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DRIVING POINT IMPEDANCE SYNTHESIS USING IMPEDANCE OPERATORS

BY

JOSEPH BERT MURDOCH

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ERRATA

Page	Correction
5	In the second equation of (1-5), change $- I_2$ to $- I_2 \times 1$ to show that the equation is dimensionally correct and also change I_{22} to I_2.
39	In eqs. (2-8) and (2-9), change num EvV to num EvV_1.
42	Change all impedances denoted by ζ to ζ_2.
43-44	Theorem E and the discussion following it in Section 2.4 may be better stated as follows:

Theorem E

The numerators of the even parts of a series of cascaded V operators are related by

$$\text{num } EvV = (\text{num } EvV_1)(\text{num } EvV_2) \; \ldots \; (\text{num } EvV_n)^{*} \qquad (2\text{-}15)$$

and, assuming that no common (surplus) factors have been cancelled in the numerator and denominator of Z, which is given by

$$Z = V_1 V_2 \; \ldots \; V_n \zeta_n \qquad (2\text{-}16)$$

*The proof appears in Appendix 1.B.

it follows that

$$\text{num } EvZ = (\text{num } EvV_1)(\text{num } EvV_2) \; \ldots . \; (\text{num } EvV_n)(\text{num } Ev\zeta_n)^{*} \qquad (2\text{-}17)$$

Thus the zeros of num EvZ are split between the V operators and the terminating impedance as described in section 1.9.

*The invalidity of eq. (2-17) if common factors have been cancelled from Z is discussed in the following section.

Page	Correction
48	The sentence in lines 4 and 5 and the subsequent V parameters should be changed to read as follows: The components of V_1 and V_2 are derived from eq. (2-8). For example $\text{num Ev}V_1 = 1 - 2s^2 = (1 + \sqrt{2}\,s)(1 - \sqrt{2}\,s)$; $V_{11} = \frac{1}{s}$, $V_{22} = \frac{1}{s}$, $V_{\substack{12\\21}} = \frac{1 \pm \sqrt{2}\,s}{s} = \frac{1}{s} \pm \sqrt{2}$
56	The second term in the denominator of eq. (2-41) is $s(\frac{b}{Z_a} + \frac{a}{\zeta_{1b}})$.
86	In the line above eq. (3-34), change (eq. 3-2) to eq. (3-12).
92	Change ζ_{1b} to ς_{1b} in line 7.
96	The denominator of the third term of V_{22} in eq. (3-58) is $s^2 + \frac{B_1}{B_3}$.
97	The last sentence in Section 3.9 should be changed to read as follows: .... ζ_4 is eight less in rank than Z whereas, if they are of first order, ζ_4 is four less in rank than Z.
100	The last equation should be changed to read $\frac{k^2}{\zeta_2} = \frac{4}{25}\left(\frac{\frac{5}{3}s + 1}{\frac{3}{5}s + 2}\right)$
105	In the first sentence in Section 4.2, change (3-49) to (3-48).
128	In line 8, change eq. (5-17) to eq. (5-15).
132	The value of the resistive termination in Fig. (5-5) is 2 ohms.

Page	Correction
153	In line 5, remove the s from procedures.
164	In the second line from the bottom, change Fig. (6-2b) to Fig. (6-3b).
168	In the first line of Appendix 1.D, change eq. (2-35) to eq. (2-36).
170	Label the second equation (A-21) and change the third equation to (A-22).

ABSTRACT

An approach to driving point impedance synthesis is developed, using the concept of an impedance operator, which is general, systematic, flexible and easily applied. It is shown that the synthesis procedures of Brune, Darlington, Bott-Duffin and Miyata readily lend themselves to this impedance operator approach. It is further shown that, through the impedance operator and the flexibility it provides, new driving point impedance realizations can be achieved and existing realizations can be made more general.

The mathematical properties of the impedance operator are investigated in detail. Specific impedance operators of rank 2, 4 and 6, derived from repeated applications of Richards' Theorem and extension thereof, are examined. Through these operators, it is shown that thirteen realizable network sections containing one or more arbitrary constants may always be removed from a prf driving point impedance function leaving, in cascade, a terminating impedance which is realizable and contains the same arbitrary constants.

Using the impedance operator approach, three cascade synthesis procedures are developed. The first is an extension of the Bott-Duffin procedure. The second is a general reciprocal synthesis procedure applicable to any prf driving point impedance. The third is a general non-reciprocal synthesis procedure not requiring transformers.

ACKNOWLEDGEMENT

The research of this thesis, and indeed much of the author's Ph.D. program, was made possible by the National Science Foundation, for which the author is deeply grateful. The author wishes t express his thanks to Dr. Dov Hazony for his friendship, guidance and encouragement throughout the entire Ph.D. program and for always being able to provide the right suggestion at the right time when the author became bogged down in his thesis research; to Dr. James D. Schoeffler for his timely suggestions and assistance at the conclusion of the research; to Miss Janet Leonard for her help in typing a portion of the manuscript; and to Miss Florence Alaimo for typing the bulk of the manuscript and then patiently making the necessary corrections.

TABLE OF CONTENTS

Page

CHAPTER I

FORMULATION OF THE PROBLEM AND A REVIEW AND EXTENSION OF EXISTING TECHNIQUES

1.1 Statement of the Problem

In the study of driving point impedance synthesis procedures, one encounters many methods and techniques which are interrelated. Each method has certain limitations, advantages and disadvantages. It is the purpose of this thesis to develop an approach to driving-point impedance synthesis using the concept of an impedance operator (to be defined presently) which is general, systematic, flexible and easy to apply. It is to be shown that the well-known synthesis procedures of Brune,[17] Darlington,[1] Bott and Duffin,[6] and Miyata[13] readily lend themselves to this impedance operator approach. It is also to be shown that, through the impedance operator and the flexibility which it provides, additional driving point impedance realizations can be achieved and existing realizations can be made more general.

The impedance operator concept to be developed herein stems from the Darlington synthesis procedure.* This procedure realizes a prf** driving point impedance given by

*This procedure is reviewed in detail in Section 1.2.

**Positive real function.

$$Z = \frac{m_1 + n_1}{m_2 + n_2}{}^{*} \qquad (1\text{-}1)$$

in terms of a lossless network terminated in a pure resistance (usually one ohm) as shown in Fig. (1-1). The procedure is

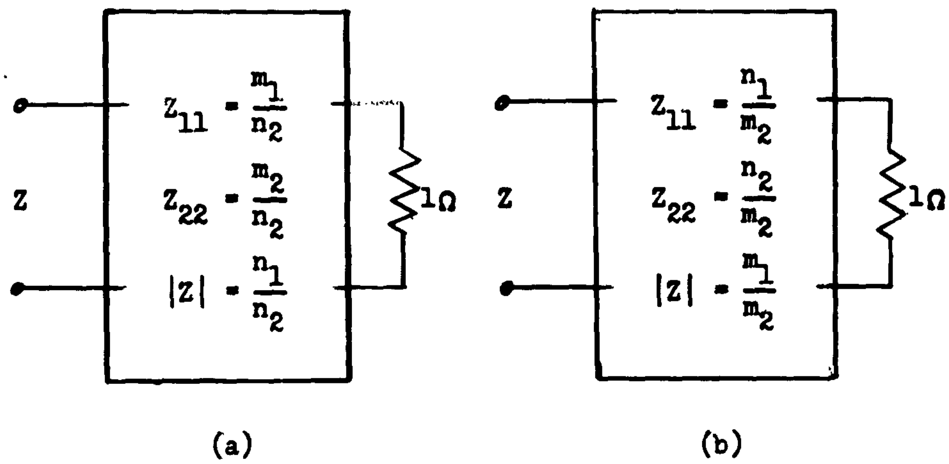

Fig. (1-1) Darlington Realizations

a cascade rather than a distributed one in that only a single termination is included.

A greater flexibility can be obtained in the synthesis of Z if the constraint of a resistive termination is relaxed to permit a prf terminating impedance, ζ. To investigate this possibility, let the following transformation be constructed:

---

* m_1 and m_2 are even polynomials in s while n_1 and n_2 are odd polynomials in s. These polynomials are further restricted by the fact that Z is prf.

$$Z = \frac{m_1\zeta + n_1}{m_2 + n_2\zeta} \qquad (1\text{-}2)$$

It is necessary to investigate the conditions under which the right hand side of eq. (1-2) represents a prf driving point impedance in the physical sense of Fig. (1-1) with the one-ohm termination replaced by ζ. Define an impedance operator, V, which is equal to Z when ζ is a one-ohm resistance and which operates on ζ to give Z. Thus V is given by

$$V = \frac{m_1 + n_1}{m_2 + n_2} \qquad (1\text{-}3)$$

and

$$Z = V\zeta \qquad (1\text{-}4)$$

where eq. (1-4) is to be interpreted as "V operating on ζ".

<u>Theorem A</u>

There is a theorem due to Hazony[4] which states that if a prf V can be constructed such that ζ has no right half plane poles, then ζ is prf if Z is prf.

Theorem A relates only to the transformation given by eq. (1-2) and does not, in itself, quarantee the cascade representation of Fig. (1-1) with a ζ termination. To achieve this cascade representation, it is necessary in addition that V in eq. (1-3) represent a lossless network terminated in one ohm. Then V may be synthesized using Darlington's procedure and the ζ termination added. Thus, in Fig. (1-1), the Z parameters in the boxes are replaced by V_{11}, V_{22} and $|V|$, respectively, and the one-ohm termination is replaced by ζ in Fig. (1-1a) and $1/\zeta$ in Fig. (1-1b).*

V and ζ can take many forms in the representation of a given Z, since arbitrary constants may be incorporated in both V and ζ. These constants may be chosen to produce desired characteristics in either the removed network sections (V) or the terminating impedance (ζ). The constants build a definite flexibility into the impedance operator approach.

The similarity of eqs. (1-1) and (1-3) is important. In effect, this similarity means that the networks derived by Darlington's procedure with a resistive termination are also applicable in the case of a prf ζ termination. It is appropriate, therefore, to review in detail the basic Darlington synthesis procedure and the types of lossless networks which it gives and to seek generalizations which will yield additional useful networks. Also, since the impedance operator approach is generally applicable to the cascade synthesis of driving point impedance functions through the removal of network sections, a brief review of existing cascade synthesis techniques is in order.

In view of the above considerations, the objectives for the remainder of Chapter I may be formulated as follows:

*The mechanics of this procedure are discussed in detail in Chapter II.

A) A review of the reciprocal and non-reciprocal Darlington synthesis procedures.

B) A discussion of the residue and extended residue conditions and their use in cascade syntheses.

C) The derivation of the Z parameters of certain loaded gyrator networks which are useful in the non-reciprocal Darlington syntheses of driving point impedances of rank* 4 and above.

D) The derivation of non-reciprocal cascade Darlington syntheses for prf driving point impedances of rank 2, 4 and 6.

E) A review of existing cascade synthesis procedures and the network sections which result therefrom.

1.2 The Reciprocal Darlington Synthesis Procedure[1,3,14]

The conventional Z-parameter four-terminal network equations for the case of a one-ohm resistive termination are:

$$E_1 = Z_{11} I_1 + Z_{12} I_2$$
$$-I_2 = Z_{21} I_1 + Z_{22} I_{22} \qquad (1\text{-}5)$$

Solving for the driving point impedance yields

*sum of degrees of numerator and denominator.

$$Z = Z_{11} \frac{1 + \frac{Z_{11}\,Z_{22} - Z_{12}\,Z_{21}}{Z_{11}}}{Z_{22} + 1} \qquad (1\text{-}6)$$

Eq. (1-1) may be rearranged in two ways to match eq. (1-6).

$$Z = \frac{m_1}{n_2} \frac{1 + \frac{n_1}{m_1}}{\frac{m_2}{n_2} + 1} \qquad (1\text{-}7)$$

$$Z = \frac{n_1}{m_2} \frac{1 + \frac{m_1}{n_1}}{\frac{n_2}{m_2} + 1} \qquad (1\text{-}8)$$

Eq. (1-7) corresponds to Fig. (1-1a) and suggests the identification

$$Z_{11} = \frac{m_1}{n_2}, \quad Z_{22} = \frac{m_2}{n_2}, \quad \frac{Z_{11}\,Z_{22} - Z_{12}\,Z_{21}}{Z_{11}} = \frac{n_1}{m_1} \qquad (1\text{-}9)$$

while Eq. (1-8), corresponding to Fig. (1-1b), suggests that

$$Z_{11} = \frac{n_1}{m_2}, \quad Z_{22} = \frac{n_2}{m_2}, \quad \frac{Z_{11}\,Z_{22} - Z_{12}\,Z_{21}}{Z_{11}} = \frac{m_1}{n_1} \qquad (1\text{-}10)$$

The third relation in eq. (1-9) reduces to

$$Z_{12}\,Z_{21} = \frac{m_1\,m_2 - n_1\,n_2}{n_2^2} = \frac{\text{num EvZ}^*}{n_2^2} \qquad (1\text{-}11)$$

---

*num EvZ = numerator of the even part of Z.

while the third relation in eq. (1-10) becomes

$$Z_{12}\, Z_{21} = \frac{-\, m_1\, m_2 + n_1\, n_2}{m_2^2} = \frac{-\text{ num EvZ}}{m_2^2} \qquad (1\text{-}12)$$

Hereafter, eq. (1-7) is referred to as the "n-type" and eq. (1-8) as the "m-type" Darlington procedure, the n and m denoting the term in the denominators of the Z parameters in each case.

In the reciprocal Darlington procedure, $Z_{12} = Z_{21}$ so that eqs. (1-11) and (1-12) become

$$Z_{12} = Z_{21} = \frac{\sqrt{\text{num EvZ}}}{n_2} \qquad (1\text{-}13)$$

$$Z_{12} = Z_{21} = \frac{\sqrt{-\text{num EvZ}}}{m_2} \qquad (1\text{-}14)$$

The conditions for realizability of a lossless reciprocal four-terminal network are that Z_{11} and Z_{22} be reactance functions and that the residue condition

$$k_{11}\, k_{22} - k_{12}^2 \geq 0 \qquad (1\text{-}15)$$

be satisfied at all poles. Since $m_1 + n_1$, $m_2 + n_2$, $m_1 + n_2$ and $m_2 + n_1$ are all Hurwitz polynomials, Z_{11} and Z_{22} are necessarily reactance functions. Furthermore, assuming that num EvZ is a perfect square, the residue condition holds with the equal sign at all finite poles. For a pole at infinity, the equal sign applies only if n_1 is not greater in rank than n_2.

In order that num EvZ be a perfect square so that Z_{12} is a rational function of s, it is necessary that the zeros of num EvZ be of even multiplicity, a condition which is not true in general. It is possible to avoid this difficulty by multiplying the numerator and denominator of Z by an auxiliary Hurwitz polynomial so chosen that num EvZ becomes a perfect square.[3] However, since this procedure is not in general permissable in the case of the impedance operator, it will not be considered further here.* Rather, the restriction that $Z_{12} = Z_{21}$ is relaxed and attention is refocused on eqs. (1-11) and (1-12).

1.3 Non-Reciprocal Darlington Synthesis

In this and the following four sections, a non-reciprocal Darlington synthesis procedure applicable to any prf driving point impedance is developed and applied to impedances of rank 2, 4 and 6. The resulting networks may be considered not only as syntheses of Z with a one-ohm termination but also as syntheses of specific impedance operators. This latter feature receives considerable attention in Chapters III and IV.

The starting point in the development of the non-reciprocal Darlington synthesis procedure is a consideration of the zeros of an even polynomial in s. In terms of s^2, these zeros may be real and positive, real and negative or complex. In terms of s, these

*The fact that this procedure is not applicable in the case of the V operator is justified in Appendix II.

zeros must have quadrantal symmetry[3]. This requirement, coupled with the restriction that the even part of a prf impedance be positive everywhere on the $j\omega$ axis, necessitates that num EvZ have only the following types of terms (or powers thereof)

$$\begin{aligned}&(a^2 - s^2)\\ &(b^2 + s^2)^2 \qquad (1\text{-}16)\\ &(s^2 + cs + d)(s^2 - cs + d)\end{aligned}$$

Whatever combinations of these terms may occur, it is always possible to write num EvZ in the form

$$\text{num EvZ} = m_o^2 - n_o^2 = (m_o + n_o)(m_o - n_o) \qquad (1\text{-}17)$$

where m_o and n_o are even and odd polynomials in s, respectively. Because of eq. (1-17), eqs. (1-11) and (1-12) may now be separated to yield*

$$Z_{12} = \frac{m_o + n_o}{n_2}, \quad Z_{21} = \frac{m_o - n_o}{n_2} \qquad (1\text{-}18)$$

$$Z_{12} = \frac{n_o + m_o}{m_2}, \quad Z_{21} = \frac{n_o - m_o}{m_2} \qquad (1\text{-}19)$$

Theorem B

Using the forms of Z_{12} and Z_{21} in eqs. (1-18) and (1-19), it is possible to achieve a Darlington synthesis of any prf driving

*This split assigns the left-half plane zeros of num EvZ to Z_{12} and those in the right half plane to Z_{21}. The reasons for this particular choice are indicated by Theorem B.

point impedance without the use of surplus factors.

To begin the proof of Theorem B, Z_{12} and Z_{21} in eqs. (1-18) and (1-19) may be expanded by partial fraction expansions. The result has two general forms,* where β, α and K are positive real constants.

$$Z_{\substack{12\\21}} = 2\,\frac{{}_1k_{12}s \pm \beta_1}{s^2 + \omega_1^2} + 2\,\frac{{}_2k_{12}s \pm \beta_2}{s^2 + \omega_2^2} + \ldots. \pm K_1 \tag{1-20}$$

$$Z_{\substack{12\\21}} = 2\,\frac{{}_1k_{12}s \pm \alpha_1 s^2}{s^2 + \omega_1^2} + 2\,\frac{{}_2k_{12}s \pm \alpha_2 s^2}{s^2 + \omega_2^2} + \ldots. \pm K_2 \tag{1-21}$$

The denominators of eqs. (1-20) and (1-21) are correct since the zeros of the even and odd parts of a Hurwitz polynomial must all lie on the $j\omega$ axis. The numerators are justified by considering a particular example.

<u>Example 1</u>

Let

$$Z = \frac{s^4 + 11s^3 + 9s^2 + 34s + 2}{s^4 + \frac{5}{2}s^3 + \frac{27}{2}s^2 + \frac{7}{2}s + 8}$$

$$\text{num Ev}Z = (-s^2 + 1)(-s^2 + 4)(s^4 + 4)$$

$$= (s^4 + 10s^2 + 2)^2 - s^2(5s^2 + 10)^2$$

*Terms of the form $k_\infty s$ and k_o/s may also occur in eqs. (1-20) and (1-21), depending on the rank of Z.

Attempting to synthesize Z by the m-type Darlington procedure gives the following Z parameters:

$$Z_{11} = \frac{s(11s^2 + 34)}{s^4 + \frac{27}{2}s^2 + 8}\,, \quad Z_{22} = \frac{s(\frac{5}{2}s^2 + \frac{7}{2})}{s^4 + \frac{27}{2}s^2 + 8}$$

$$Z_{\substack{12\\21}} = \frac{s(5s^2 + 10) \pm (s^4 + 10s^2 + 2)}{s^4 + \frac{27}{2}s^2 + 8}$$

Each of these expressions may be expanded by partial fractions.

$$Z_{11} = \frac{2\,{}_1k_{11}s}{s^2 + \omega_1^2} + \frac{2\,{}_2k_{11}s}{s^2 + \omega_2^2}$$

$$Z_{22} = \frac{2\,{}_1k_{22}s}{s^2 + \omega_1^2} + \frac{2\,{}_2k_{22}s}{s^2 + \omega_2^2}$$

$$Z_{\substack{12\\21}} = \frac{2\,{}_1k_{12}s}{s^2 + \omega_1^2} + \frac{2\,{}_2k_{12}s}{s^2 + \omega_2^2} \pm \frac{s^4 + 10s^2 + 2}{s^4 + \frac{27}{2}s^2 + 8}$$

where $(s^2 + \omega_1^2)(s^2 + \omega_2^2) = s^4 + \frac{27}{2}s^2 + 8$.

The last term in $Z_{\substack{12\\21}}$ may be expanded in several ways to yield

$$\frac{s^4 + 10s^2 + 2}{s^4 + \frac{27}{2}s^2 + 8} = \frac{2\beta_1}{s^2 + \omega_1^2} + \frac{2\beta_2}{s^2 + \omega_2^2} + K_1$$

$$= \frac{2\alpha_1 s^2}{s^2 + \omega_1^2} + \frac{2\alpha_2 s^2}{s^2 + \omega_2^2} + K_2$$

$$= \frac{2\gamma_1 s^2}{s^2 + \omega_1^2} + \frac{2\gamma_2}{s^2 + \omega_2^2} + K_3$$

$$= \frac{2\delta_1}{s^2 + \omega_1^2} + \frac{2\delta_2 s^2}{s^2 + \omega_2^2} + K_4$$

Considering only the first two forms, since the last two are not essentially different, $Z_{\substack{12\\21}}$ may be expressed in the following two general ways which are seen to agree with eqs. (1-20) and (1-21).

$$Z_{\substack{12\\21}} = 2\,\frac{{}_1k_{12}s \pm \beta_1}{s^2 + \omega_1^2} + 2\,\frac{{}_2k_{12}s \pm \beta_2}{s^2 + \omega_2^2} \pm K_1$$

$$Z_{\substack{12\\21}} = 2\,\frac{{}_1k_{12}s \pm \alpha_1 s^2}{s^2 + \omega_1^2} + 2\,\frac{{}_2k_{12}s \pm \alpha_2 s^2}{s^2 + \omega_2^2} \pm K_2$$

For impedances of higher rank, the form of $Z_{\substack{12\\21}}$ is the same except that more α and β (but not K) terms are present.

1.4 Extended Residue Conditions

Returning now to eqs. (1-20) and (1-21), the question of the validity of the residue condition in eq. (1-15) for the non-reciprocal case arises. Near a $j\omega$ axis pole, the Z parameters have the general form

$$Z_{11} = 2\,\frac{k_{11}s}{s^2 + \omega_o^2} \qquad Z_{22} = 2\,\frac{k_{22}s}{s^2 + \omega_o^2}$$

$$Z_{\substack{12\\21}} = 2\,\frac{k_{12}s \pm \alpha s^2}{s^2 + \omega_o^2} \text{ or } 2\,\frac{k_{12}s \pm \beta}{s^2 + \omega_o^2} \qquad (1\text{-}22)$$

Consider the function $\frac{Z_{11}\,Z_{22} - Z_{12}\,Z_{21}}{Z_{11}}$. This is given by either eq. (1-9) or eq. (1-10) as $\frac{n_1}{m_1}$ or $\frac{m_1}{n_1}$. Since $m_1 + n_1$ is a Hurwitz polynomial, $\frac{n_1}{m_1}$ and $\frac{m_1}{n_1}$ are positive real reactance functions. It follows that

$$\mathrm{Re}\,\frac{Z_{11}\,Z_{22} - Z_{12}\,Z_{21}}{Z_{11}} = 0 \quad \text{for} \quad \mathrm{Re}\; s = 0$$

$$\mathrm{Re}\,\frac{Z_{11}\,Z_{22} - Z_{12}\,Z_{21}}{Z_{11}} > 0 \quad \text{for} \quad \mathrm{Re}\; s > 0 \qquad (1\text{-}23)$$

Now let $s = j\omega_o + \varepsilon$, where ε is a small positive real quantity which approaches zero, and substitute this value of s into eqs. (1-22) and (1-23), considering the first form of $Z_{\substack{12\\21}}$ in eq. (1-22). After simplification, eq. (1-23) reduces to

$$\frac{(k_{11}k_{22} - k_{12}^2 - \alpha^2\omega_o^2)(\varepsilon^2 + 2\omega_o^2) + 4\alpha^2\omega_o^2\varepsilon^2}{k_{11}\varepsilon(\varepsilon^2 + 4\omega_o^2)} \geq 0 \qquad (1\text{-}24)$$

In order that this expression be positive for all positive real ε, it is necessary that

$$k_{11}\,k_{22} - k_{12}^2 - \alpha^2\omega_o^2 \geq 0 \qquad (1\text{-}25)$$

A similar development using the second form of $Z_{\substack{12\\21}}$ in eq. (1-22) yields

$$\frac{\left(k_{11}k_{22} - k_{12}^2 - \frac{\beta^2}{\varepsilon^2 + \omega_o^2}\right)(\varepsilon^2 + 2\omega_o^2) + \frac{2\varepsilon^2\beta^2}{\varepsilon^2 + \omega_o^2}}{k_{11}\varepsilon(\varepsilon^2 + 4\omega_o^2)} \geq 0 \tag{1-26}$$

In order that this expression be positive for all positive real ε, it is necessary that

$$k_{11}\, k_{22} - k_{12}^2 - \frac{\beta^2}{\omega_o^2} \geq 0^* \tag{1-27}$$

Eqs. (1-25) and (1-27) are called extended residue conditions[2,14] and are more severe than the residue condition of eq. (1-15).

