a)}{M\omega_1^2 \left[1 - (\frac{\omega}{\omega_1})^2\right]} + \dfrac{K_2 \phi_2 (a)}{M\omega_2^2 \left[1 - (\frac{\omega}{\omega_2})^2\right]} + \cdots}{1 - \dfrac{\frac{m}{M}\left[1 + i 2\zeta_c \frac{\omega}{\omega_0}\right]}{\left[1 - (\frac{\omega}{\omega_0})^2 + i 2\zeta_0 \frac{\omega}{\omega_0}\right]} \left[\dfrac{(\frac{\omega}{\omega_1})^2 \phi_1^2(a)}{1 - (\frac{\omega}{\omega_1})^2} + \dfrac{(\frac{\omega}{\omega_2})^2 \phi_2^2(a)}{1 - (\frac{\omega}{\omega_2})^2} + \cdots\right]} \right\}$$

It is of interest to examine this equation for two values of ω near the first resonant frequency.

At $\omega \approx \omega_1$ the terms containing the factor $\dfrac{1}{1 - (\frac{\omega}{\omega_1})^2}$ will dominate the expression which becomes

$$\frac{\xi}{\sin \omega t} = \frac{-K_1}{M\omega_1^2 (\frac{m}{M}) \phi_1 (a)} \tag{9.107}$$

Thus, the amplitude ξ decreases as (M/m). It is noted also that the result is independent of damping since the frequency ω_1 is off from resonance, and damping influences only the peak amplitude at resonance.

Consider next the peak response at the first resonant frequency of the loaded structure. Rewriting Eq. (9.106) as

$$\frac{\xi}{\sin \omega t} = \frac{\dfrac{K_1 \phi_1}{M\omega_1^2 \left[1 - \left(\frac{\omega}{\omega_1}\right)^2\right]} + \dfrac{K_2 \phi_2}{M\omega_2^2 \left[1 - \left(\frac{\omega}{\omega_2}\right)^2\right]} + \cdots}{\dfrac{1 - \left(\frac{\omega}{\omega_0}\right)^2 + i 2\zeta_0\left(\frac{\omega}{\omega_0}\right)}{1 + i 2\zeta_0\left(\frac{\omega}{\omega_0}\right)} - \dfrac{m}{M}\dfrac{\left(\frac{\omega}{\omega_1}\right)^2 \phi_1^2}{\left[1 - \left(\frac{\omega}{\omega_1}\right)^2\right]} - \dfrac{m}{M}\dfrac{\left(\frac{\omega}{\omega_2}\right)^2 \phi_2^2}{\left[1 - \left(\frac{\omega}{\omega_2}\right)^2\right]} \cdots}$$

$$(9.108)$$

$$\cong \frac{\dfrac{K_1 \phi_1}{M\omega_1^2 \left[1 - \left(\frac{\omega}{\omega_1}\right)^2\right]} + \dfrac{K_2 \phi_2}{M\omega_2^2 \left[1 - \left(\frac{\omega}{\omega_2}\right)^2\right]} + \cdots}{i 2\zeta_0\left(\frac{\omega}{\omega_0}\right) + \left\{1 - \left(\frac{\omega}{\omega_0}\right)^2 - \dfrac{m}{M}\dfrac{\left(\frac{\omega}{\omega_1}\right)^2 \phi_1^2}{\left[1 - \left(\frac{\omega}{\omega_1}\right)^2\right]} - \dfrac{m}{M}\dfrac{\left(\frac{\omega}{\omega_2}\right)^2 \phi_2^2}{\left[1 - \left(\frac{\omega}{\omega_2}\right)^2\right]} - \cdots\right\}}$$

the approximation coming from assuming $2\zeta_0(\omega/\omega_0) \ll 1$, one can now equate the real part of the denominator to zero for the peak response. Again, assuming $\omega_1^2 \ll \omega_2^2$ the real part of the denominator which is equated to zero becomes

$$1 - \left(\frac{\omega}{\omega_0}\right)^2 - \frac{m}{M}\frac{\left(\frac{\omega}{\omega_1}\right)^2 \phi_1^2}{\left[1 - \left(\frac{\omega}{\omega_1}\right)^2\right]} = 0$$

or

$$\left(1 + \frac{m}{M}\phi_1^2\right)\left(\frac{\omega}{\omega_1}\right)^2 - 1 + \left(\frac{\omega}{\omega_1}\right)^2 \left(\frac{\omega_1}{\omega_0}\right)^2 \left[1 - \left(\frac{\omega}{\omega_1}\right)^2\right] = 0 \qquad (9.109)$$

For the previous case where m is attached directly to the primary structure, $\omega_0 = \infty$, and

$$\left(\frac{\omega}{\omega_1}\right)^2 = \frac{1}{1 + \frac{m}{M}\phi_1^2} \qquad (9.110)$$

the same result as Eq. (9.83).

In general, from Eq. (9.109), as $(\omega_1/\omega_0)^2$ increases from zero, the frequency $(\omega/\omega_1)^2$ splits off to two frequencies of the quadratic equation in $(\omega/\omega_0)^2$ and the problem again becomes more complicated. However, for small values of $(\omega_1/\omega_0)^2$ one can assume the resonant frequency to differ only slightly from Eq. (9.110), and the peak response from Eq. (9.108) becomes

$$\left(\frac{\xi}{\sin \omega t}\right)_{peak} \cong \frac{K_1 \phi_1(a)}{M\omega_1^2 \dfrac{\dfrac{m}{M}\phi_1^2(a)}{\left[1 + \dfrac{m}{M}\phi_1^2(a)\right]}} \quad \frac{1}{\left|2\zeta_0\left(\dfrac{\omega_1}{\omega_0}\right)\right|}$$

$$= \frac{K_1 \phi_1(a)}{M\omega_1^2} \quad \frac{1}{\left|2\zeta_0\dfrac{\omega_1}{\omega_0}\right|} \quad \frac{\left[1 + \dfrac{m}{M}\phi_1^2(a)\right]}{\dfrac{m}{M}\phi_1^2(a)} \qquad (9.111)$$

It is noted now that the peak response as given by Eq. (9.111) is inversely proportional to the damping ζ_0 as it should be. Also the last factor indicates a reduction of the peak response with increase of (m/M) from zero. The infinite amplitude at $(m/M) = 0$ is explained by the fact that as m approaches zero the system reduces to the undamped primary structure.

