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Pontryagin's Maximum Principle and Optimal Control

I. Flügge-Lotz *and*
H. Halkin

September 15, 1961

This research was partially supported by the Air Force Office of Scientific Research, Air Research and Development Command under Contract AF 49(638)-513

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INTRODUCTION

The mathematical formulation of the most general problem of Optimal Control can be considered as a problem of Mayer subjected to unilateral constraints, i.e., to certain restrictions expressible in terms of inequalities [ref. 2, 10, 12, 17 and 19].

The results of the classical calculus of variations in their usual forms cannot give a general solution to this problem because, among other things, the fundamental relation of the calculus of variations, i.e., the equation of Euler-Lagrange, is valid only in the case of points interior to the set of admissible points.

To date, the most general solution to this problem is given by the Maximum Principle of Pontryagin, but in its present form this principle cannot be applied in certain situations, and its validity has been proved in particular cases only [ref. 2, 8, 15, 16 and 20]. It is our intention to give a derivation of this principle for the most general case.

This derivation corresponds to a very simple and very intuitive geometrical interpretation in the event-space, i.e., the state-space with an extra axis for the time [ref. 22]. In this derivation we will take for granted different existence theorems when the geometrical interpretation will be strong enough to motivate these assumptions. The existence theorems will be given explicitly in another paper* [ref. 13].

With the help of this method we will also treat some examples which are for the most part the already "classical" problems of Optimal Control.

---

*We also do not intend to give in this report a complete analysis of the connections between these considerations and other parts of Mathematics, as Calculus of Variations, Theory of Semi-Groups, etc., or some applications to Theoretical Physics, Operations Research, etc. All these aspects will be considered later.

SECTION I. GENERAL FORMULATION OF THE PROBLEM

We assume that the system under consideration can be completely described by the point $x = (x^1, \ldots, x^n)$ in the n-dimensional state-space X^n and that its evolution is given by the system of ordinary differential equations:

$$\dot{x}^i = f^i(x^1, \ldots, x^n; u^1, \ldots, u^r) = f^i(x,u) \quad i=1, \ldots, n \tag{1}$$

The vector $u = (u^1, \ldots, u^r)$ in U^r is called the control vector. If $u = u(t)$ is known and if appropriate initial conditions are given the system (1) can be integrated in a unique way when the Lipschitz conditions are satisfied.

The problem under consideration is to select a particular vector function u belonging to a given class F in order to meet certain requirements which will be discussed later in detail.

The specification of the class F is given by the particular problem involved. This specification usually takes the following form:

$$u(t) \in F \quad \text{if and only if}$$

(i) $u(t)$ is piecewise continuous

(ii) $g_i(u) \geq 0 \qquad i = 1, \ldots, k$

Usually we will write the condition (ii) under the form $u \in \Omega$. For instance if (ii) is

$$\sum_{i=1}^{r} u^{i^2} \leq 1$$

Ω will be the unit hypersphere in U^r, or if (ii) is

$$|u^i| \leq 1 \ , \quad i = 1, \ldots, r$$

Ω will be the unit hypercube in U^r, etc.

We assume that the initial value $x(0) = \xi_1$ and the final value $x(T) = \xi_2$ of the solution of system (1) are given.

The problem is to find a vector function $u(t) \epsilon F$ such that:

(i) there exists a $T > 0$ for which the integration of (1) with the control $u(t)$ and the initial condition $\xi_1 = x(0)$ satisfies $x(T) = \xi_2$

(ii) a chosen performance criterion

$$\int_0^T f^o\big(x, u(t)\big)dt \tag{2}$$

is minimum.

Some particular criteria of (ii) will be discussed

1) If we consider the case $f^o(x, u(t)) = 1$ we will have

$$\int_0^T f^o\big(x, u(t)\big)dt = T$$

i.e., we will require the process to take place in the minimum time.

2) In the more general case

$$\int_0^T f^o\big(x, u(t)\big)dt$$

can for instance represent the total energy consumption, cost, etc.

3) The particular problem for which T is fixed beforehand can be treated with this formulation by considering a state variable x^{n*} for which

$$f^{n*} = 1 \quad , \quad x^{n*}(0) = 0 \quad \text{and} \quad x^{n*}(T) = T$$

Let us return to the most general case (ii). If we introduce the new function

$$x^{0}(t) = \int_{0}^{t} f^{0}\big(x,\ u(t)\big)\,dt$$

we can add the differential equation

$$\dot{x}^{0} = f^{0}\big(x,\ u(t)\big)$$

to the system (1) and we will then have the extended system,

$$\dot{x} = f\big(x,\ u(t)\big) \qquad (3)$$

with

$$x = (x^{0},\ x^{1},\ \ldots,\ x^{n})$$

$$f = (f^{0},\ f^{1},\ \ldots,\ f^{n})$$

$$u = (u^{1},\ u^{2},\ \ldots,\ u^{r})$$

We assume that the functions

$$f^{i}\big(x,\ u(t)\big) \qquad i = 0,\ \ldots,\ n$$

are defined and sufficiently differentiable for all

$$(x,u) \epsilon X^{n}U^{r}$$

Then our problem can be stated:

Find a function $u(t) \epsilon F$ such that

(i) there exists a $T > 0$ and a $x^{0}(T) = X^{0}$ for which the integration of equation (3) with the control $u(t)$ and the initial condition $x(0) = (0, \xi_{1})$ satisfies $x(T) = (X^{0},\ \xi_{2})$

(ii) $X^{0} = \int_{0}^{T} f\big(x,\ u(t)\big)\,dt$ is minimum.

SECTION II. SET OF REACHABLE EVENTS*

Let us consider the (n+2) dimensional space $T X^{n+1}$ of the points $(t, x^o, x^1, \ldots, x^n)$.

A reachable event is a point of $T X^{n+1}$ defined by a function $u(t) \epsilon F$ and a value $\tau \geq 0$ in the following way

$$\begin{cases} t = \tau \\ x^i = \int_0^\tau f^i\Big(x,\ u(t)\Big)\,dt \qquad i = 0, 1, \ldots, n \\ \text{subject to the initial conditions } x(0) = (0, \xi_1) \end{cases}$$

The set of all reachable points will be called $R(\xi_1)$. We will assume this set to be dense everywhere and its boundary hypersurface to belong to $R(\xi_1)$. We will call this boundary hypersurface $S(\xi_1)$.

Example

A system $\dot{x}^1 = u^2$ shall move with the condition

$$\int_0^t u^1\, dt \rightarrow \text{minimum} \quad \text{and} \quad (u^1)^2 + (u^2)^2 \leq 1$$

this means:

$$\begin{cases} \dot{x}^o = u^1 \\ \dot{x}^1 = u^2 \end{cases}$$

with $u \epsilon \Omega \Longleftrightarrow (u^1)^2 + (u^2)^2 \leq 1$

and $x(0) = (0,0)$, i.e., $x^o(0) = 0$ and $x^1(0) = 0$.

---

*The application of this concept in Control Theory has been introduced independently by E. Roxin (private communication). See also ref. 21.

The set of reachable events $R(0)$ is here the set of points (t, x^o, x^1) such that

$$\begin{cases} {x^o}^2 + {x^1}^2 \leq t^2 \\ t \geq 0 \end{cases}$$

in other words $R(0)$ is the interior, surface included, of the semicone of revolution around the t axis with the generatrix $x^o = t$. The boundary hypersurface $S(0)$ is in this case the surface of this semicone. See Fig. 1.

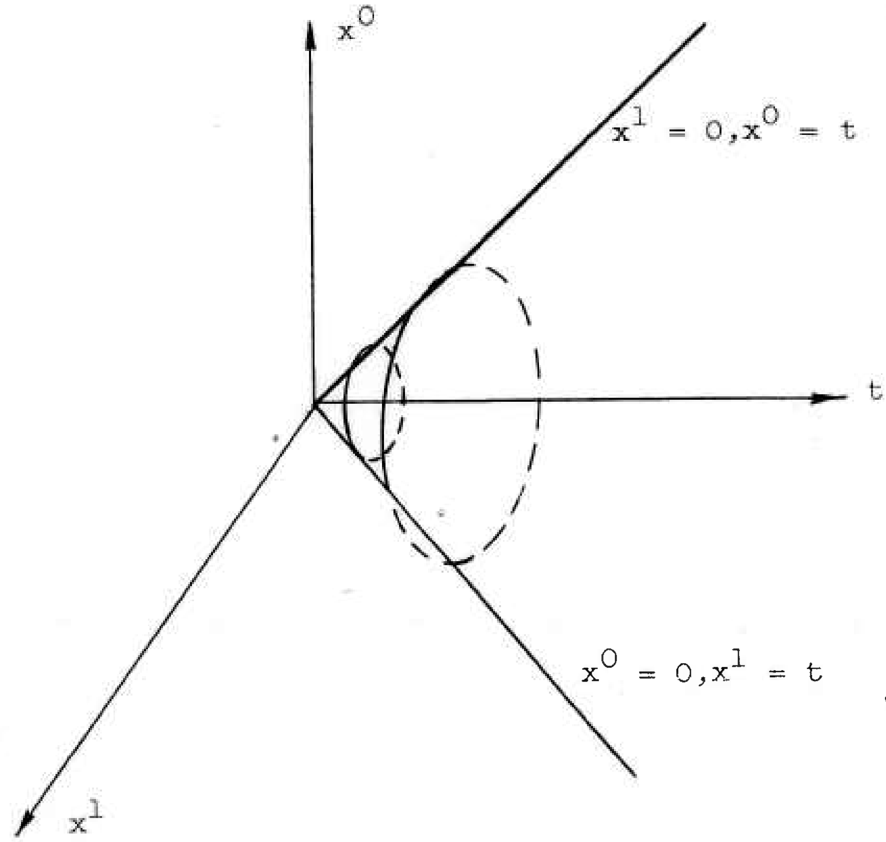

Figure 1

SECTION III. PRINCIPLE OF OPTIMAL EVOLUTION

Let us call $G(\xi_1, \xi_2)$ the set of reachable events satisfying the end conditions. By definition $G(\xi_1, \xi_2)$ is a subset of $R(\xi_1)$. More formally this is equivalent to

$$G(\xi_1, \xi_2) = \left\{ P : P \in R(\xi_1) \quad \text{and} \quad (x^1, x^2, \ldots, x^n)_P = \xi_2 \right\}$$

Among the elements of $G(\xi_1, \xi_2)$ we will assume that A is the one with the smallest x^o. We will assume that such a point exists, in other words that the infimum of $G(\xi_1, \xi_2)$ with respect to x^o belongs to $G(\xi_1, \xi_2)$.

By definition A is the end point of the optimal trajectory and the function $u(t)$ associated with this trajectory is the solution of our problem.

We see immediately that by definition A belongs to the boundary hypersurface $S(\xi_1)$ of $R(\xi_1)$.

Theorem I

Every event of the optimal trajectory belongs to $S(\xi_1)$. The proof, by contradiction, is immediate.

This theorem constitutes what we call the Principle of Optimal Evolution. In the next paragraph we will express analytically how to construct trajectories belonging to $S(\xi_1)$.

Examples

1) Let us first consider the example introduced at the end of Section II. In that case $G(0,a)$ where a is a given value of x^1 will be the set of all points to the right of the right branch of the hyperbola obtained by intersecting the semicone $R(0)$ by the plane $x^1 = a$. It is obvious that in such a case there is no point of $G(0,a)$ for which x^o is minimum. Then we conclude that this particular problem has no solution.

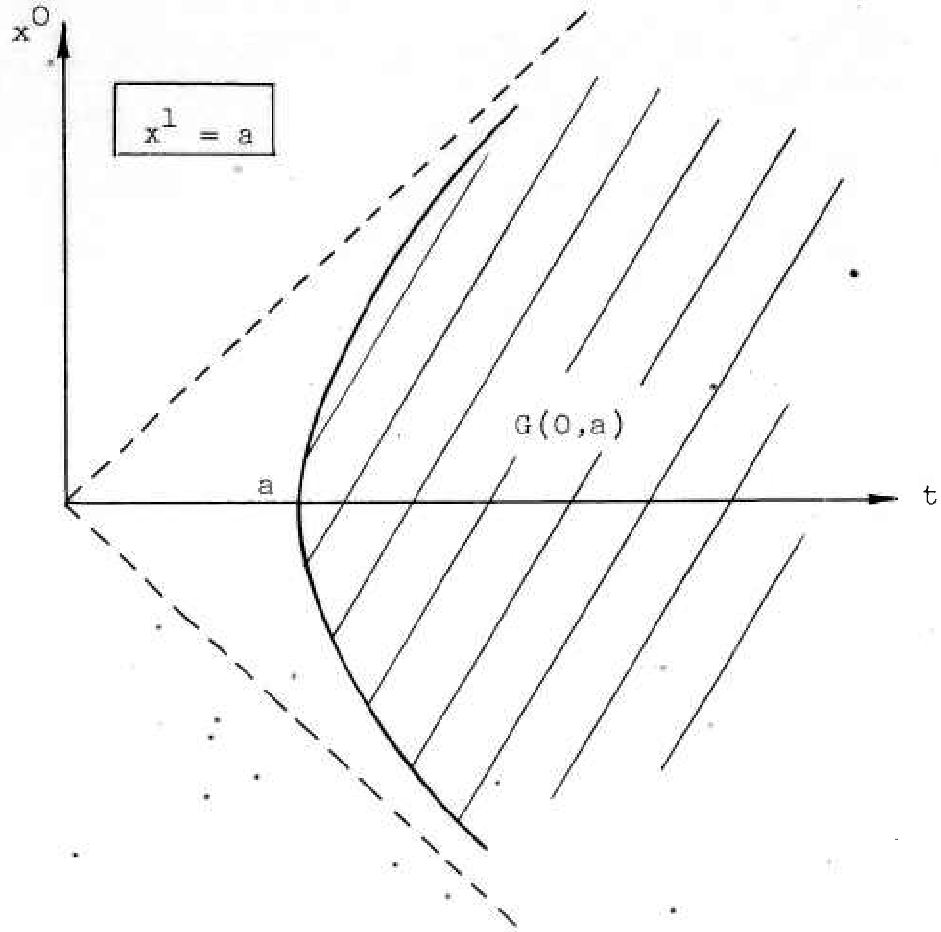

Figure 2

2) Let us now consider the system

$$\begin{cases} \dot{x}^0 = u^1 + 1 \\ \dot{x}^1 = u^2 + 1 \end{cases}$$

with $u \in \Omega \Longleftrightarrow (u^1)^2 + (u^2)^2 \leq 1$

and $\xi_1 = 0$ i.e., $x^1(0) = 0$

$\xi_2 = 1$ i.e., $x^1(T) = 1$

The set of reachable events is defined by

$$\begin{cases} (x^0-t)^2 + (x^1-t)^2 \leq t^2 \\ t \geq 0 \end{cases}$$

i.e., a semicone entirely situated in the octan

$$t \geq 0 \quad , \quad x^0 \geq 0 \quad , \quad x^1 \geq 0 \quad .$$

$G(0,1)$ is the set of events defined by

$$\begin{cases} (x^0-t)^2 + (1-t)^2 \leq t^2 \\ t \geq 0 \end{cases}$$

These equations describe a parabola shown in Figure 3.