Theorem C

The two extended residue conditions of eqs. (1-25) and (1-27) are equivalent.

To verify Theorem C, let K_1 and K_2 in eqs. (1-20) and (1-21) be expanded to yield

$$\begin{aligned} K_1 &= 2K_{11} + 2K_{12} + \cdots \\ K_2 &= 2K_{21} + 2K_{22} + \cdots \end{aligned} \tag{1-28}$$

Now eqs. (1-20) and (1-21) may be rewritten in the form

*This form is derived in References 2 and 14.

$$Z_{\substack{12\\21}} = 2\,\frac{{}_1k_{12}s \pm (K_{11}s^2 + \beta_1 + K_{11}\,\omega_1^2)}{s^2 + \omega_1^2} + \ldots. \qquad (1\text{-}29)$$

$$Z_{\substack{12\\21}} = 2\,\frac{{}_1k_{12}s \pm (\alpha_1 s^2 + K_{21}s^2 + K_{21}\omega_1^2)}{s^2 + \omega_1^2} + \ldots. \qquad (1\text{-}30)$$

These two expressions for $Z_{\substack{12\\21}}$ must be equal and therefore

$$\begin{aligned} K_{11} &= \alpha_1 + K_{21} \\ K_{11}\omega_1^2 + \beta_1 &= K_{21}\omega_1^2 \end{aligned} \qquad (1\text{-}31)$$

Solving these equations for β_1 gives

$$\beta_1 = -\,\omega_1^2\alpha_1$$

or

$$\frac{\beta_1^2}{\omega_1^2} = \omega_1^2\alpha_1^2 \qquad (1\text{-}32)$$

A similar result follows at all other poles of $Z_{\substack{12\\21}}$.

1.5 Gyrator - Transformer Networks*

To synthesize the expressions in eqs. (1-20) and (1-21), certain gyrator - transformer networks are employed. The Z parameters of these networks are now derived. Consider the network of Fig. (1-2).

*The networks in Figs. (1-2) and (1-3) have T and π equivalents which are sometimes more useful than the transformer forms. These equivalent forms are developed and utilized in Chapter III.

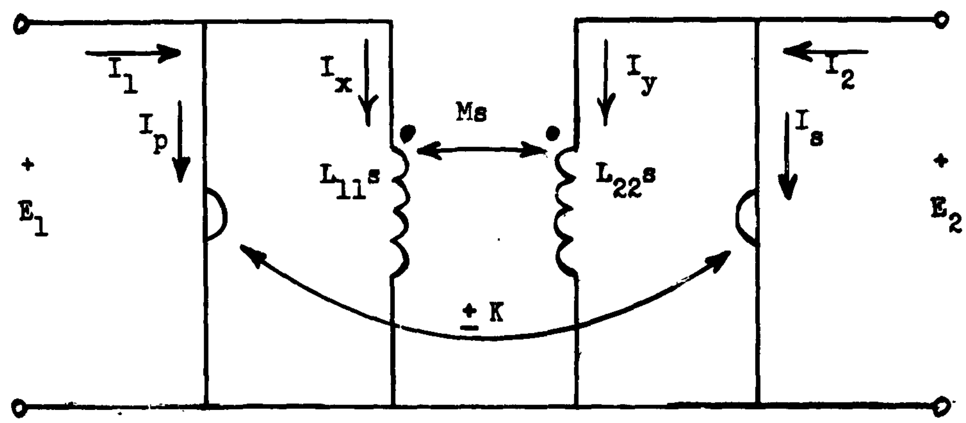

Fig. (1-2) Gyrator-Inductive Transformer Network

The following relations apply:

$$
\begin{aligned}
E_1 &= sL_{11}I_x + sMI_y = KI_s \\
E_2 &= sMI_x + sL_{22}I_y = -KI_p \\
I_1 &= I_p + I_x \\
I_2 &= I_s + I_y
\end{aligned}
\qquad (1\text{-}33)
$$

Expressing E_1 and E_2 in terms of I_1 and I_2 gives the Z parameters as

$$
Z_{11} = \frac{\omega_o^2 L_{11} s}{s^2 + \omega_o^2}, \quad Z_{22} = \frac{\omega_o^2 L_{22} s}{s^2 + \omega_o^2}
$$

$$
Z_{\substack{12\\21}} = \frac{\omega_o^2 Ms \pm Ks^2}{s^2 + \omega_o^2}, \quad \omega_o^2 = \frac{K^2}{L_{11}L_{22} - M^2} \qquad (1\text{-}34)
$$

Now consider the network of Fig. (1-3).

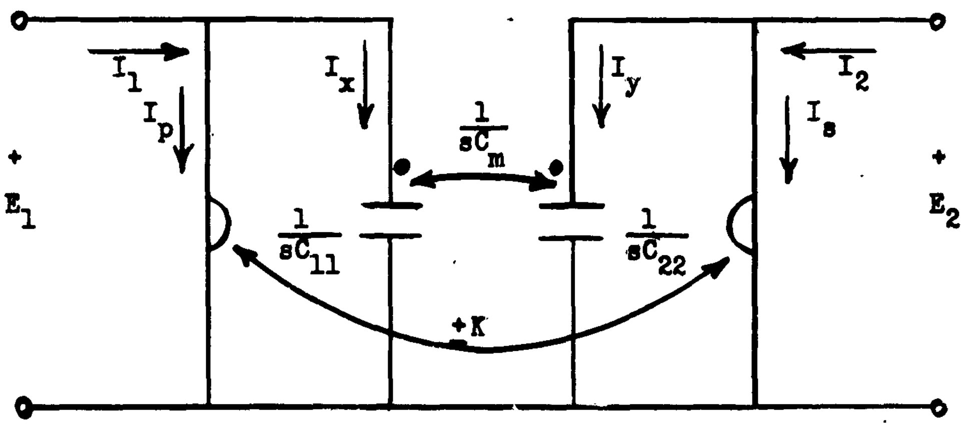

Fig. (1-5) Gyrator-Capacitive Transformer Network

The pertinent relations are

$$
\begin{aligned}
E_1 &= \frac{1}{sC_{11}} I_x + \frac{1}{sC_m} I_y = KI_s \\
E_2 &= \frac{1}{sC_m} I_x + \frac{1}{sC_{22}} I_y = -KI_p \\
I_1 &= I_p + I_x \\
I_2 &= I_s + I_y
\end{aligned}
\qquad (1\text{-}35)
$$

which yield

$$
Z_{11} = \frac{\frac{s}{C_{11}}}{s^2 + \omega_o^2}, \quad Z_{22} = \frac{\frac{s}{C_{22}}}{s^2 + \omega_o^2}
$$

$$
Z_{\substack{12\\21}} = \frac{\frac{s}{C_m} \pm K\omega_o^2}{s^2 + \omega_o^2}, \quad \omega_o^2 = \frac{1}{K^2}\left(\frac{1}{C_{11}C_{22}} - \frac{1}{C_m^2}\right) \qquad (1\text{-}36)
$$

The networks of Figs. (1-2) and (1-3) exhibit certain unusual properties.

A) The transformers do not have unity coupling. This can be seen from an examination of the extended residue conditions.

B) If the coefficients of coupling become unity, the effect of the gyrators disappears $\left(\omega_o^2\right.$ becomes infinite in eq. (1-34) and zero in eq. (1-36)$\left.\right)$.

C) If K —> 0 in eqs. (1-34) and (1-36), all impedance parameters vanish unless the coefficients of coupling become unity at the same time. In this latter case, ω_o^2 remains finite and non-zero, the residue condition of eq. (1-15) applies and the networks reduce to loaded perfect transformers.

D) The networks of Figs. (1-2) and (1-3) satisfy the extended residue condition with the equal sign as may be verified by substituting the residues from eq. (1-34) into eq. (1-25) and those from eq. (1-36) into eq. (1-27) to obtain zero in each case.

1.6 Non-Reciprocal Syntheses for Impedances of Rank 2 and 4[2,4,14]

The non-reciprocal Darlington syntheses of impedances of rank 2 and rank 4 are well-covered in the references and are therefore only briefly reviewed here. For the case of a rank 2 impedance, let

$$Z = \frac{a_o + s}{b_o + s}, \quad \text{num EvZ} = a_o b_o - s^2 \tag{1-37}$$

The n-type and m-type non-reciprocal Darlington syntheses are shown in Figs. (1-4) and (1-5) respectively, where, in Fig. (1-4) the termination is scaled to avoid the use of a capacitive transformer.

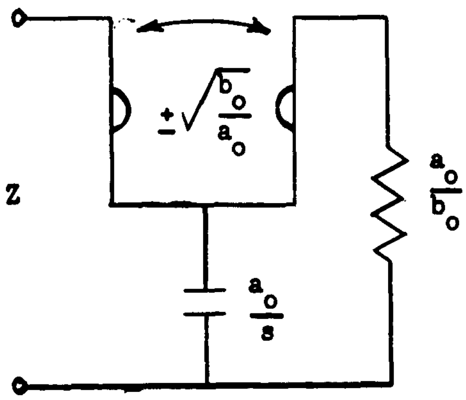

Fig. (1-4) n-Type Rank 2

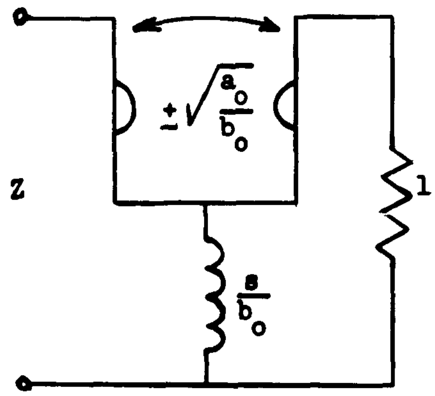

Fig. (1-5) m-Type Rank 2

For the rank 4 case, let Z be given by

$$Z = \frac{a_o + a_1 s + s^2}{b_o + b_1 s + s^2}$$

$$\text{num EvZ} = \left(s^2 + \sqrt{a_o b_o}\right)^2 - s^2\left[a_1 b_1 - \left(\sqrt{a_o} - \sqrt{b_o}\right)^2\right] \tag{1-38}$$

$$= \left(s^2 + \sqrt{a_o b_o}\right)^2 - e^2 s^2$$

where e^2 is positive from the requirement that num EvZ ≥ 0 everywhere on the $j\omega$ axis. The n-type synthesis is straightforward (since n_2 has only one zero) and the result appears in Fig. (1-5), where again the terminating impedance has been scaled to avoid the use of a capacitive transformer.

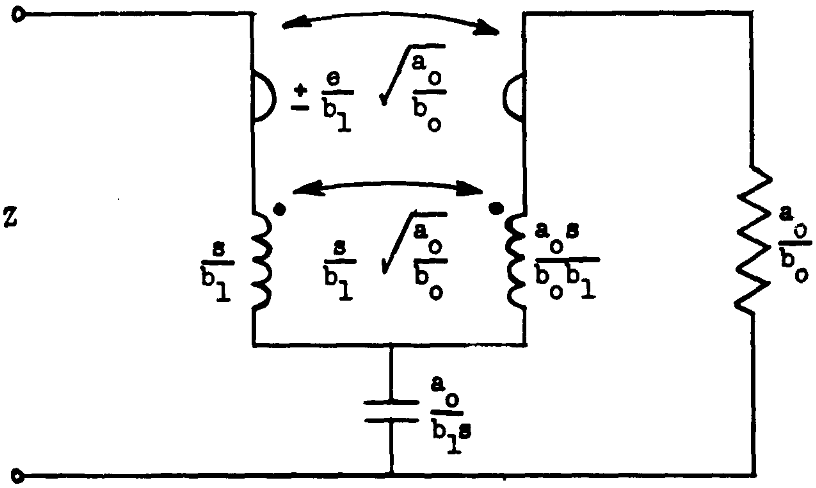

Fig. (1-6) n-Type Rank 4

For the m-type synthesis, the Z parameters are

$$Z_{11} = \frac{a_1 s}{s^2 + b_o}, \quad Z_{22} = \frac{b_1 s}{s^2 + b_o}$$

$$Z_{\substack{12\\21}} = \frac{es \pm s^2\left(s^2 + \sqrt{a_o b_o}\right)}{s^2 + b_o} \tag{1-39}$$

The $Z_{\substack{12\\21}}$ expression may be rewritten as

$$Z_{\substack{12\\21}} = \frac{es \pm s^2\left(1 - \sqrt{\frac{a_o}{b_o}}\right)}{s^2 + b_o} \pm \sqrt{\frac{a_o}{b_o}} \tag{1-40}$$

or

$$Z_{\substack{12\\21}} = \frac{es \pm b_o \left(\sqrt{\frac{a_o}{b_o}} - 1\right)}{s^2 + b_o} \pm 1 \qquad (1\text{-}41)$$

Eqs. (1-39) and (1-40) may be readily identified with eq. (1-34) to yield the m-type rank 4 inductive transformer network shown in Fig. (1-7). A similar capacitive transformer network can be derived using eqs. (1-39), (1-41) and (1-36).

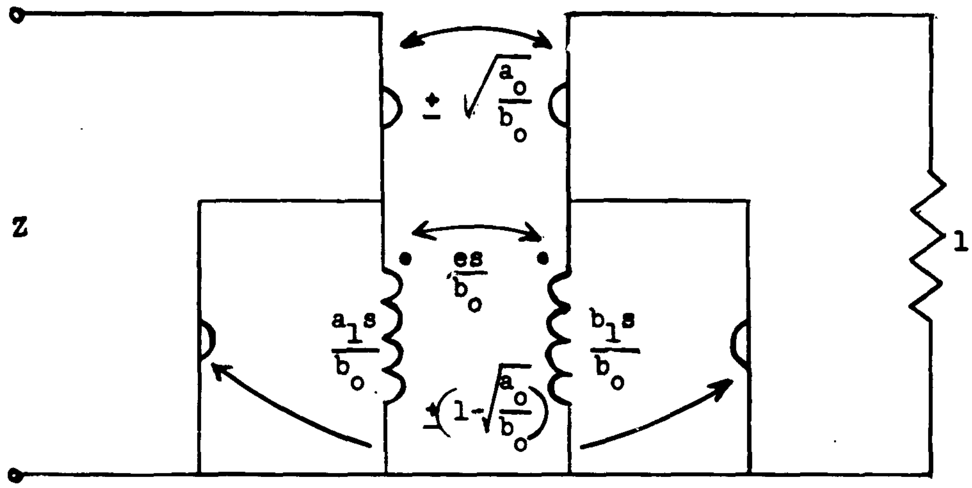

Fig. (1-7) m-Type Rank 4 Inductive

The transformer in Fig. (1-7) may be replaced by its T or π equivalent. It is then possible, by scaling the termination, to eliminate one element in the T or π network. This matter is considered in detail in the discussion of the impedance operator of rank 4 in Chapter III.

1.7 Non-Reciprocal Synthesis for Impedances of Rank 6

Let a general rank 6 impedance, assumed to be non-minimum resistance, be given by

$$Z = \frac{a_o + a_1 s + a_2 s^2 + s^3}{b_o + b_1 s + b_2 s^2 + s^3} \qquad (1\text{-}42)$$

$$\text{num EvZ} = (a_o + a_2 s^2)(b_o + b_2 s^2) - s^2(a_1 + s^2)(b_1 + s^2) \qquad (1\text{-}43)$$

$$= (A^2 - s^2)\left[(s^2 + B)^2 - C^2 s^2\right] \qquad (1\text{-}44)$$

$$= \left[s^3 + (A + C)s^2 + (AC + B)s + AB\right]$$

$$\left[-s^3 + (A + C)s^2 - (AC + B)s + AB\right] \qquad (1\text{-}45)$$

The forms of eqs. (1-44) and (1-45) are justified from the discussion in Section (1-3). For the n-type synthesis, the Z parameters are (using eqs. (1-9) and (1-18) and letting the termination be $\frac{a_o}{b_o}$ to eliminate a transformer)

$$Z_{11} = \frac{a_2 s^2 + a_o}{s(s^2 + b_1)} = \frac{a_o}{b_1 s} + \frac{\left(a_2 - \frac{a_o}{b_1}\right)s}{s^2 + b_1}$$

$$Z_{22} = \frac{\frac{b_2 a_o}{b_o} s^2 + a_o}{s(s^2 + b_1)} = \frac{a_o}{b_1 s} + \frac{\left(\frac{b_2 a_o}{b_o} - \frac{a_o}{b_1}\right)s}{s^2 + b_1} \qquad (1\text{-}46)$$

$$Z_{\substack{12\\21}} = \frac{\left[(A + C)s^2 + AB \pm s(s^2 + AC + B)\right]\sqrt{\frac{a_o}{b_o}}}{s(s^2 + b_1)}$$

The equation for $Z_{\substack{12\\21}}$ may be separated to give

$$Z_{\substack{12\\21}} = \frac{a_o}{b_1 s} + \frac{\left[\sqrt{\frac{a_o}{b_o}}(A + C) - \frac{a_o}{b_1}\right]s \pm \frac{s^2}{b_1}(b_1 - AC - B)\sqrt{\frac{a_o}{b_o}}}{s^2 + b_1} \pm \frac{AC + B}{b_1}\sqrt{\frac{a_o}{b_o}} \qquad (1\text{-}47)$$

or

$$Z_{\substack{12\\21}} = \frac{a_o}{b_1 s} + \frac{\left[\sqrt{\frac{a_o}{b_o}}(A + C) - \frac{a_o}{b_1}\right]s \pm (AC + B - b_1)\sqrt{\frac{a_o}{b_o}}}{s^2 + b_1} \pm \sqrt{\frac{a_o}{b_o}} \qquad (1\text{-}48)$$

The first terms in Z_{11}, Z_{22} and $Z_{\substack{12\\21}}$ form a capacitor. The last terms in eqs. (1-47) and (1-48) are non-loaded gyrators. The residues of the second terms in Z_{11} and Z_{22}, along with the residues of the middle terms in eqs. (1-47) and (1-48) satisfy the extended residue conditions with equal signs in eqs. (1-25) and (1-27). This may be verified by direct substitution and recognition of the identities

$$\begin{aligned} 2AB(A + C) - (AC + B)^2 &= a_2 b_o + a_o b_2 - a_1 b_1 \\ (A + C)^2 - 2(AC + B) &= a_2 b_2 - a_1 - b_1 \end{aligned} \qquad (1\text{-}49)$$

which are obtained by matching coefficients of like terms in eqs. (1-43) and (1-45). Thus eqs. (1-46) and (1-47) may be realized using the network of Fig. (1-2). The complete synthesis is shown in Fig. (1-8). A similar capacitive transformer synthesis can be developed using eq. (1-48) and Fig. (1-3).

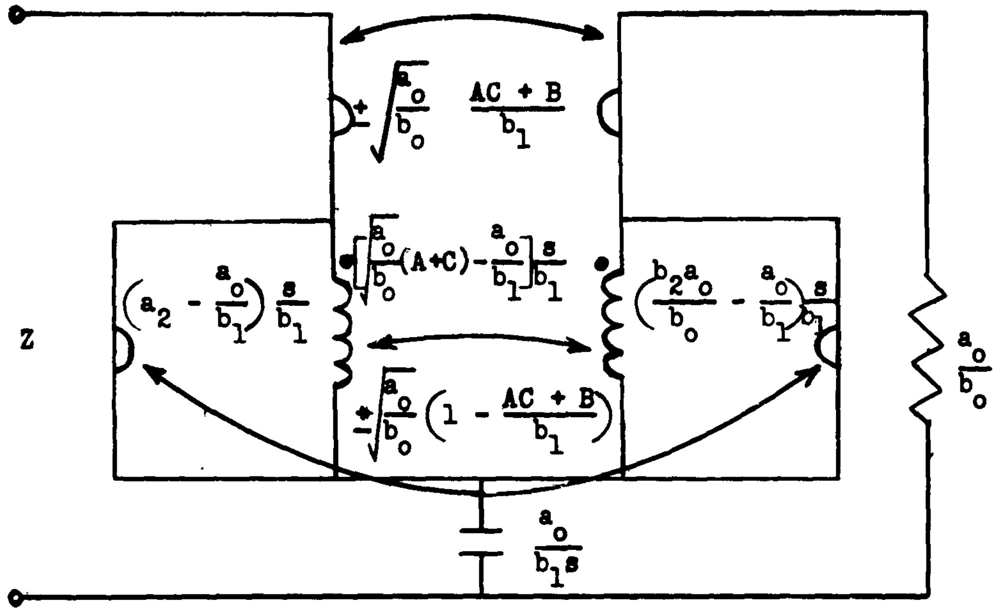

Fig. (1-8) n-Type Rank 6 Inductive

The m-type synthesis proceeds in exactly the same fashion.

The Z parameters are

$$Z_{11}=\frac{s(s^2+a_1)}{b_2s^2+b_o}=\frac{s}{b_2}+\frac{\frac{1}{b_2}\left(a_1-\frac{b_o}{b_2}\right)s}{s^2+\frac{b_o}{b_2}}$$

$$Z_{22}=\frac{s(s^2+b_1)}{b_2s^2+b_o}=\frac{s}{b_2}+\frac{\frac{1}{b_2}\left(b_1-\frac{b_o}{b_2}\right)s}{s^2+\frac{b_o}{b_2}} \qquad (1\text{-}50)$$

$$Z_{\substack{12\\21}}=\frac{s(s^2+AC+B)\pm\left[(A+C)s^2+AB\right]}{b_2s^2+b_o}$$

Separating $Z_{\substack{12\\21}}$ gives

$$Z_{\substack{12\\21}} = \frac{s}{b_2} + \frac{\frac{1}{b_2}\left(AC + B - \frac{b_o}{b_2}\right) s \pm \frac{s^2}{b_o}\left[\frac{b_o}{b_2}(A + C) - AB\right]}{s^2 + \frac{b_o}{b_2}} \pm \frac{AB}{b_o} \qquad (1\text{-}51)$$

and

$$Z_{\substack{12\\21}} = \frac{s}{b_2} + \frac{\frac{1}{b_2}\left(AC + B - \frac{b_o}{b_2}\right) s \pm \frac{1}{b_2}\left[AB - \frac{b_o}{b_2}(A + C)\right]}{s^2 + \frac{b_o}{b_2}} \pm \frac{A+C}{b_2} \qquad (1\text{-}52)$$

The complete synthesis of eqs. (1-50) and (1-51) appears in Fig. (1-9). A similar capacitive transformer synthesis can be derived using eq. (1-52).