9.3.5 Response to Random Excitation

For random excitation, one should replace the generalized force $\sin \omega t \int_0^L f(x) \phi_n(x) \, dx$ in Eq. (9.72) by $\int_0^L f(x, t) \phi_n(x) \, dx = f_n(t)$ where $f(x, t)$ is the random force. By taking the Fourier transform of the differential equations of the previous sections. Eq. (9.104) is replaced by the equation

$$Y(a, \omega) = \sum_n \frac{\dfrac{F_n(\omega)\, \phi_n(a)}{M \omega_n^2 \left[1 - \left(\dfrac{\omega}{\omega_n}\right)^2 + i\, 2\zeta_n \dfrac{\omega}{\omega_n} \right]}}{1 - \dfrac{\dfrac{m}{M}\left[1 + i\, 2\zeta_o \dfrac{\omega}{\omega_o} \right]}{\left[1 - \left(\dfrac{\omega}{\omega_o}\right)^2 + i\, 2\zeta_o \dfrac{\omega}{\omega_o} \right]} \dfrac{\left(\dfrac{\omega}{\omega_n}\right)^2 \phi_n^2(a)}{\left[1 - \left(\dfrac{\omega}{\omega_n}\right)^2 + i\, 2\zeta_n \dfrac{\omega}{\omega_n} \right]}}$$

$$= \sum_n \frac{F_n(\omega)}{A_n(\omega)} \tag{9.112}$$

where $Y(a, \omega)$ and $F_n(\omega)$ represent the Fourier transforms of $y(a, t)$ and $F_n(t)$, respectively. Then, from Eq. (9.105), the Fourier transform of ξ is

$$\xi(\omega) = \sum_n \frac{F_n(\omega)}{\underbrace{\dfrac{\left[1 - \left(\dfrac{\omega}{\omega_o}\right)^2 + i\, 2\zeta_o \dfrac{\omega}{\omega_o} \right]}{\left[1 + i\, 2\zeta_o \dfrac{\omega}{\omega_o} \right]} A_n(\omega)}} = \sum_n \frac{F_n(\omega)}{Z_n(\omega)} \tag{9.113}$$

where $A_n(\omega)$ and $Z_n(\omega)$ are defined by Equations (9.112) and (9.113).

Following the procedure outlined in Section 9.1.2, the mean square response for the secondary system can be shown to be given by

$$\overline{\xi^2} = \frac{1}{2\pi} \sum_i \sum_j \int_0^\infty \frac{\widetilde{F_{ij}(\omega)}}{Z_i(\omega)\, Z_j^*(\omega)}\, d\omega \tag{9.114}$$

where

$$\widetilde{F_{ij}(\omega)} = \lim_{T \to \infty} \frac{1}{T} \int_0^L \int_0^L K(u, \omega)\, F^*(u', \omega) \phi_i(u) \phi_j(u')\, du\, du' \tag{9.115}$$

$$F(\alpha, \omega) = \int_{0}^{\infty} e^{-i\omega t} f(\omega, t)\, dt \qquad (9.115)$$

As in Eq. (9.54), the factor $\widetilde{F_{ij}(\omega)}$ is called the generalized power spectrum of the excitation.

Example

Application of the equations of this section can be illustrated by considering a case where the random force is separable to the form

$$f(x, t) = w(x)\, p(t) \qquad (9.117)$$

The equation for $\widetilde{F_{ij}(\omega)}$ then becomes

$$\widetilde{F_{ij}(\omega)} = 2 \int_{0}^{L} \int_{0}^{L} w(x)\, w(x') \left\{ \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{T} e^{-i\omega(t-t')} p(t)\, p(t')\, dt\, dt' \right\} \phi_i(x)\, \phi_j(x')\, dx\, dx' \qquad (9.118)$$

Letting $t' = t + \tau$, the quantity $\left\{ \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{T} e^{-i\omega(t-t')} p(t)\, p(t')\, dt\, dt' \right\}$

becomes

$$\int_{-\infty}^{\infty} e^{i\omega t} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} p(t)\, p(t+\tau)\, dt\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{i\omega\tau}\, \overline{p(t)\, p(t+\tau)}\, d\tau \qquad (9.119)$$

where

$$\overline{p(t)\, p(t+\tau)} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} p(t)\, p(t+\tau)\, dt$$

is the time correlation function of the quantity $p(t)$.

In most cases the time correlation function drops off rapidly with τ. If one assumes

$$\overline{p(t)\,p(t+\tau)} = \delta(\tau) \tag{9.120}$$

where $\delta(\tau)$ is the usual unit impulse function, which means that the correlation is zero for all τ except $\tau = 0$, the last integral becomes

$$\int_{-\infty}^{\infty} e^{i\omega\tau}\,\delta(\tau)\,d\tau = 1 \tag{9.121}$$

and there occurs the special case of white noise. The generalized power spectrum $\widetilde{F_{ij}(\omega)}$ becomes here

$$\widetilde{F_{ij}(\omega)} = 2 \int_{0}^{L} \int_{0}^{L} w(u)\,w(u')\,\varphi_i(u)\,\varphi_j(u')\,du\,du' \tag{9.122}$$

which is independent of ω. The mean square response is now obtained from the equation

$$\overline{\xi^2} = \frac{1}{2\pi} \sum_i \sum_j \widetilde{F_{ij}(\omega)} \int_{0}^{\infty} \frac{d\omega}{Z_i(\omega)\,Z_j^*(\omega)} \tag{9.123}$$

where the factor $\widetilde{F_{ij}(\omega)}$ is a constant as regards the integral over ω. Since the resonant peak amplitudes from the previous analysis all show a decrease with increase of (m/M) [i. e., $1/Z_n(\omega)$ is reduced due to increase of (m/M)] the above integral is also reduced. Thus, the mean square response to random excitation under loading is a smaller number than it would be without loading.

9.4 VIBRATION INDUCED STRUCTURAL FATIGUE

One of the most important reasons for interest in the vibration response in modern flight vehicles is the concern over structural fatigue damage. Considerable work has been done over the last 15 years to correlate the vibration response of a structure with fatigue life. However, most of the efforts have been no more than an extension of procedures suggested by Palmgren in 1924 followed independently by Miner in 1945 (Ref. [15]). The basic hypothesis, often called Miner's Rule, may be stated as follows: Assume a structure is subjected to a repetitive load resulting in a maximum stress level of S_1, and the number of cycles of the repetitive load S_1, required to produce fatigue failure is N_1. Then if the structure is subjected to a repetitive load S_1 for n_1 cycles, it is stated that a fraction of the total fatigue life of the structure equal to n_1/N_1 will be consumed or used up. If the structure were then subjected to a repetitive load of S_2 for n_2 cycles, where N_2 cycles would produce failure, an additional fraction of the total fatigue life equal to n_2/N_2 will be consumed. The consumption of fatigue life in this manner is considered cumulative such that

$$\sum_i \frac{n_i}{N_i} = 1 \tag{9.124}$$

where the term on the left is sometimes called the cycle ratio.

Considerable experimental data has been gathered in attempts to substantiate Miner's Rule. Ref. [4] presents a good summary of such experiments. Considerable scatter exists in the experimental data with cycle ratios varying as much as one order of magnitude from the cycle ratio of 1 suggested by Miner. However, the vast majority of experiments did produce cycle ratios between 0.5 and 2.0.

From Ref. [4] (Chapter 6), Brooks reported a series of studies from which he found that one could have 75 percent confidence that 90 percent or more of the specimens tested would have a cycle ratio between 0.4 and 1.8. Similar results were reported by Smith, Howard and Smith who tested 600 specimens. They found the order of application of the high and low stress levels was not important. On the other

hand, many other investigators believed the order of application was influential on the resulting cycle ratio.

An area of concern to some investigators is the fact that Miner's Rule assumes linear accumulation of damage, i.e., damage accumulation for repetitive loads with a maximum stress level S_1 is directly proportional to the number of stress cycles n_1, applied. As a result, a number of more complex empirical correlations have been presented as typified by Refs. [3] and [12], but there appears to be considerable question as to whether or not they have enough more validity over Miner's Rule to be practical.