The minimum acceptable value for x^0 is $x^0 = 0$ and we see that in that case

$$T = 1$$

$$u^1(t) \equiv -1$$

$$u^2(t) \equiv 0$$

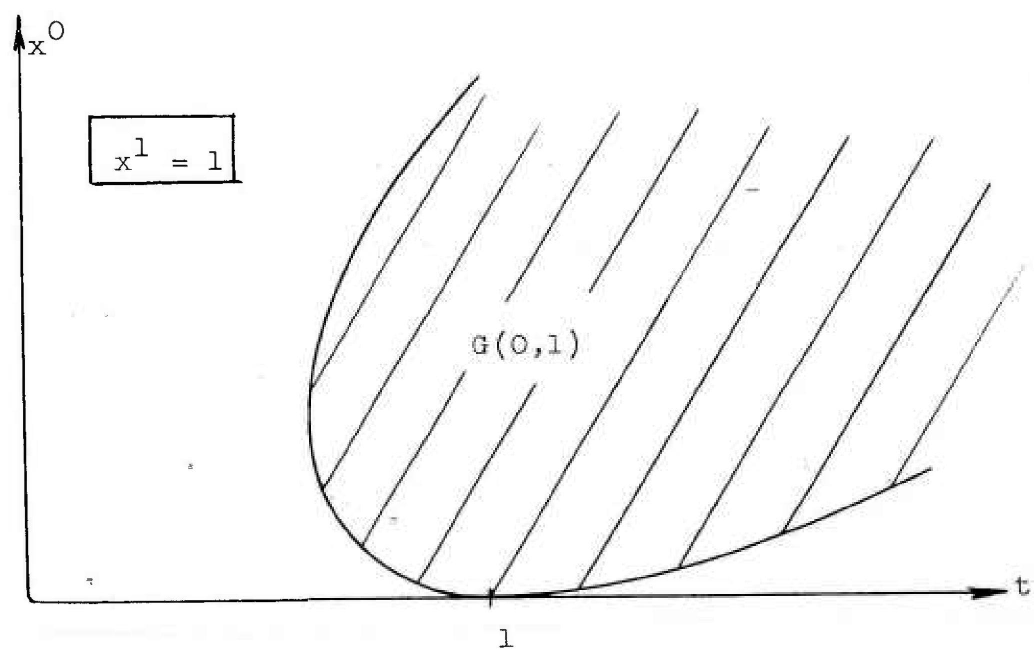

Figure 3

SECTION IV. GENERALIZED HUYGENS' PRINCIPLE

In geometrical optics there is a simple construction based on Huygens' Principle, which gives the wavefront at $t + dt$, i.e., $W(t + dt)$, when the wavefront at t, i.e., $W(t)$, is known [ref.11].

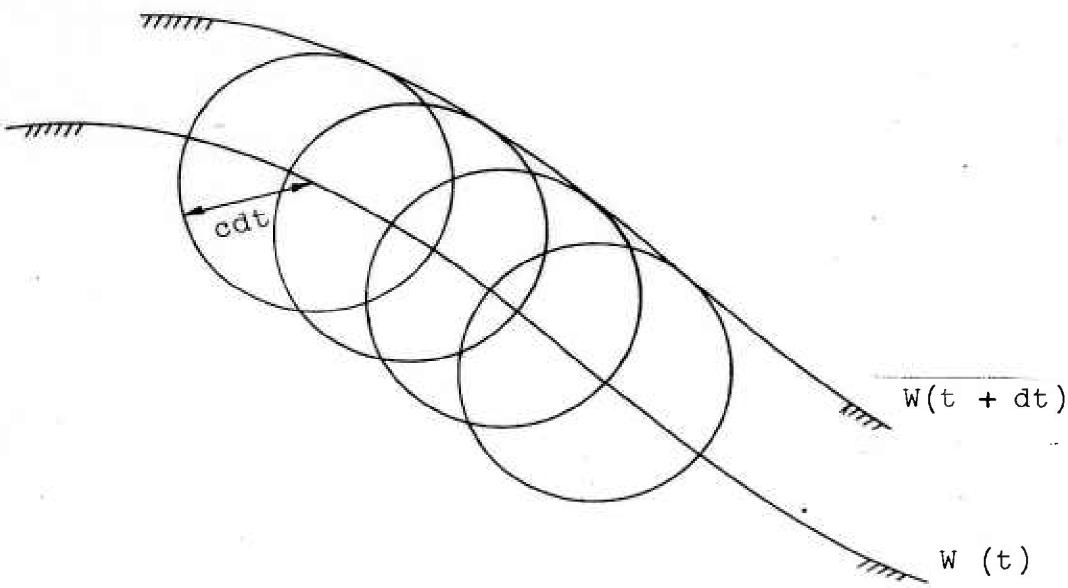

Figure 4

From every point of $W(t)$ a small circle of radius cdt, called wavelet, is drawn and the exterior envelope of the small circles is $W(t + dt)$. (See figure 4).

This is the procedure in the case of an homogeneous isotropic medium, where c is the light velocity. This construction can be generalized when these wavelets are no more circular, for instance in the case of the propagation in a homogeneous anisotropic medium.

The extension to the case of wavelets which are connected, continuous, and differentiable has been extensively studied (Theory of Contact Transformation, Hamilton-Jacobi Partial Differential Equation, etc.), [Refs. 1, 4, 5, 6, 14, 18, 22].

We will generalize this construction to the case of arbitrary wavelets, i.e., wavelets for which the conditions of connectedness, continuity and differentiability have been dropped.

Examples

1) Let us assume that the wavelet associated to the point A is the line segment BC (see fig. 5).

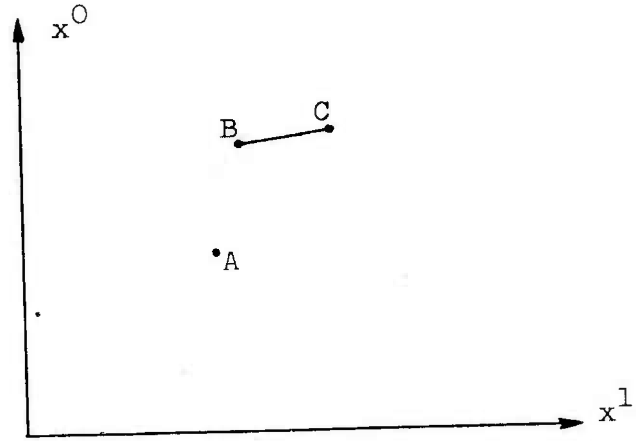

Figure 5

The construction of the new wavefront is described in fig. 6. One starts with a wavefront A A' A" .... and one obtains the new wavefront C C' C" ....

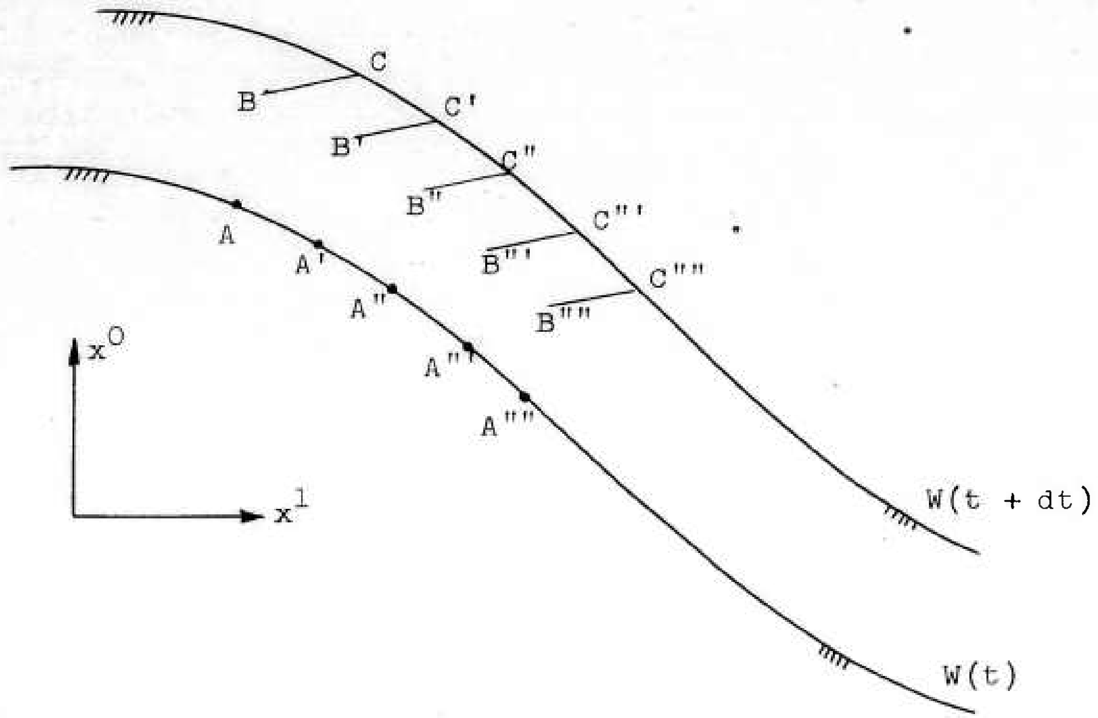

Figure 6

2) As the next example let us assume that the wavelet associated to the point A is represented by two points B and C (see fig. 7).

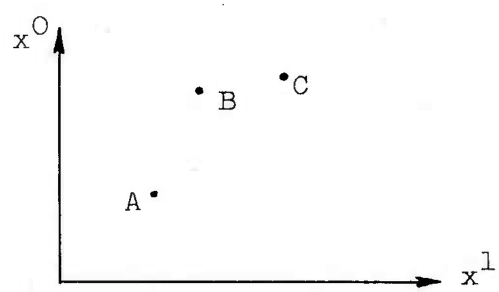

Figure 7

The construction of the new wavefront is described in fig. 8.

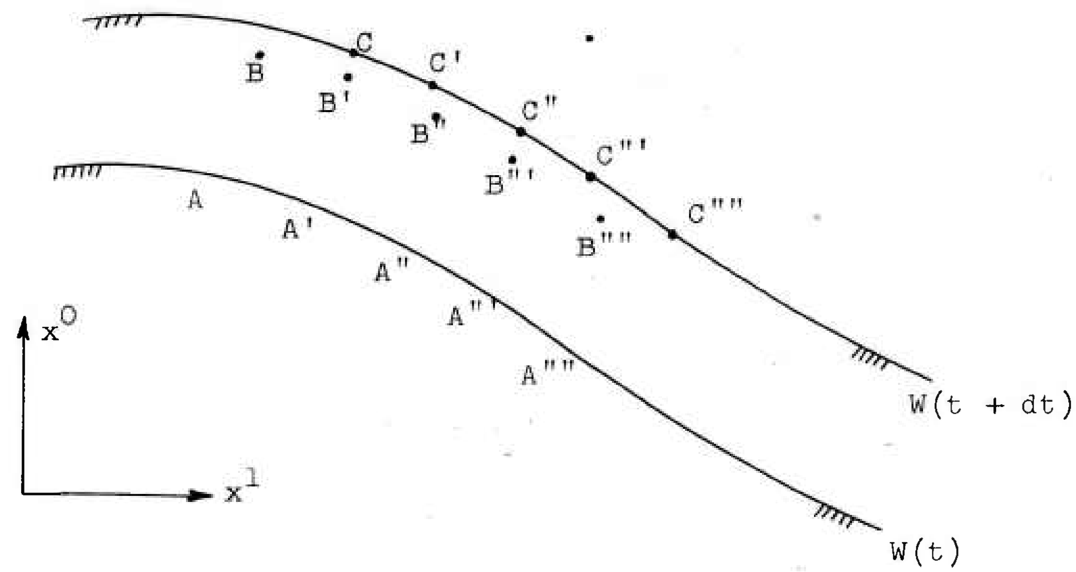

Figure 8

In our particular problem we can consider the intersection of the boundary hypersurface $S(\xi_1)$ by an hyperplane $t = \tau$ as a wavefront $W(\tau)$. Then the wavefront $W(\tau + dt)$ can be constructed by the above described method. In fact this will allow us to construct the whole surface $S(\xi_1)$ from the point $(0,\xi_1)$ which is the intersection of $S(\xi_1)$ by the hyperplane $t = 0$.

<u>Example</u>

In the second example of Section III, $W(\tau)$ is a circle in the plane (x^0,x^1) with the center at (τ,τ) and a radius equal to τ. (See fig. 9).

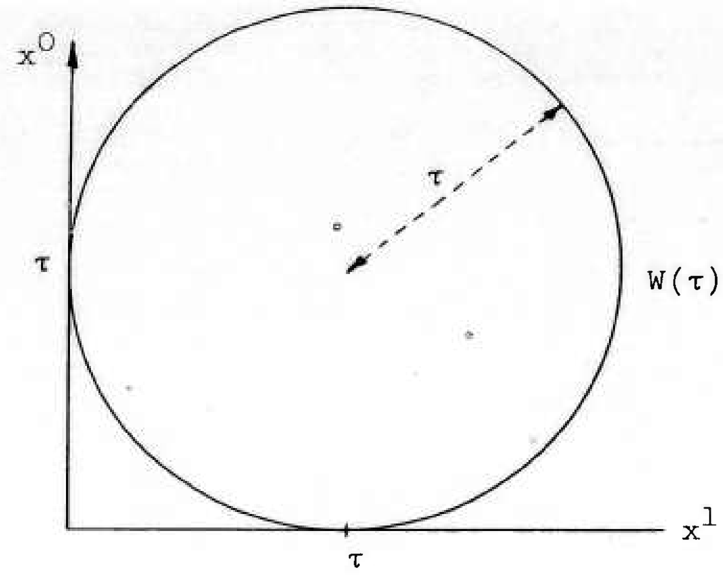

Figure 9

The wavelet corresponding to a point A will be a circle of radius dt and of center (dt,dt) relative to A. (See fig. 10).

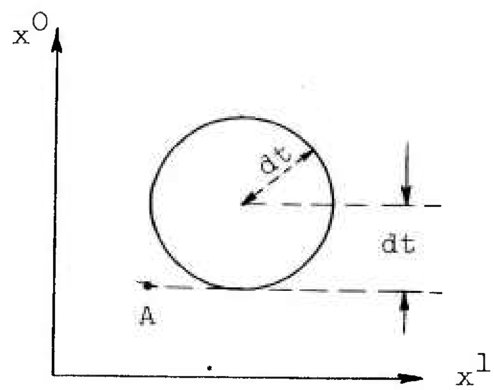

Figure 10

The construction of $W(\tau + dt)$ is given in fig. 11.

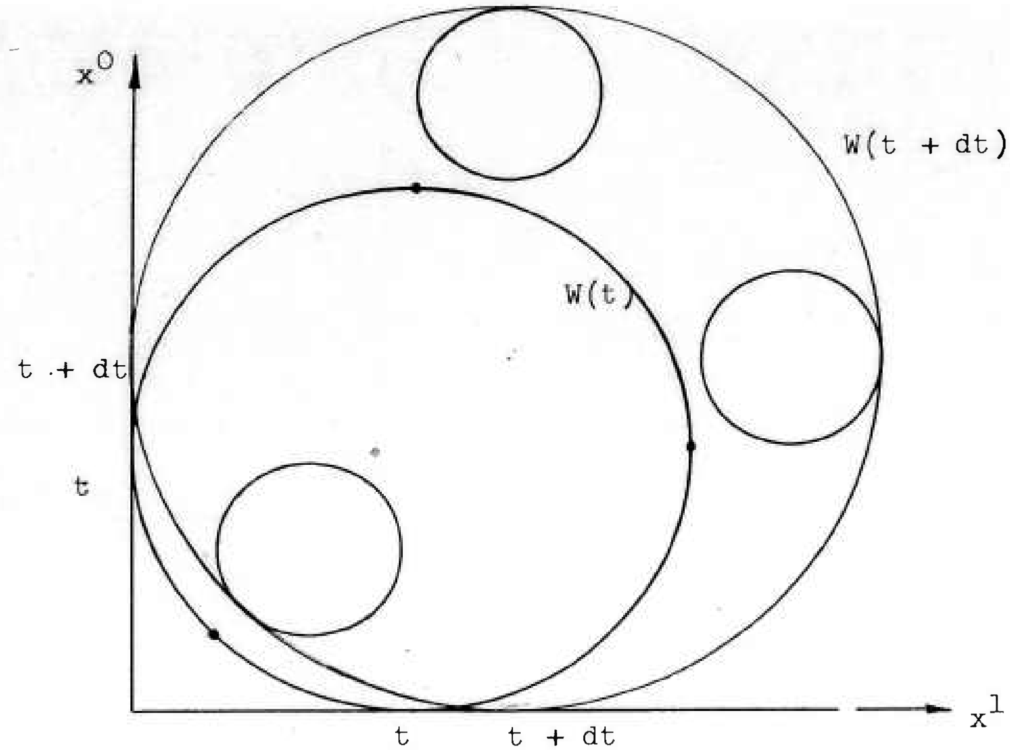

Figure 11

Usually we will draw the wavelet in a space $\dot{X}^n$ without the factor dt. This wavelet is then the mapping of Ω (set of allowable u) into $\dot{X}^n$ by the relation

$$\dot{x} = f(x,u)$$

The example under discussion is described by the differential equations

$$\left.\begin{aligned}\dot{x}^o &= u^1 + 1\\ \dot{x}^1 &= u^2 + 1\end{aligned}\right\} \text{ with } u \in \Omega \iff (u^1)^2 + (u^2)^2 \leq 1$$

This means that a circle in the u^1u^2 plane (fig. 12) is mapped into a circle in the $\dot{x}^o\ \dot{x}^1$ plane (fig. 13).