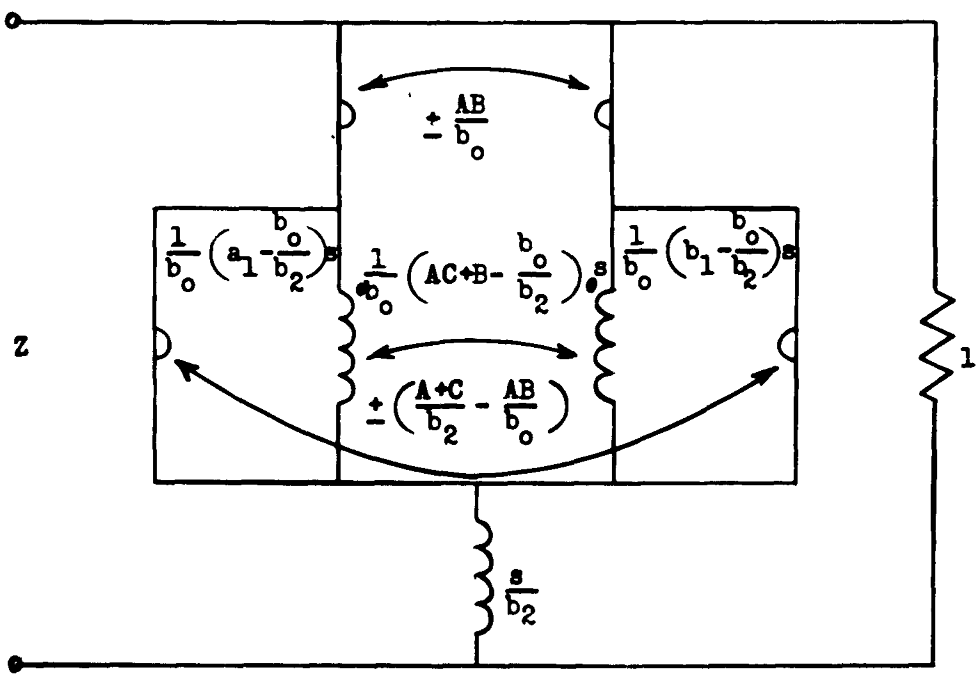

Fig. (1-9) m-Type Rank 6 Inductive

Example 2

To illustrate the rank 6 synthesis procedure, let it be required to synthesize the following driving point impedance:

$$Z = \frac{s^3 + 2s^2 + 9s + 1}{s^3 + 9s^2 + 5s + 16}$$

$$\text{num } EvZ = (s^3 + 4s^2 + 6s + 4)(-s^3 + 4s^2 - 6s + 4)$$

$$A + C = 4, \quad AC + B = 6, \quad AB = 4$$

The network corresponding to Fig. (1-8), obtained by direct substitution of the known quantities, is shown in Fig. (1-10). The network corresponding to Fig. (1-9) can be obtained in a similar manner. The extended residue condition from eq. (1-25) yields

$$\omega_o^4(L_{11}L_{22} - M^2) - \alpha^2\omega_o^2 = 25\left(\frac{9}{25} \times \frac{29}{400} - \frac{16}{625}\right) - \frac{1}{400} \times 5 = 0$$

The coefficient of coupling of the transformer is

$$k_c = \frac{M}{\sqrt{L_{11}\, L_{22}}} = \frac{16}{5\sqrt{29}} = .99$$

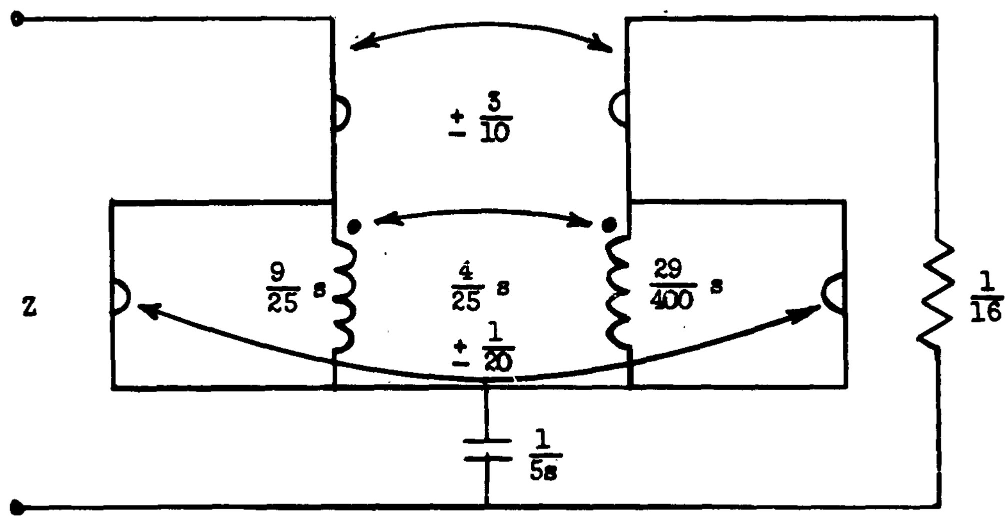

Fig. (1-10) Network Corresponding to Fig. (1-8)

1.8 Impedances of Higher Rank

The extension of the foregoing synthesis procedure to impedances of higher rank is straightforward with eqs. (1-20) and (1-21) serving as a guide. Thus in the case of a rank 8 impedance, the m-type synthesis yields networks similar to those in Fig. (1-9), but with the inductor replaced by a second gyrator-transformer network. The n-type synthesis yields networks similar to Fig. (1-8), except for the addition of a transformer to synthesize the pole of the Z parameters at infinity.

For impedances of odd rank, the poles and zeros at the origin and infinity may be removed until the impedance becomes even in rank and the remainder synthesized by the foregoing procedure.

Thus it is possible to synthesize any prf driving point impedance through the non-reciprocal Darlington procedure and the networks of Figs. (1-2) and (1-3).

1.9 Existing Cascade Synthesis Techniques[7,3]

A cascade synthesis procedure in which the zeros of Z_{12} (denoted as transmission zeros) are controlled by individual network sections has been developed by Guillemin[7], leading to a number of network structures, among which are the Darlington A, B, C and D sections. Balabanian[3] also discusses the development of these four network sections.

In the derivation of cascade impedance operator syntheses in Chapters III, IV and V, networks similar to these Darlington sections, but containing arbitrary constants, result. Thus it is pertinent at this point to review and summarize the results of Guillemin and Balabanian.*

Consider the configuration of Fig. (1-11), where each box is a lossless reciprocal network.**

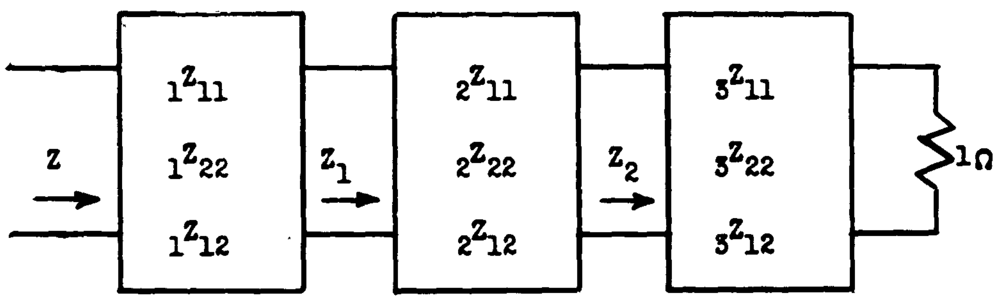

Fig. (1-11) Cascade Representation of Z

*The following discussion summarizes material presented in Chapters 6,7,9 and 10 of Reference 7 and Chapter 6 of Reference 3.

**It is assumed that surplus factors have been added so that all zeros of num EvZ are of even multiplicity.

The following relation applies

$$Z_{12} = \frac{{}_1Z_{12}}{{}_1Z_{22} + Z_1} \times \frac{{}_2Z_{12}}{{}_2Z_{22} + Z_2} \times \frac{{}_3Z_{12}}{{}_3Z_{22} + 1} \qquad (1\text{-}53)$$

Eq. (1-53) points out that the zeros of Z_{12} are made up of the zeros of ${}_1Z_{12}$, ${}_2Z_{12}$ and ${}_3Z_{12}$.* The zeros of Z_{12} are the same as the zeros of the even part of Z given by eq. (1-11). As discussed in Section 1.3, the zeros of num EvZ have quadrantal symmetry and thus only three types of terms, as given by eq. (1-16) are permitted. If all three of these terms were present in the even part of a given driving point impedance, one term could be assigned to each of the boxes in Fig. (1-11). The syntheses of Z would thereby become the syntheses of the three boxes.

In the realization of Z with its types of even part zeros given by eq. (1-16), four types of network sections are useful. These are the Darlington A, B, C and D sections shown in Fig. (1-12). The branches in the A and B sections are single inductances or capacitances or series or parallel resonant circuits. These two sections realize jω axis zeros and poles of Z. The type C section is similar to the Brune network but the transformer polarity is additive. It is used to realize real axis zeros of Z_{12}.** The type D section

*Guillemin points out that no additional zeros are introduced because of the denominator poles in eq. (1-53).

**By using the Brune form of this network (with a subtractive transformer) jω axis zeros of Z_{12} can be realized. Or by using Guillemin's method of "zero shifting", these zeros may also be realized using a ladder network development.

is employed in the realization of complex zeros of Z_{12}.

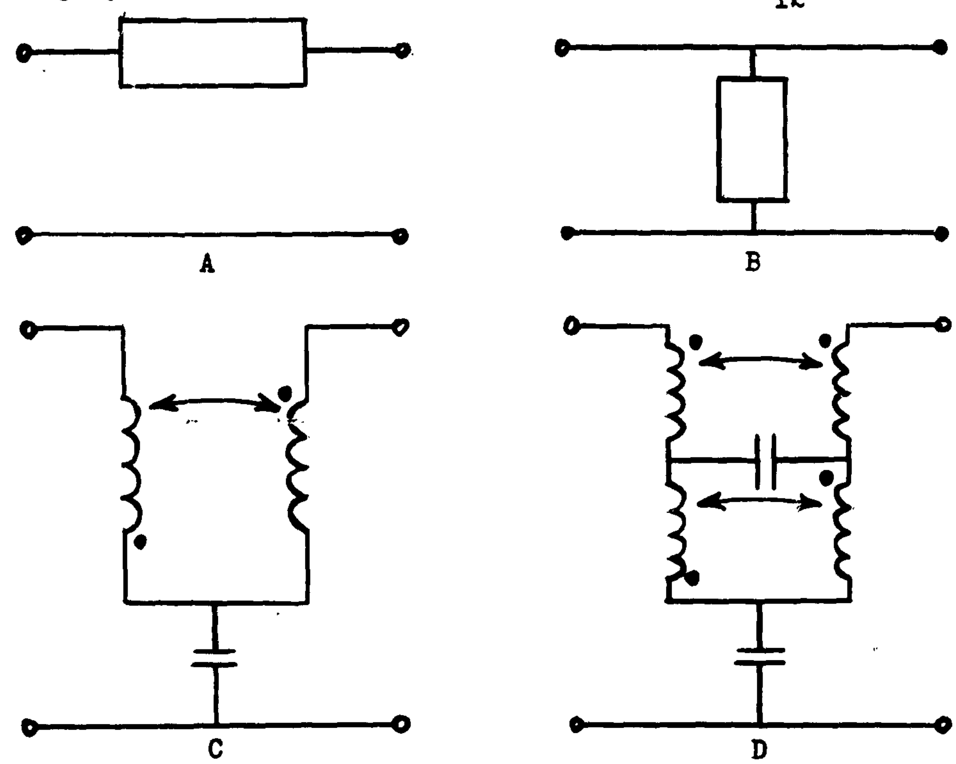

Fig. (1-12) Darlington Sections

The synthesis of a given driving point impedance, Z, would proceed as follows. Sections A and B are first employed to remove poles and zeros of Z on the $j\omega$ axis. The remaining impedance is prf and its even part has the three types of zeros of eq. (1-16). If these zeros are not initially of even order, it is necessary to use surplus factors to create these forms. Next a type C section is removed to realize a pair of real axis zeros of Z_{12}(or a Brune section to remove an imaginary axis pair). The remaining prf impedance is reduced in rank by 2, because of cancellation of a term of the form $a^2 - s^2$(or $s^2 + \omega_o^2$) from the numerator and denominator of this impedance, and its even part is missing the pair of real axis (imaginary axis) zeros. In a similar manner, the Type D section may be removed to realize a quadruplet of complex zeros. Once again surplus factors may be required. The remaining prf impedance is reduced in rank by four.

Very little has been said as yet about the actual syntheses of the networks represented by the boxes in Fig. (1-11). Following Guillemin's approach, assume that the first box is to realize a quadruplet of Z_{12} zeros.* The Z-parameters of this box may be written in the general form

$$
\begin{aligned}
{}_1Z_{12} &= \frac{K(s^2 + as + b)(s^2 - as + b)}{s(s^2 + \omega_o^2)} \\
&= \frac{KT(s)\ T(-s)}{s(s^2 + \omega_o^2)} \qquad (1\text{-}54) \\
&= K_{12}s + \frac{k_o}{s} + \frac{2k_{12}s}{s^2 + \omega_o^2}
\end{aligned}
$$

$$ {}_1Z_{11} = K_{11}s + \frac{k_o}{s} + \frac{2k_{11}s}{s^2 + \omega_o^2} \qquad (1\text{-}55) $$

$$ {}_1Z_{22} = K_{22}s + \frac{k_o}{s} + \frac{2k_{22}s}{s^2 + \omega_o^2} \qquad (1\text{-}56) $$

*This generally leads to the Darlington D section but, as Guillemin points out, can also often be made to yield unbalanced networks in the form of lattice, bridged T or twin T structures.

The functions Z, Z_1, ${}_1Z_{12}$, ${}_1Z_{11}$ and ${}_1Z_{22}$ are related by the customary driving point impedance equation

$$Z = \frac{{}_1Z_{11}\, Z_1 + {}_1Z_{11}\, {}_1Z_{22} - {}_1Z_{12}^2}{{}_1Z_{22} + Z_1} \qquad (1\text{-}57)$$

which can be rearranged to yield

$$({}_1Z_{11} - Z)\quad ({}_1Z_{22} + Z_1) = {}_1Z_{12}^2 = \frac{K^2T^2(s)T^2(-s)}{s^2(s^2 + \omega_o^2)^2} \qquad (1\text{-}58)$$

The impedance Z may be written in the general form*

$$Z = \frac{A\ P(s)}{Q(s)} \qquad (1\text{-}59)$$

where P(s) and Q(s) are polynomials of the same rank in s and A is a constant. The next step is a key point in the Guillemin procedure. The two terms on the left hand side of eq. (1-58) are separately expressed as

$${}_1Z_{11} - Z = \frac{K^2T(s)^2\ T(-s)^2\ H(s)}{s(s^2 + \omega_o^2)\ Q(s)} \qquad (1\text{-}60)$$

$${}_1Z_{22} + Z_1 = \frac{Q(s)}{s(s^2 + \omega_o^2)\ H(s)} \qquad (1\text{-}61)$$

---

*It is assumed that any required surplus factors are included in Z.

where H(s) is a specified polynomial in s.* The synthesis problem is thus reduced to the construction of H(s) and the determination of ω_0, after which $_1Z_{11}$, $_1Z_{22}$, $_1Z_{12}$ and Z_1 may be found. The result is generally the Darlington D section of Fig. (1-12) with a termination Z_1 reduced in rank by four.

Thus the Guillemin synthesis procedure, like the impedance operator synthesis procedure to be developed in Chapter III, provides a method of obtaining a cascade synthesis of any driving point impedance for any configuration of transmission (even part) zeros. The two methods start from the same point (the idea of removing sections as in Fig. (1-11)) and arrive at the same general results (the Darlington sections) but the actual procedures are quite different as the development in Chapter III will show. Also, an additional flexibility is included in the impedance operator approach in that non-reciprocal elements are permitted and arbitrary constants are present in the removed sections and the terminating impedances. These constants may or may not be chosen for rank reduction at the discretion of the designer. These features could also undoubtedly be included in the Guillemin procedure but are not discussed in Reference 7. It is also possible to derive the Brune[17] synthesis procedure using either Guillemin's approach or the impedance operator approach. This is discussed in Section 3.4.

As mentioned, it is always possible to realize a quadruplet

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*The philosophy behind eqs. (1-60) and (1-61) is explained in detail on page 251 of Reference 7.

of transmission zeros by the Darlington D section using either of the two cascade synthesis procedures. The D section contains mutually coupled elements. Guillemin points out that it is sometimes possible to arrive at alternative unbalanced network structures not requiring mutual coupling. The requirements are that the set of impedances ${}_1Z_{11}$, ${}_1Z_{12}$ and ${}_1Z_{22}$ be analytic in the right half plane, satisfy the residue condition of eq. (1-15), satisfy the real part condition[3] given by

$$ {}_1r_{11}\ {}_1r_{22} - {}_1r_{12}^2 \geq 0 \qquad (1\text{-}62) $$

and in addition, satisfy the Fialkow[3] condition that the numerator coefficients of ${}_1Z_{12}$ be positive and no greater than the corresponding ones in ${}_1Z_{11}$ and ${}_1Z_{22}$. Applying these requirements to the Darlington D section, Guillemin shows that the additive transformer may be eliminated to yield the structure of Fig. (1-13) if, in eq. (1-54), $2\ k_{12} \leq k_o$, which in turn requires that the transmission zeros lie in the shaded region of Fig. (1-14)*.

*Guillemin also discusses in detail the application of the Fialkow condition to other two-port structures to achieve unbalanced structures without mutual coupling.

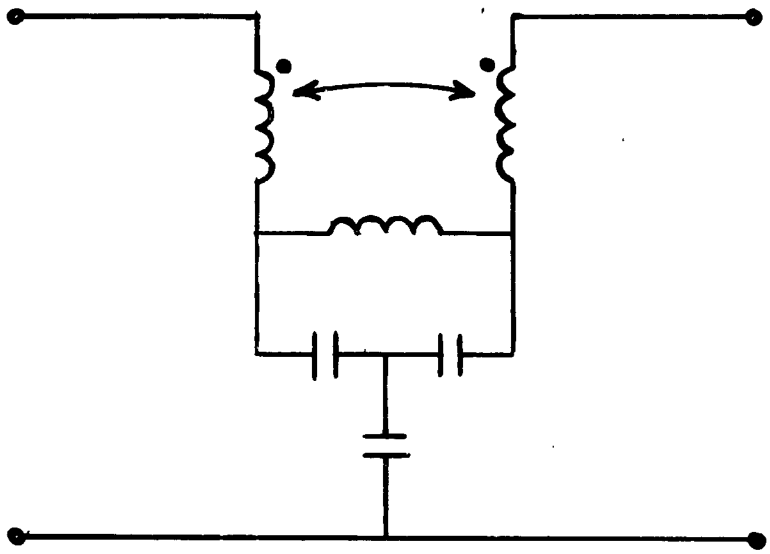

Fig. (1-13) Variation in the Darlington D Section

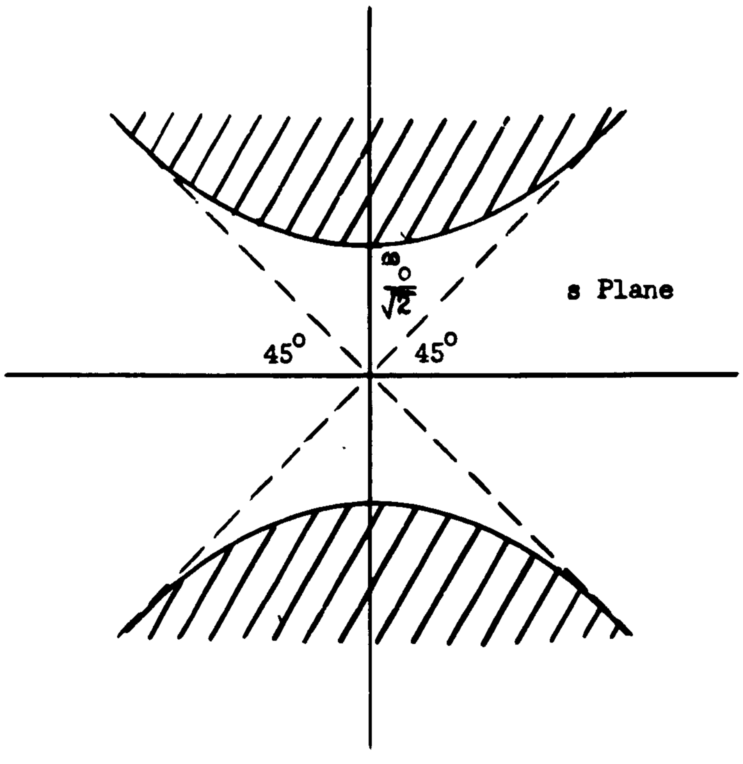

Fig. (1-14) Location of Transmission Zeros of Fig. (1-13)

One further observation will complete the work of Chapter I. Throughout the discussion in this section, it has been assumed that Z has already been augmented so that num EvZ has zeros of even multiplicity only. Let it now be assumed that this is not the case. Consider the non-reciprocal syntheses of Sections 1.6 and 1.7. The lossless networks in these syntheses still realize the real and complex zeros of num EvZ but these zeros are no longer identical with the zeros of Z_{12}. Rather the zeros of Z_{12} are the left-half plane zeros of num EvZ, the remaining zeros being assigned to Z_{21}. For example, the lossless network sections in Figs. (1-4) and (1-5) realize a pair of real zeros of num EvZ where Z_{12} contains the one in the left half plane. The sections in Figs. (1-6) and (1-7) realize a quadruplet of complex zeros of num EvZ, whereas the sections in Figs. (1-8) and (1-9) realize one pair of real zeros and one quadruplet of complex zeros of num EvZ.

CHAPTER II

PROPERTIES OF THE IMPEDANCE OPERATOR

2.1 Introduction

In Chapter I, several non-reciprocal lossless networks were derived using the Darlington synthesis procedure with each network being terminated in a pure resistance. It was pointed out that the constraint of a resistive termination could be relaxed to permit a general prf termination, ζ. Then Z is represented in terms of an impedance operator, V, operating on ζ, as shown in Fig. (2-1). Sufficient conditions for this representation are that Theorem A be satisfied and that V be a lossless network.

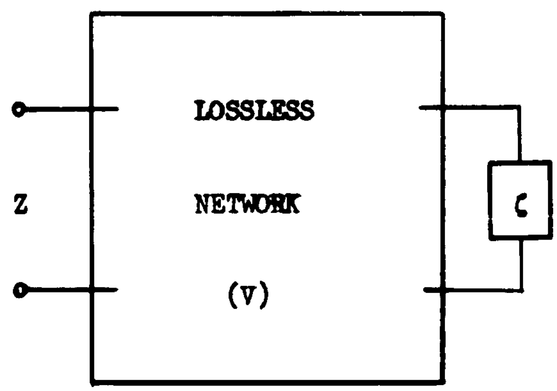

Fig. (2-1) $Z = V\zeta$

The Darlington synthesis of V, the mathematical properties of V and ζ and the utilization of V in the analysis and synthesis of networks are the subject matter of this chapter.