Thus far, only fatigue damage for steady state loads has been discussed. Miner's Rule may be readily extended for non-steady state loading if certain statistics of the response are known. A simple development is presented in Ref. [4] as follows:

> Let $f(S)dS$ denote the probability that a given stress cycle will have an amplitude between S and $S + dS$. Then the number of cycles dn at a stress level between $S + dS$ will be given in terms of the total number of cycles N_t by

$$dn = N_t \, f(S) \, dS \tag{9.125}$$

Miner's Rule now becomes

$$\int_{S_{(min)}}^{S_{(max)}} \frac{N_t f(S) \, dS}{N(S)} = 1 \tag{9.126}$$

The above equation requires that the probability density function of peak amplitudes $f(x)$ be known. It also requires that the number of cycles to failure as a function of stress level be known. The S-N curve for many materials can be approximated by a straight line on log-log paper. In this case,

$$N(S) = N_1 (S_1/S)^b \tag{9.127}$$

where N_i is the number of cycles to failure at a stress level of S_i and b is a slope constant. The previous equation is then simplified as follows:

$$N_i \int_{S_{(min)}}^{S_{(max)}} \frac{f(S)}{N_1} \left(\frac{S}{S_i}\right)^b \, dS = 1 \qquad (9.128)$$

Consider the meaning of the above statement of Miner's Rule in terms of the response of a typical flight vehicle structural panel. Vehicle structure is usually very lightly damped (damping coefficient less than 3 percent). Then for any broad band random input spectrum, the spectral power of the structural response will be concentrated in narrow frequency bands centered at the normal modes of the structure. Correspondingly, the peak stress levels of concern will be limited to the frequencies of the normal modes. In a paper by Miles, Ref. [14], it is pointed out that when a random function is filtered by a narrow band, the response becomes more normal as the bandwidth becomes narrower even if the input does not have a normal probability distribution. If the response through a narrow band has a normal distribution of instantaneous amplitudes, it will have a Rayleigh distribution of peak amplitudes. The rms value of the response would, of course, be a function of the input power spectrum and the damping coefficient of the panel. When the response function is measured, a knowledge of the source and panel characteristics is not necessary. Then for a single degree of freedom linear structure with a natural frequency of f_0 and a mean square stress response of ψ, the distribution of peak amplitudes, $f(S)$, may be replaced by the Rayleigh distribution. Miner's Rule is then extended to the following:

$$T f_0 \int_0^\infty \frac{S e^{-(S^2/2\psi)}}{\psi} \left(\frac{S}{S_0}\right)^b \, dS = 1 \qquad (9.129)$$

where T is total time to failure and S_0 is the fictitious stress level for which one cycle would cause failure. The above general equation, as modified and extended by specific investigators, is widely used to predict fatigue life of structures. In particular, a more general Rayleigh-Gaussian distribution for peak amplitudes is presented in Section 4.9.3 which might be substituted into Eq. (9.128).

Miles in Ref. [14] presents a detailed development of the cumulative fatigue damage for a single degree of freedom panel starting with a jet noise source using a generalized nonlinear cumulative damage correlation. A pertinent conclusion of the paper is that nonlinear damage correlations do not produce results significantly different from those obtained using the simple linear correlation of Miner.

The general procedures for fatigue life prediction discussed thus far were developed for a single degree of freedom system. For the real case of an elastic structure subjected to broad band random excitation, the response will reflect the participation of many modes and will not be a single degree of freedom response. There is considerable question as to whether the Rayleigh Distribution is an acceptable definition of the distribution of peak stress amplitudes for a complex multi-mode response. Experimental analysis of some missile vibration data by Douglas Aircraft, Ref. [7], indeed indicated that the distribution of peak acceleration levels in missiles deviated rather widely from a Rayleigh Distribution.

One concluding point should be noted. Many metals possess the characteristic of having S-N curves which are a straight line on log-log paper only above some specific stress level. Below this stress level, the S-N curves become nearly flat; i.e., an extremely large number of load cycles approaching infinity are required to produce failure. This stress level at which the slope of the S-N curve changes is called the endurance limit. Steel in particular has a very distinct endurance limit. Nonlinear cumulative damage correlations have been derived which take the endurance limit into account. However, Miles and others believe that if the rms stress response is high compared with the endurance limit, it may be ignored with little error.

9.5 EFFECT OF NONLINEARITIES ON RESPONSE STATISTICS

Nonlinearities in the transfer characteristics of transducers and data processing instrumentation will influence the statistics of structural vibration response measurements. Of course, the nonlinear characteristics of the structure itself will influence the statistics of the actual response as well. An ability to recognize the effects of such nonlinearities in vibration response data would be valuable to the data processing engineer.

When vibration response data is tested for normality (a Gaussian amplitude density), and that data deviates from normality, three primary possibilities exist.

1. The source of excitation was not a stochastic process with a Gaussian probability density.

2. The structural transfer function was nonlinear.

3. The amplitude transfer characteristics of the transducer or data processing instrumentation were nonlinear.

Little can be done about the first two possibilities, but something might be done about the third possibility if there were reason to believe that the nonlinear transfer characteristics of instruments were distorting the measured response statistics.

Assuming that the input source is Gaussian and based upon the limited information available, it appears there is at least one case where an instrumentation nonlinear effect could be distinguished from structural nonlinear effects by examination of the acceleration response probability density. Clipping of the response signal by an electronic device can cause distinct clustering on the tails of the amplitude probability density of the response. Since common structural nonlinearities would probably not be sufficiently abrupt to produce this result, the presence of such clustering would point to an instrumentation problem.

Another possible area of distinction would be an acceleration response probability density which displayed large kurtosis (a density function with thick tails and sharp clustering about the origin). The results of an experiment on a single degree of freedom system with a

nonlinear hardening spring indicated such an effect. Since the common amplitude nonlinearities associated with instruments do not produce anything resembling large kurtosis, the presence of such an effect in an acceleration response probability density curve may point directly to a structural nonlinearity.

9.5.1 Nonlinear Transfer Characteristics of Instruments

Consider first the case of nonlinear amplitude transfer characteristics (nonlinear gain) which might be associated with transducers and data processing instrumentation. Included in this category is the problem of limited dynamic range. When a random signal is passed through an amplitude transfer characteristic which displays amplitude linearity, the probability density function of the signal will not be changed. If however an amplitude transfer characteristic is not linear, the probability density function of the signal will be altered by the transfer characteristic. If the probability density function of the signal is altered, the mean value and the mean square value (and other statistics) of the signal will probably be changed also. The effect of nonlinear amplitude transfer characteristics on random signals is briefly discussed with illustrations by T. P. Rona in Chapter 7, Ref. [4]. The discussion given there of the effect of amplitude nonlinearities on random signals requires careful interpretation on the part of the reader. Because limited dynamic range is such an important problem in the measurement and reduction of random vibration data, a critical discussion of the effects of limited dynamic range on a random input signal is presented in Section 9.5.4 herein. A related general treatment of the effects of nonlinear transfer characteristics on the statistics of random signals is available from Ref. [5], Chapters 12 and 13.

As discussed in Section 9.5.4, the term "limited dynamic range" might be interpreted in two ways. The interpretation implied in Ref. [4] is a transfer characteristic (gain function) with a gain of one for amplitudes with absolute values below some specified level, and a gain of zero for amplitudes greater than that specified level. In other words, an instantaneous input amplitude with a value outside the dynamic range limits would result in an instantaneous output amplitude of zero.

