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Tables of Lebedev, Mehler, and Generalized Mehler Transforms

Fritz Oberhettinger

T.P. Higgins

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TABLES OF

LEBEDEV, MEHLER, AND GENERALIZED MEHLER TRANSFORMS

by

Professor Fritz Oberhettinger
Oregon State University

and

T. P. Higgins

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Introduction

Inversion formulas with kernels containing Bessel functions of purely imaginary order [1, vol. 2] and Legendre functions of complex index with the real part $-\frac{1}{2}$ (conical functions) [1, vol. 1] as variable have become prominent in recent times as methods in solving certain boundary value problems of the wave or heat conduction equation involving wedge or conically shaped boundaries. [3], [4], [6], [10], [11], [13].

These inversion formulas are:

A. Lebedev transform [7]

$$g(y) = \int_0^\infty f(x)\, K_{ix}(y)\, dx,$$

(1)

$$f(x) = 2\pi^{-2} x \sinh(\pi x) \int_0^\infty y^{-1} K_{ix}(y)\, g(y)\, dy.$$

$K_{ix}(y)$ is the modified Hankel function $[K_{ix}(y) = \int_0^\infty \exp(-y \cosh t)\cos(xt)\,dt]$.

B. Mehler transform [5], [8], [9]

$$g(y) = \int_0^\infty f(x)\, P_{ix-\frac{1}{2}}(y)\, dx,$$

(2)

$$f(x) = x \tanh(\pi x) \int_1^\infty P_{ix-\frac{1}{2}}(y)\, g(y)\, dy.$$

C. Generalized Mehler transform [12]

(3)

$$g(y) = \int_0^\infty f(x)\, P^k_{ix-\frac{1}{2}}(y)\, dx,$$

$$f(x) = \pi^{-1} x \sinh(\pi x)\Gamma(\tfrac{1}{2} - k + ix)\Gamma(\tfrac{1}{2} - k - ix) \int_1^\infty g(y)\, P^k_{ix-\frac{1}{2}}(y)\, dy.$$

For the condition of validity of the above formulas see the quoted literature. From formula 26 [1, vol. 1, p. 129] (see also the list of notations at the end of this report),

(4)
$$P^k_{ix-\frac{1}{2}}(\cosh a) = (2\pi \sinh a)^{-\frac{1}{2}}\{\exp(-iax)\,\frac{\Gamma(-ix)}{\Gamma(\frac{1}{2}-k-ix)}\,{}_2F_1[\tfrac{1}{2}+k, \tfrac{1}{2}-k; 1+ix; -\tfrac{1}{2}\exp(-a)\operatorname{csch} a] + \exp(iax)\,\frac{\Gamma(ix)}{\Gamma(\frac{1}{2}-k+ix)} \cdot {}_2F_1[\tfrac{1}{2}+k, \tfrac{1}{2}-k; 1-ix; -\tfrac{1}{2}\exp(-a)\operatorname{csch} a]\}.$$

This function obviously is an even function of x and is real for real parameters and real x.

The special case $k = 0$ in (3) yields the case (2). Furthermore from (4)

(5)

$$P^{\frac{1}{2}}_{ix-\frac{1}{2}}(\cosh a) = (2/\pi)^{\frac{1}{2}}(\sinh a)^{-\frac{1}{2}}\cos(ax),$$

$$P^{-\frac{1}{2}}_{ix-\frac{1}{2}}(\cosh a) = (2/\pi)^{\frac{1}{2}}(\sinh a)^{-\frac{1}{2}}\sin(ax).$$

Therefore from (3) for $k = \frac{1}{2}$ putting $y = \cosh a$,

(6)

$$(\sinh a)^{\frac{1}{2}} g(\cosh a) = (2/\pi)^{\frac{1}{2}} \int_0^\infty f(x) \cos(xa)\, dx,$$

$$f(x) = (2/\pi)^{\frac{1}{2}} \int_0^\infty g(\cosh a)(\sinh a)^{\frac{1}{2}} \cos(xa)\, da.$$

For $k = -\frac{1}{2}$

(7)
$$(\sinh a)^{\frac{1}{2}} g(\cosh a) = (2/\pi)^{\frac{1}{2}} \int_0^\infty x^{-1} f(x) \sin(xa)\, dx,$$

$$x^{-1} f(x) = (2/\pi)^{\frac{1}{2}} \int_0^\infty g(\cosh a)\,(\sinh a)^{\frac{1}{2}} \sin(xa)\, da.$$

But these are the Fourier cosine and the Fourier sine transformation formulas which are therefore a special case of (3).

The behavior of the kernel functions in (1), (2), (3) for large positive values of x and fixed argument y is of great importance. One has [1, vol. 2, p. 88],

(8)
$$K_{ix}(y) \sim (2\pi/x)^{\frac{1}{2}} \exp(-\tfrac{\pi}{2} x) \sin[x \log(2x/y) - x + \tfrac{\pi}{4}]$$

for large positive x and fixed y. Furthermore, from (4),

(9)
$$P^k_{ix-\frac{1}{2}}(y) \sim (2\pi \sinh a)^{-\frac{1}{2}} x^{k-\frac{1}{2}} [\exp(-iax - i\tfrac{\pi}{2}k + i\tfrac{\pi}{4}) + \exp(iax + i\tfrac{\pi}{2}k - i\tfrac{\pi}{4})]$$

for large positive x and fixed $y = \cosh a$.

Of further importance are representations of the different types of waves in the form of an integral transform of the kind expressed in (1), (2), and (3). Such representations are:

Cylindrical wave

(10)
$$K_0[\beta(r^2 + r'^2 - 2rr' \cos \Phi)^{\frac{1}{2}}] = \frac{2}{\pi} \int_0^\infty K_{ix}(\beta r) K_{ix}(\beta r') \cosh[x(\pi - |\Phi|)]\, dx,$$

$$0 \le \Phi \le 2\pi.$$

Spherical wave

$$(11)\quad (R^2 + R'^2 - 2RR'\cos\theta)^{-\frac{1}{2}}\exp[-\beta(R^2 + R'^2 - 2RR'\cos\theta)^{\frac{1}{2}}]$$

$$= \frac{2}{\pi}(RR')^{-\frac{1}{2}} \int_0^\infty x \tanh(\pi x)\, P_{ix-\frac{1}{2}}(-\cos\theta)\, K_{ix}(\beta R)\, K_{ix}(\beta R')\, dx,$$

$$0 \le \theta \le 2\pi$$

Generalized spherical wave

$$(12)\quad (R^2 + R'^2 - 2RR'\cos\theta)^{-\frac{1}{2}\alpha} K_\alpha[\beta(R^2 + R'^2 - 2RR'\cos\theta)^{\frac{1}{2}}]$$

$$= 2^{\frac{1}{2}}\pi^{-\frac{3}{2}}(RR')^{-\alpha}(\sin\theta)^{\frac{1}{2}-\alpha} \int_0^\infty x \sinh(\pi x)\Gamma(\alpha + ix)\Gamma(\alpha - ix) \cdot$$

$$\cdot P^{\frac{1}{2}-\alpha}_{ix-\frac{1}{2}}(-\cos\theta)\, K_{ix}(\beta R)\, K_{ix}(\beta R')\, dx,$$

$$\operatorname{Re}\alpha > -1, \quad 0 \le \theta \le 2\pi$$

The following tables (A), (B), (C) represent a list of integral transforms of the type (1), (2), (3). Most of the results displayed here are new and have been taken from unpublished material of the authors.

Certain combinations of Bessel functions which occur on the r.h.s. of these tables can be replaced by other combinations such as:

$$(13)\quad J_\alpha(x)\cos(\tfrac{1}{2}\pi\alpha) - Y_\alpha(x)\sin(\tfrac{1}{2}\pi\alpha) = \tfrac{1}{2}\sec(\tfrac{1}{2}\pi\alpha)\,[J_\alpha(x) + J_{-\alpha}(x)]$$

$$= -\tfrac{1}{2}\csc(\tfrac{1}{2}\pi\alpha)\,[Y_\alpha(x) - Y_{-\alpha}(x)],$$

$$(14)\quad J_\alpha(x)\sin(\tfrac{1}{2}\pi\alpha) + Y_\alpha(x)\cos(\tfrac{1}{2}\pi\alpha) = \tfrac{1}{2}\csc(\tfrac{1}{2}\pi\alpha)\,[J_\alpha(x) - J_{-\alpha}(x)]$$

$$= \tfrac{1}{2}\sec(\tfrac{1}{2}\pi\alpha)\,[Y_\alpha(x) + Y_{-\alpha}(x)],$$

$$(15)\quad J_\alpha(x)Y_{-\alpha}(y) + J_{-\alpha}(y)Y_\alpha(x) = \csc(\pi\alpha)[J_\alpha(x)J_\alpha(y) - J_{-\alpha}(x)J_{-\alpha}(y)]$$

$$= \csc(\pi\alpha)[Y_{-\alpha}(x)Y_{-\alpha}(y) - Y_\alpha(x)Y_\alpha(y)],$$

$$(16)\quad J_\alpha(x)Y_\alpha(y) - Y_\alpha(x)J_\alpha(y) = J_{-\alpha}(x)Y_{-\alpha}(x) - Y_{-\alpha}(x)J_{-\alpha}(y)$$

$$= \csc(\pi\alpha)[J_\alpha(y)J_{-\alpha}(x) - J_\alpha(x)J_{-\alpha}(y)],$$

$$(17)\quad J_\alpha(x)Y_{-\alpha}(y) - J_{-\alpha}(x)Y_\alpha(y) = \sin(\pi\alpha)[J_\alpha(x)J_\alpha(y) + Y_\alpha(x)Y_\alpha(y)]$$

$$= \sin(\pi\alpha)[J_{-\alpha}(x)J_{-\alpha}(y) + Y_{-\alpha}(x)Y_{-\alpha}(y)].$$

Table A

Lebedev Transform

$$g(y) = \int_0^\infty f(x)K_{ix}(y)dx$$

$$f(x) = 2\pi^{-2}\sinh(\pi x)\int_0^\infty y^{-1}K_{ix}(y)g(y)dy$$

$f(x)$	$g(y) = \int_0^\infty f(x)\, K_{ix}(y)\, dx$
x^2	$\frac{1}{2}\pi y \exp(-y)$
x^{2n}	$\frac{1}{2}\pi (-1)^n \left[\frac{d^{2n}}{dz^{2n}} \exp(-y \cosh z) \right]_{z=0}$
$(a^2 + x^2)^{-1}$	$\frac{1}{2}\pi a^{-1} \int_0^\infty \exp(-y \cosh t - at)\, dt$
$(a^2 + x^2)^{-\frac{1}{2}}$	$\int_0^\infty \exp(-y \cosh t) K_0(at)\, dt$
$\exp(-ax)$	$a \int_0^\infty (a^2 + t^2)^{-1} \exp(-y \cosh t)\, dt$
$\cos(ax)$	$\frac{1}{2}\pi \exp(-y \cosh a)$
$x \sin(ax)$	$\frac{1}{2}\pi y \sinh a \exp(-y \cosh a)$
$\sin(ax) \sinh(bx)$	$\frac{1}{2}\pi \exp(-y \cos b \cosh a) \sin(y \sin b \sinh a)$
$\cos(ax) \cosh(bx)$	$\frac{1}{2}\pi \exp(-y \cos b \cosh a) \cos(y \sin b \sinh a)$
$\sinh(ax) \sinh(bx)$	$\frac{1}{2}\pi \exp(-y \cos a \cos b) \sinh(y \sin a \sin b)$ $a + b \leq \frac{1}{2}\pi$
$\cosh(ax) \cosh(bx)$	$\frac{1}{2}\pi \exp(-y \cos a \cos b) \cosh(y \sin a \sin b)$ $a + b \leq \frac{1}{2}\pi$

$f(x)$	$g(y) = \int_0^\infty f(x)\, K_{ix}(y)\, dx$
$\mathrm{sech}(\frac{1}{2}\pi x)$	$\frac{1}{2}\pi\{1 - y[K_0(y)L_{-1}(y) + L_0(y)K_1(y)]\}$
$\mathrm{sech}(\pi x)\cosh(ax)$	$\frac{1}{2}\pi \exp(y\cos a)\mathrm{Erfc}[(2y)^{\frac{1}{2}}\lvert\cos(\frac{1}{2}a)\rvert]$ $a \leq \frac{3\pi}{2}$
$\mathrm{sech}(\frac{1}{2}\pi x)\cosh(ax)$	$y\int_0^\infty (y^2+t^2)^{-\frac{1}{2}}\exp(-t\cos a)K_1[(y^2+t^2)^{\frac{1}{2}}]$ $a \leq \pi$
$\mathrm{csch}(\frac{1}{2}\pi x)\sinh(ax)$	$\sin a\int_0^\infty \exp(-t\cos a)K_0[(y^2+t^2)^{\frac{1}{2}}]$ $a \leq \pi$
$\mathrm{csch}(\pi x)\sinh(ax)$	$\frac{1}{2}\sin a\int_0^\infty \exp(-t\cos a)K_0(y+t)\,dt$ $a \leq \frac{3\pi}{2}$
$\tanh(\pi x)\sinh(ax)$	$\frac{1}{2}\pi \exp(-y\cos a)\mathrm{Erf}[(2y)^{\frac{1}{2}}\sin(\frac{1}{2}a)]$ $a \leq \frac{1}{2}\pi$
$\mathrm{sech}(\pi x)\sinh(ax)\sinh(bx)$	$\frac{1}{4}\pi\{\exp[y\cos(a+b)]\mathrm{Erfc}[(2y)^{\frac{1}{2}}\cos(\frac{1}{2}a+\frac{1}{2}b)]$ $-\exp[y\cos(a-b)]\mathrm{Erfc}[(2y)^{\frac{1}{2}}\cos(\frac{1}{2}a-\frac{1}{2}b)]\}$ $a+b \leq \frac{3\pi}{2}$
$\mathrm{sech}(\pi x)\cosh(ax)\cosh(bx)$	$\frac{1}{4}\pi\{\exp[y\cos(a+b)]\mathrm{Erfc}[(2y)^{\frac{1}{2}}\cos(\frac{1}{2}a+\frac{1}{2}b)]$ $+\exp[y\cos(a-b)]\mathrm{Erfc}[(2y)^{\frac{1}{2}}\cos(\frac{1}{2}a-\frac{1}{2}b)]\}$ $a+b \geq \frac{3\pi}{2}$
$x\tanh(\frac{1}{2}\pi x)$	$yK_0(y)$