2.2 Darlington Synthesis of the V Operator

Let eqs. (1-2), (1-3) and (1-4) be rewritten slightly to give*

$$Z = \frac{m_1\zeta_1 + n_1}{m_2 + n_2\zeta_1} \tag{2-1}$$

$$V_1 = \frac{m_1 + n_1}{m_2 + n_2} \tag{2-2}$$

$$Z = V_1\zeta_1 \tag{2-3}$$

To relate the right hand side of eq. (2-1) to the network of Fig. (2-1), the equations for the latter are written as

$$\begin{aligned} E_1 &= I_1Z_{11} + I_2Z_{12} \\ E_2 &= I_1Z_{21} + I_2Z_{22} = - I_2\zeta \end{aligned} \tag{2-4}$$

Solving for the driving point impedance yields

$$Z = Z_{11}\,\frac{\zeta + \dfrac{Z_{11}Z_{22} - Z_{12}Z_{21}}{Z_{11}}}{Z_{22} + \zeta} \tag{2-5}$$

Eq. (2-1) may be rearranged in two ways to match eq. (2-5) in the same manner as was done in Section 1.2.

*The subscripts on V and ζ are introduced to distinguish this impedance operator and termination from others to be presented later in the chapter.

$$Z = \frac{m_1}{n_2} \frac{\zeta_1 + \frac{n_1}{m_1}}{\frac{m_2}{n_2} + \zeta_1} \qquad (2\text{-}6)$$

$$Z = \frac{n_1}{m_2} \frac{\frac{1}{\zeta_1} + \frac{m_1}{n_1}}{\frac{n_2}{m_2} + \frac{1}{\zeta_1}} \qquad (2\text{-}7)$$

where, in eq. (2-6)

$$V_{11} = \frac{m_1}{n_2}, \quad V_{22} = \frac{m_2}{n_2} \text{ and } V_{12}V_{21} = \frac{\text{num EvV}}{n_2^2} \qquad (2\text{-}8)$$

and, in eq. (2-7)

$$V_{11} = \frac{n_1}{m_2}, \quad V_{22} = \frac{n_2}{m_2} \text{ and } V_{12}V_{21} = \frac{-\text{num EvV}}{m_2^2} \qquad (2\text{-}9)$$

Eqs. (2-6) and (2-7) represent extensions of the two Darlington synthesis procedures discussed in Chapter I whereby the usual one-ohm resistive termination is replaced by ζ_1 in eq. (2-6) and $1/\zeta_1$ in eq. (2-7). V_{11}, V_{22}, V_{12} and V_{21} are the "Z parameters" of the lossless V_1 operator network.

The cascade nature of the V operator may be developed by letting ζ_1 in eq. (2-1) be expressed by

$$\zeta_1 = \frac{m_3 \zeta_2 + n_3}{m_4 + n_4 \zeta_2} = V_2 \zeta_2 \qquad (2\text{-}10)$$

where

$$V_2 = \frac{m_3 + n_3}{m_4 + n_4} \qquad (2\text{-}11)$$

Introducing eq. (2-10) into eq. (2-3) gives

$$Z = V_1(V_2\zeta_2) \qquad (2\text{-}12)$$

The form of eq. (2-12) suggests a cascade representation of Z in terms of two V operators and a terminating impedance. The result can be extended to include additional V operators.

The concept of the V operator is useful in both the analysis and synthesis of networks. In an analysis problem, the network is subdivided into four-terminal lossless cascaded sections and a terminating impedance. The driving point impedance of the overall network is calculated by considering that each four-terminal section operates on the ones following it. Each V is derived by determining the driving point impedance of its four-terminal section with a one-ohm resistive termination.

In a synthesis problem, Z is given as the ratio of two polynomials in s and from this function, the V and ζ functions must be found. Then each V is synthesized by either eq. (2-6) or eq. (2-7) with a one-ohm resistive termination and the resulting networks are cascaded and terminated in ζ.

These concepts of analysis and synthesis are illustrated by examples in the following sections. The use of impedance operators in cascade synthesis procedures is developed in detail in Chapters

III through V.

2.3 The Associative Law

Theorem D

The impedance operator V obeys the associative law of multiplication.

The proof of Theorem D involves the straightforward expansion of three general V operators and appears in Appendix 1.A. Because of the theorem, the parentheses in eq. (2-12) are unnecessary. This is illustrated by the following example.

Example 1

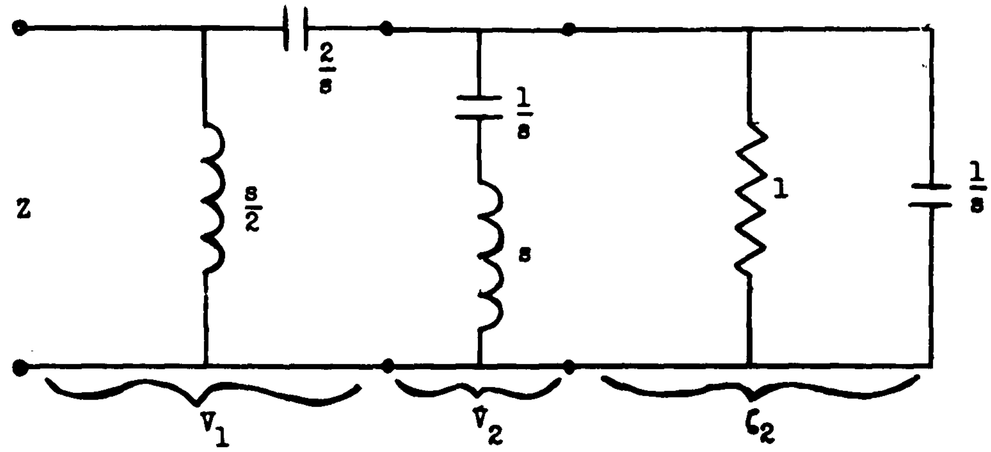

Fig. (2-2) Analysis Using the V Operator

It is desired to compute the driving point impedance Z of the network of Fig. (2-2) by using the impedance operators V_1 and V_2. This is done in two ways, to illustrate the associative property of the impedance operators. Each operator may be derived by placing a one-ohm resistance at its output terminals and computing its in-put impedance. The procedure yields

$$V_1 = \frac{s^2 + 2s}{s^2 + 2s + 4}, \quad V_2 = \frac{s^2 + 1}{s^2 + s + 1}, \quad \zeta_2 = \frac{1}{s + 1}$$

Since each operator network is purely reactive, the operation indicated by eqs. (2-1) and (2-10) may be applied.

$$V_1 V_2 = \frac{s^2 \left(\frac{s^2 + 1}{s^2 + s + 1}\right) + 2s}{s^2 + 4 + 2s \left(\frac{s^2 + 1}{s^2 + s + 1}\right)}$$

$$= \frac{s^4 + 2s^3 + 3s^2 + 2s}{s^4 + 3s^3 + 5s^2 + 6s + 4}$$

$$Z = (V_1 V_2)\zeta = \frac{(s^4 + 3s^2) \frac{1}{s + 1} + 2s^3 + 2s}{s^4 + 5s^2 + 4 + (3s^3 + 6s) \frac{1}{s + 1}}$$

$$= \frac{3s^4 + 2s^3 + 5s^2 + 2s}{s^5 + s^4 + 8s^3 + 5s^2 + 10s + 4}$$

$$V_2 \zeta = \frac{(s^2 + 1) \frac{1}{s + 1}}{s^2 + 1 + s \frac{1}{s + 1}} = \frac{s^2 + 1}{s^3 + s^2 + 2s + 1}$$

$$Z = V_1(V_2 \zeta) = \frac{s^2 \left(\frac{s^2 + 1}{s^3 + s^2 + 2s + 1}\right) + 2s}{s^2 + 4 + 2s \left(\frac{s^2 + 1}{s^3 + s^2 + 2s + 1}\right)}$$

$$Z = \frac{3s^4 + 2s^3 + 5s^2 + 2s}{s^5 + s^4 + 8s^3 + 5s^2 + 10s + 4}$$

The two Z functions are identical, illustrating the validity of the associative property.

2.4 Even Part Relationships

The even part of V_1 in eq. (2-2) is given by

$$EvV_1 = \frac{m_1m_2 - n_1n_2}{m_2^2 - n_2^2} \qquad (2\text{-}13)$$

and similarly for all other V operators. Let V be given by

$$V = V_1V_2 \ldots. V_n \qquad (2\text{-}14)$$

Theorem E

The numerators of the even parts of a series of cascaded V operators are related by

$$\text{num } EvV = (\text{num } EvV_1)(\text{num } EvV_2) \ldots. (\text{num } EvV_n)^* \qquad (2\text{-}14)$$

It follows from eq. (2-15) that if Z is given by

$$Z = V_1V_2 \ldots. V_n\zeta_n \qquad (2\text{-}16)$$

*The proof appears in Appendix 1.B.

then

$$\text{num } EvZ = (\text{num } EvV_1)(\text{num } EvV_2) \ \ldots\ (\text{num } EvV_n)(\text{num } Ev\zeta_n)^* \tag{2-17}$$

Thus the zeros of num EvZ are split between the V operators and the terminating impedance in somewhat the same fashion as described in Section 1.9. The difference lies in the fact that, generally, arbitrary constants are incorporated in each operator and these can cause the elimination of the zeros of num EvV_1, ... num EvV_n from num EvZ as explained in the following section.

2.5 Specific V Operators

In the general synthesis problem, Z is known but V and ζ are not. Eqs. (2-6) and (2-7) provide a useful synthesis procedure only if V and ζ can be separated and ζ can be reduced in rank compared with Z. The relationship given by eq. (2-17) is necessary in the separation of V and ζ and in reducing the rank of ζ, but is not sufficient by itself. To pursue the problem further, let Z in eq. (2-1) be given specifically by

$$Z = \frac{a\zeta_1 + sZ_a}{a + \frac{s}{Z_a}\zeta_1} \tag{2-18}$$

*Assuming no common factors have been cancelled from the numerator and denominator of Z after the expansion $Z = V\zeta$.

and let ζ_1 in eq. (2-10) be given specifically by

$$\zeta_1 = \frac{b\zeta_2 + s\zeta_{1b}}{b + \frac{s}{\zeta_{1b}}\zeta_2} \qquad (2\text{-}19)$$

where a and b are positive real constants and Z_a and ζ_{1b} are the values of Z and ζ_1 at s = a and s = b, respectively. The V operators corresponding to eqs. (2-18) and (2-19) are

$$V_1 = \frac{a + sZ_a}{a + \frac{s}{Z_a}} \qquad (2\text{-}20)$$

$$V_2 = \frac{b + s\zeta_{1b}}{b + \frac{s}{\zeta_{1b}}} \qquad (2\text{-}21)$$

<u>Theorem F</u>

According to a theorem by P.I. Richards,[5] ζ_1 in eq. (2-18) is prf if a is a positive real constant and Z is prf. Also ζ_2 in eq. (2-19) is prf if b is a positive real constant and ζ_1 is prf.

Thus Richards' Theorem is a special case of Theorem A in Section (1.1). Eq. (2-18) may be solved for ζ_1 to yield

$$\zeta_1 = \frac{aZ - sZ_a}{a - \frac{s}{Z_a} Z} \qquad (2\text{-}22)$$

The term s - a is a factor of both numerator and denominator in eq. (2-22); therefore ζ_1 is the same rank as Z. But if $EvZ_a = 0$, then s + a is also a factor of both the numerator and denominator in eq. (2-22) and ζ_1 is two less in rank than Z. Similarly, if $Ev\zeta_{1b} = 0$, ζ_2 is two less in rank than ζ_1 and thus four less in rank than Z. But $Ev\zeta_{1b} = 0$ requires that $EvZ_b = 0$.* Thus making $EvZ_a = EvZ_b = 0$ insures that ζ_2 is four less in rank than Z. This is the principle of zero-cancellation synthesis.[4]

Conversely, let s = - a in eq. (2-22). Then, if $EvZ_a \neq 0$, $\zeta_{1(-a)} = Z_a$. In this case, Z in eq. (2-18) has the factor s + a in both its numerator and denominator.** Cancellation of this factor causes eq. (2-17) to be invalid as mentioned in the footnote to that equation. Effectively num EvV_1 is eliminated from num EvZ when the s + a factor is cancelled.

---

*

$$\zeta_{1b} = \frac{aZ_b - bZ_a}{a - \frac{b}{Z_a} Z_b} \quad \text{and} \quad -\zeta_{1(-b)} = \frac{-a\, Z_{(-b)} - b\, Z_a}{a + \frac{b}{Z_a} Z_{(-b)}}$$

To make $Ev\zeta_{1b} = 0$, ie $\zeta_{1b} = -\zeta_{1(-b)}$, requires that

$$(a^2 - b^2)\, Z_{(-b)} = -(a^2 - b^2) Z_b$$

This is guaranteed if $Z_b = -Z_{(-b)}$ or $EvZ_b = 0$

**

$$\zeta_{1(-a)} = \frac{a\, Z_{(-a)} + a\, Z_a}{a + \frac{a}{Z_a} Z_{(-a)}} = Z_a \frac{\text{num } EvZ_a}{\text{num } EvZ_a}$$

Thus $\zeta_{1(-a)} = Z_a$ if num $EvZ_a \neq 0$

Then s = - a makes $Z_{(-a)}$ in eq. (2-18) indeterminate $(\frac{0}{0})$ and thus a common factor can be cancelled from both the numerator and denominator of Z.

Example 2

To illustrate these principles and the use of the V operator in network synthesis, the following driving point impedance is synthesized using the operators of eqs. (2-20) and (2-21) and the form of eq. (2-12).

$$Z = \frac{2s^4 + 2s^3 + 9s^2 + 5s + 1}{2s^3 + 2s^2 + 5s + 2}$$

The constants a and b are chosen such that $EvZ_a = EvZ_b = 0$. Then, since Z is of rank 7, ζ_1 is of rank 3.

$$\text{num } EvZ = (2 - s^2)(1 - 2s^2)$$

choosing $a = \frac{1}{\sqrt{2}}$ and $b = \sqrt{2}$ gives

$$Z_a = \sqrt{2} \qquad Z_b = \frac{3\sqrt{2}}{2}$$

$$V_1 = \frac{\frac{1}{\sqrt{2}} + \sqrt{2}\, s}{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} s} = \frac{1 + 2s}{1 + s}^{*}$$

$$\zeta_1 = \frac{\frac{1}{\sqrt{2}} Z - s\sqrt{2}}{\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} sZ} = \frac{Z - 2s}{1 - sZ}$$

$$= \frac{s^2 + s + 1}{s^3 + s^2 + 4s + 2}$$

*It is permissable to multiply the numerator and denominator of V by a constant to simplify the computations.

$$\zeta_{1b} = \frac{\sqrt{2}}{4}$$

$$V_2 = \frac{\sqrt{2} + s\frac{\sqrt{2}}{4}}{\sqrt{2} + s\frac{4}{\sqrt{2}}} = \frac{1 + \frac{s}{4}}{1 + 2s}$$

$$\zeta_2 = \frac{\sqrt{2}\,\zeta_1 - s\frac{\sqrt{2}}{4}}{\sqrt{2} - \frac{4s\zeta_1}{\sqrt{2}}} = \frac{s^2 + s + 2}{4s + 4}$$

The complete synthesis of Z is shown in Fig. (2-3). The components of V_1 are derived from eq. (2-8).*

$$\text{num Ev}V_1 = 1 - 2s^2 = (1 + \sqrt{2}\,s)(1 - \sqrt{2}\,s)$$

$$V_{11} = \frac{1}{2s}, \quad V_{22} = \frac{1}{2s}, \quad V_{\substack{12\\21}} = \frac{1 \pm \frac{s}{\sqrt{2}}}{2s} = \frac{1}{2s} \pm \frac{1}{2\sqrt{2}}$$

Note that in the syntheses of V_1 and V_2, non-reciprocal gyrators are required since $m_1m_2 - n_1n_2$ is not a perfect square. Methods of eliminating such gyrators are discussed in Chapter III.

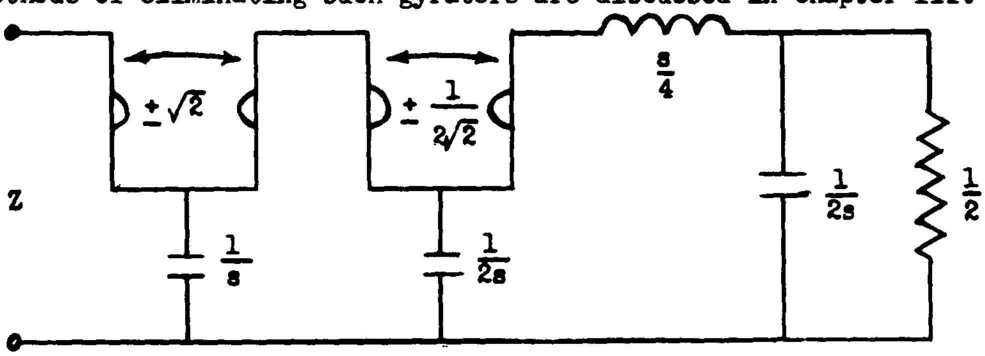

Fig. (2-3) Synthesis by Cascaded V Operators

*Eq. (2-9) could also have been used.

Since the parentheses in eq. (2-12) have no significance, it is permissable to combine V_1 and V_2 into a single operator and synthesize the combination.

$$V_1V_2 = \frac{1\,\dfrac{1+\frac{s}{4}}{1+2s} + 2s}{1 + s\,\dfrac{1+\frac{s}{4}}{1+2s}} = \frac{4s^2 + \frac{9s}{4} + 1}{\frac{s^2}{4} + 3s + 1}$$

$$\text{num Ev } V_1V_2 = (s^2+1)^2 - \frac{9}{2}s^2$$

The components of the combined operator are obtained from eq. (2-8).*

$$V_{11} = \frac{4s^2+1}{3s} \qquad V_{22} = \frac{\frac{s^2}{4}+1}{3s}$$

$$V_{\substack{12\\21}} = \frac{s^2 + 1 \pm \frac{3s}{\sqrt{2}}}{3s}$$

The complete synthesis of Z using the combined operator appears in Fig. (2-4).

*Eq. (2-9) could also be used and would lead to a network of the form of Fig. (1-7).

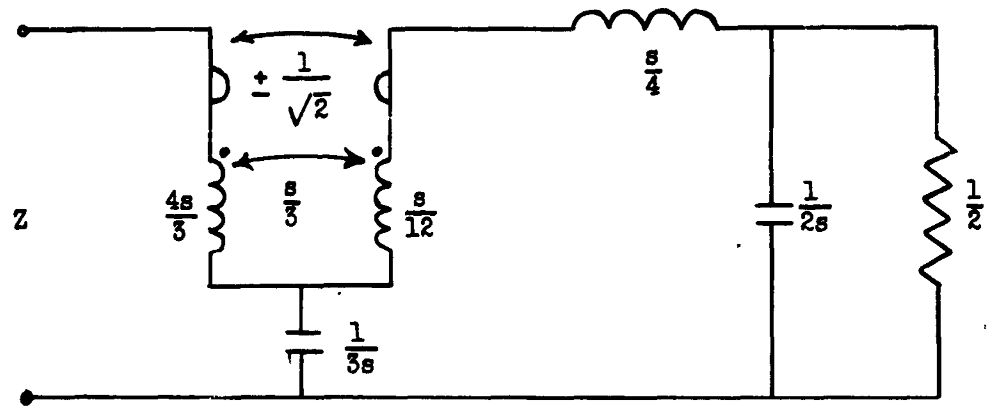

Fig. (2-4) Synthesis using the Combined V Operator

2.6 Inverse, Unit and Squared V Operators

Eq. (2-1) may be solved for ζ_1 to yield

$$\zeta_1 = \frac{m_2 Z - n_1}{m_1 - n_2 Z} \tag{2-23}$$

Eqs. (2-1) and (2-23) permit the definition of an inverse V operator such that

$$V_1^{-1} = \frac{\dfrac{m_2 - n_1}{m_1 m_2 - n_1 n_2}}{\dfrac{m_1 - n_2}{m_1 m_2 - n_1 n_2}} \tag{2-24}$$

From this definition, it follows that

$$V_1 V_1^{-1} = V_1^{-1} V_1 = \frac{1}{1} \tag{2-25}$$

which defines a unit V operator.

It should be noted that any operator which is the ratio of two equal even polynomials in s, when operating on ζ_1, yields ζ_1.

Furthermore, any operator which is the ratio of two equal odd polynomials in s, when operating on ζ_1, gives $1/\zeta_1$. The former acts like a unit operator and the latter is an inverting operator.

The squared V operator may be derived directly from eq. (2-2).

$$V_1^2 = V_1 V_1 = \frac{(m_1^2 + n_1 n_2) + n_1(m_1 + m_2)}{(m_2^2 + n_1 n_2) + n_2(m_1 + m_2)} \qquad (2\text{-}26)$$

If a prf impedance is squared in the usual sense, the result is meaningless. However, if the squaring is done in the sense of eq. (2-26), the result does have meaning in that V_1^2 is prf if V_1 is prf.

2.7 The Commutative Law

Generally the commutative law does not hold for the V operator. However, under certain conditions, $V_1 V_2 = V_2 V_1$. These conditions are now derived using the V operators in eqs. (2-2) and (2-11).

$$V_1 V_2 = \frac{(m_1 m_3 + n_1 n_4) + (m_1 n_3 + n_1 m_4)}{(m_2 m_4 + n_2 n_3) + (m_2 n_4 + n_2 m_3)} \qquad (2\text{-}27)$$

$$V_2 V_1 = \frac{(m_1 m_3 + n_2 n_3) + (m_2 n_3 + n_1 m_3)}{(m_2 m_4 + n_1 n_4) + (m_1 n_4 + n_2 m_4)} \qquad (2\text{-}28)$$

For V_1 and V_2 to commute, it is necessary that

$$\begin{aligned} n_2 n_3 &= n_1 n_4 \\ n_1(m_4 - m_3) &= n_3(m_2 - m_1) \\ n_2(m_4 - m_3) &= n_4(m_2 - m_1) \end{aligned} \qquad (2\text{-}29)$$

Theorem G

The necessary and sufficient conditions for the commutation of two V operators is

$$\frac{n_1}{n_3} = \frac{n_2}{n_4} = \frac{m_2 - m_1}{m_4 - m_3} \qquad (2\text{-}30)$$

Eq. (2-30) follows directly from eq. (2-29).

With V_1 restricted to the form of eq. (2-2), it is pertinent to compose a general form of V_2 which satisfies eq. (2-30) and is prf. This general form is

$$V_2 = \frac{\left[m_1F(s) + G(s)\right] + n_1F(s)}{\left[m_2F(s) + G(s)\right] + n_2F(s)} \tag{2-31}$$

where F(s) and G(s) are even polynomials in s, $\frac{G(s)}{F(s)n_1}$ and $\frac{G(s)}{F(s)n_2}$ are reactance functions, and $G(s)F(s)(m_1 + m_2) - G(s)^2 + F(s)(m_1m_2 - n_1n_2) \geq 0$ on the $j\omega$ axis.*

One case of interest results if V_1 and V_2 are represented by the equations

$$V_1 = \frac{m_2 + \alpha^2 n_2}{m_2 + n_2} \tag{2-32}$$

$$V_2 = \frac{m_4 + \alpha^2 n_4}{m_4 + n_4} \tag{2-33}$$

where α is a positive real constant. These two operators satisfy the conditions of eq. (2-30) and thus are commutative. Also their form is such that each may be synthesized by the non-reciprocal Darlington procedure without transformers. Furthermore the combined operator, V_1V_2, may also be synthesized without transformers.**

---

*These statements are verified in Appendix 1.C

**For example, num EvV_1 = $m_2^2 - \alpha^2 n_2^2$

$$\therefore V_{11} = V_{22} = \frac{m_2}{n_2} \text{ and } V_{\substack{12\\21}} = \frac{m_2}{n_2} \pm \alpha$$

The same general form results for the synthesis of V_1V_2.