Such an interpretation of limited dynamic range has little physical significance. In reality, limited dynamic range usually implies a transfer characteristic which limits amplitudes with absolute values above some specified level to that specified level. This second interpretation is more representative of such phenomena as clipping, magnetic saturation, and similar physical limitations on amplitude linearity.

The specific effect of both interpretations of limited dynamic range on a random signal with a uniform input probability density function is developed here in Section 9.5.4. The delta functions in the output probability density functions shown in Fig. 9.3 are, of course, the result of an idealized dynamic range limitation. In reality, physical limitations on signal amplitudes would not occur so abruptly and the delta functions would actually appear as clusters on the output probability density function. The height and width of the clusters would be dependent upon how sharply the transfer device would limit amplitudes, but the areas under the clusters would be approximately the same as the areas of the delta functions shown in Fig. 9.3. Note that the rms value of the signal is reduced by the transfer characteristic for either interpretation, but not to the same value. Clipping will result in the higher rms output. Also note that for the example considered, the mean value of the signal is unaltered. This is true only because the signal probability density function and transfer characteristics selected for the example were considered symmetrical about zero for simplicity. In general, lack of symmetry in the input probability density and/or transfer characteristic will produce a shift in the mean value of the output.

Presented in Fig. 9.4 are qualitative illustrations of the effects of common nonlinear amplitude transfer characteristics on a random signal with a Gaussian input probability function. Case (1) demonstrates the clustering on the tails of the amplitude density caused by symmetrically limited dynamic range. Case (2) illustrates the effect of a nonlinear transfer characteristic in the form of a gain that falls off gradually with amplitude. The general effect is clustering about the origin. For both Cases (1) and (2), the rms value of the signal is reduced but the mean value is not altered. Case (3) demonstrates the effect of an asymmetrical nonlinear transfer characteristic such as might be representative of an

asymmetrical transducer. For this case, not only is the rms value of the signal reduced, but the mean value also is altered.

Of the above three types of nonlinearities, limited dynamic range as represented by Case (1) constitutes the most common problem in vibration measurement work. Cases (2) and (3) will usually not be a problem if high quality transducers and instrumentation are used. The majority of vibration response data being gathered today are obtained using piezoelectric crystal acceleration transducers. Commercial crystal accelerometers have a dynamic range of up to $\pm 10,000$ g's with a maximum amplitude nonlinearity of ± 1 percent over the entire amplitude range. For normal vibration measurements, amplitude non-linearity in crystal type transducers may be considered negligible. The associated instrumentation including data transmission and processing equipment will usually display equally good linearity within the dynamic range limits of the instruments. However, even with the highest quality instrumentation, a test engineer may unduly limit the dynamic range of random signal measurements by improper use of the measurement equipment. A common example is the improper use of instruments which were designed primarily for harmonic signal work.

Many instruments used in association with vibration measurement and analysis are equipped with input attenuators and signal level meters which permit input signals levels to be appropriately adjusted for the dynamic range limits of the particular instrument. To keep the signal to noise ratio as high as possible, it is not uncommon for the test engineer to adjust the input attenuators so that signal levels are near the upper limits of the instrument's dynamic range. If the input signal level meter is a conventional average sensing voltmeter calibrated in rms for sine waves, the maximum input level to the instrument as defined on the meter would probably correspond to that voltage level where the instrument begins to clip sine waves. In other words, the instrument would clip voltage levels of about 1.4 times the maximum rms voltage input as defined on the meter. Consider the case of input signals which are random and have a Gaussian amplitude probability density. The rms value of a Gaussian signal (1σ) would be $(2/\sqrt{\pi})$ or approximately 1.3 times the reading of an average sensing voltmeter calibrated in rms for sine waves, Ref. [9]. Then if a Gaussian input

signal was adjusted to the maximum indicated input level for such an instrument, clipping would occur for signal amplitudes above only 1.25σ.

Sometimes the problem of limited dynamic range is unavoidable as in the case of the engineer who has no control over sensitivity or gain settings of the instruments during the actual measurements. For missile vibration measurements in particular, unless the test engineer has some prior knowledge of the environment, he often must estimate what the vibration levels will be and pre-set the gain of the measurement instruments accordingly. If the estimate of the environment is poor, the data may be obscured by background noise or distorted by clipping. The problem is further aggravated when the desired vibration response measurements are to cover several different flight phases characterized by different sources of excitation producing widely varying response levels.

9.5.2 Nonlinear Transfer Characteristics of Structures

Separate from the problem of nonlinear amplitude transfer characteristics of instruments is the problem associated with the response of structures having nonlinear transfer functions. Real elastic structures such as aircraft panels may demonstrate the nonlinear characteristics of a hardening spring (a spring whose stiffness increases with amplitude). For large bending deflection of a flat plate, the small slope assumption used to linearize the plate flexure problem becomes invalid, and the plate deflections are less than would be predicted by linear plate deflection formulae; i.e., the plate becomes stiffer as bending deflections increase, Ref. [21]. Furthermore, when clamped edge conditions exist, the flexure of the plate is not independent of tension loads for large deflections, and these tension loads also produce an effective stiffness to flexure which increases with deflection. The analytical development of the nonlinear spring force problem for continuous structures is rather involved, but the related although simpler problem of the response of a nonlinear string is treated in Refs. [11] and [17] for those interested in details. In general, the frequency response curve for a mechanical system with the nonlinear characteristics of a hardening spring is typified by the well known "jump phenomena", Ref. [18].

Another nonlinear parameter of real elastic structures is damping. In elementary vibration theory, the damping forces are assumed to be a function of velocity and only velocity (viscous damping) so that linear equations of motion will evolve. In real elastic structures, the damping forces are not linear. In complex structures where riveted joints are included, the dissipation of energy through damping is primarily due to relative motion in the joints. In the case of a simple continuous structure, the damping results from internal friction and is often called solid or hysteresis damping. The actual mechanics of solid damping is not clearly defined although it has been the object of considerable research. The extent of investigations in the field of solid damping is readily seen from Ref. [6] which reviews almost 900 technical papers published on this subject. In general, it can be said that the energy dissipated per cycle by solid damping is independent of frequency and proportional to the second power of the displacement amplitude.

The topic of nonlinear structural response to stochastic forces does not readily lend itself to analytical treatment. The most productive investigations of this subject have been empirical in nature. Of particular interest is an analog computer study of the response of nonlinear mechanical systems to random inputs by McIntosh, Ref. [13].

Three types of nonlinearities in a single degree of freedom system were studied by McIntosh:

1. Combined linear and displacement cubed spring force (hardening spring)

2. Combined linear and displacement cubed damping force (hysteresis damping)

3. Combined linear and velocity-squared damping force (air damping)

Each of the three nonlinear models was subjected to a random input acceleration with a Gaussian probability density function and a wide band (as compared to the computer response) uniform spectral density, and the response of the mass was measured. The results are presented as plots of the probability of exceeding ratios of output acceleration to rms output acceleration for various degrees of nonlinearity. Plots of relative

rms response versus degree of nonlinearity are also shown. The conclusions implied by the graphical results may be generalized as follows:

1. For the nonlinear hardening spring, as the degree of nonlinearity is increased:

a) The rms acceleration of the response increases.

b) The probability of exceeding acceleration ratios of greater than 1.6 increases, while the probability of exceeding acceleration ratios of less than 1.6 decreases.

2. For the nonlinear hysteresis damper, as the degree of nonlinear damping is increased;

a) The rms acceleration of the response decreases.

b) The probability of exceeding any given acceleration ratio increases.