$f(x)$	$g(y) = \int_0^\infty f(x)\, K_{ix}(y)\, dx$
$\operatorname{csch}(bx)\sinh(\pi x)\cosh(ax)$	$\frac{1}{2}\pi^2 b^{-1} \sum_{n=0}^{\infty} (-1)^n \varepsilon_n I_{n\frac{\pi}{b}}(y)\cos(n\pi a/b)$ $b - a \geq \frac{1}{2}\pi$
$\tanh(\pi x)\sinh(bx)\operatorname{csch}(ax)$	$\frac{1}{4}\pi\{\exp[-y\cos(b+a)]\operatorname{Erf}[(2y)^{\frac{1}{2}}\sin(\frac{1}{2}b + \frac{1}{2}a)]$ $+ \exp[-y\cos(b-a)]\operatorname{Erf}[(2y)^{\frac{1}{2}}\sin(\frac{1}{2}b - \frac{1}{2}a)]\}$ $a + b \leq \frac{1}{2}\pi$
$x\sinh(\pi x)\Gamma(k + \frac{1}{2}ix)\Gamma(k - \frac{1}{2}ix)$	$2^{1-2k}\pi^2 y^{2k}$ $0 \leq \operatorname{Re} k \leq \frac{1}{4}$
$x\sinh(\pi x)\,\Gamma(k - \frac{1}{2}ix)\Gamma(k + \frac{1}{2}ix)\,\cdot$ $\cdot\,\Gamma(\frac{1}{2} - k - \frac{1}{2}ix)\Gamma(\frac{1}{2} - k + \frac{1}{2}ix)$	$2^{\frac{3}{2}}\pi^{\frac{5}{2}}y^{\frac{1}{2}}K_{\frac{1}{2}-2k}(y)$
$x\tanh(\frac{1}{2}\pi x)\, P_{\frac{1}{2}ix-\frac{1}{2}}(z)$	$yK_0[y(\frac{1}{2} + \frac{1}{2}z)^{\frac{1}{2}}]$
$x\tanh(\pi x)\, P_{ix-\frac{1}{2}}(z)$	$(\frac{1}{2}\pi y)^{\frac{1}{2}}\exp(-zy)$
$x\operatorname{sech}(\pi x)\,\tanh(\pi x)\, P_{ix-\frac{1}{2}}(z)$	$-(2\pi)^{-\frac{1}{2}}y^{\frac{1}{2}}\exp(zy)\operatorname{Ei}(-zy - y)$
$x\sinh(\pi x)\operatorname{sech}(2\pi x)\, P_{i2x-\frac{1}{2}}(z)$	$2^{-\frac{7}{4}}y^{\frac{1}{4}}\exp(\frac{1}{2}z^2 y - \frac{1}{2}y)D_{-\frac{3}{2}}[z(2y)^{\frac{1}{2}}]$
$x\sinh(\frac{1}{2}\pi x)[P_{ix-\frac{1}{2}}(z)]^2$	$\frac{1}{2}\pi y\{J_0[\frac{1}{2}y(z^2 - 1)^{\frac{1}{2}}]\}^2$
$x\sinh(\frac{1}{2}\pi x)\, P^k_{\frac{1}{2}ix-\frac{1}{2}}(z)$	$\pi 2^{-\frac{5}{2}k-1}(1 + z)^{\frac{1}{2}k}y^{1+k}J_{-k}[y(\frac{1}{2}z - \frac{1}{2})^{\frac{1}{2}}]$ $\operatorname{Re} k \leq 0$

$f(x)$	$g(y) = \int_0^\infty f(x)\, K_{ix}(y)\, dx$
$x \sinh(\frac{1}{2}\pi x)\Gamma(\frac{1}{2}-k+\frac{1}{2}ix)\Gamma(\frac{1}{2}-k-\frac{1}{2}ix) \cdot$ $\cdot P^{k}_{\frac{1}{2}ix-\frac{1}{2}}(z)$	$\pi 2^{\frac{3}{2}k}(z-1)^{-\frac{1}{2}k} y^{1-k} K_k[y(\frac{1}{2}+\frac{1}{2}z)^{\frac{1}{2}}]$ $\mathrm{Re}\ k \le \frac{1}{2}$
$x \sinh(\pi x)\Gamma(\frac{1}{2}-k+ix)\Gamma(\frac{1}{2}-k-ix) P^{k}_{ix-\frac{1}{2}}(z)$	$2^{-\frac{1}{2}}\pi^{\frac{3}{2}}(z^2-1)^{-\frac{1}{2}k} y^{\frac{1}{2}-k}\exp(-zy)$ $\mathrm{Re}\ k \ge \frac{1}{2}$
$x \sinh(\pi x)\Gamma(\frac{1}{2}-k+2ix)\Gamma(\frac{1}{2}-k-2ix) \cdot$ $\cdot P^{k}_{i2x-\frac{1}{2}}(z)$	$\pi 2^{\frac{3}{2}-k} y^{\frac{1}{4}-\frac{1}{2}k}\Gamma(\frac{3}{2}-k)(z^2-1)^{-\frac{1}{2}k}\exp(z^2 y - y) \cdot$ $\cdot D_{k-\frac{3}{2}}(2zy^{\frac{1}{2}})$, $\mathrm{Re}\ k \le \frac{1}{2}$
$x \sinh(\pi x)\Gamma(\frac{1}{2}-k+\frac{1}{2}ix)\Gamma(\frac{1}{2}-k-\frac{1}{2}ix) \cdot$ $\cdot P^{k}_{i\frac{1}{2}x-\frac{1}{2}}(z)$	$2^{-\frac{1}{2}k}\pi^2(1+z)^{-\frac{1}{2}k} y^{1-k} J_{-k}[y(\frac{1}{2}z-\frac{1}{2})^{\frac{1}{2}}]$ $0 \le \mathrm{Re}\ k \le \frac{1}{2}$
$x \sinh(\pi x)\Gamma(\frac{1}{4}-\frac{1}{2}k+\frac{1}{2}ix)\Gamma(\frac{1}{4}-\frac{1}{2}k-\frac{1}{2}ix) \cdot$ $\cdot P^{k}_{ix-\frac{1}{2}}(z)$	$2^{\frac{1}{2}+k}\pi^2 y^{\frac{1}{2}} J_{-k}[y(z^2-1)^{\frac{1}{2}}]$ $\mathrm{Re}\ k \le \frac{1}{2}$
$x \sinh(\pi x)\Gamma(\frac{3}{4}-\frac{1}{2}k+\frac{1}{2}ix)\Gamma(\frac{3}{4}-\frac{1}{2}k-\frac{1}{2}ix) \cdot$ $\cdot P^{k}_{ix-\frac{1}{2}}(z)$	$2^{k-\frac{1}{2}}\pi^2 y^{\frac{3}{2}} z J_{-k}[y(z^2-1)^{\frac{1}{2}}]$ $\mathrm{Re}\ k \le \frac{3}{2}$
$x \sinh(\frac{1}{2}\pi x) P^{k}_{\frac{1}{2}ix-\frac{1}{2}}(z) P^{-k}_{\frac{1}{2}ix-\frac{1}{2}}(z)$	$\frac{1}{2}\pi y J_k[\frac{1}{2}y(z^2-1)^{\frac{1}{2}}] J_{-k}[\frac{1}{2}y(z^2-1)^{\frac{1}{2}}]$
$x \sinh(\pi x)\Gamma(k+\frac{1}{2}+\frac{1}{2}ix)\Gamma(k+\frac{1}{2}-\frac{1}{2}ix) \cdot$ $\cdot [P^{-k}_{\frac{1}{2}ix-\frac{1}{2}}(z)]^2$	$\pi^2 y\{J_k[\frac{1}{2}y(z^2-1)^{\frac{1}{2}}]\}^2$ $\mathrm{Re}\ k \ge -\frac{1}{2}$

$f(x)$	$g(y) = \int_0^\infty f(x)\, K_{ix}(y)\, dx$
$x \sinh(\pi x)\Gamma(\frac{1}{2}-k+ix)\Gamma(\frac{1}{2}-k-ix)\cdot$ $\cdot[P^{k}_{ix-\frac{1}{2}}(z)]^2$	$\pi(\frac{1}{2}\pi/y)^{\frac{1}{2}}\exp(-z^2 y)I_{-k}[(z^2-1)\, y]$ $\operatorname{Re} k \le \frac{1}{2}$
$x \sinh(\frac{1}{2}\pi x)[Q_{\frac{1}{2}ix-\frac{1}{2}}(z)+Q_{-\frac{1}{2}ix-\frac{1}{2}}(z)]$	$-\frac{1}{2}\pi^2 y Y_0[y(\frac{1}{2}z-\frac{1}{2})^{\frac{1}{2}}]$
$J_{ix}(a) + J_{-ix}(a)$	$\pi J_0[(a^2-y^2)^{\frac{1}{2}}]$, $y < a$ 0 , $y > a$
$i \sin(ax)[J_{ix}(b) - J_{-ix}(b)]$	$\frac{1}{2}\pi J_0(z_1)-\frac{1}{2}\pi J_0(z_2)$, $2by \sinh a < b^2-y^2$ $\frac{1}{2}\pi J_0(z_1)$, $2by \sinh a > b^2-y^2$ $y < b$ 0 , $2by \sinh a < y^2-b^2$ $\frac{1}{2}\pi J_0(z_1)$, $2by \sinh a > y^2-b^2$ $y > b$ $z_{1,2} = (b^2 - y^2 \pm 2by \sinh a)^{\frac{1}{2}}$
$\cos(ax)[J_{ix}(b) + J_{-ix}(b)]$	$\frac{1}{2}\pi J_0(z_1)+\frac{1}{2}\pi J_0(z_2)$, $2by \sinh a < b^2-y^2$ $\frac{1}{2}\pi J_0(z_1)$, $2by \sinh a > b^2-y^2$ $y < b$ 0 , $2by \sinh a < y^2-b^2$ $\frac{1}{2}\pi J_0(z_1)$, $2by \sinh a > y^2-b^2$ $y > b$ $z_{1,2} = (b^2 - y^2 \pm 2by \sinh a)^{\frac{1}{2}}$

$f(x)$	$g(y) = \int_0^\infty f(x)\, K_{ix}(y)\, dx$
$x \begin{Bmatrix} \sinh(\frac{1}{2}\pi x)[J_{ix}(a)+J_{-ix}(a)] \\ i \cosh(\frac{1}{2}\pi x)[Y_{ix}(a)-Y_{-ix}(a)] \end{Bmatrix} K_{ix}(a)$	$\frac{1}{2}\pi \sin(\frac{1}{2}a^2/y)$
$x \begin{Bmatrix} \sinh(\frac{1}{2}\pi x)[Y_{ix}(a)+Y_{-ix}(a)] \\ -i \cosh(\frac{1}{2}\pi x)[J_{ix}(a)-J_{-ix}(a)] \end{Bmatrix} K_{ix}(a)$	$-\frac{1}{2}\pi \cos(\frac{1}{2}a^2/y)$
$[J_{i\frac{1}{2}x}(a)]^2 + [Y_{i\frac{1}{2}x}(a)]^2$	$2\pi^{-1} K_0\{\frac{1}{2}[y+(y^2-4a^2)^{\frac{1}{2}}]\} K_0\{\frac{1}{2}[y-(y^2-4a^2)^{\frac{1}{2}}]\}$
$-ix \operatorname{sech}(\pi x)[J_{ix}(a)Y_{-ix}(a)-J_{-ix}(a)\cdot$ $\cdot Y_{ix}(a)]$	$\exp(-y+\frac{1}{2}a^2/y)\operatorname{Erfc}[a(2y)^{-\frac{1}{2}}]$
$x \sinh(\frac{1}{2}\pi x)\{[J_{i\frac{1}{2}x}(a)]^2+[Y_{i\frac{1}{2}x}(a)]^2\}$	$2y(4a^2 + y^2)^{-\frac{1}{2}}$
$x \sinh(\frac{1}{2}\pi x)[J_{i\frac{1}{2}x}(a)J_{i\frac{1}{2}x}(b) +$ $+ Y_{i\frac{1}{2}x}(a)Y_{i\frac{1}{2}x}(b)]$	$2y(y^2+4ab)^{-\frac{1}{2}}\cos\{\frac{1}{2}[(a/b)^{\frac{1}{2}}-(b/a)^{\frac{1}{2}}](y^2+4ab)^{\frac{1}{2}}\}$
$x \sinh(\frac{1}{2}\pi x)[J_{i\frac{1}{2}x}(a)Y_{i\frac{1}{2}x}(b) -$ $- J_{i\frac{1}{2}x}(b)Y_{i\frac{1}{2}x}(a)]$	$-2y(4ab+y^2)^{-\frac{1}{2}}\sin\{\frac{1}{2}[(a/b)^{\frac{1}{2}}-(b/a)^{\frac{1}{2}}](4ab+y^2)^{\frac{1}{2}}\}$
$x \sinh(\frac{1}{2}\pi x)[J_{i\frac{1}{2}x}(a)Y_{-i\frac{1}{2}x}(b) +$ $+ J_{-i\frac{1}{2}x}(b)Y_{i\frac{1}{2}x}(a)]$	$-2y(4ab-y^2)^{-\frac{1}{2}}\cos\{\frac{1}{2}[(a/b)^{\frac{1}{2}}+(b/a)^{\frac{1}{2}}](4ab-y^2)^{\frac{1}{2}}\}$, $y < 2(ab)^{\frac{1}{2}}$ 0, $y > 2(ab)^{\frac{1}{2}}$
$x[J_{k+i\frac{1}{2}x}(a)Y_{k-i\frac{1}{2}x}(a)-Y_{k+i\frac{1}{2}x}(a)\cdot$ $\cdot J_{k-i\frac{1}{2}x}(a)]$	$i2^{2k+1}a^{2k}y(4a^2+y^2)^{-\frac{1}{2}}[y+(y^2+4a^2)^{\frac{1}{2}}]^{-2k}$

$f(x)$	$g(y) = \int_0^\infty f(x)\, K_{ix}(y)\, dx$
$\cosh(ax) K_{ix}(b)$	$\frac{1}{2}\pi K_0[(b^2+y^2+2by\cos a)^{\frac{1}{2}}]$, $a \le \pi$
$x \sinh(\pi x)\, K_{i2x}(a)$	$\frac{1}{8}a\pi(2y/\pi)^{-\frac{1}{2}}\exp[-y - a^2/(8y)]$
$x \tanh(\pi x) K_{ix}(a)$	$\frac{1}{2}\pi(ay)^{\frac{1}{2}}(a+y)^{-1}\exp(-a-y)$
$x \sinh(bx)\, K_{ix}(a)$	$\frac{1}{2}\pi ay \sin b(a^2+y^2+2ay\cos b)^{-\frac{1}{2}} \cdot$ $\cdot K_1[(a^2+b^2+2ay\cos b)^{\frac{1}{2}}]$, $b \le \pi$
$x(c^2 + x^2)^{-1}\sinh(\pi x)\, K_{ix}(a)$	$\frac{1}{2}\pi^2 I_c(y) K_c(a)$, $y < a$ $\frac{1}{2}\pi^2 I_c(a) K_c(y)$, $y > a$
$x \sinh(\pi x)\Gamma(c+i\frac{1}{2}x)\Gamma(c-i\frac{1}{2}x) K_{ix}(a)$	$2^{1-2c}\pi^2(ay/z)^{2c} K_{2c}(z)$ Re $c \ge 0$ $z = (y^2+a^2)^{\frac{1}{2}}$
$x \sinh(2\pi x)\Gamma(c+ix)\Gamma(c-ix) K_{ix}(a)$	$2^c\pi^{\frac{5}{2}}[\Gamma(\frac{1}{2}-c)]^{-1}(\lvert a^{-1}-y^{-1}\rvert) K_c(\lvert y-a\rvert)$. $0 \le$ Re $c \le \frac{1}{2}$
$x \sinh(\pi x)\Gamma(c+ix)\Gamma(c-ix) K_{ix}(a)$	$2^{c-1}\pi^{\frac{3}{2}}(a^{-1}+y^{-1})^{-c}\Gamma(\frac{1}{2}+c) K_c(y+a)$ Re $c \ge 0$
$[\Gamma(\frac{3}{4}+i\frac{1}{2}x)\Gamma(\frac{3}{4}-i\frac{1}{2}x)]^{-1} x \tanh(\pi x) K_{ix}(a)$	$\frac{1}{2}(\pi ay/z)^{\frac{1}{2}}\exp(-z)$ $z = (a^2+y^2)^{\frac{1}{2}}$