These syntheses are shown in Fig. (2-5).

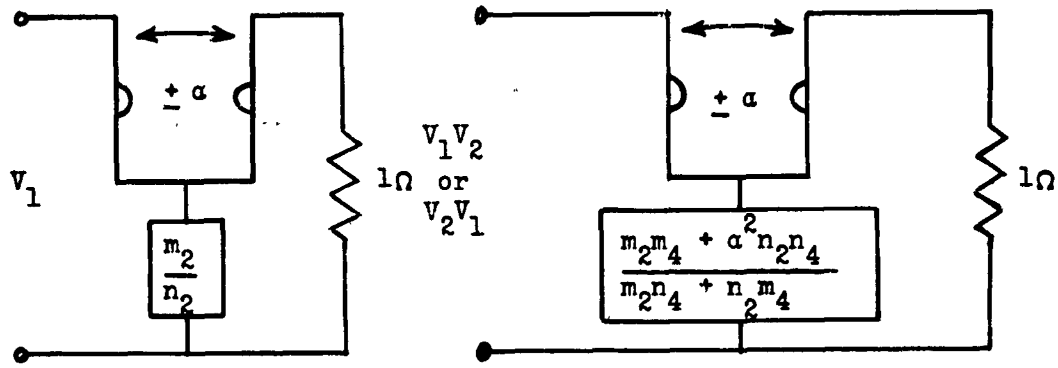

Fig. (2-5) Commutative Operator Syntheses

A second case of interest illustrates a pseudo-commutative property possessed by two prf impedance functions represented by γ_1 and γ_2 as follows:

$$\gamma_1 = \frac{m_1 + n_1}{m_2 + n_2} \tag{2-34}$$

$$\gamma_2 = \frac{M_1 + N_1}{M_2 + N_2} \tag{2-35}$$

Define two new operators V_x and V_y such that

$$V_x = \frac{(m_1 + n_1) + K^2(m_2 + n_2)}{(m_1 + n_1) + (m_2 + n_2)} \tag{2-36}$$

$$V_y = \frac{(M_1 + N_1) + K^2(M_2 + N_2)}{(M_1 + N_1) + (M_2 + N_2)} \tag{2-37}$$

where K is an arbitrary positive real constant. By rearranging

V_x and V_y, it is easy to show that each is prf if γ_1 and γ_2 are prf.* Let V_x operate on γ_2 in the following manner to yield a new function denoted by Z_1

$$Z_1 = \frac{(m_1 + n_1)\,\gamma_2 + K^2(m_2 + n_2)}{m_1 + n_1 + \gamma_2(m_2 + n_2)} \qquad (2\text{-}38)$$

Similarly, let V_y operate on γ_1 in the same way to yield a second new function Z_2

$$Z_2 = \frac{(M_1 + N_1)\,\gamma_1 + K^2(M_2 + N_2)}{M_1 + N_1 + \gamma_1(M_2 + N_2)} \qquad (2\text{-}39)$$

Eq. (2-38) may be rearranged to yield

$$Z_1 = \frac{m_1 + n_1}{m_2 + n_2}\,\frac{\gamma_2 + K^2\,\dfrac{m_2 + n_2}{m_1 + n_1}}{\dfrac{m_1 + n_1}{m_2 + n_2} + \gamma_2} \qquad (2\text{-}40)$$

Since all terms on the right hand side of eq. (2-40) are prf, eq. (2-40) may be matched term-for-term with eq. (2- 5) and thus Z_1 represents a prf driving point impedance. A similar proof may be applied to eq. (2-39) to show that Z_2 represents a prf driving point impedance. Alternatively, from the definitions of γ_1 and γ_2, Z_2 and Z_1 are identical and therefore $V_x\gamma_2 = V_y\gamma_1 = Z_1 = Z_2 = Z$. Darlington-type syntheses of $V_x\gamma_2$ and $V_y\gamma_1$ yield the two networks of Fig. 2-6.**

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* This is proved in Appendix 1.D.

** These syntheses are presented in Appendix 1.E.

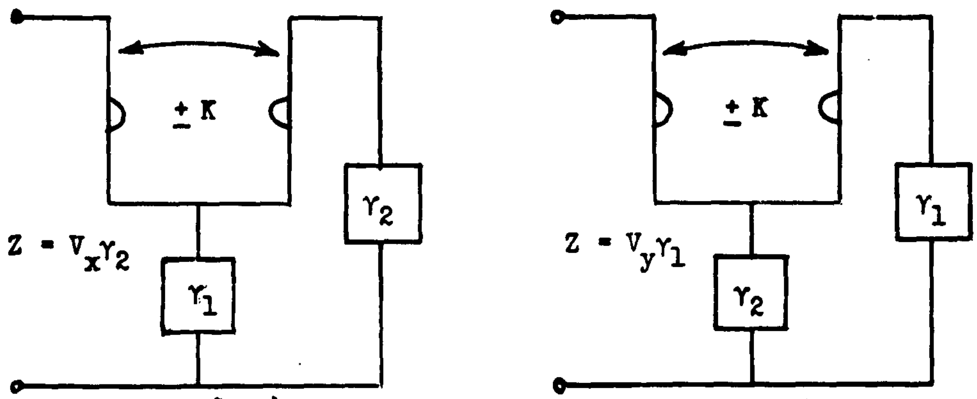


Fig. (2-6) Pseudo-Commutative Operator Syntheses

The operations defined by eqs. (2-38) and (2-39) are considerably different from the original operation defined in eq.(2-1) since each of the terms in eqs. (2-38) and (2-39) is a mixture of even and odd polynomials in s. Stated another way, the operations in eqs. (2-38) and (2-39) do not require that resistance be present only in the terminating impedance. γ_1 and γ_2 in Fig. (2-6) are generally RLC impedances and thus resistance is permitted in both the operator and the termination. The two networks in Fig. (2-6) illustrate the pseudo-commutative property in that γ_1 and γ_2 may be interchanged without changing the driving point impedance, Z.

It should be understood that the intention of this discussion was to show the existence of the pseudo-commutative property. No study has been made of ways in which a given driving point impedance might be separated to obtain V_x and γ_2 or V_y and γ_1. This latter problem is discussed further as a proposed topic for future investigation in Chapter VI.

A third case of interest is the possible commutation of the Richards' Theorem operators of eqs. (2-20) and (2-21). Combining these operators yields

$$V_1V_2 = \frac{s^2 \frac{Z_a}{\zeta_{1b}} + s(bZ_a + a\zeta_{1b}) + ab}{s^2 \frac{\zeta_{1b}}{Z_a} + s\left(\frac{b}{Z_a} + \frac{a}{\zeta_{1b}}\right) + ab} \qquad (2\text{-}41)$$

$$V_2V_1 = \frac{s^2 \frac{\zeta_{1b}}{Z_a} + s(bZ_a + a\zeta_{1b}) + ab}{s^2 \frac{Z_a}{\zeta_{1b}} + s\left(\frac{b}{Z_a} + \frac{a}{\zeta_{1b}}\right) + ab} \qquad (2\text{-}42)$$

The requirement for commutation is $\zeta_{1b} = Z_a$ and, when this requirement is met, no transformers are required in the synthesis of V_1V_2. The necessary and sufficient conditions for satisfying this requirement and its use in cascade synthesis procedures are developed in Chapters III and V.

2.8 The Distributive Law

The distributive law for the V operator is written as

$$(V_1 + V_2)\zeta_1 = V_1\zeta_1 + V_2\zeta_1 \qquad (2\text{-}43)$$

where V_1 and V_2 are given by eqs. (2-2) and (2-11), respectively. Eq. (2-43) does not hold for all V_1 and V_2 unless ζ_1 is a one-ohm resistance. For a general ζ_1, a constraint is needed on V_1 and V_2 in order that eq. (2-43) be valid. To obtain this constraint, the two sides of eq. (2-43) are expanded to give

$$(V_1 + V_2)\zeta_1 = \left(\frac{m_1 + n_1}{m_2 + n_2} + \frac{m_3 + n_3}{m_4 + n_4}\right)\zeta_1 \qquad (2\text{-}44)$$

$$V_1\zeta_1 + V_2\zeta_1 = \frac{m_1\zeta_1 + n_1}{m_2 + n_2\zeta_1} + \frac{m_3\zeta_1 + n_3}{m_4 + n_4\zeta_1} \qquad (2\text{-}45)$$

In eqs. (2-44) and (2-45), make the substitution

$$m_4 = km_2, \quad n_4 = kn_2 \qquad (2\text{-}46)$$

where k is a positive real constant. This requires that the ratio of the denominators of V_2 and V_1 equal k. The result is

$$(V_1 + V_2)\zeta_1 = V_1\zeta_1 + V_2\zeta_1 = \frac{\left(m_1 + \frac{m_3}{k}\right)\zeta_1 + n_1 + \frac{n_3}{k}}{m_2 + n_2\zeta_1} \qquad (2\text{-}47)$$

The result of eq. (2-47) may be extended as follows:

<u>Theorem H</u>

The V operator obeys the distributive law given by

$$(V_1 + V_2 + \ldots\ldots V_n)\zeta_1 = V_1\zeta_1 + V_2\zeta_1 + \ldots\ldots V_n\zeta_1 \qquad (2\text{-}48)$$

only if the denominators of V_1, V_2 .... V_n are equal or differ by a positive real constant.

Theorem H can be verified by expressing the left side of eq. (2-48) in the form

$$\frac{\sum_{\ell=1}^{n} (m_{1\ell}\zeta_1 + n_{1\ell}) \frac{1}{k_\ell}}{m_2 + n_2\zeta} \qquad (2\text{-}49)$$

which, when expanded, is identical to the right side of eq. (2-48).

Theorem I

With the constraint of Theorem H, the even part numerators are related by

$$\text{num EvV} = \text{num EvV}_1 + \text{num EvV}_2 + \ldots\ \text{num EvV}_n \qquad (2\text{-}50)$$

Theorem I may be verified by direct expansion.

The distributive law, under the constraint of Theorem H, is useful in synthesis procedures using impedance operators. For example, let V in eq. (2-2) be distributed to give

$$V_1 = \frac{m_1}{m_2 + n_2} + \frac{n_1}{m_2 + n_2} = {}_1V_1 + {}_2V_1 \qquad (2\text{-}51)$$

${}_1V_1$ and ${}_2V_1$ satisfy eq. (2-46) with k = 1. Thus they may be separately synthesized, each terminated in ζ_1, and then summed to give Z, assuming ${}_1V_1$ and ${}_2V_1$ are prf.

V_1 may be distributed in an infinite number of ways to suit the requirements of the particular synthesis desired. However the individual parts of V_1 must each be prf if they are to represent realizable networks.

Example 3

Consider the impedance given by

$$Z = \frac{(s^2 + 2)\zeta_1 + s}{s^2 + 1 + s\zeta_1}$$

$$V_1 = \frac{s^2 + s + 2}{s^2 + s + 1}$$

V_1 is not a minimum resistance function. However it may be easily synthesized without first removing resistance if it is distributed as follows:

$$V_1 = \frac{s^2 + 1}{s^2 + s + 1} + \frac{s + 1}{s^2 + s + 1} \quad *$$

$$\text{num } EvV_1 = (s^2 + 1)^2 \quad + 1$$

The parts of V_1 are each prf and may be synthesized directly from eqs. (2-6) and (2-8) without the use of gyrators (since both even part numerators are perfect squares). The result appears in Fig. (2-7).

*Guillemin[7] describes a method known as "resistance padding" which permits certain impedances to be realized by simple networks. Application of this method to V_1 yields the separation indicated and the network of Fig. (2-7).

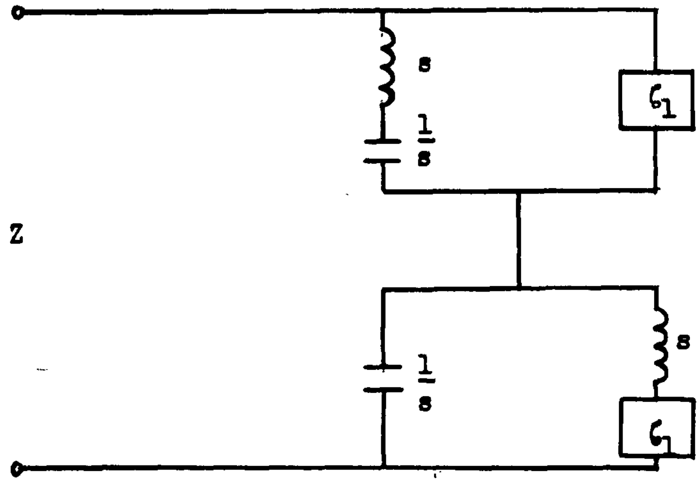


Fig. (2-7) Distributed Operator Synthesis

2.9 Impedance Operators in Matrix Form

The operations described in the previous sections may be more concisely stated using matrix notation. These relations, all of which may be verified by direct matrix expansion, are now presented.

$$V_1 = \begin{bmatrix} m_1 & n_1 \\ n_2 & m_2 \end{bmatrix} \tag{2-52}$$

$$V_2 = \begin{bmatrix} m_3 & n_3 \\ n_4 & m_4 \end{bmatrix} \tag{2-53}$$

$$V_1\zeta_1 = \frac{\begin{bmatrix} m_1 & n_1 \end{bmatrix}}{\begin{bmatrix} n_2 & m_2 \end{bmatrix}} \begin{bmatrix} \zeta_1 \\ 1 \end{bmatrix}^{*} \tag{2-54}$$

*The partitioning of the first operator in this and other equations is necessary to make the matrices compatible and to obtain a ratio of polynomials as a final result.

$$V_1V_2\zeta_1 = \frac{\begin{bmatrix} m_1 & n_1 \end{bmatrix}}{\begin{bmatrix} n_2 & m_2 \end{bmatrix}} \begin{bmatrix} m_3 & n_3 \\ n_4 & m_4 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ 1 \end{bmatrix} \qquad (2\text{-}55)$$

$$(V_1 + V_2)\zeta_1 = \left\{ \frac{\begin{bmatrix} m_1 & n_1 \end{bmatrix}}{\begin{bmatrix} n_2 & m_2 \end{bmatrix}} + \frac{\begin{bmatrix} m_3 & n_3 \end{bmatrix}}{\begin{bmatrix} n_4 & m_4 \end{bmatrix}} \right\} \begin{bmatrix} \zeta_1 \\ 1 \end{bmatrix} \qquad (2\text{-}56)$$

When $m_4 \neq km_2$ and/or $n_4 \neq kn_2$,

$$(V_1 + V_2)\zeta_1 = \frac{\left\{ \begin{bmatrix} m_1 & n_1 \end{bmatrix} \begin{bmatrix} m_4 & n_4 \\ n_4 & m_4 \end{bmatrix} + \begin{bmatrix} m_2 & n_2 \end{bmatrix} \begin{bmatrix} m_3 & n_3 \\ n_3 & m_3 \end{bmatrix} \right\}}{\begin{bmatrix} n_2 & m_2 \end{bmatrix} \begin{bmatrix} m_4 & n_4 \\ n_4 & m_4 \end{bmatrix}} \begin{bmatrix} \zeta_1 \\ 1 \end{bmatrix} \qquad (2\text{-}57)$$

When $m_4 = km_2$ and $n_4 = kn_2$,

$$(V_1 + V_2)\zeta_1 = \frac{\begin{bmatrix} m_1 + \frac{m_3}{k} & n_1 + \frac{n_3}{k} \end{bmatrix}}{\begin{bmatrix} n_2 & m_2 \end{bmatrix}} \begin{bmatrix} \zeta_1 \\ 1 \end{bmatrix} = V_1\zeta_1 + V_2\zeta_1 \qquad (2\text{-}58)$$

The numerator of the even part of V_1, in matrix form, is

$$\text{num Ev } V_1 = \left| V_1 \right|^* = \begin{vmatrix} m_1 & n_1 \\ n_2 & m_2 \end{vmatrix} = m_1m_2 - n_1n_2 \qquad (2\text{-}59)$$

---

* $\left| V_1 \right|$ = determinant of V_1

The syntheses of V_1 by the two Darlington procedures using eqs. (2-8) and (2-9) may also be formulated in matrix notation (assuming $|V|$ is a perfect square, ie $V_{12} = V_{21}$).

$$\begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} = \frac{1}{n_2} \begin{bmatrix} m_1 & |V_1|^{1/2} \\ |V_1|^{1/2} & m_2 \end{bmatrix} \tag{2-60}$$

$$\begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} = \frac{1}{m_2} \begin{bmatrix} n_1 & (-|V_1|)^{1/2} \\ (-|V_1|)^{1/2} & n_2 \end{bmatrix} \tag{2-61}$$

The inverse and unit operators may also be expressed in matrix form.

$$V_1^T = \begin{bmatrix} m_1 & n_2 \\ n_1 & m_2 \end{bmatrix} \tag{2-62}$$

$$\text{adj } V_1 = \begin{bmatrix} m_2 & -n_1 \\ -n_2 & m_1 \end{bmatrix} \tag{2-63}$$

$$V_1^{-1} = \frac{\begin{bmatrix} m_2 & -n_1 \\ -n_2 & m_1 \end{bmatrix}}{|V_1|} \tag{2-64}$$

$$V_1 V_1^{-1} = \frac{\begin{matrix} \frac{1}{|V_1|} \begin{bmatrix} m_1 & n_1 \end{bmatrix} \\ \frac{1}{|V_1|} \begin{bmatrix} n_2 & m_2 \end{bmatrix} \end{matrix}}{} \begin{bmatrix} m_2 & -n_1 \\ -n_2 & m_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{2-65}$$

$$V_1^{-1}V_1 = \dfrac{\frac{1}{|V_1|}\begin{bmatrix} m_2 & -n_1 \end{bmatrix}}{\frac{1}{|V_1|}\begin{bmatrix} -n_2 & m_1 \end{bmatrix}} \begin{bmatrix} m_1 & n_1 \\ n_2 & m_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{2-66}$$

The Richards' Theorem operators of eqs. (2-20) and (2-21) and the combined operator operating on ζ_2 can be concisely expressed in matrix form.

$$V_1 = \begin{bmatrix} a & sZ_a \\ \frac{s}{Z_a} & a \end{bmatrix} \tag{2-67}$$

$$V_2 = \begin{bmatrix} b & s\zeta_{1b} \\ \frac{s}{\zeta_{1b}} & b \end{bmatrix} \tag{2-68}$$

$$V_1V_2\zeta_2 = \dfrac{\begin{bmatrix} a & sZ_a \end{bmatrix}}{\begin{bmatrix} \frac{s}{Z_a} & a \end{bmatrix}} \begin{bmatrix} b & s\zeta_{1b} \\ \frac{s}{\zeta_{1b}} & b \end{bmatrix} \begin{bmatrix} \zeta_2 \\ 1 \end{bmatrix}$$

$$= \dfrac{\begin{bmatrix} ab + s^2\frac{Z_a}{\zeta_{1b}} & s(bZ_a + a\zeta_{1b}) \end{bmatrix}}{\begin{bmatrix} s\left(\frac{b}{Z_a} + \frac{a}{\zeta_{1b}}\right) & ab + s^2\frac{\zeta_{1b}}{Z_a} \end{bmatrix}} \begin{bmatrix} \zeta_2 \\ 1 \end{bmatrix} \tag{2-69}$$

CHAPTER III

CASCADE SYNTHESIS USING IMPEDANCE OPERATORS OF RANK 2 AND 4

3.1 Introduction

In this chapter impedance operators of rank 2 and 4 are investigated in detail. Through these operators, it is shown that nine realizable network sections (Fig. 3-1)* containing one or more arbitrary constants may always be removed from an RLC driving point impedance function leaving, in cascade, a terminating impedance which is realizable and contains the same arbitrary constants. These constants may be used to produce desired characteristics in either the removed sections or the terminating impedance.

Seven of the removed sections contain gyrators. Methods are developed whereby, through proper choice of one or more of the arbitrary constants, the gyrators may be eliminated so that the removed sections are purely reactive and reciprocal.