3. For the nonlinear air damper, as the degree of nonlinear damping is increased;

a) The rms acceleration of the response decreases.

b) The probability of exceeding any given acceleration ratio decreases.

The conclusions under a) are clear and quite meaningful. The conclusions under b), however, require further investigation. The "probability of exceeding ratio" plots are based upon ratios of acceleration to the rms accelerations for each condition or degree of nonlinearity. Since each degree of nonlinearity produced a different response rms acceleration level, the scale on the plots does not have a common denominator for all curves and thus has no significance in terms of actual acceleration levels.

For example, Fig. 17 in Ref. [13], indicates that the probability of accelerations exceeding the rms value for a linear system is about 0.32, while the probability of accelerations exceeding the rms value for a specific degree of spring force nonlinearity (identified as $b = 1.0$) is about 0.22. However, Fig. 18 shows that the rms acceleration

response for that specific degree of nonlinearity is 1.6 times the rms response for the linear case. Then the probability of the acceleration response levels exceeding some actual acceleration level, say the rms g's for the linear response, is about 0.32 for both the linear and the nonlinear case. In future studies, the data should be presented in the form of probability density curves on a common abscissa so that more explicit statistical conclusions could be drawn. Nevertheless, the report includes pertinent information if the results are interpreted with caution.

Based upon a brief examination and without attempting a detailed analysis, it is believed that the data in Ref. [13] suggests the effect of the nonlinearities discussed on the response acceleration probability density would be qualitatively as illustrated below.

_____ probability density of acceleration response without
nonlinearity (Gaussian probability density function)

- - - - - - - - - - - - probability density of acceleration response with
nonlinearity as noted

1. hardening spring   2. hysteresis damping   3. air damping
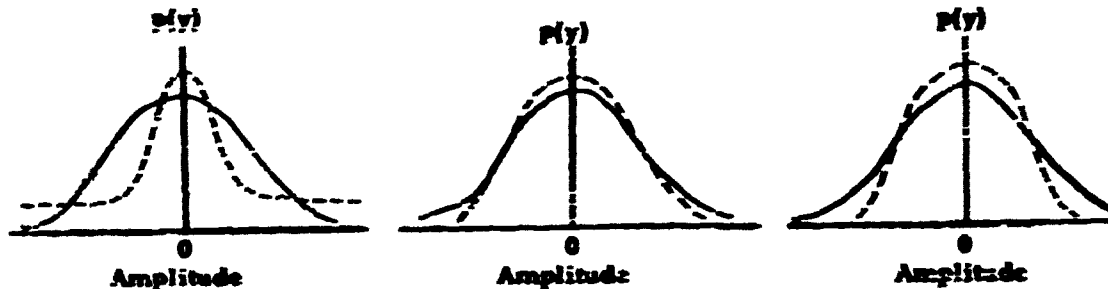
Note that the effect of nonlinear damping of the two types considered for a single degree of freedom system appears to be quite similar to the effect of amplitude nonlinearity in instruments of the type illustrated by Case (2) of Fig. 9.4. The effect shown for the hardening spring is sometimes called large kurtosis. It should also be

mentioned that the results of Ref. [13] apply only to
acceleration response probabilities and cannot be in-
terpreted in terms of other parameters such as
displacement.

9.5.3 Mathematical Derivations

Consider the general case of a stationary random signal with an
amplitude probability density function $p_i(y)$ when passed through an
amplitude transfer characteristic $G(y)$ with limited dynamic range.

$$p_i(y)$$

$$\bar{y}_1 \quad 0 \quad y_2 \quad y$$

$$G(y)$$

$$\frac{y_1}{y} \quad (b) \quad \frac{y_2}{y}$$

$$y_1 \quad 0 \quad y_2 \quad y$$
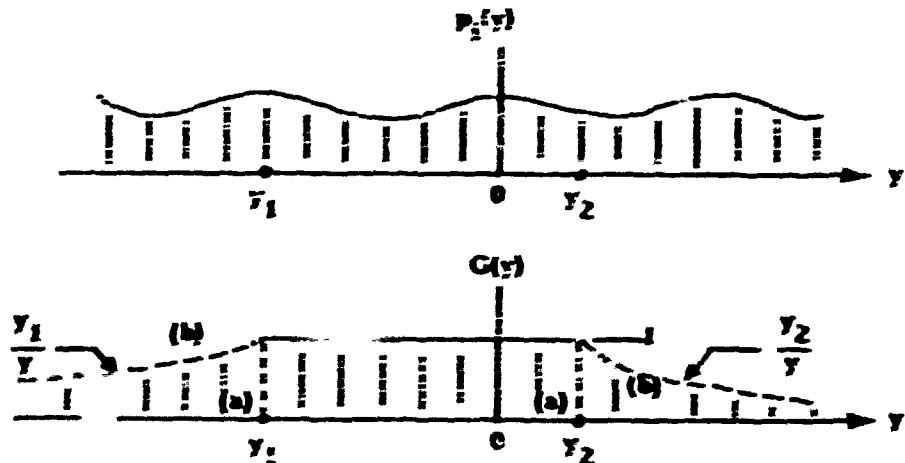
Hypothetically, two interpretations of limited dynamic range are
possible. The amplitude transfer characteristic $G(y)$ may be such that
amplitudes of y less than y_1 or greater than y_2 are:

(a) excluded (equal to zero)

(b) limited to values of y_1 or y_2

Case (a) would occur in a device which "open-circuits" when input signals
are above a level y_2 or below a level y_1.

Case (b) would occur in a device which clips or limits when input signals
are above a level y_2 or below a level y_1.

The transfer characteristic $G(y)$ associated with each of the two
interpretations for limited dynamic range may be thought of graphically
as shown above. The vertical dashed lines labeled (a) complete the

transfer characteristic for case (a) and the dashed lines labeled (b) complete the transfer characteristic for case (b). The two transfer characteristics may also be presented in the form of input-output plots as shown below.
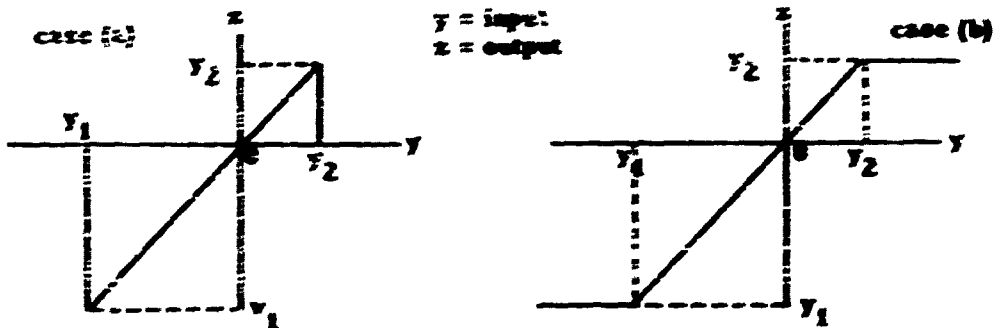
Consider the probability density function of the output $p_0(z)$ for each of the two transfer characteristics discussed above.