In general the wavelet will vary with x, but since we assumed that the functions f(x,u) are sufficiently differentiable in x and u, we can nevertheless be sure that the successive construction of the wavefronts W(t) is always possible.

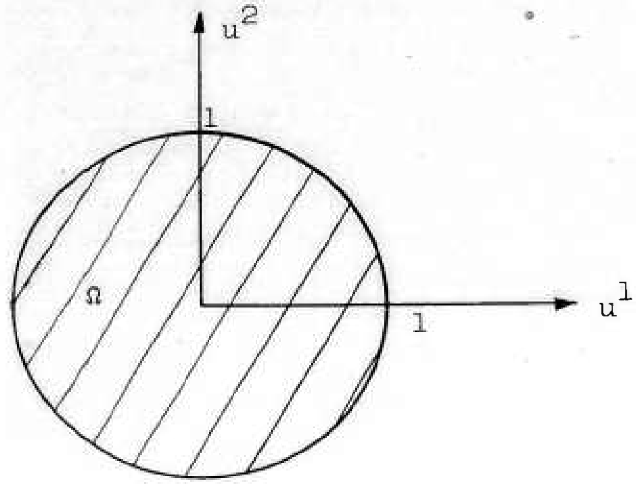

Figure 12

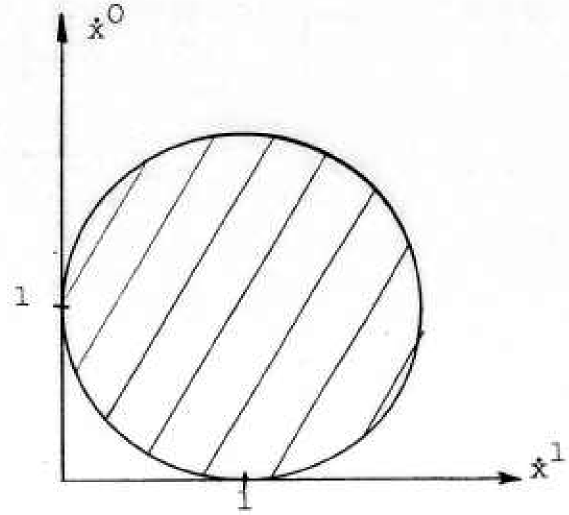

Figure 13

SECTION V. ANALYTICAL FORMULATION OF THE PRINCIPLE OF OPTIMAL EVOLUTION

Let A be a point of $W(t)$ where $W(t)$ is differentiable and where p is the normal to $W(t)$. Then the question arises, by which value of u will the point A be transferred to a point B of $W(t + dt)$?

If we define $H(x,p,u) = \langle p|f(x,u)\rangle$, that means the scalar product of p and f, then the appropriate control function u is determined by*

$$u(x,p) = \underset{u \in \Omega}{\operatorname{argmax}}\, H(x,p,u) \qquad \text{V}$$

In other words the point $A \in W(t)$ will be transferred into a point $B \in W(t + dt)$ if and only if we choose the control u for which $H(x,p,u)$ is maximum.

Justification: We see immediately in fig. 14 that $B \in W(t + dt)$ is the point corresponding to $\max \langle p|f(x,u)\rangle$ and that to C, not on $W(t + dt)$ corresponds a u^* such that:

$$\langle p|f(x,u^*)\rangle < \langle p|f(x,u)\rangle$$

For the time being we will assume that the conditions V determine one and only one u.

<u>Theorem II</u>

If there exists a control $u(t)$ transferring $A \in W(t)$ into $B \in W(t + h)$ and $C \in W(t + h + k)$ where h and $k > 0$, then the "topology" of $W(t + h)$ at B is larger or equal to the "topology" of $W(t)$ at A.

---

* We define the symbol "argmax" by

$$X = \underset{x \in \Xi}{\operatorname{argmax}}\, f(x) \quad \text{if and only if} \quad f(X) = \max_{x \in \Xi} f(x)$$

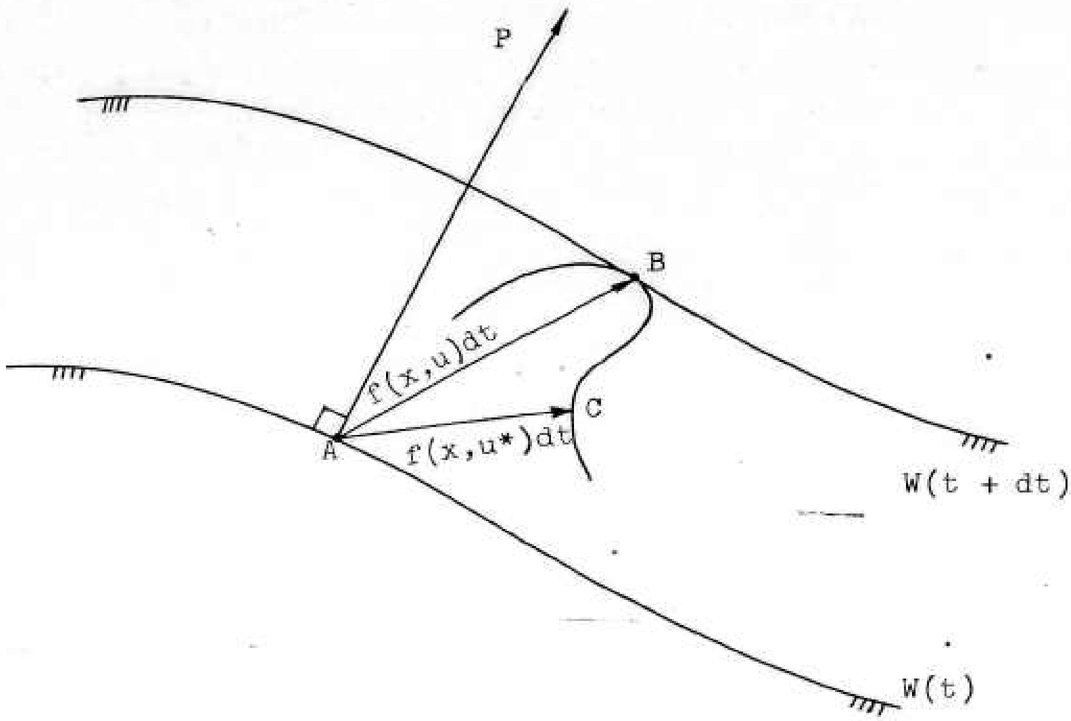

Figure 14

Topology of a wavefront W at a point P is defined as the set of properties of W in the neighborhood of P, for instance: existence, continuity, differentiability of W in the neighborhood of P.

The topology at B is said to be larger or equal to the topology at A, if all such properties of W(t) at A are also properties of W(t + h) at B.

The validity of this theorem is easily checked in the previous examples. The general proof will be given in another paper. The practical importance of this theorem will be stressed in Section XII of this report.

<u>Corollary II</u>

Let W*(t + dt) be the hypersurface obtained from W(t) by using for all its points the same control u which transfers $A \in W(t)$ into $B \in W(t + dt)$.

By definition $W^*(t + dt)$ and $W(t + dt)$ intersect at the point B, and by application of Theorem II, $W^*(t + dt)$ and $W(t + dt)$ are even tangent at the point B.

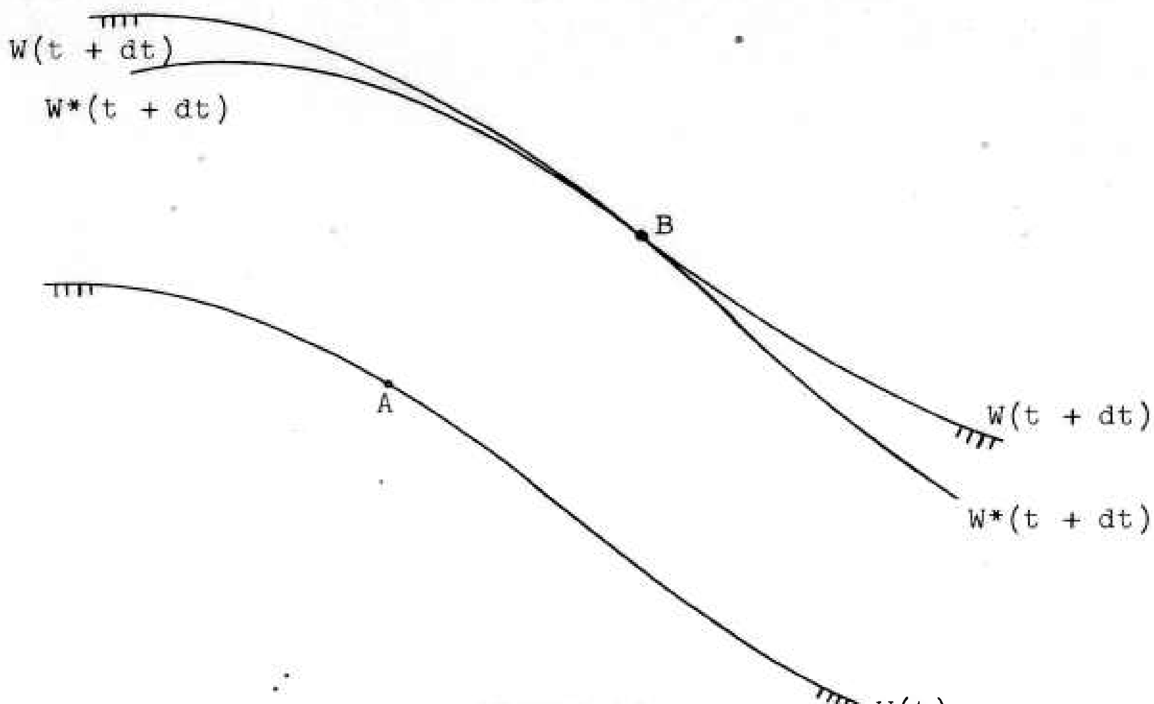

Figure 15

In particular if $W(t)$ admits a normal $\pi(A)$ at A, so will $W^*(t + dt)$ admit a normal $\pi(B)$ at B, by definition. Moreover the Corollary II states that $W(t + dt)$ will also have a normal at B and that this normal will also be $\pi(B)$.

SECTION VI. GENERALIZED HAMILTONIAN FORMULATION OF THE PROBLEM

The results obtained in the previous section allow us to construct a trajectory belonging to $S(\xi_1)$ if we know the normal p to $W(t)$ for all points of the trajectory.

For this reason we will now establish a system of differential equations for p.

Let us define:

$$\left.\begin{aligned} \langle p \,|\, (\delta x)_i \rangle &= 0 \qquad i = 1, 2, \ldots, n \\ \langle p \,|\, (\delta x)_o \rangle &= -1 \end{aligned}\right\} \quad \text{VI}$$

where the $(\delta x)_i$ are n independent vectors tangent to $W(t)$ at the point A and $(\delta x)_o$ is an arbitrary vector independent of the $(\delta x)_i$ and directed toward the inside of $W(t)$.

We will assume that we choose the same control vector for the point x and for all the $n+1$ points $x + (\delta x)_i$, $i = 0, 1, 2, \ldots, n$; which we will assume to be in the neighborhood of x.

We will require the invariance, along a trajectory, of the relations VI, in virture of Corollary II of the preceding section. This gives*

$$\langle p \,|\, (\delta x)_i \rangle^{\cdot} = 0 \qquad i = 0, 1, 2, \ldots, n$$

i.e.,

$$\langle \dot{p} \,|\, (\delta x)_i \rangle + \langle p \,|\, (\delta \dot{x})_i \rangle = 0$$

But

$$(\delta \dot{x})_i = A(\delta x)_i$$

---

* The "dot" indicates differentiation with respect to time.

where $$A_{jk} = \frac{\partial f^j}{\partial x^k}$$

Hence $$\langle p^{\cdot} | (\delta x)_i \rangle + \langle p | A(\delta x)_i \rangle = 0$$

or $$\langle p^{\cdot} | (\delta x)_i \rangle + \langle \tilde{A} p | (\delta x)_i \rangle = 0$$

where $\tilde{A}$ is the transpose of A, a real matrix.

Hence

$$\langle p^{\cdot} + \tilde{A} p | (\delta x)_i \rangle = 0 \qquad i = 0, 1, \ldots, n$$

i.e.,

$$p^{\cdot} + \tilde{A} p = 0$$

since the $(\delta x)_i$ are $n+1$ independent vectors in the space X^{n+1}.

Hence $$\boxed{p^{\cdot} = - \tilde{A} p}$$

This last relation can be written in the form

$$\dot{p}_i = - \sum_j (\tilde{A})_{ij} \, p_j$$

But $$(\tilde{A})_{ij} = A_{ji} = \frac{\partial f^j}{\partial x^i}$$

Hence $$\dot{p}_i = - \sum_j \frac{\partial f^j}{\partial x^i} p_j$$

If we define $$H(x,p,u) = \langle p | f(x,u) \rangle = \sum_i p_i \, f^i(x,u)$$

we see that

$$\dot{x}^i = f^i(x,u)$$

is strictly equivalent to

$$\dot{x}^i = \frac{\partial H(x,p,u)}{\partial p_i}$$

and that
$$\dot{p}^i = - \sum_j \frac{\partial f^j}{\partial x^i} p_j$$

is strictly equivalent to
$$\dot{p}_i = - \frac{\partial H(x,p,u)}{\partial x^i}$$

The complete integration along a trajectory on $S(\xi_1)$, i.e., the integration of the state variable and of the associated normal is given by

$$\boxed{\begin{aligned} \dot{x}^i &= \frac{\partial H(x,p,u)}{\partial p_i} \\ \dot{p}_i &= - \frac{\partial H(x,p,u)}{\partial x^i} \\ u &= \underset{u \in \Omega}{\operatorname{argmax}} \; H(x,p,u) \end{aligned}}$$

CHAPTER VII. PONTRYAGIN'S MAXIMUM PRINCIPLE

The results obtained at the end of the previous paragraph form what is called the Maximum Principle of Pontryagin.

The derivation developed in this report gives a geometrical interpretation of the vector function p introduced in the formulation of this principle by Pontryagin and his associates.

The interpretation of the vector function p as the normal to the boundary hypersurface $S(\xi_1)$ of the set of reachable events $R(\xi_1)$ allows us to overcome, with the help of the Theorem II previously introduced, most of the difficulties which usually arise in this application of the Maximum Principle to a particular problem.

These are generally of two different natures:

1) How to choose the initial value of the vector function p, i.e., p(0).

2) What is to be done when the relation

$$u = \underset{u \in \Omega}{\operatorname{argmax}} \, H(x,p,u)$$

does not determine a unique value for u.

With our geometrical interpretation it is easy to answer these questions as we will show in the section devoted to the synthesis of the solution.

SECTION VIII. CONTROL WITH NONZERO INERTIA

In the general formulation of the problem we have defined the class F of acceptable $u(t)$ by

(i) $u(t)$ is piecewise continuous.

(ii) $g_i(u) \geq 0 \quad i = 1, \ldots, k \quad$ i.e., $u \in \Omega$

For some particular problems the restriction (i) is not strong enough: the rate of change of the control itself is bounded, for instance for control devices with nonzero inertia, and the corresponding reinforced condition (i) should read:

(i)* $\quad u_i(t)$ is continuous and $\dot{u}_i(t) \leq M$.

From the theoretical point of view it is very easy to transform a problem with such reinforced conditions (i)* into a new problem with usual conditions (i).

<u>Example</u>

If $u_i(t)$ is to be continuous with $\dot{u}_i(t) \leq M$ we will replace this control variable $u_i(t)$ by a new one called $u_k(t)$ and we will consider $u_i(t)$ as a new state variable to which will be associated the differential equation

$$\dot{u}_i(t) = u_k(t)$$

SECTION IX. CHATTERING

In order to be sure that the solution proposed at the end of the paragraph VI satisfies the requirements expressed in the general formulation of the problem we have still to check that the control function obtained by this method belongs to F.