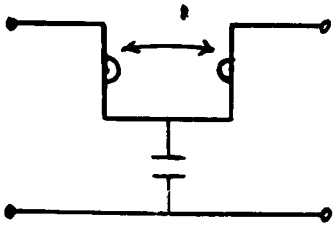
3.2 Rank 2 Operator Formulation and Synthesis

Let Z be represented by eq. (2-18) and the associated V operator (which is of rank 2) by eq. (2-20). These equations are repeated below:

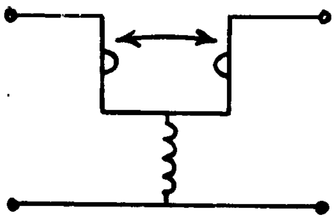
$$Z = \frac{a\zeta_1 + sZ_a}{a + \frac{s}{Z_a}\zeta_1} \tag{3-1}$$

*Some of the sections in Fig. (3-1) have non-cascade representations which do not include gyrators or transformers. These are not considered since the purpose here is to develop cascade synthesis procedures.

$$V_1 = \frac{a + sZ_a}{a + \frac{s}{Z_a}} \qquad (3\text{-}2)$$



(a)



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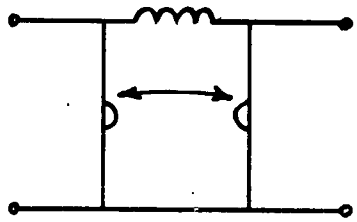

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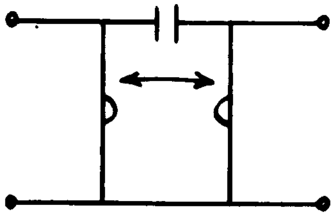

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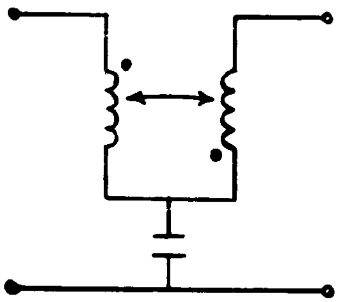

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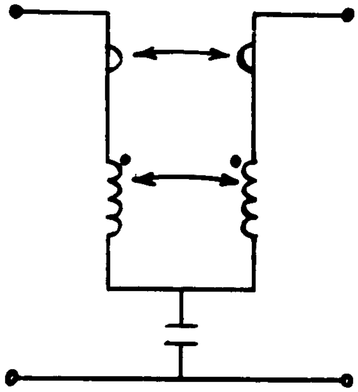

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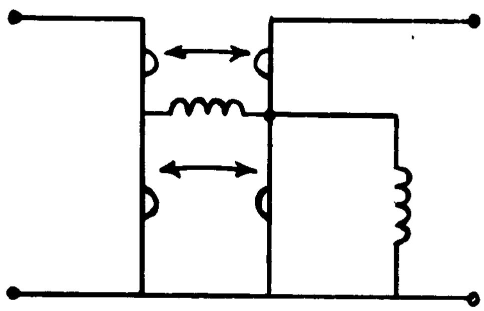

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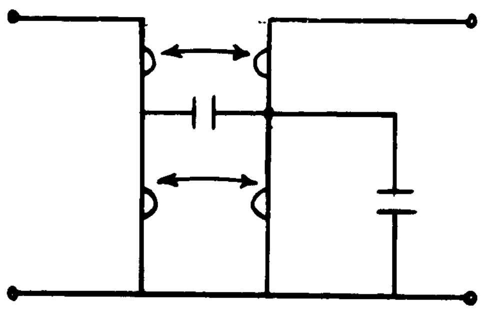

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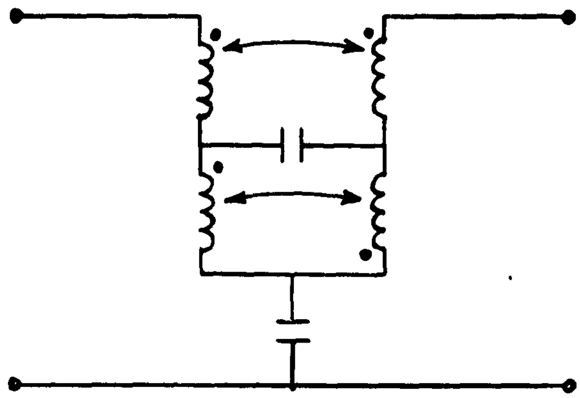


(i)

Fig. (3-1) Network Sections which may Always be Removed from Z

The n- and m-type Darlington representations of eq. (3-2) are the network sections of Figs. (3-1a) and (3-1b), respectively. These sections may always be removed from an RLC driving point impedance. The element values and terminating impedances are shown in Fig. (3-2) for the n-type and in Fig. (3-3) for the m-type synthesis. The constant appearing in each figure is completely arbitrary except that it must be positive real.*

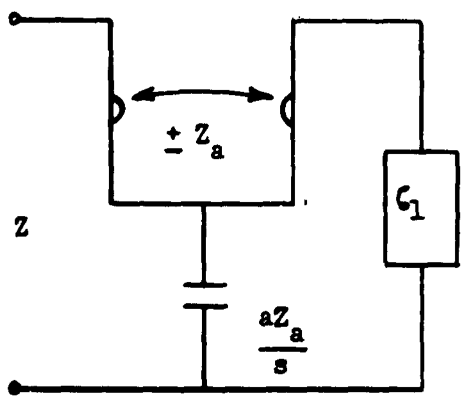


Fig. (3-2) n-Type Rank 2

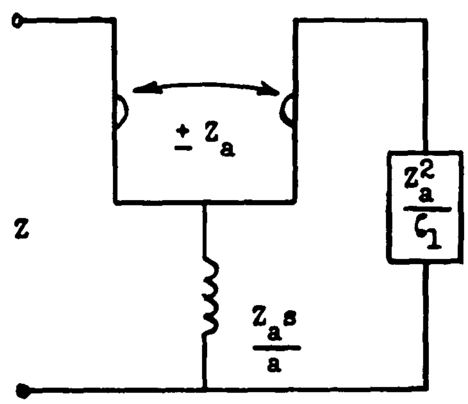


Fig. (3-3) m-Type Rank 2

*This has been discussed in Section 2.5.

The n-type network of Fig. (3-2) is derived using eq. (2-8) as follows:

$$\text{num Ev } V_1 = a^2 - s^2$$

$$V_{\substack{12\\21}} = \frac{aZ_a}{s} \pm Z_a \qquad (3\text{-}3)$$

$$V_{11} = V_{22} = \frac{aZ_a}{s}$$

Eq. (2-9) is used in obtaining the m-type network of Fig. (3-3). The terminating impedance $(1/\zeta_1)$ is scaled to Z_a^2/ζ_1 to avoid the use of a transformer in the synthesis of V_1. Thus

$$Z = \frac{aZ_a^2 \dfrac{\zeta_1}{Z_a^2} + sZ_a}{a + sZ_a \dfrac{\zeta_1}{Z_a^2}} \qquad (3\text{-}4)$$

$$V_1 = \frac{aZ_a^2 + sZ_a}{a + sZ_a} \qquad (3\text{-}5)$$

and the synthesis of V_1 is

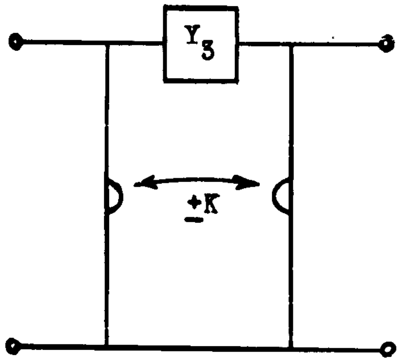
$$-\text{ num Ev } V_1 = Z_a^2(s^2 - a^2)$$

$$V_{\substack{12\\21}} = \frac{Z_a s}{a} \pm Z_a \qquad (3\text{-}6)$$

$$V_{11} = V_{22} = \frac{Z_a s}{a}$$

3.3 Equivalent Rank 2 Operator Networks

Consider the network section shown in Fig. (3-4). Its Z parameters are



$$Z_{11} = Z_{22} = K^2 Y_3$$
$$Z_{\substack{12\\21}} = K^2 Y_3 \pm K \tag{3-7}$$

Fig. (3-4)

Let $Y_3 = \frac{a}{Z_a s}$ and $K = Z_a$. Then eq. (3-7) becomes

$$Z_{11} = Z_{22} = \frac{aZ_a}{s}$$
$$Z_{\substack{12\\21}} = \frac{aZ_a}{s} \pm Z_a \tag{3-8}$$

Similarly let $Y_3 = \frac{s}{aZ_a}$ and $K = Z_a$. Then

$$Z_{11} = Z_{22} = \frac{Z_a s}{a}$$
$$Z_{\substack{12\\21}} = \frac{Z_a s}{a} \pm Z_a \tag{3-9}$$

Syntheses of eqs. (3-8) and (3-9) directly yield the sections of Figs. (3-2) and (3-3). Therefore the networks of Figs. (3-5) and (3-6) are equivalent to those of Figs. (3-2) and (3-3), respectively, and their sections may always be removed from Z. The sections appear in Figs. (3-1c) and (3-1d).

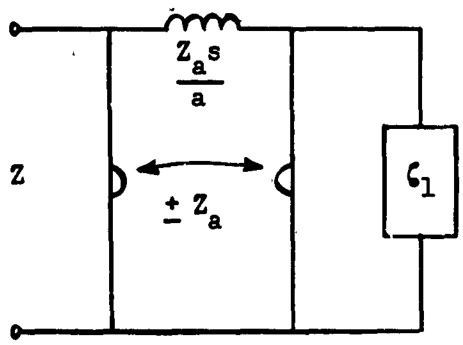


Fig. (3-5) n-Type Rank 2

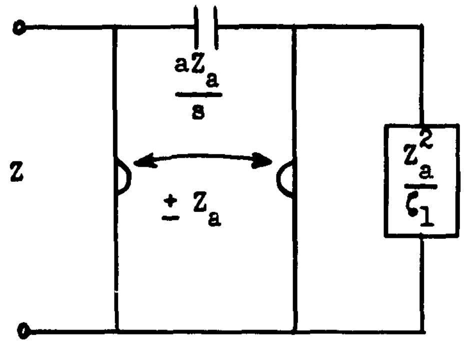


Fig. (3-6) m-Type Rank 2

3.4 Eliminating the Gyrator from a Rank 2 Operator

A method of eliminating the gyrators which appear in the network sections of Figs. (3-1a) through (3-1d) is now derived so that a network section containing only reactive reciprocal elements can always be removed from Z. Let ζ_1 be expressed by eq. (2-19) and its associated V operator by eq. (2-21). These equations are repeated below,

$$\zeta_1 = \frac{b\zeta_2 + s\zeta_{1b}}{b + \frac{s}{\zeta_{1b}}\zeta_2} \tag{3-10}$$

$$V_2 = \frac{b + s\zeta_{1b}}{b + \frac{s}{\zeta_{1b}}} \tag{3-11}$$

where b is a positive real constant. Substituting eq. (3-10) into eq. (3-1) gives Z in terms of ζ_2.

$$Z = \frac{\left(ab + s^2 \frac{Z_a}{\zeta_{1b}}\right)\zeta_2 + s(bZ_a + a\zeta_{1b})}{\left(ab + s^2 \frac{\zeta_{1b}}{Z_a}\right) + s\left(\frac{b}{Z_a} + \frac{a}{\zeta_{1b}}\right)\zeta_2} \tag{3-12}$$

The associated impedance operator is

$$V = \frac{ab + s(bZ_a + a\zeta_{1b}) + s^2 \frac{Z_a}{\zeta_{1b}}}{ab + s\left(\frac{b}{Z_a} + \frac{a}{\zeta_{1b}}\right) + s^2 \frac{\zeta_{1b}}{Z_a}} \tag{3-13}$$

Theorem J

As a corollary of Theorem A in Section 1.1, Hazony[4] has shown that, if Z is prf, then V in eq. (3-13) and ζ_2 in eq. (3-12) are prf for a and b positive real or complex conjugates with a non-negative real part.*

The numerator of the even part of V may be obtained directly from eq. (3-13) or, more easily, by the use of eq. (2-15). The result is

$$\text{num Ev } V = (a^2 - s^2)(b^2 - s^2) \tag{3-14}$$

In order to synthesize V without a gyrator, it is necessary that eq. (3-14) be a perfect square so that $V_{12} = V_{21}$. This requirement means that

$$b = \pm a$$
$$\text{num Ev } V = (a^2 - s^2)^2 \tag{3-15}$$

*ζ_1 is no longer prf if a is complex even though Z is prf.

Thus a and b must be real and equal or imaginary and opposite in sign.

For the case b = + a, V in eq. (3-13) becomes*

$$V = \frac{a^2 + as(Z_a + \zeta_{1a}) + s^2 \frac{Z_a}{\zeta_{1a}}}{a^2 + as\left(\frac{1}{Z_a} + \frac{1}{\zeta_{1a}}\right) + s^2 \frac{\zeta_{1a}}{Z_a}} \qquad (3\text{-}16)$$

The n-type reciprocal Darlington synthesis of eq. (3-16) extracts the network section of Fig. (3-le)in which the transformer polarity is additive. The element values and terminating impedance appear in Fig. (3-7). The removed section is purely reactive and contains one arbitrary positive real constant which is also contained in the ζ_2 termination.

*$\lim_{b \to a} \frac{\zeta_{1b}}{Z_a}$ is a positive real number. This may be shown by writing:

$$\frac{\zeta_{1b}}{Z_a} = \frac{a\,Z_b - b\,Z_a}{a\,Z_a - b\,Z_b}$$

For b = + a, this expression becomes indeterminate $(\frac{0}{0})$. Using L'Hospital's Rule yields

$$\lim_{b \to a} \frac{\zeta_{1b}}{Z_a} = \frac{Z_a - aZ_a'}{Z_a + aZ_a'}$$

This limit is positive real since ζ_{1a} and Z_a are positive real for any positive real value of a by Richards' Theorem.

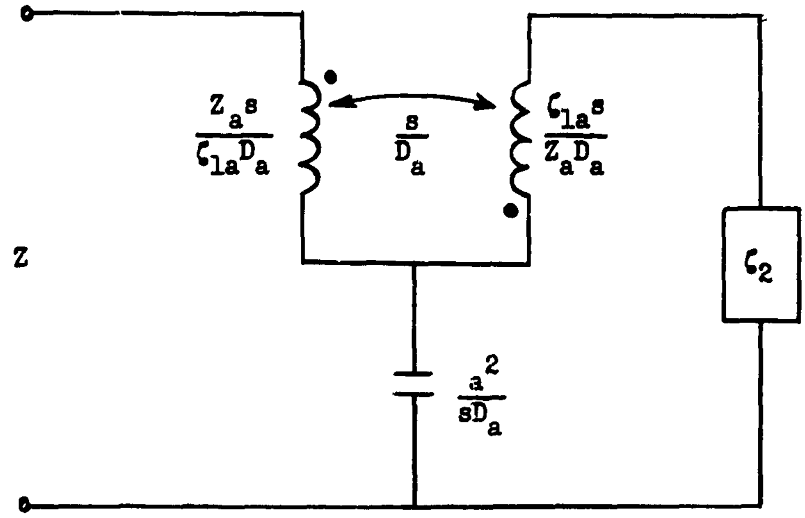


Fig. (3-7) Gyrator Elimination for a Rank 2 V, b = + a

The element values in Fig. (3-7) are obtained from eqs. (2-8) and (3-16). They are

$$V_{12} = V_{21} = \frac{a^2 - s^2}{sD_a}$$

$$V_{11} = \frac{a^2 + s^2 \frac{Z_a}{\zeta_{1a}}}{sD_a} \qquad (3\text{-}17)$$

$$V_{22} = \frac{a^2 + s^2 \frac{\zeta_{1a}}{Z_a}}{sD_a}$$

$$D_a = \left(\frac{b}{Z_a} + \frac{a}{\zeta_{1b}}\right)_{b=a} = a\left(\frac{1}{Z_a} + \frac{1}{\zeta_{1a}}\right)$$

For the case $b = -a$, two subcases result depending on whether $Ev\ Z_a$ is or is not equal to zero.* For $Ev\ Z_a \neq 0$, eq. (3-13) becomes

$$V = \frac{s^2 - a^2}{s^2 - a^2} \tag{3-18}$$

This gives the trivial result $Z = \zeta_2$.

For $Ev\ Z_a = 0$, V in eq. (3-13) becomes**

$$V = \frac{-a^2 - as(Z_a + \zeta_{1a}) - s^2 \dfrac{Z_a}{\zeta_{1a}}}{-a^2 - as\left(\dfrac{1}{Z_a} + \dfrac{1}{\zeta_{1a}}\right) - s^2 \dfrac{\zeta_{1a}}{Z_a}} \tag{3-19}$$

---

*

$$\frac{\zeta_{1b}}{Z_a} = \frac{a\ Z_b - b\ Z_a}{a\ Z_a - b\ Z_b}$$

For $b = -a$,

$$\frac{\zeta_{1(-a)}}{Z_a} = \frac{a\ Z_{(-a)} + a\ Z_a}{a\ Z_a + a\ Z_{(-a)}}$$

This expression is unity unless $Ev\ Z_a = 0$, in which case it becomes indeterminate $(\frac{0}{0})$. Using L'Hospitals' Rule in the latter case gives

$$\lim_{b \to -a} \frac{\zeta_{1b}}{Z_a} = \frac{a\ Z'_a + Z_a}{a\ Z'_a - Z_a}$$

This limit is a positive real number since the coefficients of s^2 in eq. (3-13) must be positive real. This follows from the fact that V is positive real for any a and b which are complex conjugates with a non-negative real part.

**All coefficients in eq. (3-19) are positive real since a is imaginary. Also $\zeta_{1(-a)} = -\zeta_{1a}$ since $Ev\ Z_a = 0$.

The n-type reciprocal Darlington synthesis of eq. (3-19) again yields the section of Fig. (3-1e) but, in this case, the transformer polarity is subtractive since

$$\text{num Ev } V = (s^2 - a^2)^2 = (s^2 + \omega_o^2)^2, \quad a = j\omega_o \quad (3\text{-}20)$$

which yields a positive mutual inductance term in V_{12} and V_{21}. This network is shown in Fig. (3-8). It is obtained only if Ev $Z_a = 0$.

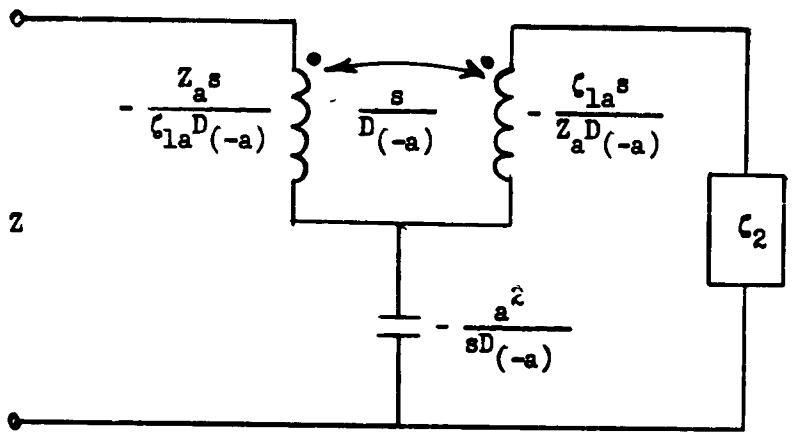


Fig. (3-8) Gyrator Elimination for a Rank 2 V, b = - a, Ev $Z_a = 0$

The element values in Fig. (3-8) are derived from eqs. (2-8) and (3-19).

$$V_{12} = V_{21} = \frac{-a^2 + s^2}{sD_{(-a)}}$$

$$V_{11} = \frac{-a^2 - s^2 \dfrac{Z_a}{\zeta_{1a}}}{sD_{(-a)}} \qquad (3\text{-}21)$$

$$V_{22} = \frac{-a^2 - s^2 \frac{\zeta_{1a}}{Z_a}}{sD_{(-a)}} \tag{3-21}$$

$$D_{(-a)} = \left(\frac{b}{Z_a} + \frac{a}{\zeta_{1b}}\right)_{b=-a} = -a\left(\frac{1}{Z_a} + \frac{1}{\zeta_{1a}}\right)^*$$

The section of Fig. (3-7) appears to be identical with the Darlington Type C section,[3,7] but there are two important differences. First, realization of the Type C section demands that num Ev Z have a factor of the form $(A^2 - s^2)^2$, requiring a pair of real even part zeros of even multiplicity. Often this requires the use of surplus factors. By contrast, the section of Fig. (3-7) may be removed from Z for any configuration of even part zeros. In essence surplus factors are already built into the impedance operator. Secondly, the section of Fig. (3-7) contains an arbitrary constant whereas the Type C section does not. If this constant is chosen so that Ev $Z_a = 0$, then, if these real zeros of Ev Z are of second order, ζ_2 is four less in rank than Z whereas, if they are of first order, ζ_2 is two less in rank than Z.

The section of Fig. (3-8) is identical to the Brune network[17] and requires that num Ev Z have a $j\omega$ axis zero (Z must be a minimum resistance function or the minimum resistance must have been removed). Thus the Brune network results from the impedance

---

*$D_{(-a)}$ is a positive real number.

operator approach for the special choices b = - a and Ev Z_a = 0.

In the event Z is not a minimum resistance function, the minimum resistance may be removed at the outset. Then the remaining impedance can be expressed by eq. (3-12). Choosing b = - a and also choosing a so that Ev Z_a = 0 again permits the removal of the section of Fig. (3-8) and the termination ζ_2 is four less in rank than Z. Now the minimum resistance may be removed from ζ_2 and the remaining impedance expressed by eq. (3-12), again permitting the removal of the section of Fig. (3-8). The process may be continued until the termination is reduced to a rank of two or less. This yields the same result as the Brune procedure, where each step using the impedance operator has its counterpart in the Brune cycle*.

To summarize the results of Sections 3.2 through 3.4, it has been shown that the network sections of Figs. (3-1a) through (3-1e) may always be removed from a prf driving point impedance and that the gyrators appearing in Figs. (3-1a) through (3-1d) may always be eliminated to yield the reactive section of Fig. (3-1e). The first four sections contain a pair of real axis even part zeros whereas the section of Fig. (3-1e) contains either a pair of real axis or a pair of imaginary axis even part zeros. In each of the

*The cascade synthesis procedure of Guillemin,[7] which was summarized in Section 1.9, yields the same result when applied to the Brune procedure as does the impedance operator approach above. In this case, T(s) in eq. (1-54) denotes imaginary, rather than complex, zeros.

five sections, one arbitrary constant is available for choice. The selection of this constant is now considered.

Assume the driving point impedance to be synthesized is not a minimum resistance function. Several synthesis paths may be followed. First, if Ev Z contains real axis zeros, the method of zero cancellation synthesis[4] can be used in which a is chosen to reduce the rank of ζ_1 of ζ_2. This choice permits a realization in terms of any of the network sections of Figs. (3-1a) through (3-1e) (excepting the case $b = -a$, Ev $Z_a = 0$). Secondly, the minimum resistance may be removed at the outset, after which the remaining impedance is synthesized by the zero cancellation method using the rank 4 V operator with $b = -a$, Ev $Z_a = 0$. This yields the Brune network. Third, assuming again that the minimum resistance has been removed from Z, a may be chosen to create a $j\omega$ axis zero or pole in ζ_1 or ζ_2. This method is essentially the Bott-Duffin procedure, but in this case is designed to yield a cascade, rather than a distributed, result.[9]

3.5 Extended Bott-Duffin Cascade Synthesis Procedure

First, the conventional Bott-Duffin procedure[3,6] is reviewed and the points essential to its extension are discussed. Let Z as given by eq. (3-1) be a minimum resistance function. Then the even part of Z has a $j\omega$ axis zero at $s = j\omega_o$ and Z appears either inductive or capacitive at that point

$$Z = sL \quad \text{or} \quad \frac{1}{sC} \quad \text{at} \quad s = j\omega_o \tag{3-22}$$

Solving eq. (3-1) for ζ_1 gives

$$\zeta_1 = \frac{a\,Z - sZ_a}{a - \frac{s}{Z_a}\,Z} \tag{3-23}$$

The zeros of the even parts of ζ_1 and Z are identical from eq. (2-17)*. Therefore Ev $\zeta_1 = 0$ at $s = j\omega_o$.

Let a be chosen so that ζ_1 has a $j\omega$ axis zero or pole at $s = j\omega_o$. This requires that

$$\begin{aligned} a\,Z - sZ_a &= 0 \quad \text{for a zero at} \quad s = j\omega_o \\ a\,Z_a - sZ &= 0 \quad \text{for a pole at} \quad s = j\omega_o \end{aligned} \tag{3-24}$$

The first form of eq. (3-24) is applicable when Z appears inductive at $s = j\omega_o$. Assume this to be the case. Then

$$\frac{Z_a}{a} = \frac{Z}{s} = L \quad \text{at} \quad s = j\omega_o \tag{3-25}$$

It follows that

$$Z_a - aL = 0 \tag{3-26}$$

Eq. (3-26) has one positive real root since the function Z - sL has one positive real zero.[3]

ζ_1 in eq. (3-23) is the same rank as Z since there is a common factor s - a in the numerator and denominator of the right

*Ev $Z_a \neq 0$. Therefore num Ev V_1 does not appear in num Ev Z (see discussion following eq. (2-22)).

hand side. Creating a $j\omega$ axis zero in ζ_1 insures that the impedance remaining after the removal of this resonant circuit is four less in rank than ζ_1(and therefore Z).