Case (a)

The output amplitudes, $z = y G(y) = \begin{cases} y & \text{for } Y_1 \leq y \leq Y_2 \\ 0 & \text{for } y < Y_1 \\ 0 & \text{for } Y_2 < y \end{cases}$ (9.130)

The output probability density function $p_0(z)$ would then be as follows:

$$p_0(z) = \begin{cases} p_i(y) & \text{for } Y_1 \leq z \leq Y_2 \\ A\delta(z) & \text{for } z = 0 \\ 0 & \text{for } Y_2 < z < Y_1 \end{cases}$$ (9.131)

where A is the area of the delta function at $z = 0$ and must have a value such that:

$$\int_{-\infty}^{\infty} p_0(z)dz = 1 = \int_{Y_1}^{Y_2} p_i(y)dy + A$$

Hence $A = 1 - \int_{Y_1}^{Y_2} p_i(y)dy$ (9.132)

Case (b)

The output amplitudes, $z = yG(y) = \begin{cases} y \text{ for } y_1 \leq y \leq y_2 \\ y_1 \text{ for } y < y_1 \\ y_2 \text{ for } y_2 < y \end{cases}$ (9.133)

The output probability density function $p_o(z)$ would then be as follows:

$$p_o(z) = \begin{cases} p_i(y) \text{ for } y_1 < z < y_2 \\ A_1 \delta(z-y_1) \text{ for } z = y_1 \\ A_2 \delta(z-y_2) \text{ for } z = y_2 \\ 0 \text{ for } y_2 < z < y_1 \end{cases}$$ (9.134)

where A_1 and A_2 are the areas of the delta functions at $z = y_1$ and $z = y_2$ and must have values such that;

$$\int_{-\infty}^{\infty} p_o(z)dz = 1 = \int_{y_1}^{y_2} p_i(y)dy + A_1 + A_2$$

Specifically, $A_1 = \int_{-\infty}^{y_1} p_i(y)dy$ and $A_2 = \int_{y_2}^{\infty} p_i(y)dy$ (9.135)

To further investigate the effect of limited dynamic range on a random signal, specialize the development for a uniform probability density function with a mean value of zero. For simplicity, consider the transfer characteristics to also be symmetrical about zero as illustrated in Fig. 9.3.

Case (1) of Figure 9.3

The area of the delta function A at $z = 0$ is as follows:

$$A = 1 - \int_{-b}^{b} \frac{1}{2a} dy = 1 - \frac{2b}{2a} = \frac{a-b}{a}$$ (9.136)

Then the output probability density function is as follows:

$$p_o(z) = \begin{cases} \dfrac{1}{2a} & \text{for } |z| < b \\[2mm] \dfrac{(a-b)}{a}\delta(z) & \text{for } z = 0 \\[2mm] 0 & \text{for } b < |z| \end{cases} \quad \text{(see Fig. 9.3)} \qquad (9.137)$$

Now consider the mean value μ and the root mean square value rms of the output probability density as compared to the input probability density for case (1). The mean value is equal to the first moment of the density function.

The mean value of the input, $\mu_i = \displaystyle\int_{-\infty}^{\infty} y p_i(y) dy = \int_{-a}^{a} \dfrac{y}{2a} dy = 0$ \qquad (9.138)

The mean value of the output, $\mu_o = \displaystyle\int_{-\infty}^{\infty} z p_o(z) dz = \int_{-b}^{b} z\left[\dfrac{1}{2a} + \dfrac{(a-b)}{a}\delta(z)\right] dz = 0$

Thus, for the simple example considered where the probability density function and the transfer characteristic are symmetrical about the origin, the mean value is not changed.

The rms value is equal to the standard deviation or the square root of the variance. Since the mean value is equal to zero, the variance is equal to the second moment of the density function about the origin.

The variance of the input, $\sigma_i^2 = \displaystyle\int_{-\infty}^{\infty} y^2 p_i(y) dy = \int_{-a}^{a} \dfrac{y^2}{2a} dy = \dfrac{a^2}{3}$

The variance of the output, $\sigma_o^2 = \displaystyle\int_{-\infty}^{\infty} z^2 p_o(z) dz = \int_{-b}^{b} z^2\left[\dfrac{1}{2a} + \dfrac{(a-b)}{a}\delta(z)\right] dz = \dfrac{b^3}{3a}$

Then $\qquad \sigma_i = \text{rms}_i = \dfrac{a}{\sqrt{3}} \quad \text{and} \quad \sigma_o = \text{rms}_o = \dfrac{b^{3/2}}{\sqrt{3a}}$ \qquad (9.139)

For the example considered, the rms value of the signal is reduced by the case (1) limited dynamic range as follows:

$$rms_o = \left[\frac{b}{a}\right]^{3/2} rms_i \qquad (9.140)$$

Case (2) of Figure 9.3

The areas of the delta functions A_1 at $z = -b$ and A_2 at $z = b$ are equal in value as follows:

$$A_1 = A_2 = \int_b^a \frac{1}{2a} dy = \frac{(a-b)}{2a} \qquad (9.141)$$

Then the output probability density function is as follows:

$$p_o(z) = \begin{cases} \frac{1}{2a} & \text{for } |z| < b \\ \frac{(a-b)}{2a} \delta(z \pm b) & \text{for } |z| = b \\ 0 & \text{for } b < |z| \end{cases} \quad \text{(see Fig. 9.3)} \qquad (9.142)$$

The mean value and root mean square value of the input probability density function are the same as in case (1).

The mean value of the output, $\mu_o = \int_{-\infty}^{\infty} z p_o(z) dz = \int_{-b}^{b} z \left[\frac{1}{2a} + \frac{(a-b)}{2a} \delta(z \pm b)\right] dz = 0$

$$(9.143)$$

Thus, for case (2) as in case (1) the mean value is not changed due to the symmetry of the example considered.

The variance of the output, $\sigma_o^2 = \int_{-\infty}^{\infty} z^2 p_o(z) dz = \int_{-b}^{b} z^2 \left[\frac{1}{2a} + \frac{(a-b)}{2a} \delta(z \pm b)\right] dz$

$$= \frac{b^3}{3a} + \frac{b^2(a-b)}{a} = \frac{b^2}{a}\left(a - \frac{2b}{3}\right)$$

Then $\quad \sigma_0^- = \text{rms}_0 = \left[\dfrac{b^2}{a}\left(a - \dfrac{2b}{3}\right)\right]^{1/2}$ (9.144)

For the example considered, the rms value of the signal is reduced by the case (2) limited dynamic range as follows:

$$\text{rms}_0 = \left[\dfrac{b^2(3a-2b)}{a^3}\right]^{1/2} \text{rms}_i \qquad (9.145)$$

Figures 9.3 and 9.4, discussed previously in Section 9.5.1, and in the above mathematical derivation, appear in the next two pages.

INPUT: A Uniform Probability Density Function With a
                          Mean Value of Zero.