$f(x)$	$g(y) = \int_0^\infty f(x)\, K_{ix}(y)\, dx$
$x \sinh(\pi x) P^{k}_{-\frac{1}{2}+ix}(z) K_{ix}(a)$	$2^{-k-2}\pi^{\frac{3}{2}}(z^2-1)^{\frac{1}{2}k} c^{\frac{1}{2}+k}(z-\tau)^{-\frac{1}{2}k-\frac{1}{4}} J_{-k-\frac{1}{2}}[c(z-\tau)^{\frac{1}{2}}]$, $z > \tau$ 0 , $z < \tau$ $c = (2ay)^{\frac{1}{2}}$, $\tau = \frac{1}{2}(a/y + y/a)$
$x \tanh(\pi x) P_{-\frac{1}{2}+ix}(z) K_{ix}(a)$	$\frac{1}{2}\pi(ay)^{\frac{1}{2}}(a^2+y^2+2azy)^{-\frac{1}{2}}\exp[-(a^2+y^2+2azy)^{\frac{1}{2}}]$
$x \sinh(\pi x)\Gamma(c+ix)\Gamma(c-ix)P^{\frac{1}{2}-c}_{-\frac{1}{2}+ix}(z)\cdot$ $\cdot K_{ix}(a)$	$2^{-\frac{1}{2}}\pi^{\frac{3}{2}}(ay/b)^{c}(z^2-1)^{\frac{1}{2}c-\frac{1}{4}}K_c(b)$ $b = (y^2+a^2+2ayz)^{\frac{1}{2}}$
$x[I_{-i\frac{1}{2}x}(a)I_{-i\frac{1}{2}x}(b) - I_{i\frac{1}{2}x}(a)I_{i\frac{1}{2}x}(b)]$	$2iy(4ab-y^2)^{-\frac{1}{2}}\cosh\{\frac{1}{2}[(a/b)^{\frac{1}{2}}-(b/a)^{\frac{1}{2}}](4ab-y^2)^{\frac{1}{2}}\}$, $y < 2(ab)^{\frac{1}{2}}$ 0 , $y > 2(ab)^{\frac{1}{2}}$
$x \sinh(\frac{1}{2}\pi x)[I_{i\frac{1}{2}x}(a)+I_{-i\frac{1}{2}x}(a)]K_{i\frac{1}{2}x}(b)$	$\pi y(4ab-y^2)^{-\frac{1}{2}}\exp\{\frac{1}{2}[(a/b)^{\frac{1}{2}}-(b/a)^{\frac{1}{2}}](4ab-y^2)^{\frac{1}{2}}\}$, $y < 2(ab)^{\frac{1}{2}}$ $\pi y(y^2-4ab)^{-\frac{1}{2}}\sin\{\frac{1}{2}[(a/b)^{\frac{1}{2}}-(b/a)^{\frac{1}{2}}](y^2-4ab)^{\frac{1}{2}}\}$, $y > 2(ab)^{\frac{1}{2}}$
$x \tanh(\pi x)[I_{ix}(a)+I_{-ix}(a)]K_{ix}(a)$	$-\frac{1}{2}i\pi\exp(-y-\frac{1}{2}a^2/y)\,\mathrm{Erf}[ia(2y)^{-\frac{1}{2}}]$
$I_{k-\frac{1}{2}ix}(a)K_{k+\frac{1}{2}ix}(a)+I_{k+\frac{1}{2}ix}(a)K_{k-\frac{1}{2}ix}(a)$	$\pi I_k\{\frac{1}{2}[(4a^2+y^2)^{\frac{1}{2}}-y]\}K_k\{\frac{1}{2}[(4a^2+y^2)^{\frac{1}{2}}+y]\}$

$f(x)$	$g(y) = \int_0^\infty f(x)\, K_{ix}(y)\, dx$
$\sinh(\pi x)[K_{\frac{1}{2}ix+\frac{1}{2}}(a)K_{\frac{1}{2}ix+\frac{1}{2}}(b) -$ $- K_{\frac{1}{2}ix-\frac{1}{2}}(a)K_{\frac{1}{2}ix-\frac{1}{2}}(b)]$	$0 \quad , y < 2(ab)^{\frac{1}{2}}$ $2i\pi^2(y^2-4ab)^{-\frac{1}{2}}\cos\{\frac{1}{2}[(b/a)^{\frac{1}{2}}-(a/b)^{\frac{1}{2}}](y^2-4ab)^{\frac{1}{2}}\}$ $, y > 2(ab)^{\frac{1}{2}}$
$x \sinh(\pi x)[K_{i\frac{1}{2}x}(a)]^2$	$0 \quad , y < 2a$ $\pi^2 y(y^2-4a^2)^{-\frac{1}{2}} \quad , y > 2a$
$x \sinh(\frac{1}{2}\pi x)K_{i\frac{1}{2}x}(a)K_{i\frac{1}{2}x}(b)$	$\frac{1}{2}\pi^2 yz^{-1}\exp[-\frac{1}{2}(ab)^{-\frac{1}{2}}(az+bz)]$ $z = (y^2 + 4ab)^{\frac{1}{2}}$
$x \sinh(\pi x)K_{i\frac{1}{2}x}(a)K_{i\frac{1}{2}x}(b)$	$0 \quad , y < 2(ab)^{\frac{1}{2}}$ $\pi^2 y(y^2-4ab)^{-\frac{1}{2}}\cos\{\frac{1}{2}[(a/b)^{\frac{1}{2}}-(b/a)^{\frac{1}{2}}](y^2-4ab)^{\frac{1}{2}}\}$ $, y > 2(ab)^{\frac{1}{2}}$
$x \sinh(\pi x)K_{ix}(a)K_{ix}(b)$	$\frac{1}{4}\pi^2 \exp[-\frac{1}{2}y(\frac{a}{b} + \frac{b}{a} + \frac{ab}{y^2})]$
$x \tanh(\pi x)K_{ix}(a)K_{ix}(b)$	$\frac{1}{4}\pi^2\exp[\frac{1}{2}(\frac{ay}{b} + \frac{by}{a} + \frac{ab}{y})]\cdot$ $\cdot \mathrm{Erfc}[2^{-\frac{1}{2}}(\frac{ay}{b} + \frac{by}{a} + \frac{ab}{y})]$
$x \sinh(\pi x)K_{\frac{1}{2}ix+c}(a)K_{\frac{1}{2}ix-c}(a)$	$0 \quad , y < 2a$ $2^{-2c-1}a^{-2c}\pi^2 z^{-1}y[(y+c)^{2c}+(y-z)^{2c}]$ $, y > 2a$ $z = (y^2-4a^2)^{\frac{1}{2}}$

$f(x)$	$g(y) = \int_0^\infty f(x)\, K_{ix}(y)\, dx$
$x \sinh(\pi x) K_{\frac{1}{2}ix+ic}(a) K_{\frac{1}{2}ix-ic}(a)$	$0 \quad , y < 2a$ $\pi^2 y(y^2-4a^2)^{-\frac{1}{2}} \cos\{2c \log[\frac{1}{2}ya^{-1}+(\frac{1}{4}y^2a^{-2}-1)^{\frac{1}{2}}]\}$ $, y > 2a$
$x \sinh(\frac{1}{2}\pi x) S_{0,ix}(a)$	$\frac{1}{2}\pi ay(a^2 + y^2)^{-1}$
$x \tanh(\pi x) S_{0,2ix}(a)$	$-\frac{1}{8}(2y/\pi)^{-\frac{1}{2}} a \exp[-y+a^2/(8y)] \mathrm{Ei}[-a^2/(8y)]$
$x \sinh(\pi x) \Gamma(\frac{1}{2}-\frac{1}{2}k-\frac{1}{2}ix) \Gamma(\frac{1}{2}-\frac{1}{2}k+\frac{1}{2}ix) \cdot$ $\cdot S_{k,ix}(a)$	$2^k a^{k+1} \pi^2 y^{1-k} (a^2 + y^2)^{-1}$ $\mathrm{Re}\, k \leq 1$
$x \sinh(\pi x) \Gamma(\frac{1}{2}-k+ix) \Gamma(\frac{1}{2}-k-ix) \cdot$ $\cdot S_{2k,2ix}(a)$	$\pi(2y/\pi)^{-\frac{1}{2}} 2^{2k-3} a \Gamma(1-k) \exp[-y+a^2/(8y)] \cdot$ $\cdot \Gamma[k, a^2/(8y)]$ $\mathrm{Re}\, k \leq \frac{1}{2}$
$x \tanh(\pi x) [D_{-\frac{1}{2}+ix}(a) D_{-\frac{1}{2}-ix}(-a) +$ $+ D_{-\frac{1}{2}+ix}(-a) D_{-\frac{1}{2}-ix}(a)]$	$\pi y^{\frac{1}{2}} \cos[a(2y)^{\frac{1}{2}}]$
$x \sinh(\pi x) \Gamma(\frac{1}{2}-k+i\frac{1}{2}x) \Gamma(\frac{1}{2}-k-i\frac{1}{2}x) \cdot$ $\cdot W_{k,\frac{1}{2}ix}(a)$	$(4a)^k \pi^2 y^{1-2k} \exp[-\frac{1}{2}a-y^2/(4a)]$ $\mathrm{Re}\, k \leq \frac{1}{2}$
$x \sinh(\pi x) \Gamma(\frac{1}{2}-k+ix) \Gamma(\frac{1}{2}-k-ix) W_{k,ix}(2a)$	$\pi(\frac{1}{2}\pi)^{\frac{1}{2}} a \Gamma(1-k) y^{\frac{1}{2}-k} (a+y)^{k-1} \exp(-a-y)$ $\mathrm{Re}\, k \leq \frac{1}{2}$

Table B

Mehler Transform

$$g(y) = \int_0^\infty f(x) P_{ix-\frac{1}{2}}(y)dx$$

$$f(x) = x\tanh(\pi x)\int_1^\infty P_{ix-\frac{1}{2}}(y)g(y)dy$$

As noted in the introduction, a Mehler transform pair can be obtained from any generalized Mehler transform by setting $k = 0$. In general, the transform pairs that can be so obtained have not been included in Table B.

$f(x)$	$g(y) = \int_0^\infty f(x)\, P_{ix-\frac{1}{2}}(y)\, dx$
$x^{-1}\tanh(\pi x)$	$2[y + (y^2 - 1)^{\frac{1}{2}}]^{-\frac{1}{2}} K\{[y + (y^2 - 1)^{\frac{1}{2}}]^{-1}\}$
$x\tanh(\pi x)(a^2 + x^2)^{-1}$	$Q_{a-\frac{1}{2}}(y)$
$\tanh(\pi x)\operatorname{sech}(\pi x)\sinh(ax)$	$2^{\frac{1}{2}}\pi^{-1}(y+\cos a)^{-\frac{1}{2}}\arctan[(1-\cos a)^{\frac{1}{2}}(y+\cos a)^{-\frac{1}{2}}]$
$\sin(ax)\tanh(\pi x)$	$(2\cosh a - 2y)^{-\frac{1}{2}}$, $y < \cosh a$ 0 , $y > \cosh a$
$(\operatorname{sech} \pi x)^2\cos(ax)$	$\pi^{-1}(\frac{1}{2}y-\frac{1}{2}\cosh a)^{-\frac{1}{2}}\arctan[(\frac{y-\cosh a}{1+\cosh a})^{\frac{1}{2}}]$, $y > \cosh a$ $2^{-\frac{1}{2}}\pi^{-1}(\frac{1}{2}\cosh a - \frac{1}{2}y)^{-\frac{1}{2}} \cdot$ $\cdot \log\left[\frac{(\cosh a + 1)^{\frac{1}{2}} + (\cosh a - y)^{\frac{1}{2}}}{(\cosh a + 1)^{\frac{1}{2}} - (\cosh a - y)^{\frac{1}{2}}}\right]$, $y < \cosh a$
$\cosh(ax)[\operatorname{sech}(\pi x)]^2$	$2^{-\frac{1}{2}}(y - \cos a)^{-\frac{1}{2}} - 2^{\frac{1}{2}}\pi^{-1}(y - \cos a)^{\frac{1}{2}} \cdot$ $\cdot \arctan[(1 + \cos a)^{\frac{1}{2}}(y - \cos a)^{-\frac{1}{2}}]$
$x\sinh(\pi x)\Gamma(\alpha - \frac{1}{2}x)\Gamma(\alpha + \frac{1}{2}x) \cdot$ $\cdot \Gamma(\frac{1}{2} - \alpha - \frac{1}{2}x)\Gamma(\frac{1}{2} - \alpha + \frac{1}{2}x)$	$2\pi^2(y^2-1)^{-\frac{1}{2}}\{[y+(y^2-1)^{\frac{1}{2}}]^{\frac{1}{2}-2\alpha}+[y-(y^2-1)^{\frac{1}{2}}]^{\frac{1}{2}-2\alpha}\}$ $0 \le \operatorname{Re}\alpha \le \frac{1}{2}$