Customarily, the distributed Bott-Duffin network is obtained from the foregoing procedure,* but it is also possible to obtain the cascade representations of Figs. (3-2) and (3-3).[9]

With these concepts in mind, the Bott-Duffin procedure can be extended. Let ζ_1 be given by eq. (3-10) and Z by eq. (3-12). Solving eq. (3-10) for ζ_2 gives

$$\zeta_2 = \frac{b\zeta_1 - s\zeta_{1b}}{b - \frac{s}{\zeta_{1b}}\zeta_1} \qquad (3\text{-}27)$$

choosing b equal to + a so that the pertinent V operator in eq. (3-13) may be synthesized without a gyrator causes eq. (3-27) to take the form

$$\zeta_2 = \frac{a\zeta_1 - s\zeta_{1a}}{a - \frac{s}{\zeta_{1a}}\zeta_1} \qquad (3\text{-}28)$$

The even part of ζ_1 is zero at $s = j\omega_o$. Therefore,

$$\zeta_1 = sL_1 \quad \text{or} \quad \frac{1}{sC_1} \quad \text{at} \quad s = j\omega_o$$

and all of the previous arguments apply with Z replaced by ζ_1, ζ_1 by ζ_2, L by L_1 and C by C_1. The even part of ζ_2 is zero at $s = j\omega_o$

*This is shown in Fig. (5-9).

and a is chosen to create a zero or pole in ζ_2 at that point.

$$a\zeta_1 - s\zeta_{1a} = 0 \quad \text{for a zero at} \quad s = j\omega_0$$
$$a\zeta_{1a} - s\zeta_1 = 0 \quad \text{for a pole at} \quad s = j\omega_0 \qquad (3\text{-}29)$$

Again considering the inductive case,

$$\frac{\zeta_{1a}}{a} = \frac{\zeta_1}{s} = L_1 \quad \text{at} \quad s = j\omega_0 \qquad (3\text{-}30)$$

$$\zeta_{1a} - aL_1 = 0 \qquad (3\text{-}31)$$

Eq. (3-31) has one positive real root. ζ_2, ζ_1 and Z are of the same rank and, when the resonant circuit is removed, the remaining termination is four less in rank than Z. The network is that of Fig. (3-7), where ζ_2 contains a removable resonant circuit.

3.6 Rank 2 Operator Examples

The syntheses which result for the various choices of a are illustrated by considering the following driving point impedance function:

$$Z = \frac{s^2 + s + 1}{s^2 + s + 4} \quad \text{where} \quad \text{num Ev } Z = (s^2 + 2)^2$$

Solution A

Since Z is a minimum resistance function, the Bott-Duffin procedure may be used with the rank 2 V operator to synthesize Z. Z behaves like $\frac{s}{2}$ at $s = j\sqrt{2}$ and a is chosen so that ζ_1 has a $j\omega$ axis zero $(aZ - sZ_a = 0)$ at that point. The resulting a is positive real and therefore ζ_1 is prf. The calculations yield

$$a = 1, \quad Z_a = 1/2, \quad \zeta_1 = \frac{s^2 + 2}{4s^2 + 6s + 8}$$

$$V_1 = \frac{1 + \frac{s}{2}}{1 + 2s}, \text{ num Ev} V_1 = 1 - s^2$$

The cascade syntheses of V_1 and ζ_1 give networks corresponding to Figs. (3-2) and (3-3). The results appear in Figs. (3-9) and (3-10).

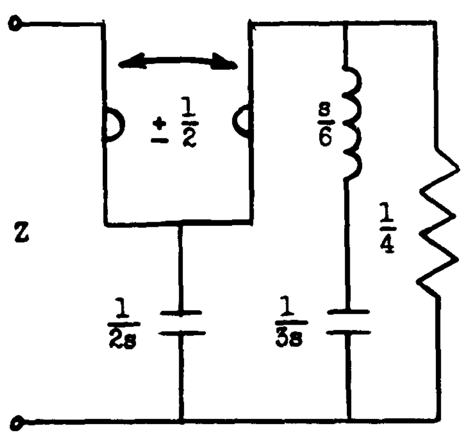


Fig. (3-9) n-Type Rank 2

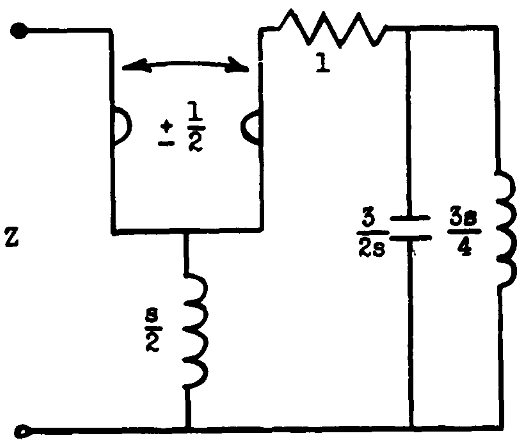


Fig. (3-10) m-Type Rank 2

Solution B

The extended Bott-Duffin procedure may be used with the Rank 4 V operator to synthesize Z. a is chosen so that ζ_2 has a $j\omega$ axis zero at $s = j\sqrt{2}$. The a obtained is again positive real and b = + a.

Carrying through the calculations yields

$$a\zeta_1 - s\zeta_{1a} = 0 \quad \text{at} \quad s = j\sqrt{2}$$

but
$$\frac{\zeta_{1a}}{Z_a} = \frac{s}{2}\,\frac{a-1}{a+2} \quad \text{at} \quad s = j\sqrt{2}$$

Therefore
$$\frac{\zeta_1}{s} = \frac{\zeta_{1a}}{a} = \frac{Z_a(a-1)}{2(a+2)}$$

Expanding and collecting terms gives

$$a^6 - a^5 + 2a^4 - 12a^3 - 8a^2 - 20a - 16 = 0$$

$$(a^2 - 2a - 2)(a^2 + a + 4)(a^2 + 2) = 0$$

The positive real root is $a = 1 + \sqrt{3}$. Then

$$Z_a = \frac{3+\sqrt{3}}{6}, \quad \zeta_{1a} = \frac{3+\sqrt{3}}{12}$$

$$\zeta_2 = \frac{s^2+2}{4(s^2+2) + 4(3-\sqrt{3})s}$$

$$V = \frac{2s^2 + \frac{3+2\sqrt{3}}{2}s + 4 + 2\sqrt{3}}{\frac{s^2}{2} + 6\sqrt{3}\,s + 4 + 2\sqrt{3}}$$

$$\text{num Ev } V = (4 + 2\sqrt{3} - s^2)^2$$

The syntheses of V and ζ_2 corresponding to Fig. (3-7) appear in Fig. (3-11).

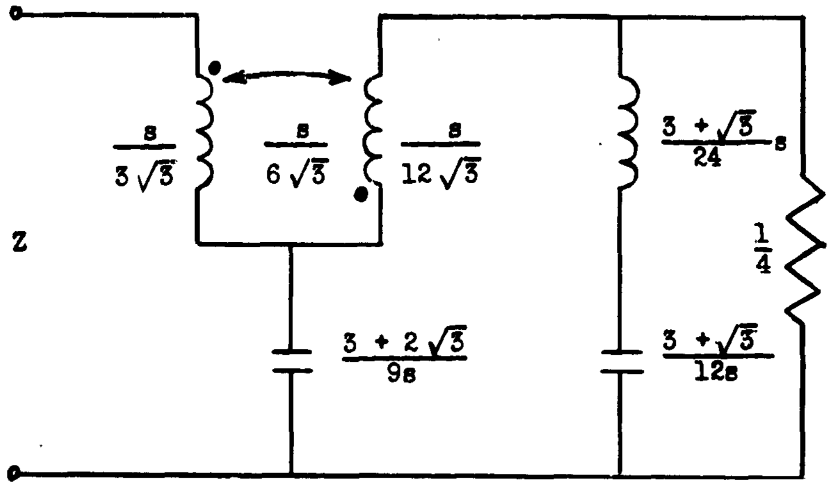


Fig. (3-11) Gyrator Elimination, b = + a

Solution C

The method of zero-cancellation synthesis may also be used to synthesize Z. In this case, $a = j\sqrt{2}$ makes Ev $Z_a = 0$ and thus ζ_1 is two less in rank than Z. The results are

$$Z_a = j\frac{1}{\sqrt{2}}, \quad \zeta_1 = \frac{1-s}{4+2s}, \quad V_1 = \frac{1+\frac{s}{2}}{1-s}$$

Because a is not positive real, ζ_1 and V_1 are not prf. Therefore no syntheses corresponding to Figs. (3-2) and (3-3) are shown. However, a synthesis corresponding to Fig. (3-8) is possible if $b = -j\sqrt{2}$. Choosing this value for b makes Ev ζ_{1b} = Ev $Z_b = 0$ which makes ζ_2 four less in rank than Z. Carrying out the calculations gives

$$\zeta_{1b} = -\zeta_{1a} = \frac{j\sqrt{2}}{4}, \quad \zeta_2 = \frac{1}{4}$$

$$V = \frac{2s^2 + \frac{s}{2} + 2}{\frac{s^2}{2} + 2s + 2}, \quad \text{num Ev } V = (s^2 + 2)^2$$

The synthesis of Z is given in Fig. (3-12) and is identical with that which results if the Brune synthesis procedure is used to synthesize Z.

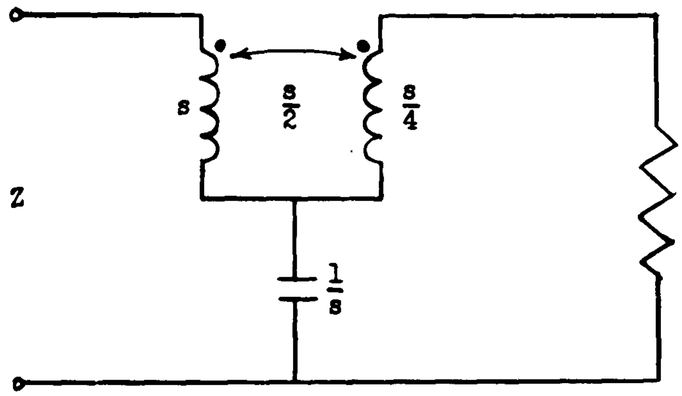


Fig. (3-12) Gyrator Elimination, b = - a, $EvZ_a = 0$

There is no loss in generality from having considered a rank 4 minimum resistance impedance in this example. For the case of a non-minimum resistance rank 4 impedance, the minimum resistance may be extracted from Z initially. For impedances of higher rank, the procedures are the same but ζ_1 and ζ_2 are correspondingly higher in rank.

3.7 Rank 4 Operator n-Type Realization

The procedures developed in Sections 3.2 through 3.6 do not consider the realization of complex even part zeros. The synthesis of the rank 4 operator in eq. (3-13), in which a and b are generally complex conjugates with a non-negative real part, fulfills this need. The two Darlington procedures are applied to eq. (3-13) resulting in additional network sections which may always be removed from Z. These appear in Fig. (3-1f) through (3-1i).

The n-type Darlington synthesis of eq. (3-13) is obtained by putting eq. (3-14) in the form

$$\text{num Ev } V = (s^2 + ab)^2 - s^2(a + b)^2 \qquad (3\text{-}32)$$

and employing eq. (2-8). The results are

$$V_{\substack{12\\21}} = \frac{(s^2 + ab)}{Ds} \pm \frac{a + b}{D}$$

$$V_{11} = \frac{s^2 \frac{Z_a}{\zeta_{1b}} + ab}{Ds}, \quad V_{22} = \frac{s^2 \frac{\zeta_{1b}}{Z_a} + ab}{Ds} \qquad (3\text{-}33)$$

$$D = \frac{b}{Z_a} + \frac{a}{\zeta_{1b}}$$

The removed section appears in Fig. (3-1f) and the complete network in Fig. (3-13). The two constants a and b are still completely arbitrary.

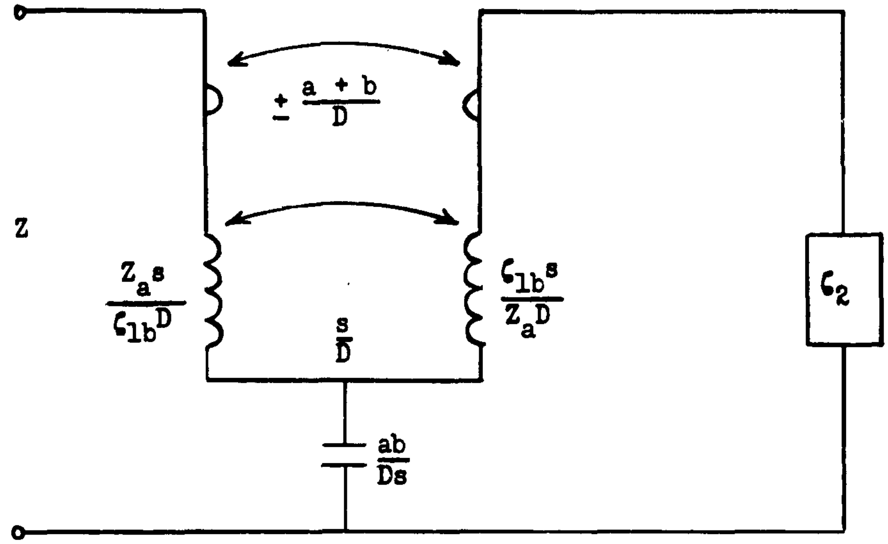


Fig. (3-13) n-Type Rank 4

It is often possible to simplify the network of Fig. (3-13) by replacing the transformer by an inductor. This requires that $Z_a = \zeta_{1b}$ in eq. (3-33) which is equivalent to requiring that the operators V_1 and V_2 be commutative. This condition is investigated in Appendix III.

3.8 Rank 4 Operator m-Type Realization

The m-type Darlington synthesis of eq. (3-13) may be obtained directly using the negative of eq. (3-32) in conjunction with eq. (2-9). However it is possible to eliminate an element if the terminating impedance is first scaled. Let Z in eq. (3-2) be rewritten as

$$Z = \frac{k^2\left(s^2 \frac{Z_a}{\zeta_{1b}} + ab\right)\frac{\zeta_2}{k^2} + s\left(bZ_a + a\zeta_{1b}\right)}{s^2 \frac{\zeta_{1b}}{Z_a} + sk^2\left(\frac{b}{Z_a} + \frac{a}{\zeta_{1b}}\right)\frac{\zeta_2}{k^2}} \qquad (3\text{-}34)$$

where k is an arbitrary positive real constant and the termination is now $\frac{\zeta_2}{k^2}$. The associated V operator is

$$V = \frac{s^2 k^2 \frac{Z_a}{\zeta_{1b}} + s(bZ_a + a\zeta_{1b}) + k^2 ab}{s^2 \frac{\zeta_{1b}}{Z_a} + sk^2\left(\frac{b}{Z_a} + \frac{a}{\zeta_{1b}}\right) + ab} \qquad (3\text{-}35)$$

and

$$\text{num Ev } V = k^2(s^2 + ab)^2 - k^2 s^2 (a + b)^2 \qquad (3\text{-}36)$$

The m-type Z parameters for eq. (3-35) are

$$V_{11} = \frac{\frac{Z_a}{\zeta_{1b}}(bZ_a + a\zeta_{1b})s}{s^2 + ab\frac{Z_a}{\zeta_{1b}}}$$

$$V_{22} = \frac{k^2 \frac{Z_a}{\zeta_{1b}}\left(\frac{b}{Z_a} + \frac{a}{\zeta_{1b}}\right)s}{s^2 + ab\frac{Z_a}{\zeta_{1b}}} \qquad (3\text{-}37)$$

$$V_{\substack{12\\21}} = \frac{k\frac{Z_a}{\zeta_{1b}}(a + b)s \pm k\frac{Z_a}{\zeta_{1b}}(s^2 + ab)}{s^2 + ab\frac{Z_a}{\zeta_{1b}}}$$

The $V_{\substack{12\\21}}$ expression in eq. (3-37) may be rewritten in two ways.

$$V_{\substack{12\\21}} = \frac{k\frac{Z_a}{\zeta_{1b}}(a + b)s \pm k\, s^2\left(\frac{Z_a}{\zeta_{1b}} - 1\right)}{s^2 + ab\frac{Z_a}{\zeta_{1b}}} \pm k \qquad (3\text{-}38)$$

$$V_{\substack{12\\21}} = \frac{k\frac{Z_a}{\zeta_{1b}}(a+b)s \pm k\,ab\frac{Z_a}{\zeta_{1b}}\left(1-\frac{Z_a}{\zeta_{1b}}\right)}{s^2 + ab\frac{Z_a}{\zeta_{1b}}} \pm k\frac{Z_a}{\zeta_{1b}} \qquad (3\text{-}39)$$

The similarity of eqs. (3-37), (3-38) and (3-39) with eqs. (1-34) and (1-36) should be noted. By direct substitution of eqs. (3-37) and (3-38) or (3-39), the extended residue condition of either eq. (1-25) or (1-27) may be shown to hold with the equal sign. Therefore Z in eq. (3-34) may be realized by networks having the form of Figs. (1-2) and (1-3) except that the termination is k^2/ζ_2 instead of one ohm and an extra gyrator is required to take care of the last term in eqs. (3-38) and (3-39) (see Fig. 1-7).

Eqs. (3-37) and (3-38) are identified with eq. (1-34) to yield

$$\begin{aligned} K &= k\left(\frac{Z_a}{\zeta_{1b}} - 1\right) \\ M &= \frac{k(a+b)}{ab} \\ L_{11} &= \frac{bZ_a + a\zeta_{1b}}{ab} \\ L_{22} &= \frac{k^2\left(\frac{b}{Z_a} + \frac{a}{\zeta_{1b}}\right)}{ab} \end{aligned} \qquad (3\text{-}40)$$

Similarly, eqs. (3-37) and (3-39) may be identified with eq. (1-36) to yield

$$K = k\left(1 - \frac{Z_a}{\zeta_{1b}}\right)$$

$$\frac{1}{C_m} = k \frac{Z_a}{\zeta_{1b}} (a + b)$$

$$\frac{1}{C_{11}} = \frac{Z_a}{\zeta_{1b}}\left(bZ_a + a\zeta_{1b}\right) \qquad (3\text{-}41)$$

$$\frac{1}{C_{22}} = k^2 \frac{Z_a}{\zeta_{1b}}\left(\frac{b}{Z_a} + \frac{a}{\zeta_{1b}}\right)$$

Each of the transformers represented by eqs. (3-40) and (3-41) has a T equivalent. For the inductive transformer of eq. (3-40) the T equivalent values are

$$L_1 = L_{11} - M$$
$$L_2 = L_{22} - M \qquad (3\text{-}42)$$
$$L_3 = M$$

while for the capacitive transformer of eq. (3-41), they are

$$\frac{1}{C_1} = \frac{1}{C_{11}} - \frac{1}{C_m}$$

$$\frac{1}{C_2} = \frac{1}{C_{22}} - \frac{1}{C_m} \qquad (3\text{-}43)$$

$$\frac{1}{C_3} = \frac{1}{C_m}$$

It is now shown that the scaling constant k may always be chosen to make either L_1 or L_2 zero in eq. (3-42) or $\frac{1}{C_1}$ or $\frac{1}{C_2}$ zero in eq. (3-43) while insuring that the remaining transformer elements are positive. Consider the inductive transformer case first. To make $L_1 = 0$ requires that

$$k = \frac{bZ_a + a\zeta_{1b}}{a + b} \tag{3-44}$$

The quantities $bZ_a + a\zeta_{1b}$ and $a + b$ are positive real. Hence a positive real k can always be found to satisfy eq. (3-44). The extended residue condition holds for all positive real k.

$$L_{11} L_{22} - M^2 > 0 \quad \text{for all} \quad +k \tag{3-45}$$

With $L_1 = 0$, eq. (3-45) becomes

$$M(L_2 - M) > 0 \tag{3-46}$$

Since M is always positive from eq. (3-40), L_2 must be positive.

Similarly k may be chosen to make L_2 zero in which case L_1 is always positive. The required value of k is

$$k = \frac{a + b}{\frac{b}{Z_a} + \frac{a}{\zeta_{1b}}} \tag{3-47}$$

A completely parallel development for the capacitive transformers yields eq. (3-44) to make $1/C_1$ zero and eq. (3-47) to make $1/C_2$ zero.

The syntheses of Z with k satisfying eq. (3-44) appear in Figs. (3-14) and (3-15). The constants a and b in these networks

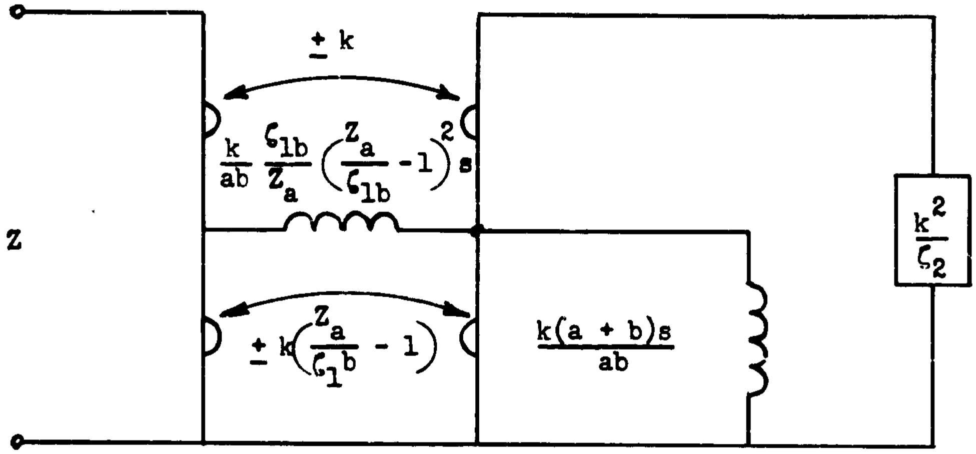


Fig. (3-14) m-Type Rank 4 Inductive

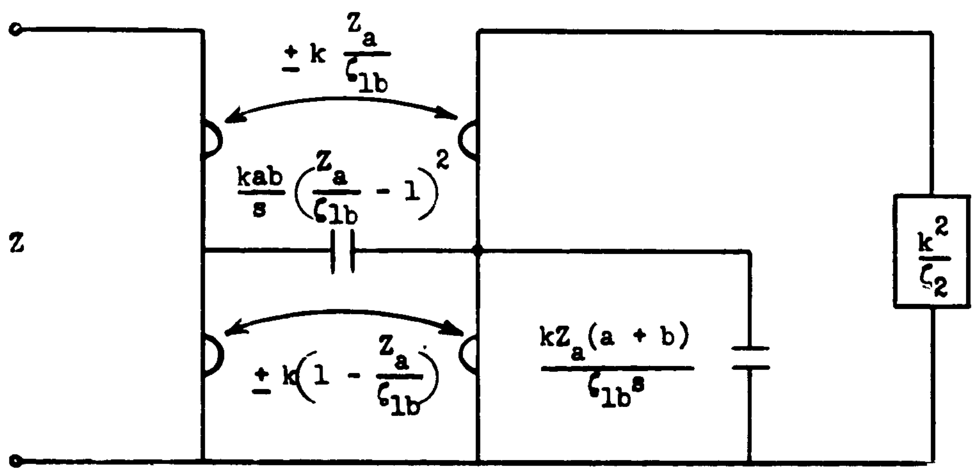


Fig. (3-15) m-Type Rank 4 Capacitive

are still completely arbitrary. The network sections in Figs. (3-14) and (3-15) may always be removed from Z and thus appear in Figs. (3-1g) and (3-1h). Note that only four elements are present in each of these removed sections, whereas five elements are necessary in the n-type section of Fig. (3-1f).

A further simplification may often be made in the m-type synthesis of V in eq. (3-13). Consider the $V_{\substack{12\\21}}$ expression in eq. (3-37). If $\zeta_1 b = Z_a$, the entire non-reciprocal term reduces to a non-loaded gyrator. This is discussed in Appendix III.

3.9 Eliminating the Gyrator from a Rank 4 Operator

Following the pattern of Section 3.4, a means of eliminating the gyrators in Figs. (3-13), (3-14) and (3-15) is now derived. The result is the network section of Fig. (3-1i), a purely reactive section with two arbitrary constants. The gyrator elimination is accomplished through the following transformations.