F was defined as the set of functions $u(t)$ such that

(i) $u(t)$ is piecewise continuous

(ii) $u \epsilon \Omega$

The condition (ii) is satisfied by construction but it could happen that the condition (i) is not.

In that case the control function given by the Maximum Principle is an oscillating function of infinite frequency characterized only by its mean value.

When such a case occurs, the problem has no solution, mathematically speaking. It corresponds to a variational problem with a bounded minimizing sequence whose infimum does not belong to the sequence.

Example

Find the shortest continuous and differentiable curve joining two given points A and B of a plane such that the tangents to the curve at A and B have given directions, one of which at least being different from $\overline{AB}$.

From a physical point of view, nevertheless, such a pseudo solution is not without interest: it gives the infimum of $x^{o}(T)$, i.e., a value impossible to actually obtain perhaps, but such that it is possible to get closer and closer to it from above.

In a specific problem when such a case arises the problem must be reformulated and the modification (i)* given in the paragraph VIII is to be performed.

SECTION X. MOVING TARGET

In some problems the terminal condition $x(T) = \xi_2$ is to be replaced by $x(T) = \xi_2(T)$ where $\xi_2(t)$ is a given continuous trajectory and T is the time when the target is hit. The interpretation in the space of events of such a case is easy and we see immediately that the Principle of Optimal Evolution is still valid with all its consequences.

In such a case T is the smallest value of t for which $\xi_2(T) \in R(\xi_1)$. Since we assume $\xi_2(t)$ to be continuous we can even say: T is the smallest value of t for which $\xi_2(t) \in S(\xi_1)$ where $S(\xi_1)$ is the boundary hypersurface of $R(\xi_1)$.

Example: Brachistochrone tracking of a target with known trajectory.

Let S be the system to be controlled and T the target. We assume that S and T are both moving in the plane x^1x^2. We know the trajectory $\xi(t)$ of the target and we want to steer the system in order to hit the target in minimum time (see fig. 16). At $t = 0$ the system X is assumed to be at the point $x^1 = 0$, $x^2 = 0$. The speed of the system is bounded: $\dot{x}^{1^2} + \dot{x}^{2^2} \leq 1$. Formally the problem is: Find the functions $u^1(t)$ and $u^2(t)$ which give the minimum value T for which

$$x(T) = \xi(T)$$

where $\xi(t)$ is a given function
$x(t)$ is the solution of the system

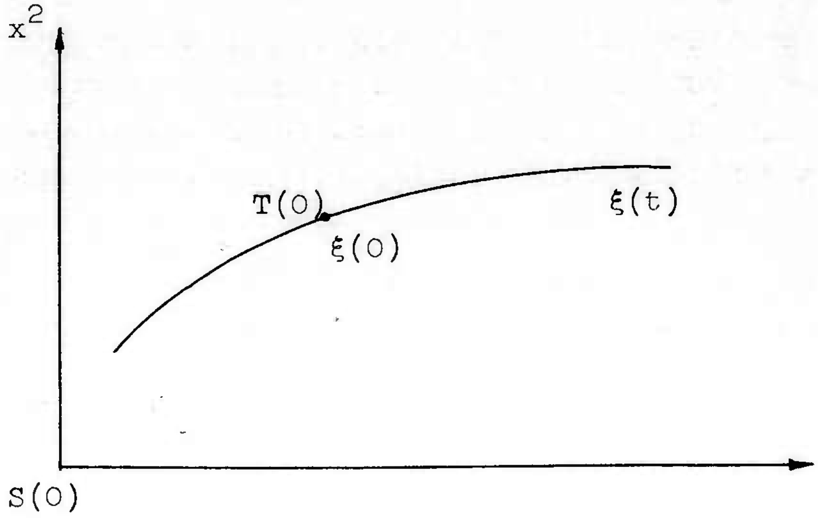

Figure 16

$$\dot{x}^1 = u^1$$

$$\dot{x}^2 = u^2$$

with $u^{1^2} + u^{2^2} \leq 1$
and $x(0) = (0,0)$

Solution: let us consider the event - space (t, x^1, x^2) where we have the trajectory ABCD of the target and the semicone of the reachable events of the system (see fig. 17).

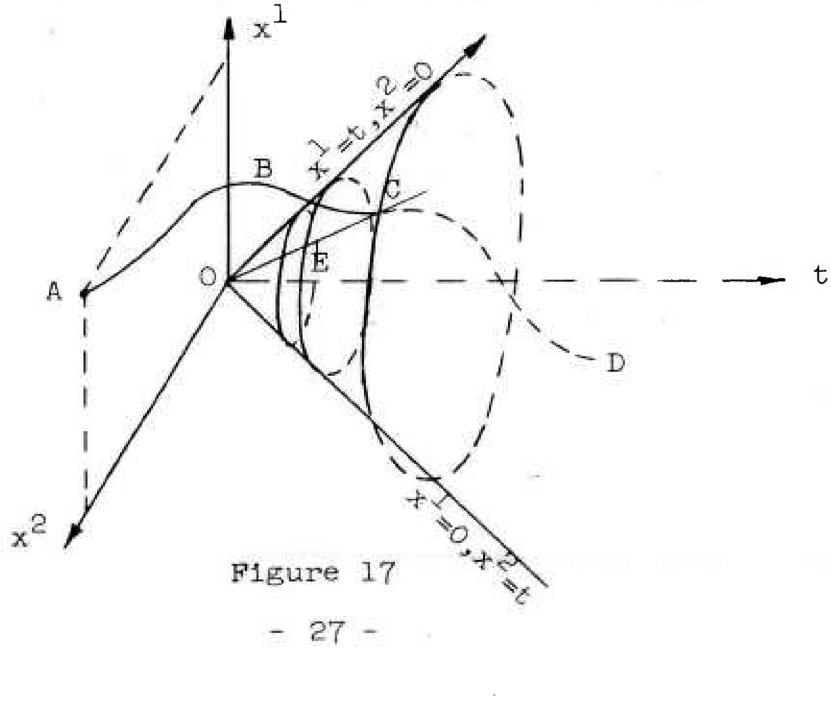

Figure 17

The problem will have a solution if and only if the trajectory ABCD intersects the semicone of reachable events. Let C be the first such intersection, we see that $T = t_C$ and that the optimal trajectory for the system is OEC.

SECTION XI. SYNTHESIS OF THE SOLUTION

Preliminary Remarks

The Maximum Principle leads to a solution which is a relatively strong minimum but not necessarily an absolute strong minimum.

Example:

The two points A and B of W(t) (see fig. 18), not in the same neighborhood on W(t), can be transferred to the same point C of W(t + dt) (see fig. 19).

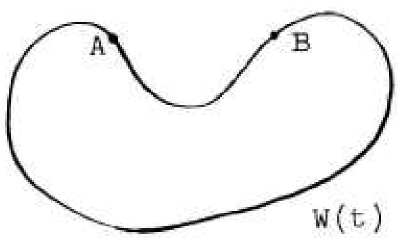

Figure 18

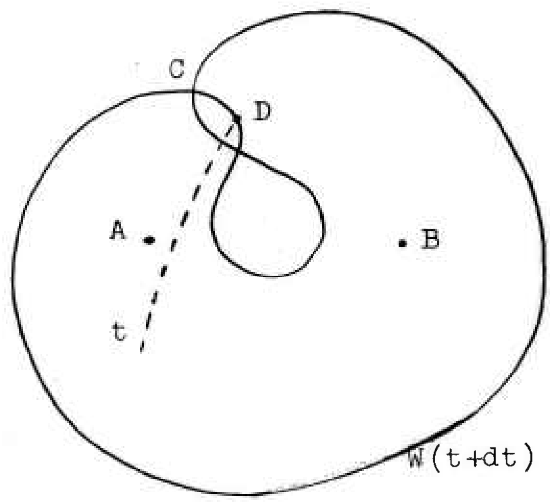

Figure 19

In such a case the trajectory t passing through the point D for instance will give a relative strong minimum but not an absolute strong minimum.*

---

*Example

$$\dot{x}^1 = u^1(\sin^2 x^1 + \sin^2 x^2)$$

$$\dot{x}^2 = u^2(\sin^2 x^1 + \sin^2 x^2)$$

with $u \in \Omega \iff (u^1)^2 + (u^2)^2 \leq 1$

In the following developments we will assume that such cases will not arise.

The next problem is to analyze the situation at a point A of $W(t)$ where $W(t)$ is not differentiable.

<u>Example</u>: Consider the wavefront of fig. 20

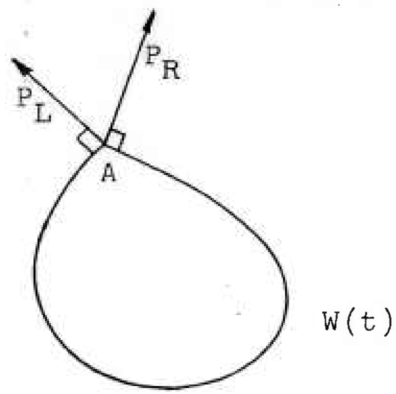

Figure 20

to which corresponds the wavelet of fig. 21.

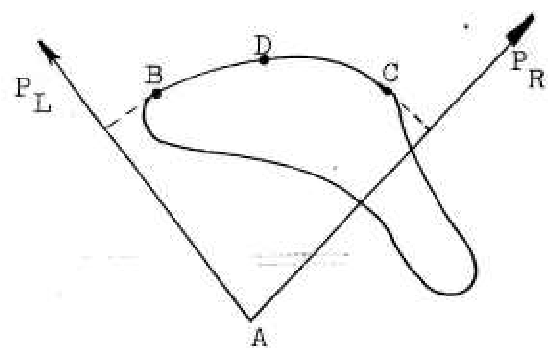

Figure 21

In such a case the maximum principle applied to P_L will give the control vector corresponding to the point B, and applied to P_R will give the control vector corresponding to the point C.

We see immediately that a control vector corresponding to any point D, on the boundary of the wavelet between B and C, will also transfer the point A to W(t + dt). In such a case we see that the transformation of W(t) into W(t + dt) is not pointwise one to one.

Such a problem could also arise in the case of a differentiable wavefront at A if the relation

$$u = \underset{u \in \Omega}{\operatorname{argmax}} H(x,p,u)$$

does not determine a unique control u.

Example: Consider the wavefront of fig. 22

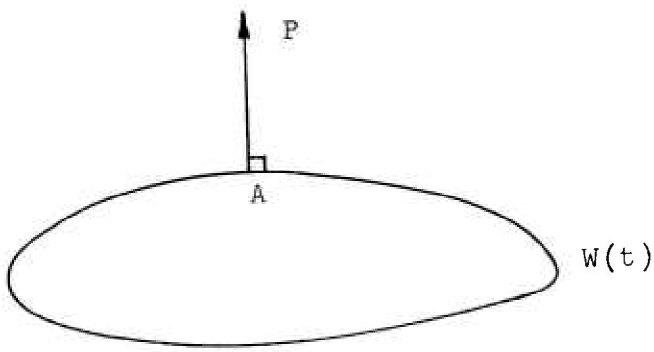

Figure 22

and wavelet at A of fig. 23.

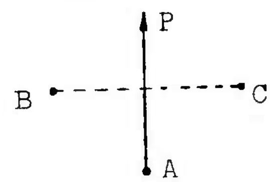

Figure 23

In this particular case, the controls corresponding respectively to B and C will both transfer A into W(t + dt).

Conclusions

If we consider the set of trajectories obtained by choosing all possible values of $p(0)$ and all possible values of the control vector satisfying the Maximum Principle, we will obtain a generalized field of trajectories, corresponding to the set of all rays in geometrical optics.

This generalized field may contain

1) branching points (see fig. 24)

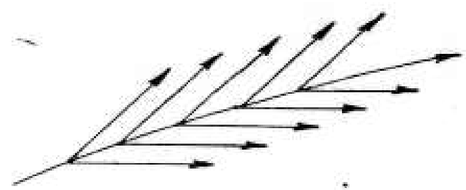

Figure 24

2) disparition points (see fig. 25)

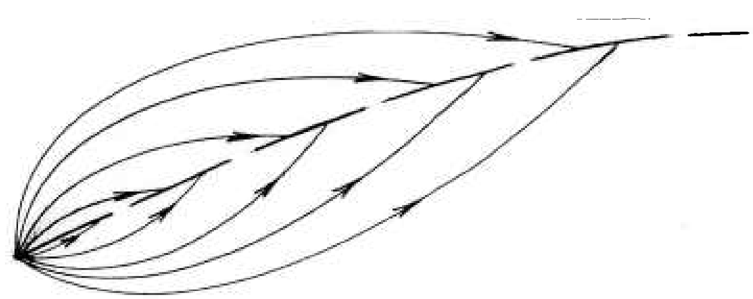

Figure 25

3) indeference regions

i.e., a subset of X^{n+1} where all trajectories passing through all its points and corresponding to all possible controls belong to the generalized field.

Example

$$\dot{x}^c = 1$$

$$\dot{x}^1 = u$$

with $\qquad x^1(0) = 0$

and $\qquad |u| \leq 1 \iff u \in \Omega$

It is easy to see that the wavefront $W(t)$ is determined by

$$x^0(t) = t$$

$$|x^1(t)| \leq t \quad \text{(see fig. 26)}$$

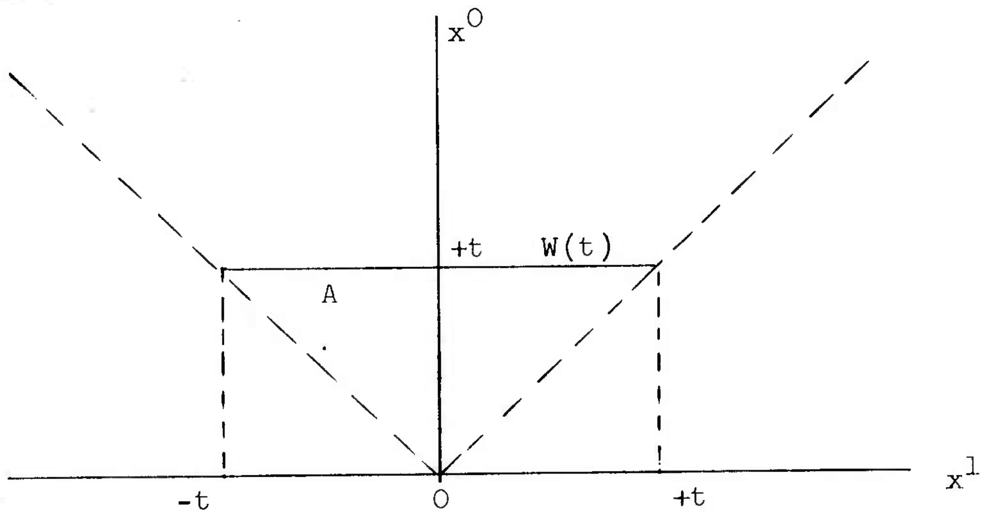

Figure 26

and that if $A \in W(t)$ any $|u| \leq 1$ will transfer A to $W(t + dt)$.

SECTION XII. IMBEDDING METHOD

We will present a general method for avoiding the difficulties arising in the case of a non-differentiable wavefront.

In the formulation given earlier the wavefront at $t = 0$, i.e., $W(0)$ is reduced to a single point, namely $(0,\xi_1)$. Let us now assume that $\overline{W}(0)$ is a n dimensional circular manifold* of center $(0,\xi_1)$ and of radius ϵ, a small positive quantity. The orientation of this $\overline{W}(0)$ is completely determined by its normal $p(0)$. By applying the Maximum Principle to this new starting wavefront $\overline{W}(0)$ we may obtain $\overline{W}(t)$ for any $t \geq 0$. Because of the Theorem II (see Section V), we know that $\overline{W}(t)$ is differentiable at A, a point of $\overline{W}(t)$ corresponding to the point $(0,\xi_1)$ of $\overline{W}(0)$.

In general the trajectory (ies) obtained by the set of points of $\overline{W}(t)$ corresponding to $(0,\xi_1)$ of $\overline{W}(0)$ for all $t \geq 0$ will belong to the generalized field defined in the previous paragraph and the whole field will be the set of all such trajectories corresponding to all possible choices of $\overline{W}(0)$ i.e., of $p(0)$. This method gives a good interpretation of the initial choice of $p(0)$ and its application will help greatly the solution of most practical problems.