$f(x)$	$g(y) = \int_0^\infty f(x)\, P_{ix-\frac{1}{2}}(y)\, dx$
$x \tanh(\pi x)\Gamma(\alpha - \frac{i}{2}x)\Gamma(\alpha + \frac{i}{2}x)\,\cdot$ $\cdot\,\Gamma(\frac{1}{2} - \alpha + \frac{i}{2}x)\Gamma(\frac{1}{2} - \alpha - \frac{i}{2}x)$	$2\pi^2 \sec(2\pi\alpha)(y^2-1)^{-\frac{1}{2}}\,\cdot$ $\cdot\,\{[y+(y^2-1)^{\frac{1}{2}}]^{\frac{1}{2}-2\alpha} - [y-(y^2-1)^{\frac{1}{2}}]^{\frac{1}{2}-2\alpha}\}$ $0 \leq \mathrm{Re}\,\alpha \leq \frac{1}{2}$
$[\psi(\frac{1}{2} + ix) + \psi(\frac{1}{2} - ix)]\cos(ax)$	$-2^{-\frac{1}{2}}\pi(\cosh a - y)^{-\frac{1}{2}}$, $y < \cosh a$ $(\frac{1}{2}y - \frac{1}{2}\cosh a)^{-\frac{1}{2}}[-\gamma - \log 4 + \frac{1}{2}\log(y^2 - 1)$ $- \log(y - \cosh a)]$, $y > \cosh a$
$x \tanh(\pi x)\Gamma(\frac{1}{2}-\alpha+ix)\Gamma(\frac{1}{2}-\alpha-ix)P^{\alpha}_{-\frac{1}{2}+ix}(z)$	$(z^2 - 1)^{-\frac{1}{2}\alpha}\Gamma(1 - \alpha)(z + y)^{\alpha-1}$ $\mathrm{Re}\,\alpha \leq \frac{1}{2}$
$x \tanh(\pi x)[\mathrm{sech}(\pi x)]^2 P_{ix-\frac{1}{2}}(z)$	$\pi^{-2}(y - z)^{-1}\log(\frac{y + 1}{z + 1})$
$x \sinh(\frac{1}{2}\pi x)\mathrm{sech}(\pi x)P_{\frac{1}{2}ix-\frac{1}{2}}(z)$	$2^{-\frac{1}{2}}\pi^{-1}\; c^{\frac{3}{2}}\{2E[(\frac{1}{2} - \frac{1}{2}yc)^{\frac{1}{2}}] - K[(\frac{1}{2} - \frac{1}{2}yc)^{\frac{1}{2}}]\}$ $c = (y^2 + \frac{1}{2}z - \frac{1}{2})^{-\frac{1}{2}}$
$x \sinh(\frac{1}{2}\pi x)\mathrm{sech}(\pi x)P^{\alpha}_{\frac{1}{2}ix-\frac{1}{2}}(z)$	$2^{-\frac{3}{2}-\frac{3}{2}\alpha}(1 + z)^{\frac{1}{2}\alpha}(y^2 + \frac{1}{2}z - \frac{1}{2})^{-\frac{3}{4}-\frac{1}{2}\alpha}\,\cdot$ $\cdot\,P^{\alpha}_{\frac{1}{2}+\alpha}[y(y^2 + \frac{1}{2}z - \frac{1}{2})^{-\frac{1}{2}}]$
$x \sinh(\frac{1}{2}\pi x)\mathrm{sech}(\pi x)\,\cdot$ $\Gamma(\frac{1}{2} - \alpha + \frac{i}{2}x)\Gamma(\frac{1}{2} - \alpha - \frac{i}{2}x)P^{\alpha}_{\frac{i}{2}x-\frac{1}{2}}(z)$	$2^{\frac{3}{2}\alpha-1}\pi^{\frac{1}{2}}\Gamma(\frac{3}{2} - 2\alpha)(z - 1)^{-\frac{1}{2}\alpha}(\frac{1}{2}z + \frac{1}{2})^{-\frac{1}{4}}\,\cdot$ $\cdot\begin{cases}(y^2 - \frac{z+1}{2})^{\frac{1}{2}\alpha-\frac{1}{2}}P^{\alpha-1}_{\alpha-\frac{1}{2}}[y(\frac{1}{2}z+\frac{1}{2})^{-\frac{1}{2}}], & y>(\frac{1}{2}z+\frac{1}{2})^{\frac{1}{2}} \\ (\frac{z+1}{2} - y^2)^{\frac{1}{2}\alpha-\frac{1}{2}}P^{\alpha-1}_{\alpha-\frac{1}{2}}[y(\frac{1}{2}z+\frac{1}{2})^{-\frac{1}{2}}], & y<(\frac{1}{2}z+\frac{1}{2})^{\frac{1}{2}}\end{cases}$ $\mathrm{Re}\,\alpha \leq \frac{1}{2}$

$f(x)$	$g(y) = \int_0^{\infty} f(x)\, P_{ix-\frac{1}{2}}(y)\, dx$
$x \tanh(\pi x)\Gamma(\frac{1}{2} - \alpha + \frac{i}{2}x)\Gamma(\frac{1}{2} - \alpha - \frac{i}{2}x) \cdot$ $\cdot\ P^{\alpha}_{i\frac{1}{2}x-\frac{1}{2}}(z)$	$2^{\frac{1}{2}-\frac{1}{2}\alpha}\pi^{\frac{1}{2}}\Gamma(\frac{3}{2} - 2\alpha)(z+1)^{-\frac{1}{2}\alpha}(y^2 + \frac{z-1}{2})^{\frac{1}{2}\alpha-\frac{3}{4}} \cdot$ $\cdot\ P^{\alpha}_{\frac{1}{2}-\alpha}[y(y^2 + \frac{z-1}{2})^{-\frac{1}{2}}]$ $\mathrm{Re}\,\alpha \leq \frac{1}{2}$
$x \tanh(\pi x)\Gamma(\frac{1}{4} - \frac{\alpha}{2} + \frac{i}{2}x)\Gamma(\frac{1}{4} - \frac{\alpha}{2} - \frac{i}{2}x) \cdot$ $\cdot\ P^{\alpha}_{ix-\frac{1}{2}}(z)$	$2^{1+\alpha}\pi^{\frac{1}{2}}(y^2+z^2-1)^{-\frac{1}{2}}(z^2-1)^{-\frac{1}{2}\alpha}[y+(y^2+z^2-1)^{\frac{1}{2}}]^{\alpha}$ $\mathrm{Re}\,\alpha \leq \frac{1}{2}$
$x \tanh(\pi x)\Gamma(\frac{3}{4} - \frac{\alpha}{2} + \frac{i}{2}x)\Gamma(\frac{3}{4} - \frac{\alpha}{2} - \frac{i}{2}x) \cdot$ $P^{\alpha}_{ix-\frac{1}{2}}(z)$	$2^{\alpha}\pi^{\frac{1}{2}}z(z^2-1)^{-\frac{1}{2}\alpha}(y + z^2 - 1)^{-\frac{3}{2}} \cdot$ $\cdot\ [y - \alpha(y^2 + z^2 - 1)^{\frac{1}{2}}][y + (y^2 + z^2 - 1)^{\frac{1}{2}}]^{\alpha}$ $\mathrm{Re}\,\alpha \leq \frac{3}{2}$
$x \tanh(\pi x)[P_{-\frac{1}{2}+ix}(a)]^2$	$\pi^{-1}(2a^2 - 1 - y)^{\frac{1}{2}}(y - 1)^{-\frac{1}{2}}$, $1 < y < 2a^2-1$ 0 , $y > 2a^2-1$ $a > 1$
$x \sinh(\frac{1}{2}\pi x)\mathrm{sech}(\pi x)P^{\alpha}_{\frac{1}{2}ix-\frac{1}{2}}(z) \cdot$ $\cdot\ P^{-\alpha}_{\frac{1}{2}ix-\frac{1}{2}}(z)$	$2^{-\frac{3}{2}}y^{-\frac{1}{2}}(z^2 - 1)^{\frac{1}{2}}c^{-1} \cdot$ $\cdot\ [(\alpha + \frac{1}{4})P^{\alpha}_{-\frac{1}{4}}(c/y)P^{-\alpha}_{\frac{1}{4}}(c/y) - (\alpha - \frac{1}{4}) \cdot$ $\cdot\ P^{\alpha}_{\frac{1}{4}}(c/y)P^{-\alpha}_{-\frac{1}{4}}(c/y)]$ $c = (y^2+z^2-1)^{\frac{1}{2}}$
$x \tanh(\pi x)\Gamma(\alpha + \frac{1}{2} + \frac{i}{2}x)\Gamma(\alpha + \frac{1}{2} - \frac{i}{2}x) \cdot$ $[P^{-\alpha}_{\frac{1}{2}ix-\frac{1}{2}}(z)]^2$	$2^{-\alpha-\frac{1}{2}}\pi^{\frac{1}{2}}\Gamma(2\alpha + \frac{3}{2})(z^2 - 1)^{\frac{1}{2}}y^{-\frac{1}{2}}c^{-1}P^{-\alpha}_{\frac{1}{4}}(c/y) \cdot$ $\cdot\ P^{-\alpha}_{-\frac{1}{4}}(c/y)$ $c = (y^2+z^2-1)^{\frac{1}{2}}$

$f(x)$	$g(y) = \int_0^\infty f(x)\, P_{ix-\frac{1}{2}}(y)\, dx$
$x \tanh(\pi x)\Gamma(\frac{1}{2} - \alpha + ix)\Gamma(\frac{1}{2} - \alpha - ix) \cdot [P^{\alpha}_{-\frac{1}{2}+ix}(z)]^2$	$(z^2 - 1)^{-\alpha}(y + 1)^{-\frac{1}{2}}(y - 1 + 2z^2)^{-\frac{1}{2}} \cdot [y + z^2 + (y + 1)^{\frac{1}{2}}(y - 1 + 2z^2)^{\frac{1}{2}}]^{\alpha}$
$x\, \mathrm{sech}(\pi x)\sinh(\frac{\pi}{2}x)[Y_{ix}(a) + Y_{-ix}(a)]$	$(2a/\pi)^{\frac{1}{2}}\sin(ay - \frac{3}{4}\pi)$
$x\, \mathrm{sech}(\pi x)\sinh(\frac{\pi}{2}x)[J_{ix}(a) + J_{-ix}(a)]$	$(2a/\pi)^{\frac{1}{2}}\cos(ay - \frac{3}{4}\pi)$
$x \tanh(\pi x)\mathrm{sech}(\frac{\pi}{2}x)[J_{ix}(a) + J_{-ix}(a)]$	$2(a/\pi)^{\frac{1}{2}}[\sin(ay) - \cos(ay)]$
$x \tanh(\pi x)\mathrm{sech}(\frac{\pi}{2}x)[Y_{ix}(a) + Y_{-ix}(a)]$	$-2(a/\pi)^{\frac{1}{2}}[\sin(ay) + \cos(ay)]$
$x \tanh(\pi x)[J_{ix}(a)Y_{-ix}(b) + Y_{ix}(a)J_{-ix}(b)]$	$-2\pi^{-1}(ab)^{\frac{1}{2}}(a^2+b^2+2aby)^{\frac{1}{2}}\sin[(a^2+b^2+2aby)^{\frac{1}{2}}]$
$x \tanh(\pi x)[Y_{ix}(a)Y_{-ix}(b) - J_{ix}(a)J_{-ix}(b)]$	$2\pi^{-1}(ab)^{\frac{1}{2}}(a^2+b^2+2aby)^{\frac{1}{2}}\cos[(a^2+b^2+2aby)^{\frac{1}{2}}]$
$x \tanh(\pi x)\{[J_{ix}(a)]^2 + [Y_{ix}(a)]^2\}$	$2^{\frac{1}{2}}\pi^{-1}(y - 1)^{-\frac{1}{2}}\exp[-a(2y - 2)^{\frac{1}{2}}]$
$x \tanh(\pi x)\exp(-\pi x)[H^{(1)}_{ix}(a)]^2$	$-2^{\frac{1}{2}}\pi^{-1}(1 + y)^{-\frac{1}{2}}\exp[ia(2 + 2y)^{\frac{1}{2}}]$
$x \tanh(\pi x)[I_{ix}(a) + I_{-ix}(a)]$	$(2y - 2)^{-\frac{1}{2}}\sin[a(2y - 2)^{\frac{1}{2}}]$
$x \tanh(\pi x)K_{ix}(a)$	$(\frac{1}{2}a\pi)^{\frac{1}{2}}\exp(-ay)$

$f(x)$	$g(y) = \int_0^\infty f(x)\, P_{ix-\frac{1}{2}}(y)\, dx$
$x \tanh(\pi x) K_{i2x}(a)$	$\frac{1}{4} a K_0[a(\frac{1}{2} + \frac{1}{2}y)^{\frac{1}{2}}]$
$x\, \mathrm{sech}(\pi x) \tanh(\pi x) K_{ix}(a)$	$-(2\pi)^{-\frac{1}{2}} a^{\frac{1}{2}} \exp(ay) \mathrm{Ei}(-ay - a)$
$x \sinh(\frac{1}{2}\pi x)\, \mathrm{sech}(\pi x) K_{i\frac{1}{2}x}(a)$	$(\frac{1}{2}\pi)^{\frac{1}{2}} a^{\frac{1}{4}} \exp(ay^2 - a) D_{-\frac{3}{2}}(2ya^{\frac{1}{2}})$
$x \sinh(\frac{1}{2}\pi x)\, \mathrm{sech}(\pi x) [J_{ix}(a) + J_{-ix}(a)] \cdot$ $\cdot K_{ix}(a)$	$(2y)^{-\frac{1}{2}} \exp(-ay^{\frac{1}{2}}) \sin(ay^{\frac{1}{2}})$
$x \sinh(\frac{1}{2}\pi x)\, \mathrm{sech}(\pi x) \cdot$ $\cdot [Y_{ix}(a) + Y_{-ix}(a)] K_{ix}(a)$	$-(2y)^{-\frac{1}{2}} \exp(-ay^{\frac{1}{2}}) \cos(ay^{\frac{1}{2}})$
$x \tanh(\pi x)[I_{ix}(a) K_{ix}(b) - K_{ix}(a) I_{ix}(b)]$	$(ab)^{\frac{1}{2}} (a^2 + b^2 - 2aby)^{-\frac{1}{2}} \cosh[(a^2 + b^2 - 2aby)^{\frac{1}{2}}]$ $y < \frac{1}{2}\frac{a}{b} + \frac{1}{2}\frac{b}{a}$ 0 otherwise
$x \sinh(\pi x)[K_{ix}(a)]^2$	$2^{-\frac{3}{2}} \pi (y - 1)^{-\frac{1}{2}} \cos[a(2y - 2)^{\frac{1}{2}}]$
$x \tanh(\pi x) K_{ix}(a) K_{ix}(b)$	$\frac{1}{2}\pi (ab)^{\frac{1}{2}} (a^2 + b^2 + 2aby)^{-\frac{1}{2}} \cdot$ $\cdot \exp[-(a^2 + b^2 + 2aby)^{\frac{1}{2}}]$

$f(x)$	$g(y) = \int_0^\infty f(x)\, P_{ix-\frac{1}{2}}(y)\, dx$
$x \tanh(\pi x) K_{ix}(ae^{i\pi/4}) K_{ix}(ae^{-i\pi/4})$	$\pi 2^{-\frac{3}{2}} y^{-\frac{1}{2}} \exp[-a(2y)^{\frac{1}{2}}]$
$x \tanh(\pi x)\Gamma(\frac{1}{4} + \frac{i}{2}y)\Gamma(\frac{1}{4} - \frac{i}{2}y) S_{\frac{1}{2},ix}(a)$	$4(a\pi)^{\frac{1}{2}}[\sin(ay)\mathrm{Ci}(ay) - \cos(ay)\mathrm{si}(ay)]$
$x \tanh(\pi x)\Gamma(\frac{1}{2} - \frac{1}{2}\alpha + ix) \cdot$ $\cdot\ \Gamma(\frac{1}{2} - \frac{1}{2}\alpha - ix) S_{\alpha,2ix}(a)$	$\frac{1}{2}a[\Gamma(1 - \frac{1}{2}\alpha)]^2 S_{\alpha-1,0}[a(\frac{1}{2} + \frac{1}{2}y)^{\frac{1}{2}}]$ $\mathrm{Re}\,\alpha \leq 1$