AMPLITUDE TRANSFER
CHARACTERISTIC: A Gain of One With Limited Dynamic Range of
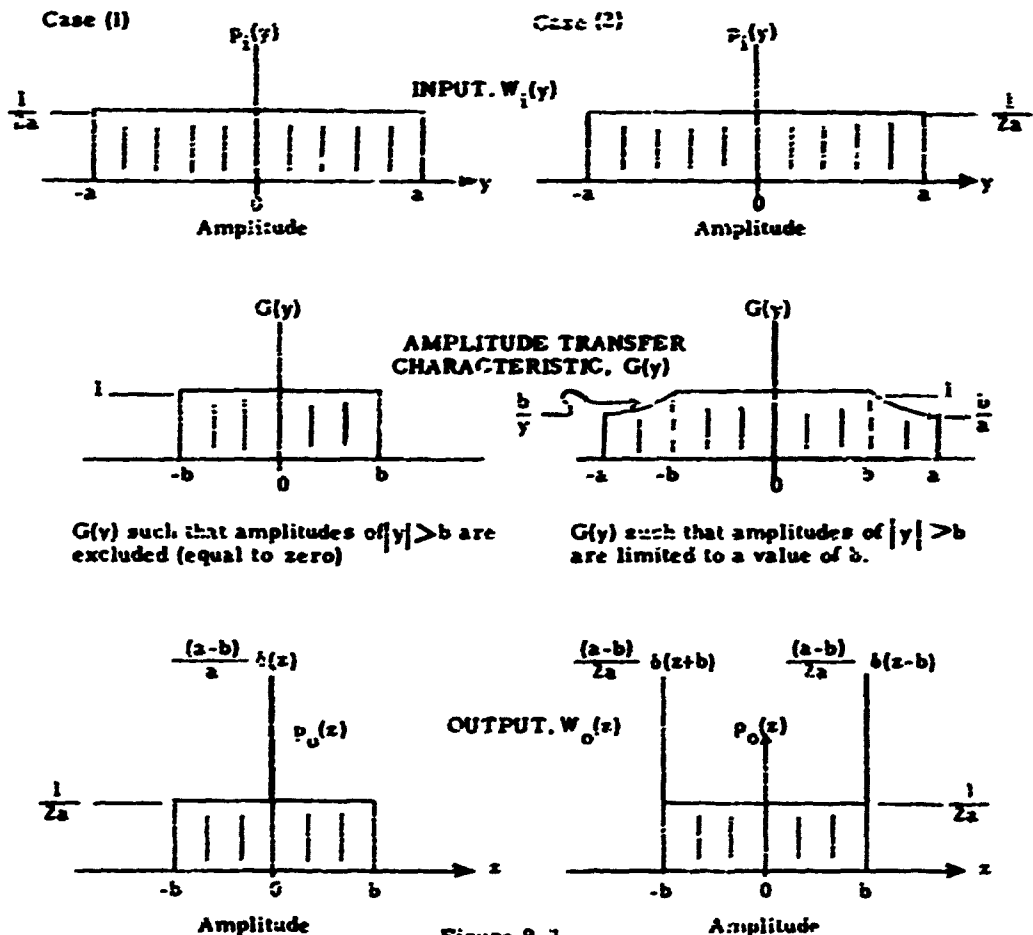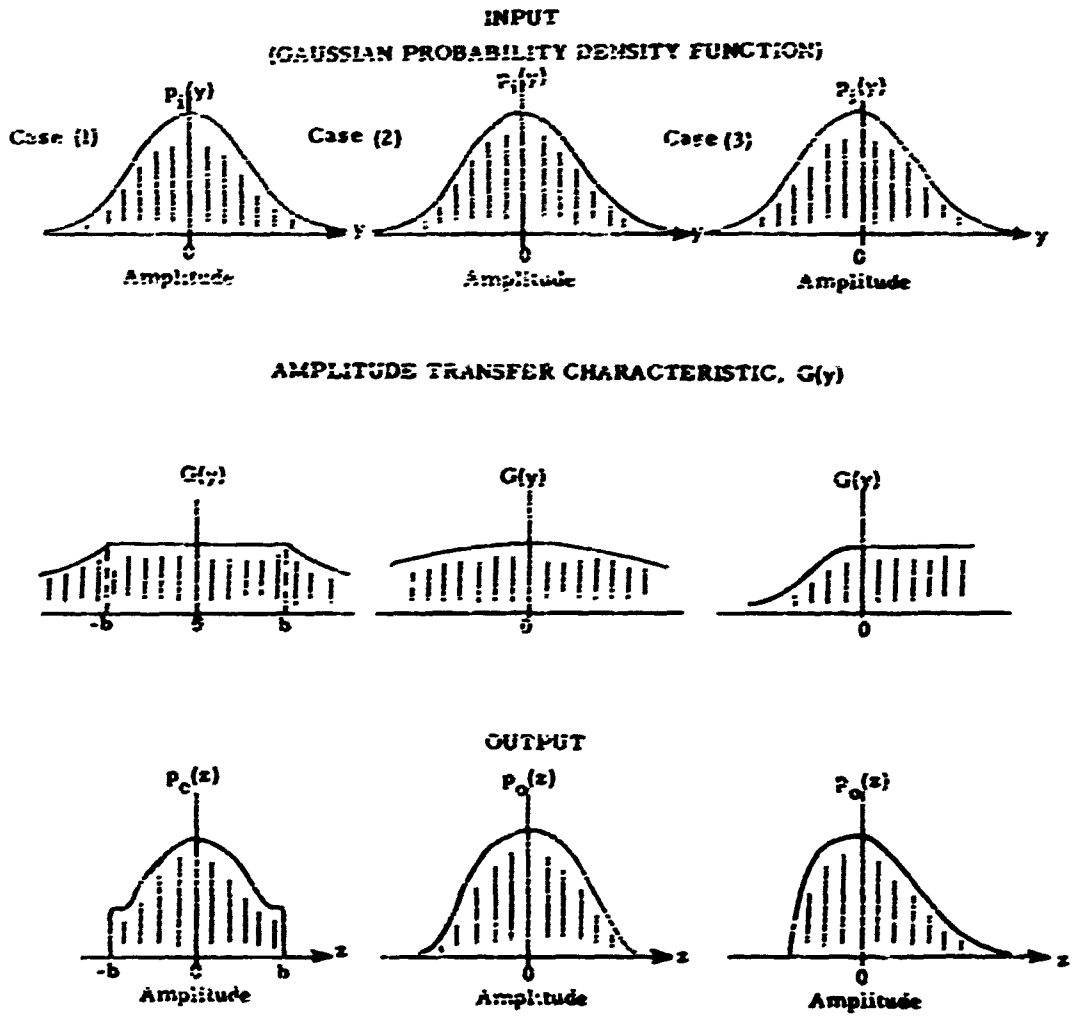   Two Types As Shown.

Figure 9. 3

Output Probability Density Function as Function of Two Different Nonlinear
Transfer Characteristics (Uniform Input Probability Density Function)

Figure 9.4

Output Probability Density Function as Function of Three
Different Nonlinear Transfer Characteristics (Gaussian Input Probability
Density Function)

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10. CONCLUSIONS AND RECOMMENDATIONS

The objectives of the contract and summary of main results are listed in Sections 1 and 2. It is unnecessary to repeat this material here. To conclude the report, a brief review will be given of some of the important results found in the other individual technical Sections 3 through 9. This will help to indicate how material from section to section is interrelated, and point out to the interested reader how many of the same topics are covered from different points of view throughout the report. This review will then be followed by some specific recommendations for future work.

10.1 REVIEW OF REPORT

Section 3. Physical Discussion of Flight Vehicle Vibration Problems

This section serves to motivate the need for later mathematical, statistical, experimental, and more advanced physical investigations which appear in the report. Various flight vehicles are broken down into different operating characteristics, and an outline is presented of many vibration excitation sources. A preliminary description is given of the response of linear structures to periodic and random excitation. This is followed by a brief discussion of some known experimental results from previous engineering analyses of vibration in flight vehicles. One becomes aware here of a lack of statistical procedures not only in analyzing vibration data after it has been gathered, but also in the original gathering of the data.