---

*By a n dimensional circular manifold of center $(0,\xi_1) = (y^0,y^1,\ldots,y^n)$ and of radius ϵ, in the space X^{n+1} we mean the set of points such that

$$\begin{cases} \sum_{i=0}^{n} p_i(x^i-y^i) = 0 \\ \sum_{i=0}^{n} (x^i-y^i)^2 \leq \epsilon^2 \end{cases}$$

where the vector p is the characteristic orientation of the manifold. Example: if $n = 2$ the circular manifold is an ordinary circular disk of radius ϵ.

SECTION XIII. APPLICATIONS

I. A second order system subject to inertial forces only shall be controlled in such a manner that a fixed target is reached in minimum time.

This problem is usually given in the form:
How to choose the forcing term u with $|u| \leq 1$ in order to find the quickest path of integration of

$$\frac{dx^2}{dt^2} = u$$

between $x = 0, \frac{dx}{dt} = 0$, and $x = a, \frac{dx}{dt} = b$.

This system is equivalent to a system described by:

$$\dot{x}^o = 1$$

$$\dot{x}^1 = x^2$$

$$\dot{x}^2 = u$$

with

$$x^1(0) = x^2(0) = 0$$

$$x^1(T) = a, \quad x^2(T) = b$$

and $|u| \leq 1 \iff u \epsilon \Omega$.

One should note that $\dot{x}^o = 1$ and $x^o(0) = 0$ imply $x^o(t) = t$. It allows us to forget the variable x^o and analyze the problem in the three dimensional space t, x^1, x^2.

Maximum Principle

$$H = \langle p | f \rangle = p_1 f^1 + p_2 f^2 = p_1 x^2 + p_2 u$$

$$u = \underset{u \epsilon \Omega}{\operatorname{argmax}} H = \operatorname{sgn} p_2$$

Differential System for P

$$\mathring{p}_1 = -\frac{\partial H}{\partial x^1} = 0$$

$$\mathring{p}_2 = -\frac{\partial H}{\partial x^2} = p_1$$

hence $\quad p_1(t) = c_1$

$$p_2(t) = -c_1 t + c_2$$

with $p(0) = (c_1, c_2)$ i.e., $p_1(0) = c_1$ and $p_2(0) = c_2$.

Construction of the trajectory corresponding to a particular p(0).

We have found $\quad u = \operatorname{sgn} p_2$

$$p_2 = -c_1 t + c_2$$

hence,

if $c_1 = 0$ and $c_2 < 0$, we will have $p_2(t) < 0$ for $t > 0$

i.e., the control will always be $u = -1$

if $c_1 = 0$ and $c_2 > 0$, we will have $p_2(t) > 0$ for $t > 0$

i.e., the control will always be $u = +1$

if $c_1 > 0$ we will have for $t > \frac{c_2}{c_1}$, $p_2(t) < 0$

and otherwise, $p_2(t) > 0$

i.e., the control will be $u = -1$ for $t > \frac{c_2}{c_1}$

$u = +1$ otherwise.

if $c_1 < 0$ we will have for $t > +\frac{c_2}{c_1}$, $p_2(t) > 0$

and otherwise, $p_2(t) < 0$

i.e., the control will be $u = +1$ for $t > +\frac{c_2}{c_1}$

$u = -1$ otherwise.

In other words, there will be two types of trajectories:

- one type will be formed by a first arc with $u = +1$ followed by a second arc with $u = -1$
- the second type will be formed by a first arc with $u = -1$ followed by a second arc with $u = +1$

with, in both cases, the possibility of having one such arc of length zero.

The integration of the initial system for

$u = +1$ gives
$$x^1(t) = \frac{1}{2} t^2 + x^2(0)t + x^1(0)$$
$$x^2(t) = t + x^2(0)$$

$u = -1$ gives
$$x^1(t) = -\frac{1}{2} t^2 + x^2(0)t + x^1(0)$$
$$x^2(t) = -t + x^2(0)$$

Construction of the wavefront for $t = T$

For an arbitrary trajectory of the first type we will have

$$u = +1 \quad \text{for} \quad 0 \leq t < \tau \quad \text{with} \quad 0 \leq \tau \leq T$$

and

$$u = -1 \quad \text{for} \quad \tau \leq t < T$$

This gives the coordinates of a point reached at time T after starting at $x^1(0) = 0$ and $x^2(0) = 0$:

$$\left.\begin{aligned} x^1(T) &= \frac{1}{2}\tau^2 - \frac{1}{2}(T-\tau)^2 + \tau(T-\tau) = \frac{1}{2}T^2 - (T-\tau)^2 \\ x^2(T) &= \tau - (T-\tau) = 2\tau - T \end{aligned}\right\} (\alpha)$$

For a given T the system α represents parametrically half of the wavefront $W(T)$, the parameter τ being restricted to $0 \leq \tau \leq T$.

The elimination of τ in system α gives a segment ABC of the parabola

$$x^1(T) = \frac{1}{2} T^2 - (T - \frac{x^2(T)+T}{2})^2$$

If we repeat the same procedure for the second type of trajectory we obtain a segment ADC of the parabola

$$-x^1(T) = \frac{1}{2} T^2 - (T - \frac{-x^2(T)+T}{2})^2$$

(see fig. 27).

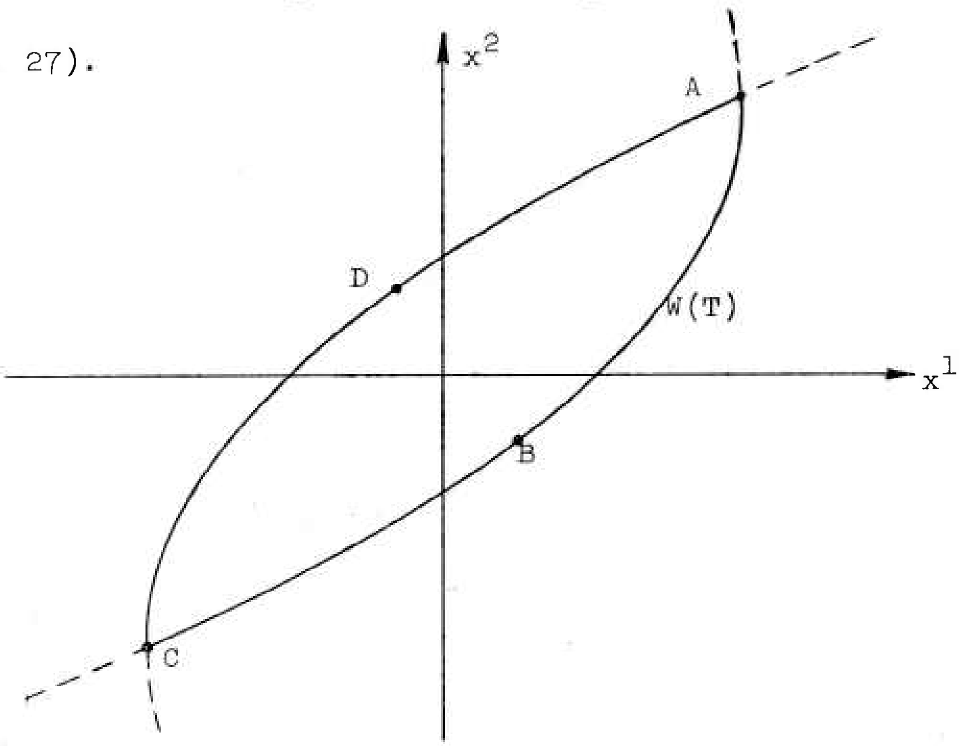

Figure 27

The trajectories for $0 \leq t \leq T$ are given in fig. 28.

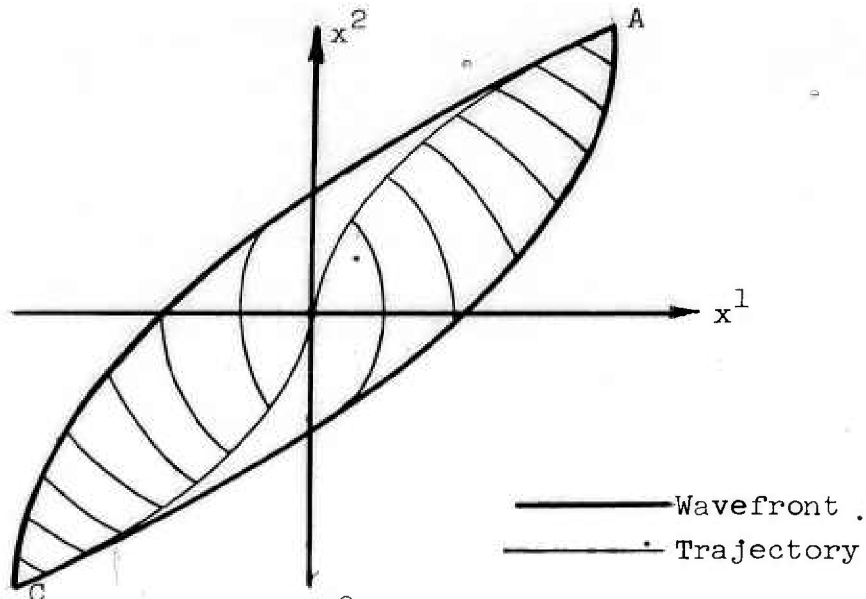

Figure 28

If we consider the successive wavefronts we will have the configuration given in fig. 29.

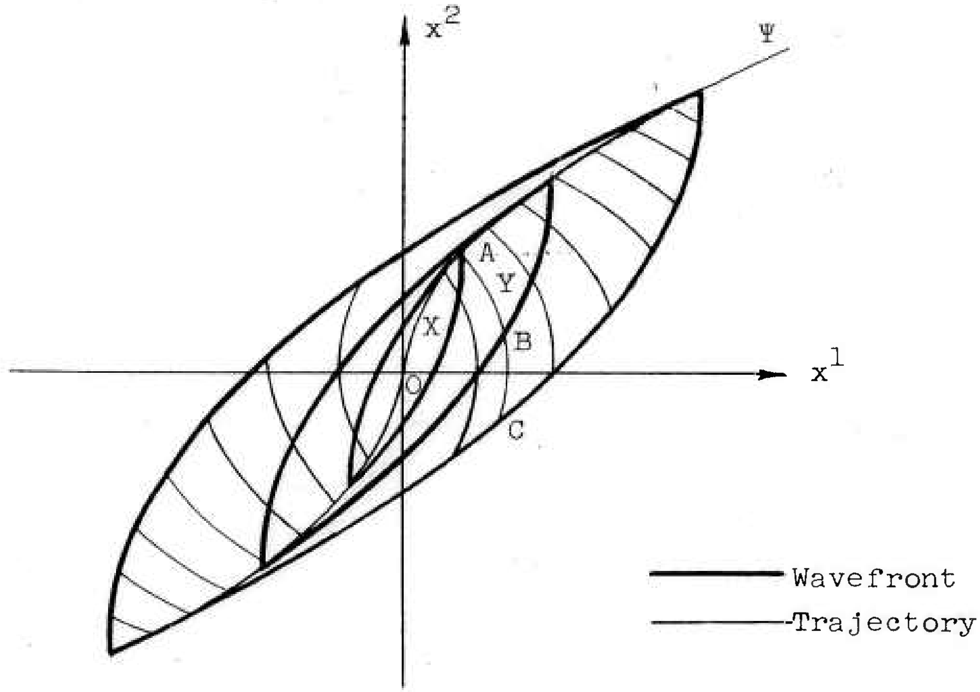

Figure 29

In this example we see clearly the structureof the covering of the whole $(x^1 x^2)$ plane by the field of trajectories from the origin. Between the origin and an arbitrary point of this plane there is one and only one such trajectory, hence the solution corresponding to the given end point exists and is unique.

If we consider the trajectory OABC we can see how the Theorem II applies: At any point X on OA the wavefront is one sided differentiable and this property is conserved for any point of the trajectory beyond X. At any point Y on AC the wavefront admits a unique normal and this property is conserved for any point of the trajectory beyond Y.

A direct construction of the field of trajectories without applying the Imbedding Method would have given the following results:

For any point X of $O\psi$ we have the choice between $u = +1$ and $u = -1$, but this possibility of choice is conserved if and only if we choose always $u = +1$.

If we compare this result with the content of Theorem II we can say: When the topology on a wavefront is high, the number of choices is low and reciprocal.

II <u>Bushaw's Problem, i.e., Problem of Minimum Settling Time for a Second Order System with Subcritical Damping</u>.

$$\begin{aligned} \dot{x}^0 &= 1 \\ \dot{x}^1 &= -x^2 \\ \dot{x}^2 &= +x^1 + u \end{aligned} \qquad (1)$$

For simplicity zero damping is assumed.

$$x^1(0) = x^2(0) = 0$$

with

$$x^1(T) = a, \quad x^2(T) = b$$

and $u \leq 1 \Longleftrightarrow u \in \Omega$.

One should note that x^0 can be ignored as in I.

Maximum Principle

$$h = \langle p | f \rangle = p_1 f^1 + p_2 f^2 = -p_1 x^2 + p_2(x^1 + u)$$

Optimal control: $$u = \underset{u \in \Omega}{\operatorname{argmax}} \; H = \operatorname{sgn} p_2 \qquad (2)$$

Differential System for p

$$\dot{p}_1 = -\frac{\partial H}{\partial x^1} = -p_2$$
$$\dot{p}_2 = -\frac{\partial H}{\partial x^2} = +p_1 \qquad (3)$$

with the solution $p_1 = A \cos(t + \phi)$

$p_2 = A \sin(t + \phi)$

But as the differential system (3) and the switching function (2) are linear and homogeneous in p_1 and p_2 we can normalize the vector p, i.e., let $A = 1$, which gives

$$p_1 = \cos(t + \phi)$$
$$p_2 = \sin(t + \phi) \qquad (4)$$

where ϕ is determined by $p(0)$ (see fig. 30).

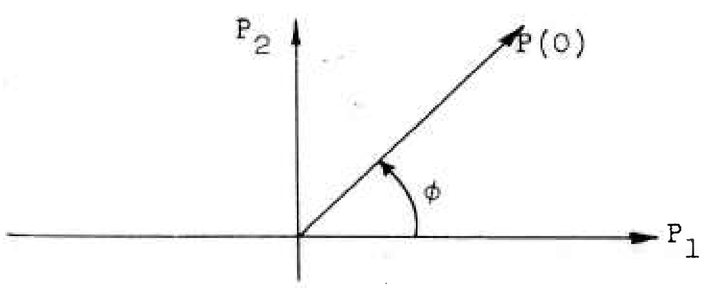

Figure 30

Analytical Forms of the Two Types of Arcs

The integration of (1) for $u = +1$ gives

$$x^1(t) = \mathcal{A}\cos(t+\psi) + 1$$
$$x^2(t) = \mathcal{A}\sin(t+\psi) \qquad (5)$$

with $\mathcal{A}$ and ψ determined by $x(0)$.
This trajectory is a circle of center $(+1,0)$ and of frequency 1 (c.p.s.). Similarly the integration of (1) for $u = -1$ gives

$$x^1(t) = \mathcal{A}\cos(t+\psi) - 1$$
$$x^2(t) = \mathcal{A}\cos(t+\psi) \qquad (5)'$$

where $\mathcal{A}$ and ψ are also determined by $x(0)$.
This trajectory is a circle of center $(-1,0)$ and of frequency 1.

Construction of the Trajectory Corresponding to a Particular $p(0)$, i.e., to a Particular ϕ.

We have found $\quad u = \operatorname{sgn} p_2$

and $\quad p_2 = \sin(t+\phi)$

hence

$$u(t) = \operatorname{sgn} \sin(t+\phi) \qquad (6)$$

In other words, the control function u will be alternatively $+1$ and -1 the length of an interval being π, with the exception of the first and the last intervals which may have a length smaller than π.
Conclusions: if $0 \leq \phi \leq \pi$ $u(t)$ will be

$+1$ on $[0, \pi-\phi]$

-1 on $[\pi-\phi, 2\pi-\phi]$

$\vdots$

$(-1)^n$ on $[n\pi-\phi, (n+1)\pi-\phi]$, etc., (see fig. 31).