Table C

Generalized Mehler Transform

$$g(y) = \int_0^\infty f(x) P^k_{ix-\frac{1}{2}}(y)\,dx$$

$$f(x) = \pi^{-1}\sinh(\pi x)\Gamma(\tfrac{1}{2}-k+ix)\Gamma(\tfrac{1}{2}-k-ix)\int_1^\infty g(y) P^k_{ix-\frac{1}{2}}(y)\,dy$$

$f(x)$	$g(y) = \int_0^\infty f(x)\, P^k_{ix-\frac{1}{2}}(y)\, dx$
a	$a(\frac{1}{2}\pi)^{\frac{1}{2}}[\Gamma(\frac{1}{2} - k)]^{-1}(y^2 - 1)^{\frac{1}{2}k}(y - 1)^{-k-\frac{1}{2}}$ $\operatorname{Re} k < \frac{1}{2}$
$\cos(ax)$	$(\frac{1}{2}\pi)^{\frac{1}{2}}[\Gamma(\frac{1}{2} - k)]^{-1}(y^2 - 1)^{\frac{1}{2}k}(y - \cosh a)^{-k-\frac{1}{2}}$, $y > \cosh a$ 0 , $y < \cosh a$ $\operatorname{Re} k < \frac{1}{2}$
$\cos(ax)\operatorname{sech}(\pi x)$	$2^{-\frac{1}{2}-k}(1+y)^{\frac{1}{2}k}(y+\cosh a)^{-\frac{1}{2}-\frac{1}{2}k} \cdot$ $P^k_k[(1+\cosh a)^{\frac{1}{2}}(y+\cosh a)^{-\frac{1}{2}}]$
$\cos(ax)\Gamma(\frac{1}{2} - k + ix)\Gamma(\frac{1}{2} - k - ix)$	$(\frac{1}{2}\pi)^{\frac{1}{2}}\Gamma(\frac{1}{2} - k)(y^2 - 1)^{-\frac{1}{2}k}(y + \cosh a)^{k-\frac{1}{2}}$ $\operatorname{Re} k \leq \frac{1}{2}$
$\cosh(\pi x)\Gamma(\frac{1}{2} - k + ix)\Gamma(\frac{1}{2} - k - ix)$	$(\frac{1}{2}\pi)^{\frac{1}{2}}\Gamma(\frac{1}{2} - k)(y^2 - 1)^{-\frac{1}{2}k}(y - 1)^{k-\frac{1}{2}}$ $-\frac{1}{2} \leq \operatorname{Re} k \leq \frac{1}{2}$
$\Gamma(\frac{1}{4} - \frac{k}{2} + \frac{i}{2}x)\Gamma(\frac{1}{4} - \frac{k}{2} - \frac{i}{2}x)\cos(ax)$	$\pi 2^{k+\frac{1}{2}}\Gamma(\frac{1}{2} - k)(y^2 + \sinh^2 a)^{-\frac{1}{4}} \cdot$ $\cdot\, P^k_{-\frac{1}{2}}[\cosh a(y^2 + \sinh^2 a)^{-\frac{1}{2}}]$
$\Gamma(\frac{3}{4} - \frac{k}{2} + \frac{i}{2}x)\Gamma(\frac{3}{4} - \frac{k}{2} - \frac{i}{2}x)\cos(ax)$	$\pi 2^{k-\frac{1}{2}} y\Gamma(\frac{3}{2} - k)(y^2 + \sinh^2 a)^{-\frac{3}{4}} \cdot$ $\cdot\, P^k_{\frac{1}{2}}[\cosh a(y^2 + \sinh^2 a)^{-\frac{1}{2}}]$

$f(x)$	$g(y) = \int_0^\infty f(x)\, P^k_{ix-\frac{1}{2}}(y)\, dx$
$x \sinh(\pi x)\Gamma(\alpha + ix)\Gamma(\alpha - ix)$	$\pi 2^{\alpha-\frac{1}{2}}\Gamma(\frac{1}{2} + \alpha)[\Gamma(\frac{1}{2} - k - \alpha)]^{-1} \cdot$ $\cdot\, (y - 1)^{-\frac{1}{2}-k-\alpha}(y^2 - 1)^{\frac{1}{2}k}$ $\mathrm{Re}(2\alpha + k - \frac{1}{2}) < 0$ $\mathrm{Re}\,\alpha \geq 0$
$x \sinh(\pi x)\Gamma(\frac{1}{2} - k + ix)\Gamma(\frac{1}{2} - k - ix) \cdot$ $\cdot\, \Gamma(\alpha + \frac{k}{2} + \frac{i}{2}x)\Gamma(\alpha + \frac{k}{2} - \frac{i}{2}x)$	$\pi^{\frac{3}{2}} 2^{\frac{3}{2}-k-2\alpha}\Gamma(\frac{1}{2} + 2\alpha) y^{-\frac{1}{2}-2\alpha}(y^2 - 1)^{-\frac{1}{2}k}$ $\mathrm{Re}(\alpha + \frac{k}{2}) \geq 0$ $\mathrm{Re}\,k \leq \frac{1}{2}$
$x \sinh(\pi x)\Gamma(\frac{1}{2} - k + ix)\Gamma(\frac{1}{2} - k - ix) \cdot$ $\cdot\, \Gamma(\frac{3}{4} - \lambda + \frac{i}{2}x)\Gamma(\frac{3}{4} - \lambda - \frac{i}{2}x)[\Gamma(\frac{3}{4} - \frac{k}{2} + \frac{i}{2}x) \cdot$ $\cdot\, \Gamma(\frac{3}{4} - \frac{k}{2} - \frac{i}{2}x)]^{-1}$	$\pi 2^{1-k}[\Gamma(\lambda - \frac{k}{2})]^{-1}\Gamma(1 - \lambda - \frac{k}{2})(y^2 - 1)^{\lambda-1}$ $\frac{1}{4} < \mathrm{Re}\,\lambda \leq \frac{3}{4}$ $\mathrm{Re}\,k \leq \frac{1}{2}$
$x \sinh(2\pi x)\Gamma(\alpha + ix)\Gamma(\alpha - ix) \cdot$ $\Gamma(\frac{1}{2} - k + ix)\Gamma(\frac{1}{2} - k - ix)$	$\pi^2 2^{\frac{1}{2}+\alpha}\Gamma(\frac{1}{2} + \alpha - k)[\Gamma(\frac{1}{2} - \alpha)]^{-1} \cdot$ $\cdot\, (y - 1)^{k-\alpha-\frac{1}{2}}(y^2 - 1)^{-\frac{1}{2}k}$ $\mathrm{Re}(2\alpha - k - \frac{1}{2}) < 0, \quad \mathrm{Re}\,\alpha \geq 0$ $\mathrm{Re}\,k \leq \frac{1}{2}$
$\Gamma(\frac{1}{2}-k+ix)\Gamma(\frac{1}{2}-k-ix)\tanh(\pi x)\sinh(ax)$	$-2^{k-1}\pi^{\frac{1}{2}}\Gamma(1-2k)(y-1)^{-\frac{1}{2}k}(y+1)^{-\frac{1}{4}}(y+\cos a)^{\frac{1}{2}k-\frac{1}{4}} \cdot$ $\cdot\, [P^{k-\frac{1}{2}}_{-k-\frac{1}{2}}(z) - P^{k-\frac{1}{2}}_{-k-\frac{1}{2}}(-z)]$ $z = (1 - \cos a)^{\frac{1}{2}}(1 + y)^{-\frac{1}{2}}$ $\mathrm{Re}\,k \leq \frac{1}{2}$

$f(x)$	$g(y) = \int_0^\infty f(x)\, P^k_{ix-\frac{1}{2}}(y)\, dx$
$\cos(ax)\operatorname{sech}(\pi x)\Gamma(\frac{1}{2}-k+ix)\Gamma(\frac{1}{2}-k-ix)$	$2^{\frac{1}{2}+k}\Gamma(1-2k)(y-1)^{-\frac{1}{2}k}(\cosh a-y)^{\frac{1}{2}k-\frac{1}{2}}\cdot$ $\cdot\, e^{-i\pi k}Q^k_{-k}[(\frac{1}{2}\cosh a-\frac{1}{2}y)^{-\frac{1}{2}}\cosh(\frac{1}{2}a)],\quad y < \cosh a$ $2^k\Gamma(1-2k)\pi^{\frac{1}{2}}(y^2-1)^{-\frac{1}{4}}(y-1)^{\frac{1}{4}-\frac{1}{2}k}(y-\cosh a)^{\frac{1}{2}k-\frac{1}{4}}\cdot$ $\underline{P}^{k-\frac{1}{2}}_{k-\frac{1}{2}}[(\frac{1}{2}+\frac{1}{2}y)^{-\frac{1}{2}}\cosh(\frac{1}{2}a)],\quad y > \cosh a$ $\operatorname{Re} k < \frac{1}{2}$
$\cosh(ax)\operatorname{sech}(\pi x)\Gamma(\frac{1}{2}-k+ix)\Gamma(\frac{1}{2}-k-ix)$	$(\frac{1}{2}\pi)^{\frac{1}{2}}\Gamma(\frac{1}{2}-k)(y^2-1)^{-\frac{1}{2}k}\{(y-\cos a)^{k-\frac{1}{2}}$ $+\, 2^{-k-\frac{1}{2}}\pi^{-\frac{1}{2}}\Gamma(1-k)[(y+1)(y-\cos a)]^{\frac{1}{2}k-\frac{1}{4}}\cdot$ $\cdot\, [\underline{P}^{k-\frac{1}{2}}_{-k-\frac{1}{2}}(z)-\underline{P}^{k-\frac{1}{2}}_{-k-\frac{1}{2}}(-z)]\}$ $z = (1+\cos a)^{\frac{1}{2}}(1+y)^{-\frac{1}{2}}$ $\operatorname{Re} k \le \frac{1}{2}$
$x\sinh(\pi x)\operatorname{sech}(2\pi x)P_{2ix-\frac{1}{2}}(z)$	$2^{-\frac{7}{2}-\frac{3}{2}k}(1+y)^{\frac{1}{2}k}(z^2+\frac{1}{2}y-\frac{1}{2})^{-\frac{3}{4}-\frac{1}{2}k}\cdot$ $\cdot\, \underline{P}^k_{-\frac{1}{2}+k}[z(z^2+\frac{1}{2}y-\frac{1}{2})^{-\frac{1}{2}}]$
$x\sinh(\pi x)\Gamma(\frac{1}{2}-k+ix)\Gamma(\frac{1}{2}-k-ix)\cdot$ $\cdot\, P_{-\frac{1}{2}+ix}(a)$	$\pi[\Gamma(k)]^{-1}(y^2-1)^{-\frac{1}{2}k}(a-y)^{k-1},\quad y > a > 1$ $0,\quad 1 < y < a$ $0 < \operatorname{Re} k \le \frac{1}{2}$
$x\sinh(\pi x)\Gamma(\frac{1}{2}-\alpha+ix)\Gamma(\frac{1}{2}-\alpha-ix)\cdot$ $\cdot\, \Gamma(\frac{1}{2}-k+ix)\Gamma(\frac{1}{2}-k-ix)P^\alpha_{-\frac{1}{2}+ix}(z)$	$\pi\Gamma(1-\alpha-k)(z^2-1)^{-\frac{1}{2}\alpha}(y^2-1)^{-\frac{1}{2}k}\cdot$ $(z+y)^{k+\alpha-1}$ $\operatorname{Re}(\alpha,k) < \frac{1}{2}$

$f(x)$	$g(y) = \int_0^\infty f(x)\, P^k_{ix-\frac{1}{2}}(y)\, dx$
$x \tanh(\pi x)\Gamma(\frac{1}{2}-k+ix)\Gamma(\frac{1}{2}-k-ix)P_{-\frac{1}{2}+ix}(a)$	$\Gamma(1-k)(y^2-1)^{-\frac{1}{2}k}(y+a)^{k-1}$ $\operatorname{Re} k \leq \frac{1}{2}$
$x \tanh(2\pi x)\Gamma(\frac{1}{2}-k+ix)\Gamma(\frac{1}{2}-k-ix)P_{2ix-\frac{1}{2}}(z)$	$2^{-\frac{3}{2}-\frac{1}{2}k}\pi^{\frac{1}{2}}\Gamma(\frac{3}{2}-2k)(z+1)^{-\frac{1}{2}k}(z^2+\frac{y-1}{2})^{\frac{1}{2}k-\frac{3}{4}}\cdot$ $\cdot\, P^k_{-\frac{1}{2}-k}[z(z^2+\frac{y-1}{2})^{\frac{1}{2}}]$ $\operatorname{Re} k \leq \frac{1}{2}$
$x \tanh(\pi x)\Gamma(\frac{1}{4}-\frac{k}{2}+\frac{i}{2}x)\Gamma(\frac{1}{4}-\frac{k}{2}-\frac{i}{2}x)\cdot$ $\cdot\, P_{ix-\frac{1}{2}}(z)$	$2^{1+k}\pi^{\frac{1}{2}}(y^2+z^2-1)^{-\frac{1}{2}}(y^2-1)^{-\frac{1}{2}k}[z+(y^2+z^2-1)^{\frac{1}{2}}]^k$ $\operatorname{Re} k \leq \frac{1}{2}$
$x \tanh(\pi x)\Gamma(\frac{3}{4}-\frac{k}{2}+\frac{i}{2}x)\Gamma(\frac{3}{4}-\frac{k}{2}-\frac{i}{2}x)\cdot$ $\cdot\, P_{ix-\frac{1}{2}}(z)$	$2^k\pi^{\frac{1}{2}}(y^2-1)^{-\frac{1}{2}k}(y^2+z^2-1)^{-\frac{3}{2}}y\cdot$ $\cdot\,[z-k(y^2+z^2-1)^{\frac{1}{2}}][z+(y^2+z^2-1)^{\frac{1}{2}}]^k$ $\operatorname{Re} k \leq \frac{3}{4}$
$x \sinh(\pi x)\Gamma(\frac{1}{4}-\frac{\alpha}{2}+\frac{i}{2}x)\Gamma(\frac{1}{4}-\frac{\alpha}{2}-\frac{i}{2}x)\cdot$ $\cdot\,\Gamma(\frac{1}{2}-k+ix)\Gamma(\frac{1}{2}-k-ix)P^\alpha_{ix-\frac{1}{2}}(z)$	$2^{1+\alpha}\pi^{\frac{3}{2}}(y^2-1)^{-\frac{1}{2}k}\Gamma(1-k-\alpha)(y^2+z^2-1)^{\frac{1}{2}k-\frac{1}{2}}\cdot$ $\cdot\, P^\alpha_{-k}[y(y^2+z^2-1)^{-\frac{1}{2}}]$ $\operatorname{Re}(\alpha,k) < \frac{1}{2}$
$x \sinh(\pi x)\Gamma(\frac{3}{4}-\frac{\alpha}{2}+\frac{i}{2}x)\Gamma(\frac{3}{4}-\frac{\alpha}{2}-\frac{i}{2}x)\cdot$ $\cdot\,\Gamma(\frac{1}{2}-k+ix)\Gamma(\frac{1}{2}-k-ix)P^\alpha_{ix-\frac{1}{2}}(z)$	$2^\alpha\pi^{\frac{3}{2}}\Gamma(2-\alpha-k)z(y^2-1)^{-\frac{1}{2}k}\cdot$ $\cdot\,(y^2+z^2-1)^{\frac{1}{2}k-1}P^\alpha_{-1-k}[y(y^2+z^2-1)^{-\frac{1}{2}}]$ $\operatorname{Re}\alpha < \frac{3}{2}$ $\operatorname{Re} k < \frac{1}{2}$