Section 4. Mathematical Background for Analyzing Vibration Phenomena

The material in this section is considerably broader than the title implies and might be more appropriately labeled as mathematical background for analyzing a large class of random phenomena, of which vibration phenomena is but one example. A short introduction is given relative to phenomena which can be described by periodic, complex, or analytic records, after which the main discussion is then devoted to random records. Special subsections explain basic ideas of probability theory, random processes, and matters of estimation theory. Quantitative formulas are derived for

predicting the mean square error to be expected in measurements of important parameters of a random process such as its mean value, autocorrelation function, and power spectral density function. The analysis treats in detail a practical engineering (analog) technique for making power spectrum measurements for a single random process, as well as cross-power spectrum measurements for a pair of random processes. Several physical applications are treated including: (a) probability distribution of instantaneous amplitude values of a random record, (b) measurement of a linear system frequency response function, (c) zero crossings and threshold crossings of a random record, (d) peak probability distribution of both narrow-band and wide-band noise, (e) statistics of extreme vibration amplitudes. The mathematical material in this section provides the basis for measurement errors of parameters of interest in individual records which are to be tested later in the experimental program.

Section 5. Statistical Techniques for Evaluating Data

As in the previous section, the material here is applicable to many other physical problems besides the vibration problem. General statistical ideas and techniques are developed which are basic to evaluating data regardless of its physical origin. By way of illustration, a number of special examples are formulated which are particularly directed towards flight vehicle vibration data. The first part of the section discusses statistical aspects of parameter estimation and hypothesis testing. This is followed by mathematical formulas, tables, and applications of four special probability distributions used in statistical tests: the normal distribution, chi-square distribution, Student "t" distribution, and the F distribution. Included among the applications is a test for normality. The final portions of the section explain more advanced statistical matters relative to: (a) statistical results from repeated experiments, (b) quality control procedures, (c) multiple regression techniques. The statistical material in this section provides the basis for estimates of sample means and variances over a (large) collection of records which are to be tested later in the experimental program.

Section 6. Analytical Procedures for Determining Vibration Environment

Block diagrams are displayed in this section for carrying out step-by-step an analytic scientific analysis of flight vehicle vibration data. The discussion is divided into two parts: Part 1 - the procedure for analyzing the pertinent statistical properties of a single vibration time history record; Part 2 - the procedure for establishing the over-all vibration environment given the statistical properties of each of a collection of vibration time history records. For single records, analog tests based on statistical considerations are described for testing fundamental assumptions of randomness, stationarity, and normality. Detailed theoretical arguments to justify these three tests are presented in this section. Other desired measurements for single records are described briefly, but actual details of instrumentation plus an experimental program to verify measurement accuracies are developed more completely in succeeding Sections 7 and 8, respectively. For analysis over a (large) collection of records, corresponding to a number of vibration records obtained from the same flight or from a number of different flights, block diagrams are displayed for carrying out desired statistical tests. A detailed discussion is given here on theoretical statistical matters relative to random sampling techniques for reducing the amount of data to be gathered. Other statistical considerations pertinent to experimental programs for testing the accuracy of these random sampling techniques, as well as the accuracy of repeated experiments over collections of records, are discussed in Section 8.

Section 7. Instrumentation to Measure Vibration Characteristics

Following the preceding three sections devoted in large measure to theoretical questions, and preliminary to the succeeding section which outlines an experimental program to verify analytical procedures developed in the report, this section represents an attempt to bridge the gap between theory and experiment by discussing some of the practical limitations inherent in instrumentation equipment commonly used to measure vibration characteristics. Emphasis is given to properties of actual available equipment and to their measurement accuracies. The discussion includes material on

transducers, transmission and recording, voltmeter measurements, power spectral density measurements, probability density measurements, and correlation measurements. Justification for most of the statements made concerning statistical accuracy of measurements may be found in earlier sections of the report.

Section 8. Experimental Program to Verify Analytical Procedures

This section ties in quite closely with the preceding two sections. It outlines in some detail an experimental program to verify important measurement accuracies and statistical procedures developed in this report. The proposed experimental program is divided into two parts: Part 1 – Laboratory Test Program; Part 2 – Flight Test Program. Laboratory test procedures are described for verifying tests of fundamental assumptions of randomness, stationarity and normality. Further laboratory tests indicate how to verify statistical estimates of root mean square values, power spectra, probability densities, and correlation functions. This is followed by a broad treatment on statistical considerations for analyzing large collections of records (repeated experiments), and random sampling techniques. The final portion of this section contains material relevant to conducting a flight test program.

Section 9. Applications to Response of Structures

This last technical section of the report contains some advanced material dealing with physical applications on the response of structures under various conditions. These investigations originated from a desire to learn what statistical information from vibration data may be pertinent to these questions. Five topics are discussed: (1) Response of linear structures to random excitation; (2) Response of continuous structures to correlated random excitation; (3) Modification of response due to loading; (4) Vibration induced structural fatigue; (5) Effect of nonlinearities on response statistics. Work in this section indicates that considerable statistical information available from vibration data is presently being neglected that would be helpful for some of these problems. Techniques for other needed statistical information have still to be developed. In particular, knowledge of the cross-correlation function between two points on a structure represents useful information which is not being explored fully, while analysis of fatigue and nonlinearities represent largely open problems.

10.2 RECOMMENDATIONS FOR FUTURE WORK

Three main areas exist for important extensions of the present work:

1. Experimental testing of the statistical techniques.
2. Theoretical investigations of more advanced problems.
3. Engineering development of special purpose statistical instruments.

Some specific recommendations for each of these areas are as follows:

1. Experimental testing of the statistical techniques

   (a) Carry out experimental laboratory program, as developed in Sections 6, 7, and 8 of report.

   (b) Carry out flight test program, as outlined in Section 8 of report.

2. Theoretical investigations on more advanced problems

   (a) Further study of joint statistical properties of vibration data at two or more points on a structure, enlarging work discussed in Sections 4 and 9 of report.

   (b) Analytical procedures for estimating joint statistical properties, including rules for selection and analysis of samples, expanding work discussed in Sections 5 and 6 of report.

   (c) Development of experimental program for verifying joint statistical properties of vibration data, extending work discussed in Sections 7 and 8 of the report.

   (d) Further statistical analysis of random sampling techniques, continuing work started in Section 6 of report.

   (e) Advanced studies on methods for evaluating nonstationary data, a topic not considered in this report.

(f) Advanced studies on effects of nonlinearities on response statistics, delving deeper into questions discussed in Section 9 of report.

(g) Advanced studies on response of structures to random excitation, extending physical applications investigated in Section 9 of report.

(h) Research in damping theory of structures and damage accumulation (fatigue) theory, for improving upon methods discussed in Section 9 of report.

(i) Further investigations on applications of various specialized statistical procedures to vibration problems, broadening the scope of material developed in Section 9 of report.

3. Engineering development of special purpose statistical instruments

(a) Develop a small-scale, special-purpose statistical computer to carry out desired data processing and statistical calculations for large quantities of vibration data quickly and economically, as discussed in Sections 5, 6, and 6 of report.

(b) Develop a simple practical random sampling device, as discussed in Section 6 of report.

(c) Develop a practical correlation computer for analysis of vibration data, as discussed in Section 7 of report.

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