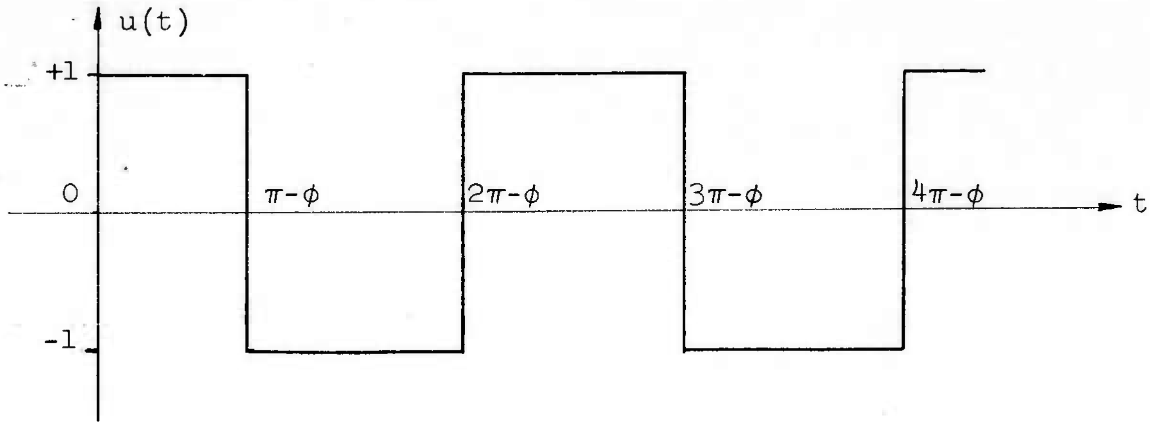

Figure 31

Similarly if $-\pi \leq \phi \leq 0$, $u(t)$ will be

-1 on $[0, -\phi]$

$+1$ on $[-\phi, -2+\pi]$

$\vdots$

$(-1)^n$ on $[-\phi+n\pi, -\phi+(n+1)\pi]$

An example for $\phi = 90^\circ$ is given in fig. 32.

Switching Curves

If we call N_1 the locus of the first switching of a solution starting with $u = -1$ (such as the point A on fig. 32), N_1 will be the upper half unit circle around $(-1,0)$ (see fig. 33). Similarly P_1, the locus of the first switching of a solution starting with $u = +1$, will be the lower half unit circle around $(+1,0)$.

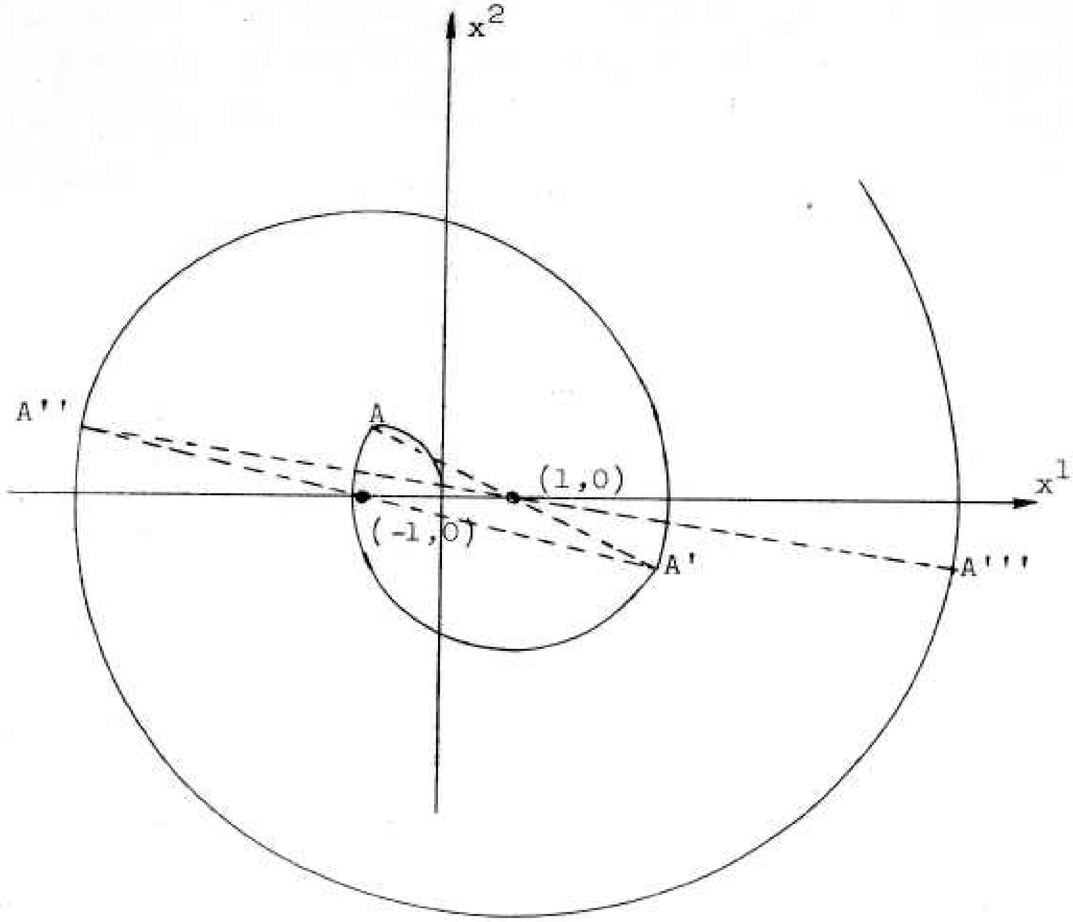

Figure 32

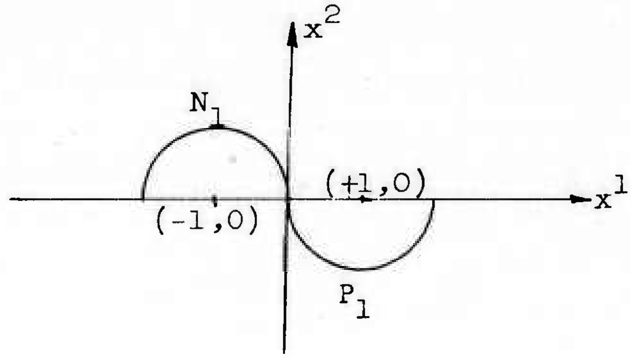

Figure 33

If N_i is the locus of the i^{th} switching of a solution starting with $u = -1$, and P_i the i^{th} switching of a solution starting with $u = +1$, and if we use the operator n to mean an inversion through (-1,0) (see dashed lines in fig. 32), and p an inversion through (+1,0), we obtain immediately:

$$N_{2n} = pN_{2n-1}$$

$$N_{2n+1} = nN_{2n}$$

$$P_{2n} = nP_{2n-1}$$

$$P_{2n+1} = pP_{2n}$$

These rules and the knowledge of N_1 and P_1 give us by induction the whole switching curve [ref. 5] (see fig. 34).

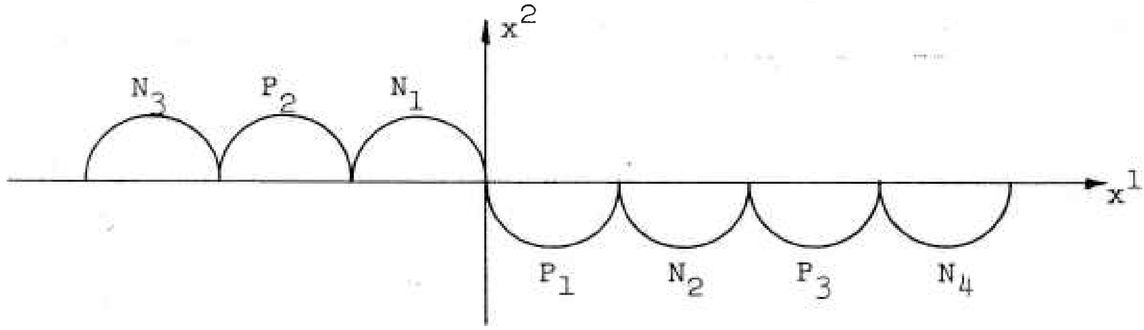

Figure 34

The whole set of trajectories is shown in fig. 35.

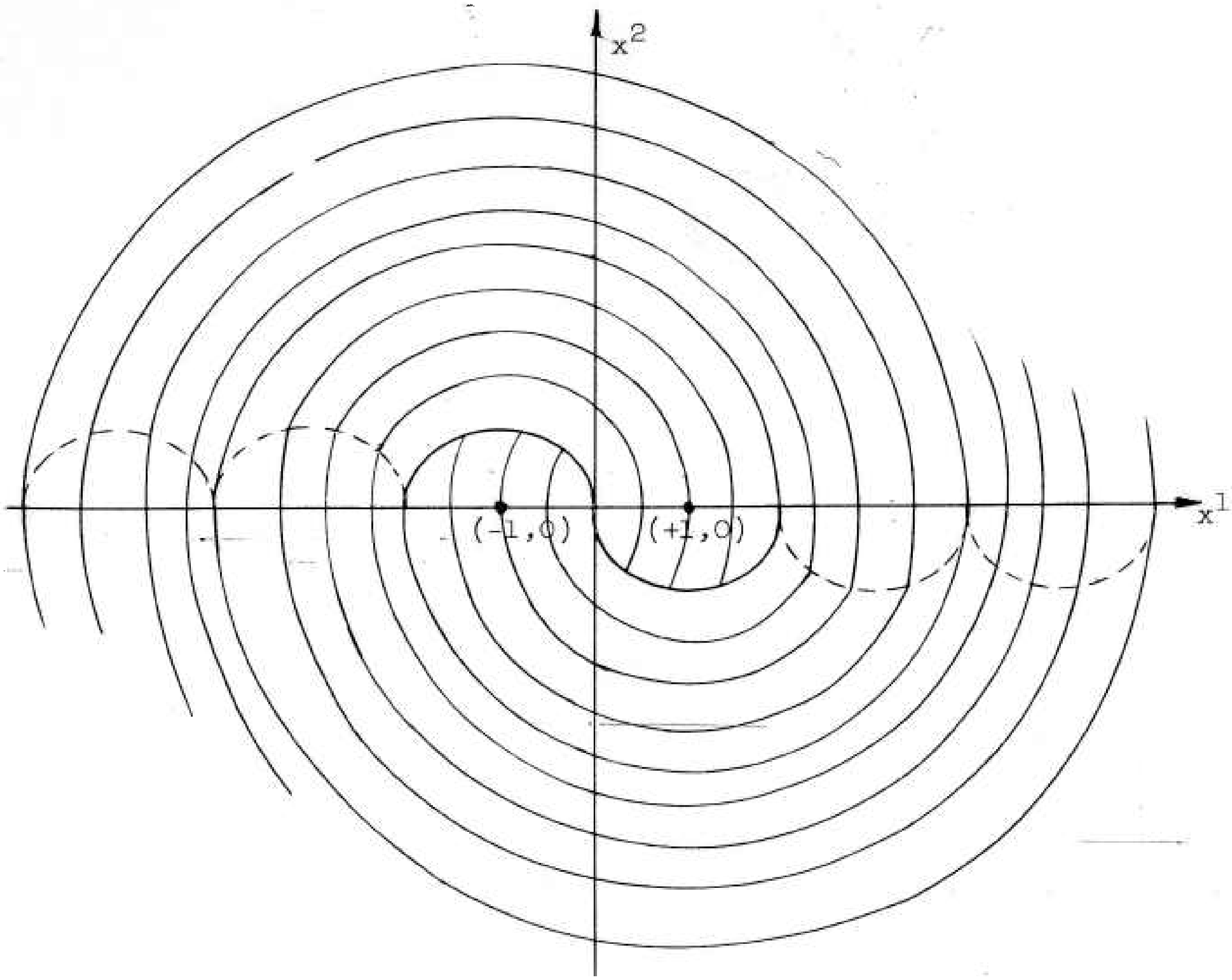

Figure 35

Obviously every point of the plane is reached by one and only by one trajectory.

Wavefront for $t = T$

Let $x(T,\phi)$ be the value of $x(T)$ corresponding to

$$p(0) = (\cos\phi, \sin\phi)$$

(see fig. 30).

The wavefront for $t = T$ is the locus of $x(T,\phi)$ for all ϕ, i.e.,

$$W(T) = \left\{ x(T,\phi) : -\pi \leq \phi \leq \pi \right\}$$

We know that

$$u(t) = \operatorname{sgn} \sin(t + \phi)$$

By integration of the initial system with such a control we find after tedious but simple computations (see Appendix) that:*

$$X(T,\phi) = x^1(T,\phi) + ix^2(T,\phi) = -2ke^{i(T+\phi)} - e^{iT} + e^{i\pi k} \qquad \text{if } 0 \leq \phi \leq \pi(1-k_1) \qquad (7a)$$

$$= -2(k+1)\, e^{i(T+\phi)} - e^{iT} - e^{i\pi k} \qquad \text{if } \pi(1-k_1) \leq \phi \leq \pi \qquad (7b)$$

$$= -2(k+1)\, e^{i(T+\phi)} + e^{iT} + e^{i\pi k} \qquad \text{if } -k_1\pi \leq \phi \leq 0 \qquad (7c)$$

$$= -2ke^{i(T+\phi)} + e^{iT} - e^{i\pi k} \qquad \text{if } -\pi \leq \phi \leq -k_1\pi \qquad (7d)$$

where k_1 and k are defined by**

---

*Please note that $i = \sqrt{-1}$, the imaginary unit, and not an integer as in other places.

**By definition [x] is the largest integer smaller or equal to x.

$$k = [\tfrac{T}{\pi}]$$

$$k_1 = \frac{T}{\pi} - [\tfrac{T}{\pi}] \qquad (8)$$

In other words $T = \pi(k_1 + k)$

with $0 \leq k_1 < 1$ and

k integer

If we analyze the expressions (7) we can see that in general W(T) will be constituted of four circular arcs joined with continuous slope (see fig. 36).

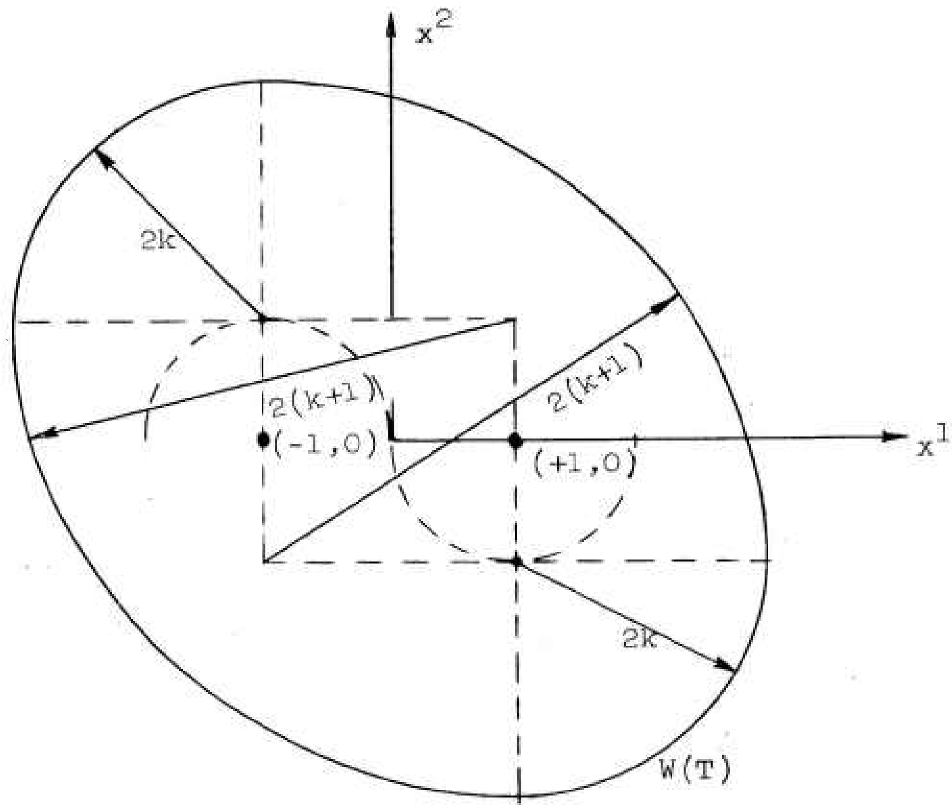

Figure 36

This drawing is made for $T = \frac{3}{2}\pi$ which can be written $T = \frac{1}{2}\pi + 1\cdot\pi$ which gives $k_1 = \frac{1}{2}$ and $k = 1$.