$f(x)$	$g(y) = \int_0^\infty f(x)\, P^k_{ix-\frac{1}{2}}(y)\, dx$
$x \sinh(\pi x)\Gamma(\frac{1}{2}-\alpha+\frac{i}{2}x)\Gamma(\frac{1}{2}-\alpha-\frac{i}{2}x) \cdot$ $\cdot\ \Gamma(\frac{1}{2}-k+ix)\Gamma(\frac{1}{2}-k-ix)P^{\alpha}_{i\frac{1}{2}x-\frac{1}{2}}(z)$	$2^{\frac{1}{2}-\frac{1}{2}\alpha}\pi^{\frac{3}{2}}(1+z)^{-\frac{1}{2}\alpha}\Gamma(\frac{3}{2}-k-2\alpha)(y^2-1)^{-\frac{1}{2}k} \cdot$ $\cdot\ (y^2+\frac{z-1}{2})^{\frac{1}{2}(\alpha+k-\frac{3}{2})}P^{\alpha}_{\frac{1}{2}-\alpha-k}[y(y^2+\frac{z-1}{2})^{\frac{1}{2}}]$ $\operatorname{Re}(\alpha,k) < \frac{1}{2}$
$x \sinh(\frac{1}{2}\pi x)\Gamma(\frac{1}{2}-k+ix)\Gamma(\frac{1}{2}-k-ix) \cdot$ $\cdot\ P^{\alpha}_{\frac{1}{2}ix-\frac{1}{2}}(z)$	$2^{-\frac{3}{2}\alpha-\frac{1}{2}}\pi^{\frac{1}{2}}\Gamma(\frac{3}{2}-k)(z+1)^{\frac{1}{2}\alpha}(y^2-1)^{-\frac{1}{2}k} \cdot$ $(y^2+\frac{z-1}{2})^{\frac{1}{2}(k-\alpha-\frac{3}{2})}P^{\alpha}_{\alpha-k+\frac{1}{2}}[y(y^2+\frac{z-1}{2})^{\frac{1}{2}}]$ $\operatorname{Re} k \le \frac{1}{2}$
$x \sinh(\pi x)[\Gamma(\frac{1}{2}-k+ix)\Gamma(\frac{1}{2}-k-ix)]^2 \cdot$ $\cdot\ [P^{k}_{-\frac{1}{2}+ix}(z)]^2$	$2^{-k}\pi^{\frac{1}{2}}\Gamma(\frac{1}{2}-k)(z^2-1)^{-k}(y-1)^{-\frac{1}{2}k}(y+1)^{\frac{1}{2}k-\frac{1}{2}} \cdot$ $\cdot\ (2z^2-1+y)^{k-\frac{1}{2}}$ $\operatorname{Re} k < \frac{1}{2}$
$x \sinh(\pi x)\operatorname{sech}(2\pi x) \cdot$ $\cdot\ \Gamma(\frac{1}{2}-k+ix)\Gamma(\frac{1}{2}-k-ix)P_{i2x-\frac{1}{2}}(z)$	$2^{\frac{3}{2}k-3}\pi^{\frac{1}{2}}\Gamma(\frac{3}{2}-2k)(y-1)^{-\frac{1}{2}k}(\frac{1}{2}y+\frac{1}{2})^{-\frac{1}{4}} \cdot$ $\cdot \begin{cases} (z^2-\frac{y+1}{2})^{\frac{1}{2}k-\frac{1}{2}}P^{k-1}_{k-\frac{1}{2}}[z(\frac{1}{2}y+\frac{1}{2})^{-\frac{1}{2}}], & z > (\frac{1}{2}y+\frac{1}{2})^{\frac{1}{2}} \\ (\frac{y+1}{2}-z^2)^{\frac{1}{2}k-\frac{1}{2}}P^{k-1}_{k-\frac{1}{2}}[z(\frac{1}{2}y+\frac{1}{2})^{-\frac{1}{2}}], & z < (\frac{1}{2}y+\frac{1}{2})^{\frac{1}{2}} \end{cases}$ $\operatorname{Re} k \le \frac{1}{2}$
$x \sinh(\frac{1}{2}\pi x)\Gamma(\frac{1}{2}-\alpha+\frac{i}{2}x)\Gamma(\frac{1}{2}-\alpha-\frac{i}{2}x) \cdot$ $\cdot\ \Gamma(\frac{1}{2}-k+ix)\Gamma(\frac{1}{2}-k-ix)P^{\alpha}_{i\frac{1}{2}x-\frac{1}{2}}(z)$	$2^{\frac{3}{2}\alpha}\pi\Gamma(\frac{3}{2}-k)\Gamma(\frac{3}{2}-k-2\alpha)(z-1)^{-\frac{1}{2}\alpha}(\frac{z+1}{2})^{-\frac{1}{4}}(y^2-1)^{-\frac{1}{2}k} \cdot$ $\begin{cases} (y^2-\frac{z+1}{2})^{\frac{1}{2}(\alpha+k-1)}P^{\alpha+k-1}_{\alpha-\frac{1}{2}}[y(\frac{z+1}{2})^{-\frac{1}{2}}], & y > (\frac{z+1}{2})^{\frac{1}{2}} \\ (\frac{z+1}{2}-y^2)^{\frac{1}{2}(\alpha+k-1)}P^{\alpha+k-1}_{\alpha-\frac{1}{2}}[y(\frac{z+1}{2})^{-\frac{1}{2}}], & y < (\frac{z+1}{2})^{\frac{1}{2}} \end{cases}$ $\operatorname{Re}(\alpha,k) < \frac{1}{2}$

$f(x)$	$g(y) = \int_0^\infty f(x)\, P^k_{ix-\frac{1}{2}}(y)\, dx$
$x\Gamma(\frac{1}{2}-k+ix)\Gamma(\frac{1}{2}-k-ix)\cdot$ $\cdot\,\{[J_{ix}(a)]^2 - [J_{-ix}(a)]^2\}$	$-\,i2^{\frac{1}{4}+\frac{k}{2}}\pi a^{\frac{1}{2}-k}(y+1)^{-\frac{1}{4}}(y-1)^{-\frac{1}{2}k}J_{\frac{1}{2}-k}[a(2y+2)^{\frac{1}{2}}]$ $\operatorname{Re} k < \frac{1}{2}$
$x\sinh(\pi x)\Gamma(\frac{1}{2}-k+ix)\Gamma(\frac{1}{2}-k-ix)\cdot$ $\cdot\,[J_{ix}(a)J_{-ix}(a)-Y_{ix}(a)Y_{-ix}(a)]$	$\pi^{\frac{1}{2}}2^{\frac{1}{2}k+\frac{1}{4}}a^{\frac{1}{2}-k}(y+1)^{-\frac{1}{4}}(y-1)^{-\frac{1}{2}k}Y_{\frac{1}{2}-k}[a(2y+2)^{\frac{1}{2}}]$ $-\frac{1}{2} \le \operatorname{Re} k \le \frac{1}{2}$
$x\sinh(\pi x)K_{i2x}(a)$	$\pi 2^{-3-\frac{3}{2}k}(1+y)^{\frac{1}{2}k}a^{1+k}J_{-k}[(\frac{1}{2}ay-\frac{1}{2}a)^{\frac{1}{2}}]$ $\operatorname{Re} k \le 0$
$x\sinh(2\pi x)\Gamma(\frac{1}{2}-k+ix)\Gamma(\frac{1}{2}-k-ix)K_{i2x}(a)$	$2^{-2-\frac{1}{2}k}\pi^2(1+y)^{-\frac{1}{2}k}a^{1-k}J_{-k}[(\frac{1}{2}ay-\frac{1}{2}a)^{\frac{1}{2}}]$ $\operatorname{Re} k \le \frac{1}{2}$
$x\sinh(\pi x)\Gamma(\frac{1}{2}-k+ix)\Gamma(\frac{1}{2}-k-ix)K_{i2x}(a)$	$2^{\frac{3}{2}k-2}\pi a^{1-k}(y-1)^{-\frac{1}{2}k}K_k[a(\frac{y+1}{2})^{\frac{1}{2}}]$ $\operatorname{Re} k \le \frac{1}{2}$
$x\tanh(\pi x)\Gamma(\frac{1}{2}-k+ix)\Gamma(\frac{1}{2}-k-ix)K_{ix}(a)$	$(\frac{1}{2}\pi a)^{\frac{1}{2}}\Gamma(1-k)(y-1)^{-\frac{1}{2}k}(y+1)^{\frac{1}{2}k}\cdot$ $\exp(ay)\Gamma(-k,ay+a)$ $\operatorname{Re} k \le \frac{1}{2}$
$x\sinh(\pi x)\Gamma(\frac{1}{4}-\frac{k}{2}+\frac{i}{2}x)\Gamma(\frac{1}{4}-\frac{k}{2}-\frac{i}{2}x)K_{ix}(a)$	$2^{\frac{1}{2}+k}\pi^2a^{\frac{1}{2}}J_{-k}[a(y^2-1)^{\frac{1}{2}}]$ $\operatorname{Re} k \le \frac{1}{2}$

$f(x)$	$g(y) = \int_0^\infty f(x)\, P^k_{ix-\frac{1}{2}}(y)\, dx$
$x \sinh(\pi x)\Gamma(\frac{1}{2}-k+ix)\Gamma(\frac{1}{2}-k-ix)K_{ix}(a)$	$2^{-\frac{1}{2}}\pi^{\frac{3}{2}}a^{\frac{1}{2}-k}(y^2-1)^{-\frac{1}{2}k}\exp(-ay)$ $\mathrm{Re}\, k \le \frac{1}{2}$
$x \sinh(\frac{1}{2}\pi x)\Gamma(\frac{1}{2}-k+ix)\Gamma(\frac{1}{2}-k-ix)K_{i\frac{1}{2}x}(a)$	$\pi 2^{\frac{1}{2}-k}a^{\frac{1}{4}-\frac{1}{2}k}\Gamma(\frac{3}{2}-k)(y^2-1)^{-\frac{1}{2}k}\exp(ay^2-a)\,\cdot$ $D_{k-\frac{3}{2}}(2ya^{\frac{1}{2}})$ $\mathrm{Re}\, k \le \frac{1}{2}$
$x \sinh(\frac{1}{2}\pi x)\Gamma(\frac{1}{2}-k+ix)\Gamma(\frac{1}{2}-k-ix)\,\cdot$ $\cdot\,[J_{ix}(a)+J_{-ix}(a)]K_{ix}(a)$	$i(\frac{1}{2}\pi)^{\frac{1}{2}}(y^2-1)^{-\frac{1}{2}k}(2y/a^2)^{\frac{1}{2}k-\frac{1}{4}}\,\cdot$ $\cdot\,\{\exp(i\frac{\pi}{4}-i\frac{\pi}{4}k)K_{k-\frac{1}{2}}[a(2iy)^{\frac{1}{2}}]$ $-\exp(i\frac{\pi}{4}k-i\frac{\pi}{4})K_{k-\frac{1}{2}}[a(-2iy)^{\frac{1}{2}}]\}$ $\mathrm{Re}\, k \le \frac{1}{2}$
$x \sinh(\frac{1}{2}\pi x)\Gamma(\frac{1}{2}-k+ix)\Gamma(\frac{1}{2}-k-ix)\,\cdot$ $\cdot\,[Y_{ix}(a)+Y_{-ix}(a)]K_{ix}(a)$	$-(\frac{1}{2}\pi)^{\frac{1}{2}}(y^2-1)^{-\frac{1}{2}k}(2y/a^2)^{\frac{1}{2}k-\frac{1}{4}}\,\cdot$ $\{\exp(i\frac{\pi}{4}-i\frac{\pi}{4}k)K_{k-\frac{1}{2}}[a(2iy)^{\frac{1}{2}}]$ $+\exp(i\frac{\pi}{4}k)K_{k-\frac{1}{2}}[a(-2iy)^{\frac{1}{2}}]\}$ $\mathrm{Re}\, k \le \frac{1}{2}$
$x \sinh(\pi x)\Gamma(\frac{1}{2}-k-ix)\Gamma(\frac{1}{2}-k+ix)\,\cdot$ $\cdot\,[K_{ix}(a)]^2$	$\pi^{\frac{3}{2}}2^{-\frac{3}{4}+\frac{1}{2}k}(y+1)^{-\frac{1}{4}}(y-1)^{-\frac{1}{2}k}K_{\frac{1}{2}-k}[a(2y+2)^{\frac{1}{2}}]$ $\mathrm{Re}\, k \le \frac{1}{2}$
$x \sinh(\pi x)K_{ix}(a)K_{ix}(b)$	$0 \qquad y < \tau$ $2^{-k-2}\pi^{\frac{3}{2}}c^{\frac{1}{2}+k}(y^2-1)^{\frac{1}{2}k}(y-\tau)^{-\frac{1}{2}k-\frac{1}{4}}J_{-k-\frac{1}{2}}[c(y-\tau)^{\frac{1}{2}}]$ $y > \tau$ $c = (2ab)^{\frac{1}{2}}$ $\tau = \frac{1}{2}(a/b+b/a)$

$f(x)$	$g(y) = \int_0^\infty f(x)\, P^k_{ix-\frac{1}{2}}(y)\, dx$
$x \sinh(\pi x)\Gamma(\frac{1}{2}-k+ix)\Gamma(\frac{1}{2}-k-ix) \cdot$ $\cdot\, [\Gamma(-k+ix)\Gamma(-k-ix)]^{-1}[K_{ix}(a)]^2$	$\pi^{\frac{3}{2}} 2^{-\frac{k}{2}-\frac{7}{4}} a^{k-\frac{1}{2}} y^{\frac{1}{2}-k} (y^2-1)^{\frac{1}{2}k} J_{-\frac{1}{2}-k}[a(2y-2)^{\frac{1}{2}}]$ Re $k \leq -\frac{1}{2}$
$x \sinh(\pi x)\Gamma(\frac{1}{2}-k+ix)\Gamma(\frac{1}{2}-k-ix) \cdot$ $\cdot\, K_{ix}(a)K_{ix}(b)$	$2^{-\frac{1}{2}}\pi^{\frac{3}{2}}(ab)^{\frac{1}{2}-k}(y^2-1)^{-\frac{1}{2}k} \cdot$ $(a^2+b^2+2aby)^{\frac{1}{2}k-\frac{1}{4}} K_{\frac{1}{2}-k}[(a^2+b^2+2aby)^{\frac{1}{2}}]$ Re $k \leq \frac{1}{2}$
$x \sinh(\pi x)\Gamma(\frac{1}{2}-k+ix)\Gamma(\frac{1}{2}-k-ix) W_{k,ix}(a)$	$\frac{1}{2}\pi a(y+1)^{\frac{1}{2}k}(y-1)^{-\frac{1}{2}k}\exp(-\frac{1}{2}ay)$ Re $k \leq \frac{1}{2}$
$x \sinh(\pi x)\Gamma(\frac{1}{2}-k+ix)\Gamma(\frac{1}{2}-k-ix) \cdot$ $W_{\frac{1}{2}k-\frac{1}{4},\frac{i}{2}x}(a)$	$2^{1-k}\pi a^{\frac{3}{4}-\frac{1}{2}k}(y^2-1)^{-\frac{1}{2}k}\exp(\frac{1}{2}a-ay^2)$ Re $k \leq \frac{1}{2}$
$x \sinh(\pi x)\Gamma(\frac{1}{2}-k+ix)\Gamma(\frac{1}{2}-k-ix) W_{\frac{1}{2}k+\frac{1}{4},\frac{i}{2}x}(a)$	$2^{1-k}\pi a^{\frac{5}{4}-\frac{1}{2}k} y (y^2-1)^{-\frac{1}{2}k}\exp(\frac{1}{2}a-ay^2)$ Re $k \leq \frac{1}{2}$
$x \sinh(\pi x)\Gamma(\frac{1}{2}-k+ix)\Gamma(\frac{1}{2}-k-ix) \cdot$ $\cdot\, \Gamma(\frac{1}{4}+\frac{k}{2}+\frac{i}{2}x)\Gamma(\frac{1}{4}+\frac{k}{2}-\frac{i}{2}x) W_{\frac{1}{4}-\frac{k}{2},\frac{i}{2}x}(a)$	$\pi^2 2^{1-k} a^{\frac{3}{4}-\frac{k}{2}}(y^2-1)^{-\frac{1}{2}k}\exp(ay^2-\frac{1}{2}a)\,\mathrm{Erfc}(ya^{\frac{1}{2}})$ Re $k < \frac{1}{2}$
$x \sinh(\pi x)\Gamma(\frac{1}{2}-k+ix)\Gamma(\frac{1}{2}-k-ix) \cdot$ $\cdot\, \Gamma(\frac{1}{2}-\alpha+\frac{i}{2}x)\Gamma(\frac{1}{2}-\alpha-\frac{i}{2}x) W_{\alpha,\frac{1}{2}ix}(a)$	$\pi^{\frac{3}{2}} 2^{\alpha+\frac{5}{4}-\frac{k}{2}} a^{\frac{3}{4}-\frac{k}{2}}\Gamma(\frac{3}{2}-2\alpha-k) \cdot$ $(y^2-1)^{-\frac{1}{2}k}\exp(\frac{1}{2}ay^2-\frac{1}{2}a) D_{2\alpha+k-\frac{3}{2}}[y(2a)^{\frac{1}{2}}]$ Re$(\alpha,k) < \frac{1}{2}$

$f(x)$	$g(y) = \int_0^\infty f(x)\, P^k_{ix-\frac{1}{2}}(y)\, dx$
$x \sinh(\pi x)\Gamma(\frac{1}{2}-k+ix)\Gamma(\frac{1}{2}-k-ix) \cdot$ $\cdot\ \Gamma(\frac{1}{4}+\frac{k}{2}+\frac{i}{2}x)\Gamma(\frac{1}{4}+\frac{k}{2}-\frac{i}{2}x)W_{k,\frac{1}{2}ix}(a)$	$\pi^2 2^{1-k} a^{\frac{3}{2}-k}(y^2-1)^{-\frac{1}{2}k}\exp(-\frac{1}{2}a^2+a^2y^2)\mathrm{Erfc}(ay)$ $-\frac{1}{2} < \mathrm{Re}\,k < \frac{1}{2}$
$x \sinh(\pi x)\Gamma(\frac{1}{2}-\alpha+ix)\Gamma(\frac{1}{2}-\alpha-ix) \cdot$ $\cdot\ \Gamma(\frac{1}{2}-k+ix)\Gamma(\frac{1}{2}-k-ix)W_{\alpha,ix}(a)$	$\pi(\frac{1}{2}a)^{\frac{1}{2}-\frac{1}{2}k}\Gamma(1-\alpha)\Gamma(1-\alpha-k)(y-1)^{-\frac{1}{2}k}(y+1)^{-\frac{1}{2}} \cdot$ $\cdot\ \exp(\frac{1}{4}ay-\frac{1}{4}a)W_{\alpha-\frac{k}{2}-\frac{1}{2},-\frac{k}{2}}(\frac{1}{2}a+\frac{1}{2}ay)$ $\mathrm{Re}(\alpha,k) < \frac{1}{2}$
$x \sinh(\pi x)\Gamma(\frac{1}{2}-\frac{\alpha}{2}+\frac{i}{2}x)\Gamma(\frac{1}{2}-\frac{\alpha}{2}-\frac{i}{2}x) \cdot$ $\cdot\ \Gamma(\frac{1}{2}-k+ix)\Gamma(\frac{1}{2}-k-ix)S_{\alpha,ix}(a)$	$2^{\frac{1}{2}+\alpha}\pi^{\frac{3}{2}}a^{1-k}\Gamma(\frac{3}{2}-\alpha-k)y^{\frac{1}{2}}(y^2-1)^{-\frac{1}{2}k}S_{\alpha+k-1,\frac{1}{2}}(ay)$ $\mathrm{Re}\,\alpha < 1$ $\mathrm{Re}\,k < \frac{1}{2}$
$x \sinh(\pi x)\Gamma(\frac{1}{2}-k+ix)\Gamma(\frac{1}{2}-k-ix) \cdot$ $\cdot\ S_{k+\frac{1}{2},ix}(a)$	$\pi a^{\frac{3}{2}} y K_k[a(y^2-1)^{\frac{1}{2}}]$ $\mathrm{Re}\,k \le \frac{1}{2}$
$x \sinh(\pi x)\Gamma(\frac{1}{2}-k+ix)\Gamma(\frac{1}{2}-k-ix) \cdot$ $\cdot\ S_{2k,2ix}(a)$	$\pi 2^{\frac{1}{2}k-2}a^{k+1}(1+y)^{\frac{1}{2}k}K_k[a(\frac{1}{2}y-\frac{1}{2})^{\frac{1}{2}}]$ $\mathrm{Re}\,k \le \frac{1}{2}$
$x \sinh(\pi x)\Gamma(\frac{1}{2}-k+ix)\Gamma(\frac{1}{2}-k-ix) \cdot$ $\cdot\ S_{k-\frac{1}{2},ix}(a)$	$\pi a^{\frac{1}{2}} K_k[a(y^2-1)^{\frac{1}{2}}]$ $\mathrm{Re}\,k \le \frac{1}{2}$

List of Abbreviations, Symbols and Notations

ε_n = Neumann's numbers, $\varepsilon_o = 1$, $\varepsilon_n = 2$, $n = 1,2,3,\ldots$

$\binom{a}{b}$ = Binomial coefficient, $\binom{a}{b} = \frac{\Gamma(a+1)}{\Gamma(b+1)\Gamma(a-b+1)}$

γ = Euler's constant, $\gamma = 0.57721\ldots$

1. Elementary functions

Trigonometric and inverse trigonometric functions:

$\sin x$, $\cos x$, $\tan x = \sin x / \cos x$, $\operatorname{ctn} x = \cos x / \sin x$,

$\sec x = 1/\cos x$, $\csc x = 1/\sin x$, $\arcsin x$, $\arccos x$, $\arctan x$,

$\operatorname{arcctn} x$

Hyperbolic functions:

$\sinh x = (e^x - e^{-x})/2$, $\cosh x = (e^x + e^{-x})/2$, $\tanh x = \sinh x / \cosh x$,

$\operatorname{ctnh} x = \cosh x / \sinh x$, $\operatorname{sech} x = 1/\cosh x$, $\operatorname{csch} x = 1/\sinh x$.

2. Orthogonal polynomials

Legendre polynomials:

$$P_n(x) = 2^{-n}(n!)^{-1} \frac{d^n}{dx^n} (x^2-1)^n = {}_2F_1(-n,n+1;1;\tfrac{1-x}{2})$$

Gegenbauer's polynomials:

$$C_n^{\alpha}(x) = [n!\Gamma(2\alpha)]^{-1}\Gamma(2\alpha+n)\ {}_2F_1(-n,2\alpha+n;\alpha+1/2;\tfrac{1-x}{2})$$

Chebycheff polynomials:

$$T_n(x) = \cos(n \arccos x) = {}_2F_1(-n,n;\tfrac{1}{2};\tfrac{1-x}{2}) = \frac{n}{2} \lim_{\alpha=0} \Gamma(\alpha)\ C_n^{\alpha}(x)$$

$$U_n(x) = (1-x^2)^{-\frac{1}{2}}\sin[(n+1)\arccos x]$$

$$= x(n+1)\,{}_2F_1(\tfrac{1-n}{2},\tfrac{3+n}{2};\tfrac{3}{2};1-x^2)$$

Jacobi polynomials:

$$P_n^{(\beta,\alpha)}(x) = [n!\Gamma(1+\beta)]^{-1}\Gamma(1+\beta+n)\,{}_2F_1(-n,n+\alpha+\beta+1;\beta+1;\tfrac{1-x}{2})$$

Laguerre polynomials:

$$L_n^{\alpha}(x) = (n!)^{-1}x^{-\alpha}e^{x}\,\frac{d^n}{dx^n}\,(e^{-x}x^{n+\alpha}) = [n!\Gamma(1+\alpha)]^{-1}\Gamma(\alpha+1+n)\,{}_1F_1(-n;1+\alpha;x$$

$$L_n(x) = L_n^{0}(x)$$

Hermite polynomials:

$$He_n(x) = (-1)^n\exp(x^2/2)\,\frac{d^n}{dx^n}\,\exp(-x^2/2)$$

$$He_{2n}(x) = (-1)^n2^{-n}(n!)^{-1}(2n)!\,{}_1F_1(-n;\tfrac{1}{2};\tfrac{1}{2}x^2)$$

$$He_{2n+1}(x) = (-1)^n2^{-n}(n!)^{-1}(2n+1)!\,x\,{}_1F_1(-n;\tfrac{3}{2};\tfrac{1}{2}x^2)$$

3. Gamma function and related functions

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \quad \text{Re } z > 0$$

ψ-function:

$$\psi(z) = \frac{d}{dz} \log \Gamma(z)$$

Beta function:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

See also under incomplete gamma function.

4. Riemann's and Hurwitz's zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad \text{Re } s > 1$$

$$\zeta(s,v) = \sum_{n=0}^{\infty} (n+v)^{-s}, \quad \text{Re } s > 1$$

5. Legendre functions (definition according to Hobson)

$$P_\alpha^\beta(z) = [\Gamma(1-\beta)]^{-1}\left(\frac{z+1}{z-1}\right)^{\beta/2} {}_2F_1(-\alpha,\alpha+1;1-\beta;\frac{1-z}{2})$$

$$Q_\alpha^\beta(z) = 2^{-\alpha-1}[\Gamma(\alpha+3/2)]^{-1} e^{i\pi\beta} \sqrt{\pi}\, \Gamma(\alpha+\beta+1) z^{-\alpha-\beta-1}(z^2-1)^{\beta/2} \cdot$$

$$\cdot\ {}_2F_1\left(\frac{\alpha+\beta+1}{2}, \frac{\alpha+\beta+2}{2}; \alpha+3/2; z^{-2}\right)$$

z is a point in the complex z plane cut along the real axis from $-\infty$ to $+1$.

$$\mathsf{P}_\alpha^\beta(x) = [\Gamma(1-\beta)]^{-1}\left(\frac{1+x}{1-x}\right)^{\beta/2} {}_2F_1(-\alpha,\alpha+1;1-\beta;\frac{1-x}{2}), \quad -1 < x < 1$$

$$\mathsf{Q}_\alpha^\beta(x) = \tfrac{1}{2} e^{-i\pi\beta} [e^{-i\pi\beta/2} Q_\alpha^\beta(x+i0) + e^{i\pi\beta/2} Q_\alpha^\beta(x-i0), \quad -1 < x < 1$$

$$P_\alpha(z) = P_\alpha^0(z); \quad Q_\alpha(z) = Q_\alpha^0(z); \quad \mathsf{P}_\alpha(x) = \mathsf{P}_\alpha^0(x); \quad \mathsf{Q}_\alpha(x) = \mathsf{Q}_\alpha^0(x)$$

6. Bessel functions

$$J_\alpha(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{\alpha+2n}}{n!\,\Gamma(\alpha+n+1)}$$

$$Y_\alpha(z) = \operatorname{ctn}(\pi\alpha) J_\alpha(z) - \csc(\pi\alpha) J_{-\alpha}(z)$$

$$H_\alpha^{(1)}(z) = J_\alpha(z) + iY_\alpha(z); \quad H_\alpha^{(2)}(z) = J_\alpha(z) - iY_\alpha(z)$$

7. Modified Bessel functions

$$I_\alpha(z) = e^{-i\pi\alpha/2} J_\alpha(ze^{i\pi/2}) = \sum_{n=0}^{\infty} \frac{(z/2)^{\alpha+2n}}{n!\,\Gamma(\alpha+n+1)}$$

$$K_\alpha(z) = \tfrac{1}{2}\pi \csc(\pi\alpha)[I_{-\alpha}(z) - I_\alpha(z)]$$

$$= \tfrac{1}{2} i\pi\, e^{i\pi\alpha/2} H_\alpha^{(1)}(ze^{i\pi/2}) = -\tfrac{1}{2} i\pi\, e^{-i\pi\alpha/2} H_\alpha^{(2)}(ze^{-i\pi/2})$$

8. Anger-Weber functions

$$\mathbf{J}_\alpha(z) = \pi^{-1} \int_0^\pi \cos(z \sin t - \alpha t)\,dt$$

$$\mathbf{E}_\alpha(z) = -\pi^{-1} \int_0^\pi \sin(z \sin t - \alpha t)\,dt$$

$$\mathbf{J}_n(z) = J_n(z), \quad n=0,1,2,\ldots$$

$$\mathbf{J}_{\frac{1}{2}}(z) = (\pi z/2)^{-\frac{1}{2}}\{\cos z\,[C(z) - S(z)] + \sin z\,[S(z) + C(z)]\} = \mathbf{E}_{-\frac{1}{2}}(z)$$

$$\mathbf{J}_{-\frac{1}{2}}(z) = (\pi z/2)^{-\frac{1}{2}}\{\cos z\,[C(z) + S(z)] - \sin z\,[C(z) - S(z)\} = \mathbf{E}_{\frac{1}{2}}(z)$$