This general structure can degenerate and give the following situations:

(1) if $k = 1$ the length of the two arcs of radius $2(k+1)$ is zero and the two other arcs of radius $2k$ join tangentially (in fact they form a circle, see fig. 37).

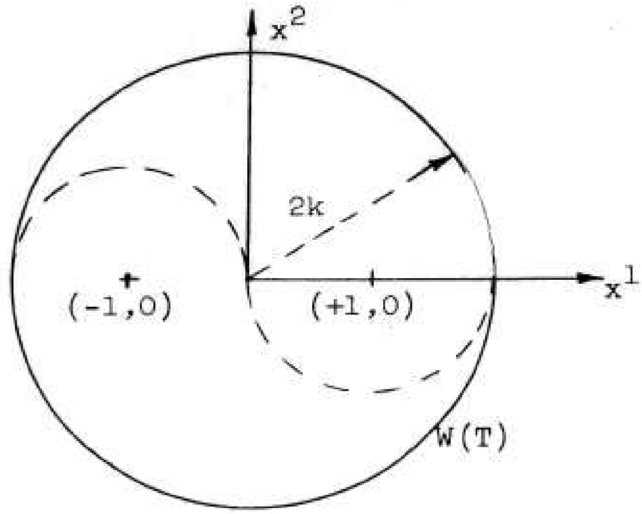

Figure 37

This drawing is made for $T = \pi$ which can be written $T = 0\cdot\pi + 1\cdot\pi$, i.e., $k_1 = 0$, $k = 1$.

(2) if $k = 0$ the two arcs of radius $2k$ disappear and the two other arcs join, but not tangentially (see fig. 38).

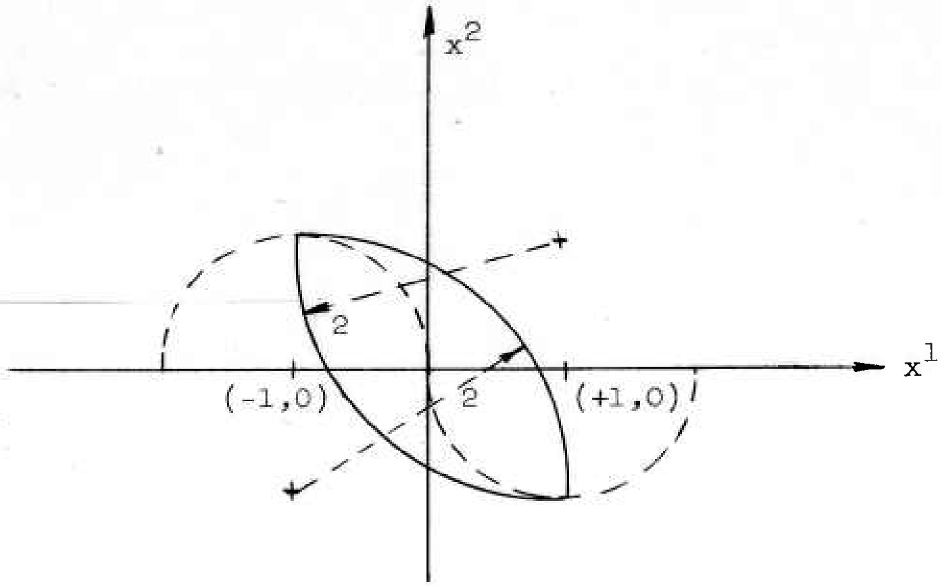

Figure 38

This drawing is made for $T = \frac{\pi}{2}$ which can be written $T = \frac{1}{2} \cdot \pi + 0 \cdot \pi$, i.e., $k_1 = \frac{1}{2}$ and $k = 0$.

<u>Situation (2) happens if and only if $T < \pi$</u>

In figure 39 we have drawn the successive wavefronts for $T = \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi, \cdots$ and the trajectories corresponding to $\phi = 0, \pm\frac{\pi}{3}, \pm\frac{\pi}{2}, \pm\frac{2}{3}\pi, \pi$.

If we consider the trajectory OABCDEF $\cdots$ corresponding to $\phi = -\frac{2}{3}\pi$ we can repeat the remarks of example I on the applications of Theorem II and Corrolary II. In short at A and B $W(t)$ is not differentiable, but from C outward, $W(t)$ is and remains differentiable.

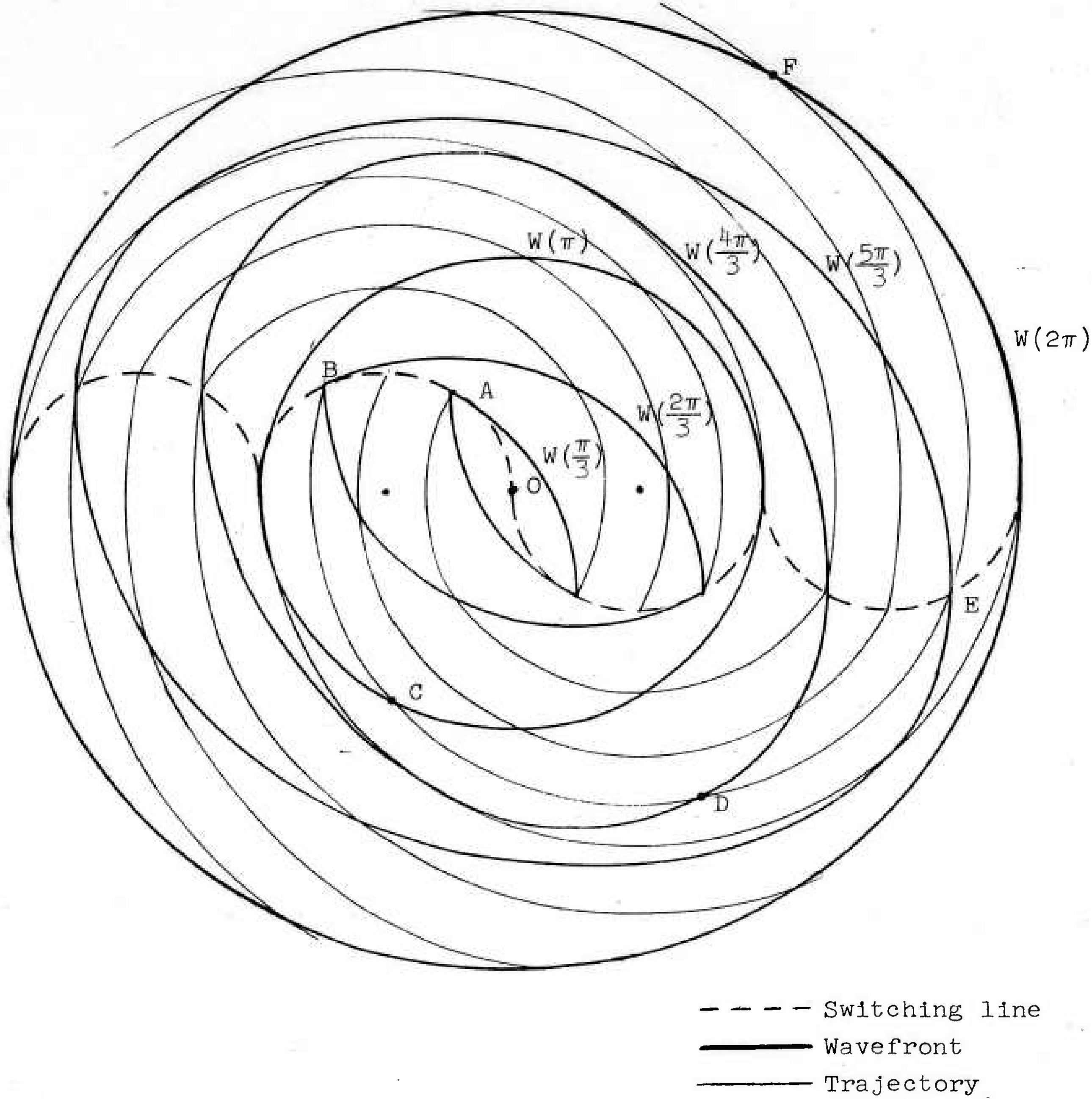
F
W(π)
W(4π/3)
W(5π/3)
W(2π)
B
A
W(2π/3)
W(π/3)
O
E
C
D
Switching line
Wavefront
Trajectory

Figure 39

APPENDIX

DERIVATION OF THE EQUATION FOR THE WAVEFRONT W(T) IN BUSHAW'S PROBLEM

As we have seen, from equation 5 in Section XIII, the integration of the system, eq. 1 Section XIII, for $u = +1$ gives

$$x^1(t) = \mathcal{A}\cos(t+\psi) + 1 \qquad (1)$$
$$x^2(t) = \mathcal{A}\sin(t+\psi)$$

With the convention

$$X(t) = x^1(t) + ix^2(t) \qquad (2)$$

(1) can be written

$$X(t+t_o) = [X(t_o) - 1]e^{it} + 1 \qquad (3)$$

and in the particular case $t = \pi$ we have

$$X(\pi+t_o) = - X(t_o) + 2 \qquad (4)$$

Similarly the integration of the system, eq.(1) Section XIII for $u = -1$ gives:

$$\left.\begin{aligned} x^1(t) &= \mathcal{A}\cos(t+\psi) - 1 \\ x^2(t) &= \mathcal{A}\sin(t+\psi) \end{aligned}\right\} \qquad (5)$$

With the convention (2), (5) can be written

$$X(t+t_o) = [X(t_o) + 1]e^{it} - 1 \qquad (6)$$

and in the particular case $t = \pi$

$$X(\pi+t_o) = - X(t_o) - 2 \qquad (7)$$

The formulae (3) and (6) can be replaced by the unique formula

$$X(t+t_o) = [X(t_o) - u]e^{it} + u \qquad (8)$$

The formula (8) is valid for a <u>constant u</u> which can be either +1 or -1.

Similarly (4) and (7) can be replaced by

$$X(\pi + t_o) = - X(t_o) + 2u \tag{9}$$

We have seen, Section XIII, eq.(6), that

$$u(t) = \text{sgn}\, \sin(t+\phi) \tag{10}$$

In Section XIII, eq.(8), we have introduced the unique decomposition

$$T = \pi(k_1 + k)$$

with $k = [\frac{T}{\pi}]$

and $k_1 = \frac{T}{\pi} - [\frac{T}{\pi}]$

In other words k is integer

and $0 \leq k_1 < 1$.

In the following computations we will have to distinguish different cases depending on the parity of k. For this purpose we will introduce further decomposition

$$k = k_2 + 2k_3$$

with

$$k_2 = 0 \text{ or } 1$$

and k_3 integer.

This gives finally

$$T = \pi(k_1 + k_2 + 2k_3) \tag{11}$$

with

$$0 \leq k_1 < 1$$

$$k_2 = 0 \text{ or } 1$$

$$k_3 \text{ integer}$$

Let $X(T,\phi)$ be the value of $X(T)$ corresponding to $p(0) = (\cos\phi, \sin\phi)$.

The wavefront for $t = T$ is the locus of $X(T,\phi)$ for all ϕ, i.e.,

$$W(T) = \left\{ X(T,\phi): -\pi \leq \phi \leq \pi \right\} \tag{12}$$

In order to solve our problem we have to exhibit a general expression for $X(T,\phi)$ for all $T \geq 0$ and all $-\pi \leq \phi \leq \pi$. We will distinguish 8 cases

$k_2 = 0$	and	$0 \leq \phi \leq \pi(1-k_1)$	Case no. 1
$k_2 = 0$	and	$\pi(1-k_1) \leq \phi \leq \pi$	Case no. 2
$k_2 = 0$	and	$-k_1\pi \leq \phi \leq 0$	Case no. 3
$k_2 = 0$	and	$-\pi \leq \phi \leq -k_1\pi$	Case no. 4
$k_2 = 1$	and	$0 \leq \phi \leq \pi(1-k_1)$	Case no. 5
$k_2 = 1$	and	$\pi(1-k_1) \leq \phi \leq \pi$	Case no. 6
$k_2 = 1$	and	$-k_1\pi \leq \phi \leq 0$	Case no. 7
$k_2 = 1$	and	$-\pi \leq \phi \leq -k_1\pi$	Case no. 8

We see immediately that

$$X(T,\phi) = - X(T,\phi+\pi) = - X(T,\phi-\pi) \quad (13)$$

because $W(T)$ is symmetric with respect to the center of coordinates $X(0,\phi) = 0$.

With the help of relation (13) the computation of the cases 3,4,7 and 8 will be particularly easy when the results of the cases 1,2,5 and 6 will be known.

Case no. 1

$k_2 = 0$ and $0 \leq \phi \leq \pi(1-k_1)$

From (10) and (11) we see that

$u = +1$	on	$(0,\pi-\phi)$	$(1\cdot 1)$
$u = -1$	on	$(\pi-\phi,2\pi-\phi)$	$(1\cdot 2)$
$u = +1$	on	$(2\pi-\phi,3\pi-\phi)$	$(1\cdot 3)$
$\vdots$			
$u = -1$	on	$((2k_3-1)\pi-\phi,2\pi k_3-\phi)$	$(1\cdot 2k_3)$
$u = +1$	on	$(2\pi k_3-\phi,T)$	$(1\cdot 2k_3+1)$

Conclusion: in the case no. 1 we will have $2k_3+1$ subarcs.

By assumption

$$X(0,\phi) = 0$$

that means the wave propagation starts at the origin of the state-space. By application of (1·1) and (8)

$$X(\pi-\phi,\phi) = [X(0,\phi)-1]e^{i(\pi-\phi)} + 1$$
$$= e^{-i\phi} + 1$$

By application of (1·2) and (9)

$$X(2\pi-\phi,\phi) = - X(\pi-\phi,\phi) - 2$$
$$= -e^{-i\phi} - 3$$

By application of (1·3) and (9)

$$X(3\pi-\phi,\phi) = - X(2\pi-\phi,\phi) + 2$$
$$= +e^{-i\phi} + 5$$

etc.

By recurrent applications of (1·i) and (9) for $i = 2, 3, \cdots, 2k_3$ we obtain

$$X(2\pi k_3-\phi,\phi) = - X((2k_3-1)\pi-\phi,\phi) - 2$$
$$= -e^{-i\phi} - 4k_3 + 1$$

(proof by induction).

By application of $(1\cdot 2k_3+1)$ and (8)

$$X(T,\phi) = [X(2\pi k_3-\phi,\phi) - 1]e^{i(T-2\pi k_3+\phi)} + 1$$
$$= - 4k_3 e^{i(\pi k_1+\phi)} - e^{i\pi k_1} + 1$$
$$= -2ke^{i(T+\phi)} - e^{iT} + e^{i\pi k}$$

(since $k_2 = 0$ in this case)

Case no. 2

$$k_2 = 0 \quad \text{and} \quad \pi(1-k_1) \leq \phi \leq \pi$$

From (10) and (11) we see that

$u = +1$	on	$(0, \pi-\phi)$	(2·1)
$u = -1$	on	$(\pi-\phi, \phi\pi-\phi)$	(2·2)
$u = +1$	on	$(2\pi-\phi, 3\pi-\phi)$	(2·3)
	⋮		
$u = -1$	on	$((2k_3-1)\pi-\phi, 2\pi k_3-\phi)$	$(2\cdot 2k_3)$
$u = +1$	on	$(2k_3-\phi, (2k_3+1)\pi-\phi)$	$(2\cdot 2k_3+1)$
$u = -1$	on	$((2k_3+1)\pi-\phi, T)$	$(2\cdot 2k_3+2)$

Hence we will have $2k_3+2$ subarcs in case no. 2.