9. Struve functions

$$\mathbf{H}_\alpha(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{\alpha+2n+1}}{\Gamma(n+3/2)\Gamma(\alpha+n+3/2)} = 2^{1-\alpha}\pi^{-\frac{1}{2}}[\Gamma(\alpha+1/2)]^{-1} s_{\alpha,\alpha}(z)$$

$$\mathbf{L}_\alpha(z) = -ie^{-i\pi\alpha/2}\, \mathbf{H}_\alpha(ze^{i\pi/2})$$

10. Lommel functions

$$s_{\alpha,\beta}(z) = [(\alpha-\beta+1)(\alpha+\beta+1)]^{-1} z^{\alpha+1} {}_1F_2(1; \tfrac{\alpha-\beta+3}{2}, \tfrac{\alpha+\beta+3}{2}; -z^2/4); \quad \alpha \pm \beta \neq -1,-2,-3,\ldots$$

$$S_{\alpha,\beta}(z) = s_{\alpha,\beta}(z) + 2^{\alpha-1}\Gamma(\tfrac{\alpha-\beta+1}{2})\Gamma(\tfrac{\alpha+\beta+1}{2})[\sin(\tfrac{\pi\alpha-\pi\beta}{2})J_\alpha(z) - \cos(\tfrac{\pi\alpha-\pi\beta}{2})Y_\alpha(z)]$$

Special cases of Lommel's functions:

$$s_{\alpha,\alpha}(z) = \pi^{\frac{1}{2}}2^{\alpha-1}\Gamma(\alpha+1/2)\,\mathbf{H}_\alpha(z)$$

$$S_{\alpha,\alpha}(z) = \pi^{\frac{1}{2}}2^{\alpha-1}\Gamma(\alpha+1/2)[\,\mathbf{H}_\alpha(z) - Y_\alpha(z)]$$

$$s_{0,\beta}(z) = \tfrac{1}{2}\pi\csc(\pi\beta)[\,\mathbf{J}_\beta(z) - \mathbf{J}_{-\beta}(z)]$$

$$S_{0,\beta}(z) = \frac{\pi}{2}\csc(\pi\beta)[\,\mathbf{J}_\beta(z) - \mathbf{J}_{-\beta}(z) - J_\beta(z) + J_{-\beta}(z)]$$

$$s_{-1,\beta}(z) = -\frac{\pi}{2}\beta^{-1}\csc(\pi\beta)[\,\mathbf{J}_\beta(z) + \mathbf{J}_{-\beta}(z)]$$

$$S_{-1,\beta}(z) = \frac{\pi}{2}\beta^{-1}\csc(\pi\beta)[J_\beta(z) + J_{-\beta}(z) - \mathbf{J}_\beta(z) - \mathbf{J}_{-\beta}(z)]$$

$$s_{1,\beta}(z) = 1 + \beta^2 s_{-1,\beta}(z); \quad S_{1,\beta}(z) = 1 + \beta^2 S_{-1,\beta}(z)$$

$$S_{\frac{1}{2},\frac{1}{2}}(z) = z^{-\frac{1}{2}}; \quad S_{\frac{3}{2},\frac{1}{2}}(z) = z^{\frac{1}{2}}$$

$$S_{-\frac{1}{2},\frac{1}{2}}(z) = z^{-\frac{1}{2}}[\sin z\,\mathrm{Ci}(z) - \cos z\,\mathrm{si}(z)]; \quad S_{-\frac{3}{2},\frac{1}{2}}(z) = -z^{-\frac{1}{2}}[\sin z\,\mathrm{si}(z) + \cos z\,\mathrm{Ci}(z)]$$

$$\lim_{\alpha=\beta}\,[\Gamma(\beta-\alpha)]^{-1}s_{\alpha-1,\beta}(z) = -2^{\beta-1}\Gamma(\beta)J_\beta(z)$$

Lommel functions of two variables:

$$U_\alpha(w,z) = \sum_{n=0}^{\infty}(-1)^n(w/z)^{\alpha+2n}J_{\alpha+2n}(z)$$

$$V_\alpha(w,z) = \cos(\tfrac{1}{2}w + \tfrac{1}{2}z^2/w + \tfrac{1}{2}\alpha\pi) + U_{2-\alpha}(w,z)$$

11. Gauss's hypergeometric function

$$ {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!}, \quad |z| < 1 $$

12. Generalized hypergeometric series

$$ {}_mF_n(a_1,a_2,\ldots,a_m;\, b_1,b_2,\ldots,b_n;z) = \frac{\Gamma(b_1)\cdots\Gamma(b_n)}{\Gamma(a_1)\cdots\Gamma(a_m)} \sum_{k=0}^{\infty} \frac{\Gamma(a_1+k)\cdots\Gamma(a_m+k)}{\Gamma(b_1+k)\cdots\Gamma(b_n+k)} \frac{z^k}{k!} $$

13. Confluent hypergeometric functions

$$ {}_1F_1(a;c;z) = \frac{\Gamma(c)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(c+n)} \frac{z^n}{n!} $$

$$ {}_1F_1(a;a;z) = e^z, \quad {}_1F_1(a;2a;2z) = 2^{a-\frac{1}{2}}\Gamma(a+\tfrac{1}{2})z^{\frac{1}{2}-a}e^z I_{a-\frac{1}{2}}(z) $$

$$ {}_1F_1(\tfrac{1}{2};\tfrac{3}{2};ix) = e^{ix}\,{}_1F_1(1;\tfrac{3}{2};-ix) = (\tfrac{1}{2}\pi/x)^{\frac{1}{2}}[C(x) + i\,S(x)] $$

Whittaker's functions:

$$ M_{\alpha,\beta}(z) = z^{\beta+\frac{1}{2}}e^{-\frac{1}{2}z}\,{}_1F_1(\beta-\alpha+\tfrac{1}{2};2\beta+1;z) $$

$$ W_{\alpha,\beta}(z) = \frac{\Gamma(-2\beta)}{\Gamma(-\alpha-\beta+\frac{1}{2})} M_{\alpha,\beta}(z) + \frac{\Gamma(2\beta)}{\Gamma(\beta-\alpha+\frac{1}{2})} M_{\alpha,-\beta}(z) $$

Special cases of Whittaker's functions:

$$ M_{0,\beta}(z) = \Gamma(1+\beta)2^{2\beta}I_\beta(z/2)\sqrt{z}\ ; \quad W_{0,\beta}(z) = (z/\pi)^{\frac{1}{2}}K_\beta(z/2) $$

$$ M_{\alpha,0}(z) = z^{\frac{1}{2}}e^{-\frac{1}{2}z}L_{\alpha-\frac{1}{2}}(z); \quad M_{\frac{1}{4},\frac{1}{4}}(z) = -i\,\tfrac{1}{2}\pi^{\frac{1}{2}}z^{\frac{1}{4}}e^{-\frac{1}{2}z}\mathrm{Erf}(iz^{\frac{1}{2}}) $$

Parabolic cylinder function:

$$ D_\alpha(z) = 2^{(\alpha+\frac{1}{2})/2}z^{-\frac{1}{2}}W_{(\alpha+\frac{1}{2})/2,\frac{1}{4}}(z^2/2) $$

$$ D_n(z) = e^{-z^2/4}He_n(z), \quad n=0,1,2,\ldots $$

$$ D_{-1}(z) = (\pi/2)^{\frac{1}{2}}e^{z^2/4}\mathrm{Erfc}(2^{-\frac{1}{2}}z) $$

$$ D_{-\frac{1}{2}}(z) = (\tfrac{1}{2}z/\pi)^{\frac{1}{2}}K_{\frac{1}{4}}(z^2/4) $$

Error integrals:

$$\mathrm{Erf}(z) = 2\pi^{-\frac{1}{2}} \int_0^z e^{-t^2} dt = 2\pi^{-\frac{1}{2}} z\, {}_1F_1(\tfrac{1}{2};\tfrac{3}{2};-z^2) = 2(\pi z)^{-\frac{1}{2}} e^{-z^2/2} M_{-\frac{1}{4},\frac{1}{4}}(z^2)$$

$$\mathrm{Erfc}(z) = 1 - \mathrm{Erf}(z) = 2\pi^{-\frac{1}{2}} \int_z^\infty e^{-t^2} dt = (\pi z)^{-\frac{1}{2}} e^{-z^2/2} W_{-\frac{1}{4},\frac{1}{4}}(z^2) = \pi^{-\frac{1}{2}} \Gamma(\tfrac{1}{2}, z^2)$$

$$\mathrm{Erf}(x^{\frac{1}{2}} e^{i\pi/4}) = 2^{\frac{1}{2}} e^{i\pi/4} [C(x) - i\, S(x)]$$

$$\mathrm{Erfc}(x^{\frac{1}{2}} e^{i\pi/4}) = 1 - C(x) - S(x) - i[C(x) - S(x)]$$

Fresnel's integrals:

$$C(x) = (2\pi)^{-\frac{1}{2}} \int_0^z t^{-\frac{1}{2}} \cos t\, dt; \quad S(x) = (2\pi)^{-\frac{1}{2}} \int_0^z t^{-\frac{1}{2}} \sin t\, dt$$

Exponential integral:

$$-\mathrm{Ei}(-z) = \int_z^\infty t^{-1} e^{-t} dt = -\gamma - \log z - \sum_{n=1}^\infty \frac{(-z)^n}{n \cdot n!} = z^{-\frac{1}{2}} e^{-z/2} W_{-\frac{1}{2},0}(z) = \Gamma(0,z),$$

$$-\pi < \arg z < \pi$$

$$\overline{\mathrm{Ei}}(x) = \tfrac{1}{2}[\mathrm{Ei}(x+i0) + \mathrm{Ei}(x-i0)] = -\mathrm{P.V.} \int_{-x}^\infty t^{-1} e^{-t} dt$$

$$= \gamma + \log x + \sum_{n=1}^\infty \frac{x^n}{n \cdot n!}, \quad x > 0$$

$$\mathrm{Ei}(-ix) = \mathrm{Ci}(x) - i\,\mathrm{si}(x); \quad \overline{\mathrm{Ei}}(ix) = \mathrm{Ci}(x) + i\pi + i\,\mathrm{si}(x)$$

Sine and cosine integral:

$$\mathrm{Si}(x) = \int_0^x t^{-1} \sin t\, dt; \quad \mathrm{si}(x) = -\int_x^\infty t^{-1} \sin t\, dt = \mathrm{Si}(x) - \frac{\pi}{2}$$

$$= \frac{i}{2} [\mathrm{Ei}(-ix) - \mathrm{Ei}(ix)]$$

$$\mathrm{Ci}(x) = -\int_x^\infty t^{-1} \cos t\, dt = \frac{1}{2} [\mathrm{Ei}(-ix) + \mathrm{Ei}(ix)] = \gamma + \log x + \sum_{n=1}^\infty \frac{(-1)^n x^{2n}}{2n(2n)!}$$

Incomplete gamma function:

$$\gamma(\alpha,z) = \int_0^z t^{\alpha-1}e^{-t}dt = \frac{1}{\alpha} z^\alpha {}_1F_1(\alpha;\alpha+1;-z)$$

$$\Gamma(\alpha,z) = \Gamma(\alpha) - \gamma(\alpha,z) = \int_z^\infty t^{\alpha-1}e^{-t}dt = z^{(\alpha-1)/2}e^{-z/2}W_{(\alpha-1)/2,\alpha/2}(z)$$

$$\Gamma(\tfrac{1}{2},z^2) = \pi^{\frac{1}{2}}\mathrm{Erfc}(z); \quad \Gamma(0,z) = -\mathrm{Ei}(-z); \quad \gamma(\tfrac{1}{2},z^2) = \pi^{\frac{1}{2}}\mathrm{Erf}(z)$$

14. Elliptic integrals and theta functions

Complete elliptic integrals:

$$K(k) = \int_0^{\pi/2}(1 - k^2\sin^2 t)^{-\frac{1}{2}}dt = \frac{\pi}{2}\, {}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;k^2)$$

$$E(k) = \int_0^{\pi/2}(1 - k^2\sin^2 t)^{\frac{1}{2}}dt = \frac{\pi}{2}\, {}_2F_1(-\tfrac{1}{2},\tfrac{1}{2};1;k^2)$$

Theta functions:

$$\theta_1(v,t) = (-it)^{-\frac{1}{2}}\sum_{-\infty}^{\infty}(-1)^n\exp[-i\pi(v + n - \tfrac{1}{2})^2t^{-1}]$$

$$\theta_2(v,t) = (-it)^{-\frac{1}{2}}\sum_{-\infty}^{\infty}(-1)^n\exp[-i\pi(v + n)^2t^{-1}]$$

$$\theta_3(v,t) = (-it)^{-\frac{1}{2}}\sum_{-\infty}^{\infty}\exp[-i\pi(v + n)^2t^{-1}]$$

$$\theta_4(v,t) = (-it)^{-\frac{1}{2}}\sum_{-\infty}^{\infty}\exp[-i\pi(v + n - \tfrac{1}{2})^2t^{-1}]$$

Symbol	Name of the Function	Listed under
$C(x)$	Fresnel's integral	13
$Ci(x)$	Cosine integral	13
$C_n^{\alpha}(x)$	Gegenbauer's polynomial	2
$D_{\alpha}(x)$	Parabolic cylinder function	13
$E(k)$	Complete elliptic integral	14
$Ei(-x)$, $\overline{Ei}(x)$	Exponential integrals	13
$Erf(z)$, $Erfc(z)$	Error integrals	13
$\mathbf{E}_{\alpha}(z)$	Anger-Weber function	8
${}_mF_n$	Hypergeometric function	11, 12, 13
$He_n(x)$	Hermite's polynomial	2
$H_{\alpha}^{(1,2)}(x)$	Hankel's functions	6
$\mathbf{H}_{\alpha}(z)$	Struve's function	9
$I_{\alpha}(z)$	Modified Bessel function	7
$J_{\alpha}(z)$	Bessel's function	6
$\mathbf{J}_{\alpha}(z)$	Anger-Weber function	8
$K(k)$	Complete elliptic integral	14
$K_{\alpha}(z)$	Modified Hankel function	7
$L_n^{\alpha}(x)$	Laguerre's polynomial	2
$\mathbf{L}_{\alpha}(z)$	Struve's function	9
$M_{\alpha,\beta}(z)$, $W_{\alpha,\beta}(z)$	Whittaker's functions	13
$P_n(x)$	Legendre's polynomials	2
$P_n^{(\alpha,\beta)}(x)$	Jacobi's polynomials	2

Symbol	Name of the Function	Listed under
$P_\alpha^\beta(z)$, $\underline{P}_\alpha^\beta(x)$	Legendre functions	5
$Q_\alpha^\beta(z)$, $\underline{Q}_\alpha^\beta(x)$	Legendre functions	5
$S(x)$	Fresnel's integral	13
$si(x)$, $Si(x)$	Sine integrals	13
$s_{\alpha,\beta}(z)$, $S_{\alpha,\beta}(z)$	Lommel's function	10
$T_n(x)$, $U_n(x)$	Chebycheff's polynomials	2
$U_\alpha(w,z)$, $V_\alpha(w,z)$	Lommel's function of two variables	10
$W_{\alpha,\beta}(z)$	Whittaker's function	13
$Y_\alpha(z)$	Neumann's function	6
$B(x,y)$	Beta function	3
$\Gamma(z)$	Gamma function	3
$\Gamma(\alpha,z)$, $\gamma(\alpha,z)$	Incomplete gamma functions	13
$\psi(z)$	Psi function	3
$\theta_\alpha(v,t)$	Theta functions	14
$\zeta(s)$	Riemann's zeta function	4
$\zeta(s,v)$	Hurwitz's zeta function	4

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