As $(2\cdot i) \equiv (1\cdot i)$ for $i = 1, \cdots, 2k_3$ the computation in case no. 2 will be the same as in case no. 1 up to the point

$$X(2\pi k_3-\phi,\phi) = -e^{i\phi} - 4k_3 + 1$$

By application of (9) and $(2\cdot 2k_3+1)$

$$X((2k_3+1)\pi-\phi,\phi) = -X(2\pi k_3-\phi,\phi) + 2$$
$$= e^{-i\phi} + 4k_3 + 1$$

By application of (8) and $(2\cdot 2k_3+2)$

$$X(T,\phi) = [X((2k_3+1)\pi-\phi,\phi) + 1]e^{i[T-((2k_3+1)\pi-\phi)]} - 1$$
$$= -2(2k_3+1)e^{i(\pi k_1+\phi)} - e^{i\pi k_1} - 1$$
$$= -2(k+1)e^{i(T+\phi)} - e^{iT} - e^{i\pi k}$$

Case no. 3

$$k_2 = 0 \quad \text{and} \quad -k_1\pi \leq \phi \leq 0$$

We use the relation

$$X(T,\phi) = -X(T,\phi+\pi) = -X(T,\phi^*) \tag{13}$$

$X(T,\phi+\pi)$ is given by case no. 2 because $-k_1\pi \leq \phi \leq 0$ implies $(1-k_1)\pi \leq \phi^* \leq \pi$ hence

$$X(T,\phi) = -[-2(k+1)e^{i(T+\phi+\pi)} - e^{iT} - e^{i\pi k}]$$

$$= -2(k+1)e^{i(T+\phi)} + e^{iT} + e^{i\pi k}$$

Case no. 4

$$k_2 = 0 \quad \text{and} \quad -\pi \leq \phi \leq -k_1\pi$$

We use again the relation

$$X(T,\phi) = -X(T,\phi+\pi) \tag{13}$$

$X(T,\phi+\pi)$ is given by case no. 1 because $-\pi \leq \phi \leq -k_1\pi$ implies $0 \leq \phi + \pi \leq (1-k_1)\pi$, hence

$$X(T,\phi) = -[-2ke^{i(T+\phi+\pi)} - e^{iT} + e^{i\pi k}]$$

$$= -2ke^{i(T+\phi)} + e^{iT} - e^{i\pi k}$$

Case no. 5

$$k_2 = 1 \quad \text{and} \quad 0 \leq \phi \leq \pi(1-k_1)$$

From (10) and (11) we see that

$$\dot{u} = +1 \quad \text{on} \quad (0,\pi-\phi) \tag{5·1}$$

$$u = -1 \quad \text{on} \quad (\pi-\phi,2\pi-\phi) \tag{5·2}$$

$$u = +1 \quad \text{on} \quad (2\pi-\phi,3\pi-\phi) \tag{5·3}$$

$$\vdots$$

$$u = -1 \quad \text{on} \quad ((2k_3-1)\pi-\phi,2\pi k_3-\phi) \tag{5·2k_3}$$

$$u = +1 \quad \text{on} \quad (2\pi k_3-\phi,\pi(2k_3+1)-\phi) \tag{5·2k_3+1}$$

$$u = -1 \quad \text{on} \quad (\pi(2k_3+1)-\phi,T) \tag{5·2k_3+2}$$

Hence we will have $2k_3+2$ subarcs in case no. 5. As $(5·i) \equiv (2·i)$ for $i = 1, \cdots, 2k_3+1$ the computation in case no. 5 will be the same as in case no. 2 up to the point

$$X((2k_3+1)\pi-\phi,\phi) = e^{-i\phi} + 4k_3+1$$

By application of (8)

$$X(T,\phi) = [X((2k_3+1)\pi-\phi,\phi) + 1]e^{i[T-((2k_3+1)\pi-\phi)]} - 1$$

$$= e^{i\pi k_3} + 2(2k_3+1)e^{i(\pi k_1+\phi)} - 1$$

$$= -2ke^{i(T+\phi)} - e^{iT} + e^{i\pi k}$$

Case no. 6

$$k_2 = 1 \quad \text{and} \quad \pi(1-k_1) \leq \phi \leq \pi$$

From (10) and (11) we see that

$$\dot{u} = +1 \quad \text{on} \quad (0,\pi-\phi) \tag{6·1}$$

$$u = -1 \quad \text{on} \quad (\pi-\phi,2\pi-\phi) \tag{6·2}$$

$$u = +1 \quad \text{on} \quad (2\pi-\phi,3\pi-\phi) \tag{6·3}$$

$$\vdots$$

$$u = +1 \quad \text{on} \quad (2\pi k_3-\phi,\pi(2k_3+1)-\phi) \tag{6·2k_3+1}$$

$$u = -1 \quad \text{on} \quad (\pi(2k_3+1)-\phi,\pi(2k_3+2)-\phi) \tag{6·2k_3+2}$$

$$u = +1 \quad \text{on} \quad (\pi(2k_3+2)-\phi,T) \tag{6·2k_3+3}$$

Hence we will have $2k_3+3$ subarcs in case no. 6. As $(6 \cdot i) \equiv (2 \cdot i)$ for $i = 1, 2, \cdots, 2k_3+1$ the computation in case no. 6 will be the same as in case no. 2 up to the point

$$X((2k_3+1)\pi-\phi,\phi) = e^{-i\phi} + 4k_1 + 1$$

By application of (9)

$$X((2k_3+2)\pi-\phi,\phi) = -X((2k_3+1)\pi-\phi,\phi) - 2$$
$$= -e^{-i\phi} - 4k_3 - 3$$

By application of (8)

$$X(T,\phi) = [X((2k_3+2)\pi-\phi,\phi) - 1]e^{i\left\{T-[\pi(2k_3+2)-\phi]\right\}} + 1$$
$$= -e^{i(k_1+1)} - 2(2k_3+2)e^{i(\pi(k_1+1)+\phi)} + 1$$
$$= -2(k+1)e^{i(T+\phi)} - e^{iT} - e^{i\pi k}$$

Case no. 7

$$k_2 = 1 \quad \text{and} \quad -k_1\pi \leq \phi \leq 0$$

$$X(T,\phi) = -X(T,\phi+\pi) \qquad (13)$$

but $X(T,\phi+\pi)$ is given by case no. 6 because $-k_1\pi \leq \phi \leq 0$ implies $(1-k_1)\pi \leq \phi+\pi \leq \pi$. Hence,

$$X(T,\phi) = -[-2(k+1)e^{i(T+\phi+\pi)} - e^{iT} - e^{i\pi k}]$$
$$= -2(k+1)e^{i(T+\phi)} + e^{iT} + e^{i\pi k}$$

Case no. 8

$$k_2 = 1 \quad \text{and} \quad -\pi \leq \phi \leq -k_1\pi$$

$$X(T,\phi) = -X(T,\phi+\pi) \qquad (13)$$

but $X(T,\phi+\pi)$ is given by case no. 5 because $-\pi \leq \phi \leq -k_1\pi$ implies $0 \leq \phi \leq (1-k_1)\pi$. Hence,

$$X(T,\phi) = -[-2ke^{i[T+\phi+\pi]} - e^{iT} + e^{i\pi k}]$$
$$= -2ke^{i(T+\phi)} + e^{iT} - e^{i\pi k}$$

These results can be summarized by:

<u>Case no. 1</u> $k_2 = 0$ and $0 \leq \phi \leq \pi(1-k_1)$

$$X(T,\phi) = -2ke^{i(T+\phi)} - e^{iT} + e^{i\pi k}$$

<u>Case no. 2</u> $k_2 = 0$ and $\pi(1-k_1) \leq \phi \leq \pi$

$$X(T,\phi) = -2(k+1)e^{i(T+\phi)} - e^{iT} - e^{i\pi k}$$

<u>Case no. 3</u> $k_2 = 0$ and $-k_1\pi \leq \phi \leq 0$.

$$X(T,\phi) = -2(k+1)e^{i(T+\phi)} + e^{iT} + e^{i\pi k}$$

<u>Case no. 4</u> $k_2 = 0$ and $-\pi \leq \phi \leq -k_1\pi$

$$X(T,\phi) = -2ke^{i(T+\phi)} + e^{iT} - e^{i\pi k}$$

<u>Case no. 5</u> $k_2 = 1$ and $0 \leq \phi \leq \pi(1-k_1)$

$$X(T,\phi) = -2ke^{i(T+\phi)} - e^{iT} + e^{i\pi k}$$

<u>Case no. 6</u> $k_2 = 1$ and $\pi(1-k_1) \leq \phi \leq \pi$

$$X(T,\phi) = -2(k+1)e^{i(T+\phi)} - e^{iT} - e^{i\pi k}$$

<u>Case no. 7</u> $k_2 = 1$ and $-k_1\pi \leq \phi \leq 0$

$$X(T,\phi) = -2(k+1)e^{i(T+\phi)} + e^{iT}, e^{i\pi k}$$

<u>Case no. 8</u> $k_2 = 1$ and $-\pi \leq \phi \leq -k_1\pi$

$$X(T+\phi) = -2ke^{i(T+\phi)} + e^{iT} - e^{i\pi k}$$

We see immediately that the results in cases 5, 6, 7, and 8 are identical with the results in cases 1, 2, 3, and 4 respectively. In other words, k_2 has no influence on the structure of the final answers. Therefore, we will thus restrict our attention to the cases 1, 2, 3, and 4 only, in the consideration of the shape of the wavefront.

<u>Case no. 1</u> $0 \leq \phi \leq \pi(1-k_1)$

$$X(T+\phi) = -2ke^{i(T+\phi)} - e^{iT} + e^{i\pi k}$$

This is an arc of a circle of radius $2k$ and of center $(-e^{iT} + e^{i\pi k})$.

Extremal values: $$X(T,0) = -(2k+1)e^{iT} + e^{i\pi k} \qquad (14)$$

$$X(T,\pi(1-k_1)) = (2k+1)e^{i\pi k} - e^{iT} \qquad (15)$$

Tangent:* $$\dot{X}(T,\phi) = -2kie^{i(T+\phi)}$$

Extremal tangents: $$\dot{X}(T,0) = -2kie^{iT} \qquad (16)$$

$$\dot{X}(T,\pi(1-k_1)) = +2kie^{i\pi k} \qquad (17)$$

<u>Case no. 2</u> $\pi(1-k_1) \leq \phi \leq \pi$

$$X(T,\phi) = -2(k+1)e^{i(T+\phi)} - e^{iT} - e^{i\pi k}$$

This is an arc of a circle of radius $2(k+1)$ and of center $(-e^{iT} - e^{i\pi k})$.

Extremal values: $$X(T,\pi(1-k_1)) = (2k+1)e^{i\pi k} - e^{iT} \qquad (18)$$

$$X(T,\pi) = (2k+1)e^{iT} - e^{i\pi k} \qquad (19)$$

---

*The dot on $\dot{X}(T,\phi)$ means a derivation with respect to ϕ

$$\dot{X}(T,\phi) = \frac{\partial}{\partial\phi} X(T,\phi)$$

Tangent: $\dot{X}(T,\phi) = -2(k+1)ie^{i(T+\phi)}$

Extremal tangents: $\dot{X}(T,\pi(1-k_1)) = 2(k+1)ie^{i\pi k}$ (20)

$\dot{X}(T,\pi) = 2(k+1)ie^{iT}$ (21)

<u>Case no. 3</u> $-k_1\pi \leq \phi \leq 0$

$$X(T,\phi) = -2(k+1)e^{i(T+\phi)} + e^{iT} + e^{i\pi k}$$

This is an arc or a circle of radius $2(k+1)$ and of center $(e^{iT} + e^{i\pi k})$.

Extremal values: $X(T,-k_1\pi) = -(2k+1)e^{ik\pi} + e^{iT}$ (22)

$X(T,0) = -(2k+1)e^{iT} + e^{i\pi k}$ (23)

Tangent: $\dot{X}(T,\phi) = -2(k+1)ie^{i(T+\phi)}$

Extremal tangents: $\dot{X}(T,-k_1\pi) = -2(k+1)ie^{ik\pi}$ (24)

$\dot{X}(T,0) = -2(k+1)ie^{iT}$ (25)

<u>Case no. 4</u> $-\pi \leq \phi \leq -k_1\pi$

$$X(T,\phi) = -2ke^{i(T+\phi)} + e^{iT} - e^{i\pi k}$$

This is an arc of a circle of radius $2k$ and of center $(+e^{iT} - e^{i\pi k})$.

Extremal values: $X(T,-\pi) = +(2k+1)e^{iT} - e^{i\pi k}$ (26)

$X(T,-k_1\pi) = -(2k+1)e^{i\pi k} + e^{iT}$ (27)

Tangent: $\dot{X}(T,\phi) = -2kie^{i(T+\phi)}$

Extremal tangents: $\dot{X}(T,-\pi) = 2kie^{iT}$ (28)

$\dot{X}(T,-k_1\pi) = -2kie^{i\pi k}$ (29)

By comparing the formulae (14) to (29) we can check the continuity of $X(T,\phi)$ and the continuity, up to a real factor, of $\dot{X}(T,\phi)$, i.e.,the continuity of the direction of the tangent.

REFERENCES

1. G. A. Bliss, Lectures on the Calculus of Variations, University of Chicago Press, 1946.

2. V. G. Boltyanskii, R. V. Gamkrelidze and L. S. Pontryagin, "Theory of Optimal Processes," (Russian), Izvestia Akad. Nauk SSSR, Ser. Mat. 24, No. 1 (1960).

3. J. V. Breakwell, "The Optimizations of Trajectories," J. Soc. Indust. Appl. Math, Vol. 7, No. 2, June 1959, pp. 215-247.

4. L. Brillouin, Les Tenseurs en Mécanique et en Elasticité, Masson, Paris, 1938.

5. D. W. Bushaw, "Optimal Discontinuous Forcing Terms," in Contributions to the Theory of Nonlinear Oscillations, Vol. IV, Ed. by S. Lefschetz, Princeton Press, 1958.

6. C. Carathéodory, Variationsrechnung und Partielle Differentialgleichungen Erster Ordnung. Teubner, Leipzig, 1935.

7. R. Courant und D. Hilbert, Methoden der Mathematischen Physik, Springer, Berlin, 1937.

8. C. A. Desoer, "Pontryagin's Maximum Principle and the Principle of Optimality," Journ. of the Franklin Inst., May 1961, Vol. 271, No. 5.

9. I. Flügge-Lotz, "Discontinuous Automatic Control," AMR Aug. 1961. This survey article gives a detailed analysis and a bibliography, not repeated here, of the present state of research in Discontinuous Control Theory.

10. B. Fraeijs de Veubeke, "Méthodes Variationnelles et Performances Optimales en Aéronautique," Bull. Soc. Math. Belgique, VIII, 2, 1956, pp. 136-157.

11. J. Hadamard, "Le principe de Huygens," Bull. Soc. Math. France; 52, 1924, pp. 610-640.

12. H. Halkin, "Trajectories Optimales," Eng. Dissertation, Université de Liége, 1960.

13. H. Halkin, "The Principle of Optimal Evolution," (to appear in Contributions to the Theory of Differential Equations. Macmillan Co., New York, 1961.)

14. E. Hille and R. S. Philipps, Functional Analysis and Semi-Groups, Am. Math. Soc., Colloquium Publications, 1957.

15. R. E. Kalman, "Contributions to the Theory of Optimal Control," Bol. Soc. Mat. Mexicana, V, 1, 1960, pp. 102-119.

16. J. P. LaSalle, "Time Optimal Control," Bol. Soc. Mat. Mexicana, V, 1, 1960, pp. 102-119.

17. G. Leitmann, "On a Class of Variational Problems in Rocket Flight," J. of the Aerospace Sciences, 26, 9, 1959, pp. 586-591.

18. S. Lie and G. Scheffers, Geometrie der Berührungstransformationen, Teubner, Leipzig, 1896.

19. A. Miele, "A Survey of the Problem of Optimizing Flight Paths of Aircrafts and Missiles," Boeing Sci. Res. Lab., July 1960.

20. L. W. Neustadt, "Synthesizing Time Optimal Control Systems," Journ. of Math. An. and Appl., I, 484-493, 1960.

21. E. D. Roxin, "Reachable Zones in Autonomous Differential Systems," Bol. Soc. Mat. Mexicana, V, 1, 1960, pp. 125-135.

22. J. L. Synge, "Classical Dynamics," in Band III/1 Handbuch der Physik, herausgegeben von S. Flügge, Springer, Berlin, 1960, pp. 1-225.

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