Let

$$\zeta_2 = \frac{c\zeta_3 + s\zeta_{2c}}{c + \frac{s}{\zeta_{2c}}\zeta_3} \qquad (3\text{-}48)$$

$$\zeta_3 = \frac{d\zeta_4 + s\zeta_{3d}}{d + \frac{s}{\zeta_{3d}}\zeta_4} \qquad (3\text{-}49)$$

Substituting eq. (3-49) into eq. (3-48) gives

$$\zeta_2 = \frac{\left(cd + s^2\frac{\zeta_{2c}}{\zeta_{3d}}\right)\zeta_4 + s(d\zeta_{2c} + c\zeta_{3d})}{\left(cd + s^2\frac{\zeta_{3d}}{\zeta_{2c}}\right) + s\left(\frac{d}{\zeta_{2c}} + \frac{c}{\zeta_{3d}}\right)\zeta_4} \qquad (3\text{-}50)$$

Eq. (3-50) should be compared with eq. (3-12). ζ_4 is prf if ζ_2 is prf (and thus Z prf) and if c and d are either positive real constants or complex conjugates with a non-negative real part.

It is now necessary to substitute eq. (3-50) into eq. (3-12) in order to express Z in terms of a new V operator (of rank 8) operating on ζ_4. The new V has four arbitrary constants. Proper choice of two of these permits a synthesis of the new V without a gyrator. The results of the substitution are

$$Z = \frac{(A_o + A_2 s^2 + A_4 s^4)\zeta_4 + s(A_1 + A_3 s^2)}{(B_o + B_2 s^2 + B_4 s^4) + s(B_1 + B_3 s^2)\zeta_4} \qquad (3\text{-}51)$$

where

$$\begin{aligned}
A_o &= abcd = B_o \\
A_1 &= ab(d\zeta_{2c} + c\zeta_{3d}) + cd(bZ_a + a\zeta_{1b}) \\
B_1 &= ab\left(\frac{d}{\zeta_{2c}} + \frac{c}{\zeta_{3d}}\right) + cd\left(\frac{b}{Z_a} + \frac{a}{\zeta_{1b}}\right) \\
A_2 &= ab\,\frac{\zeta_{2c}}{\zeta_{3d}} + cd\,\frac{Z_a}{\zeta_{1b}} + (bZ_a + a\zeta_{1b})\left(\frac{d}{\zeta_{2c}} + \frac{c}{\zeta_{3d}}\right) \\
B_2 &= ab\,\frac{\zeta_{3d}}{\zeta_{2c}} + cd\,\frac{\zeta_{1b}}{Z_a} + \left(\frac{b}{Z_a} + \frac{a}{\zeta_{1b}}\right)(d\zeta_{2c} + c\zeta_{3d}) \qquad (3\text{-}52) \\
A_3 &= \frac{Z_a}{\zeta_{1b}}(d\zeta_{2c} + c\zeta_{3d}) + \frac{\zeta_{3d}}{\zeta_{2c}}(bZ_a + a\zeta_{1b}) \\
B_3 &= \frac{\zeta_{1b}}{Z_a}\left(\frac{d}{\zeta_{2c}} + \frac{c}{\zeta_{3d}}\right) + \frac{\zeta_{2c}}{\zeta_{3d}}\left(\frac{b}{Z_a} + \frac{a}{\zeta_{1b}}\right) \\
A_4 &= \frac{Z_a\zeta_{2c}}{\zeta_{1b}\zeta_{3d}} = \frac{1}{B_4}
\end{aligned}$$

The associated V operator is

$$V = \frac{A_o + A_1 s + A_2 s^2 + A_3 s^3 + A_4 s^4}{B_o + B_1 s + B_2 s^2 + B_3 s^3 + B_4 s^4} \qquad (3\text{-}53)$$

Its even part numerator may be easily formulated using eq. (2-15).

$$\begin{aligned} \text{num Ev } V &= (a^2 - s^2)(b^2 - s^2)(c^2 - s^2)(d^2 - s^2) \\ &= \left[a^2b^2 - (a^2 + b^2)s^2 + s^4\right]\left[c^2d^2 - (c^2 + d^2)s^2 + s^4\right] \end{aligned} \qquad (3\text{-}54)$$

To synthesize V without a gyrator, it is necessary that eq. (3-54) be a perfect square. This requirement is satisfied if*

$$c = \pm a, \quad d = \pm b \qquad (3\text{-}55)$$

which causes eq. (3-54) to become

$$\text{num Ev } V = \left[a^2b^2 - (a^2 + b^2)s^2 + s^4\right]^2 \qquad (3\text{-}56)$$

V in eq. (3-53) may now be synthesized by the n-type Darlington procedure to yield the section of Fig. (3-11) and the network of Fig. (3-16).

---

*The minus signs are applicable only if a and b are pure imaginaries since c and d may not have negative real parts. If a and b are imaginary, then b = - a and the conditions of eqs. (3-18) through (3-21) apply. Thus either the synthesis is trivial or the gyrator has already been removed. Therefore only the plus signs in eq. (3-55) are of importance.

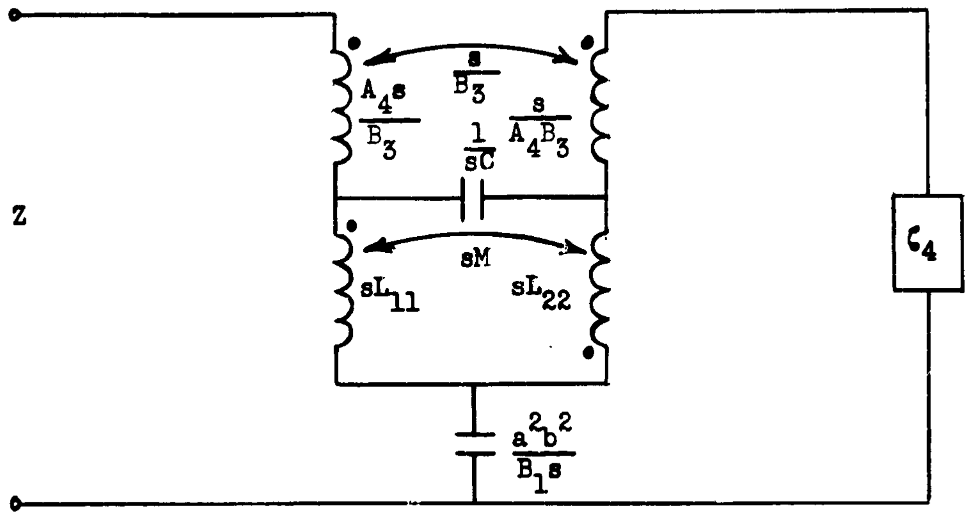


Fig. 3-16 Gyrator Elimination for a Rank 4 V

The components of V are obtained from

$$V_{11} = \frac{A_4 s^4 + A_2 s^2 + a^2 b^2}{B_3 s\left(s^2 + \frac{B_1}{B_3}\right)}$$

$$V_{22} = \frac{B_4 s^4 + B_2 s^2 + a^2 b^2}{B_3 s\left(s^2 + \frac{B_1}{B_3}\right)} \tag{3-57}$$

$$V_{12} = V_{21} = \frac{s^4 - (a^2 + b^2) + a^2 b^2}{B_3 s\left(s^2 + \frac{B_1}{B_3}\right)}$$

Performing partial fraction expansions, eq. (3-57) becomes

$$V_{11} = \frac{a^2 b^2}{B_1 s} + \frac{A_4 s}{B_3} + \frac{\left(\frac{A_2}{B_3} - \frac{a^2 b^2}{B_1} - \frac{A_4 B_1}{B_3^2}\right) s}{s^2 + \frac{B_1}{B_3}} \tag{3-58}$$

$$V_{22} = \frac{a^2b^2}{B_1 s} + \frac{B_4 s}{B_3} + \frac{\left(\frac{B_2}{B_3} - \frac{a^2b^2}{B_1} - \frac{B_4B_1}{B_3^2}\right)}{}$$

$$V_{12} = V_{21} = \frac{a^2b^2}{B_1 s} + \frac{s}{B_3} + \frac{\left(-\frac{a^2+b^2}{B_3} - \frac{a^2b^2}{B_1} - \frac{B_1}{B_3^2}\right)}{s^2 + \frac{B_1}{B_3}} \tag{3-58}$$

The first terms yield a capacitor. The second terms give an ideal subtractive transformer. The third terms yield a capacitive-loaded ideal additive transformer. This last statement may be verified by considering the network in Fig. (3-17) and its Z parameters in eq. (3-59).

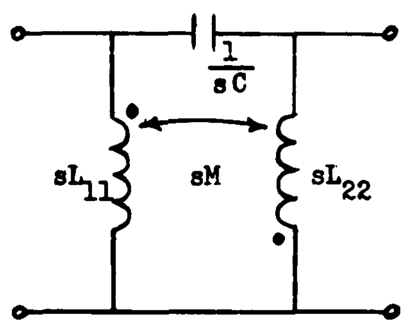


Fig. (3-17) Capacitive Loaded Transformer Network

$$Z_{11} = \frac{s\omega_o^2 L_{11}}{s^2 + \omega_o^2}$$

$$Z_{22} = \frac{s\omega_o^2 L_{22}}{s^2 + \omega_o^2}$$

$$Z_{12} = Z_{21} = \frac{s\omega_o^2 M}{s^2 + \omega_o^2} \tag{3-59}$$

$$\omega_o^2 = \frac{1}{C(L_{11} + L_{22} + 2M)}$$

Matching terms between eq. (3-59) and the last terms in eq. (3-58) gives

$$L_{11} = \frac{A_2}{B_1} - \frac{a^2b^2B_3}{B_1^2} - \frac{A_4}{B_3}$$

$$L_{22} = \frac{B_2}{B_1} - \frac{a^2b^2B_3}{B_1^2} - \frac{B_4}{B_3} \tag{3-60}$$

$$M = -\frac{a^2+b^2}{B_1} - \frac{a^2b^2B_3}{B_1^2} - \frac{1}{B_3}$$

$$\frac{1}{C} = \frac{B_1}{B_3}(L_{11} + L_{22} + 2M)$$

The network sections of Figs. (3-13) through (3-16) each have a quadruplet of even part zeros. The section of Fig. (3-16) is identical in form to the Darlington D section but again there are two differences. First, realization of the D section requires that num EvZ have a term $(s^4 + As^2 + B)^2$, which often necessitates the use of surplus factors. These surplus factors are effectively built into the impedance operator. Secondly, the section of Fig. (3-16) contains two arbitrary constants also contained in ζ_4. If these constants are chosen so that Ev Z_a = Ev Z_b = 0, then, if these complex zeros of Ev Z are of second order, ζ_4 is four less in rank than Z.

3.10 A General Cascade Reciprocal Synthesis Procedure*

*The specific procedure is new but the philosophy behind it is that of Guillemin as discussed in Section 1.9.

The combination of the principle of zero cancellation synthesis with the syntheses developed in Sections (3-4) and (3-9) results in a general cascade reciprocal synthesis procedure applicable to any prf driving point impedance. The overall procedure has been outlined in Section 1.9. The network sections of Figs. (3-7), (3-8) and (3-16) are used to realize real, imaginary and complex zeros, respectively, of num Ev Z. In each case the arbitrary constants are chosen to make Ev Z_a = Ev Z_b = 0 in order to reduce the rank of the terminating impedance. The procedure is illustrated by the following examples.

3.11 Rank 4 Operator Examples

The principles developed in Sections 3.7 through 3.10 are illustrated by considering the following driving point impedance function.

$$Z = \frac{s^3 + \frac{14}{3}s^2 + 2s + 4}{s^3 + 4s^2 + \frac{44}{3}s + 2}$$

where

$$\text{num Ev } Z = (2 - s^2)(s^2 + 2s + 2)(s^2 - 2s + 2)$$

Solution A

Zero cancellation synthesis may be used in conjunction with the rank 4 V operator to synthesize Z. a and b are chosen to make Ev Z_a = Ev Z_b = 0 and thus ζ_2 is reduced in rank by four. The calculations yield

$$a = 1 - j \quad , \quad b = 1 + j$$

$$Z_a = \frac{8 - 2j}{17} , \; Z_b = \frac{8 + 2j}{17}$$

Then, from eq. (3-1),

$$\zeta_{1b} = \frac{3}{5} Z_a$$

From eqs. (3-13) and (3-32),

$$V = \frac{\frac{5}{3} s^2 + \frac{4}{5} s + 2}{\frac{3}{5} s^2 + \frac{17}{3} s + 2}$$

$$\text{num Ev } V = (s^2 + 2)^2 - 4s^2$$

and using eq. (3-12),

$$\zeta_2 = \frac{\frac{3}{5} s + 2}{\frac{5}{3} s + 1} , \quad \text{num Ev } \zeta_2 = 2 - s^2$$

The n-type synthesis of V is obtained by means of eq. (3-33). The network appears in Fig. (3-18). Note that the removed section realizes the quadruplet of even part complex zeros and that V_{12} contains the left-half plane pair and V_{21} the right-half plane pair.

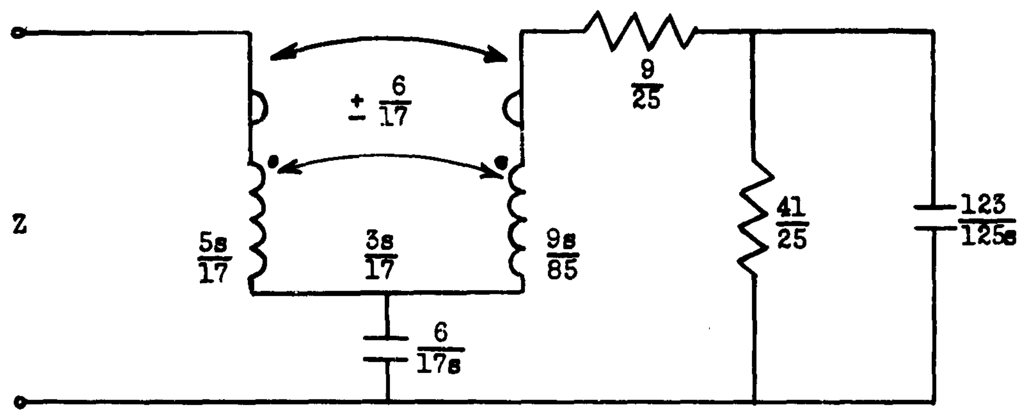


Fig. (3-18) n-Type Rank 4

To obtain the m-type networks of Figs. (3-14) and (3-15), k must be chosen to satisfy eq. (3-44). The required value is

$$k = \frac{2}{5}$$

Then, from eqs. (3-35) and (3-36),

$$V = \frac{\frac{4}{15} s^2 + \frac{4}{5} s + \frac{8}{25}}{\frac{3}{5} s^2 + \frac{68}{75} s + 2}$$

$$\text{num Ev } V = \frac{4}{25} (s^2 + 2)^2 - \frac{16}{25} s^2$$

and the termination is

$$\frac{k^2}{\zeta_2} = \frac{4}{25}\left(\frac{\frac{5}{3}s + 1}{\frac{5}{3}s + 2}\right)$$

V may now be synthesized using eqs. (3-40) and (3-42) or eqs. (3-41) and (3-43). The networks appear in Figs. (3-19) and (3-20), respectively. Once again the removed section realizes the quadruplet of complex even part zeros and the set is split between V_{12} and V_{21} as in the n-type case.

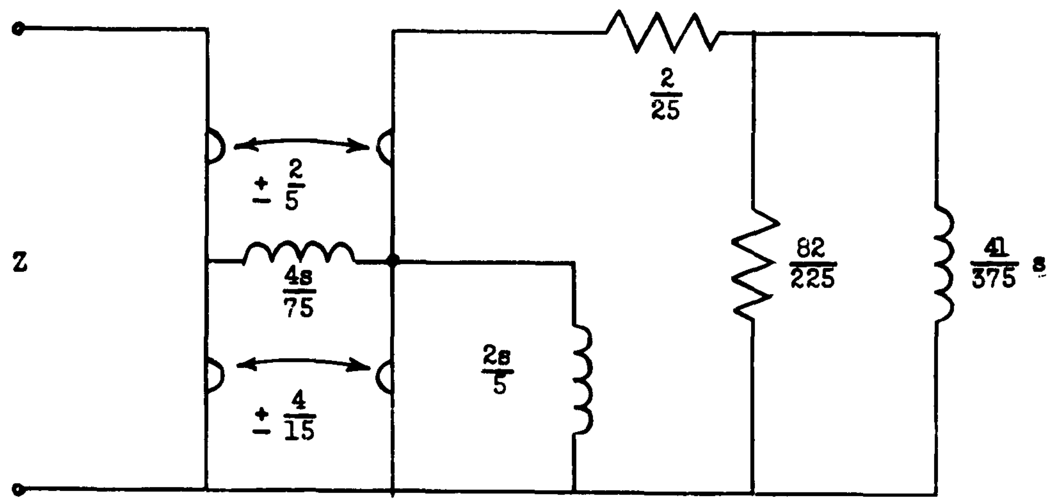


Fig. (3-19) m-Type Rank 4 Inductive

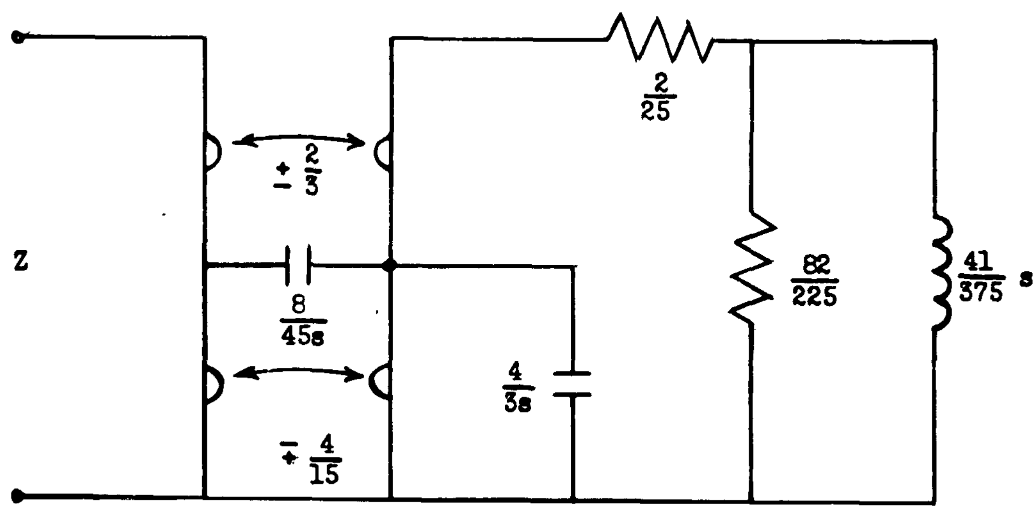


Fig. (3-20) m-Type Rank 4 Capacitive

Solution B

Zero cancellation synthesis may also be used in conjunction with the Rank 8 V operator to synthesize Z. a and b are chosen as before to make the ζ_4 termination four less in rank than Z, and c and d are chosen equal to a and b, respectively, to synthesize without a gyrator. The results are

$$\zeta_{2a} = \frac{357 + j123}{445}, \quad \zeta_{2b} = \frac{357 - j123}{445}$$

$$\frac{\zeta_{3d}}{\zeta_{2c}} = \frac{80}{39}$$

Then, from eqs. (3-52), (3-53) and (3-54)

$$V = \frac{\frac{13}{16} s^4 + \frac{242}{39} s^3 + \frac{139}{24} s^2 + \frac{92}{13} s + 4}{\frac{16}{13} s^4 + \frac{31}{8} s^3 + \frac{812}{39} s^2 + \frac{361}{24} s + 4}$$

$$\text{num Ev } V = (s^4 + 4)^2$$

and from eq. (3-50)

$$\zeta_4 = \frac{\frac{16}{13} s + 2}{\frac{13}{16} s + 1}, \quad \text{num Ev } \zeta_4 = 2 - s^2$$

The components of Fig. (3-16) may be computed from eqs. (3-58) and (3-60).

$$V_{11} = \frac{96}{361s} + \frac{13s}{62} + \frac{.415s}{s^2 + \frac{361}{93}}$$

$$V_{22} = \frac{96}{361s} + \frac{128s}{403} + \frac{3.88s}{s^2 + \frac{361}{93}}$$

$$V_{12} = V_{21} = \frac{96}{361s} + \frac{8s}{31} - \frac{1.27s}{s^2 + \frac{361}{93}}$$

The network appears in Fig. (3-21). Note that the removed section is reactive and reciprocal and that the termination is four less in rank than Z, as anticipated. Once again the quadruplet of even part zeros is realized by the removed section but in this case $V_{12} = V_{21}$ so that V_{12} contains the entire quadruplet instead of just the left-half plane pair.

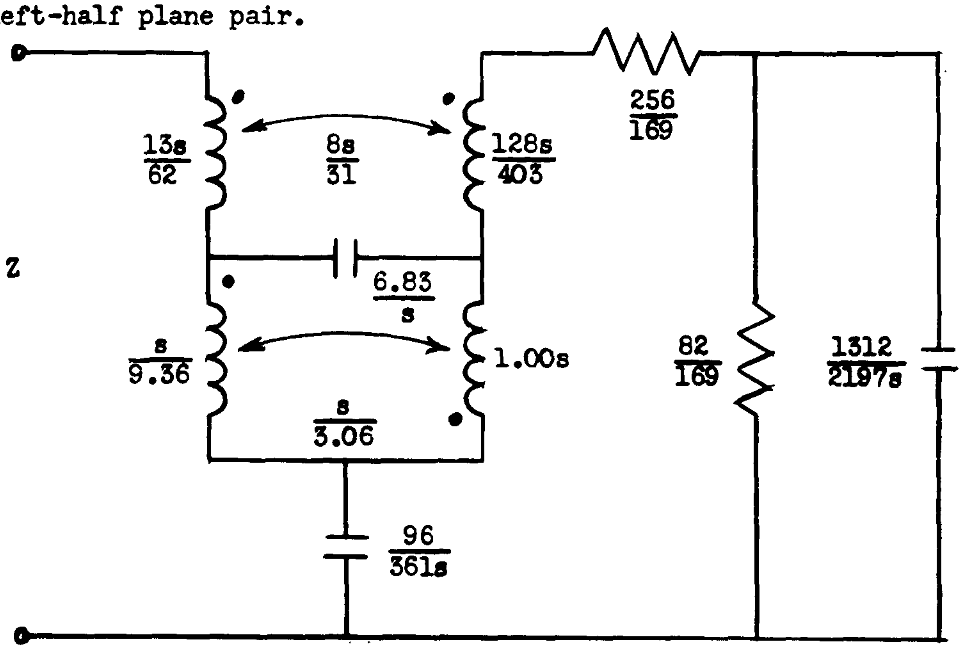


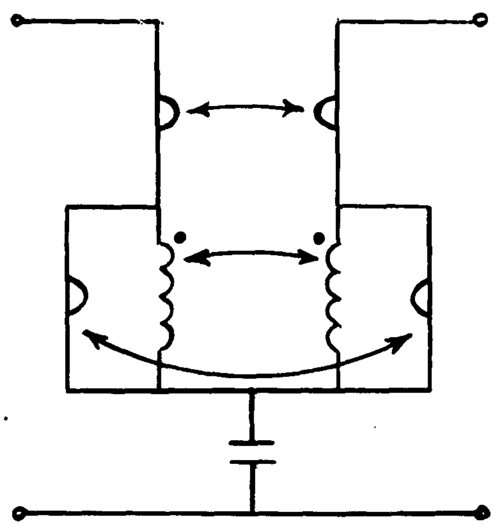
Fig. (3-21) Gyrator Elimination, c = a and d = b

C H A P T E R IV

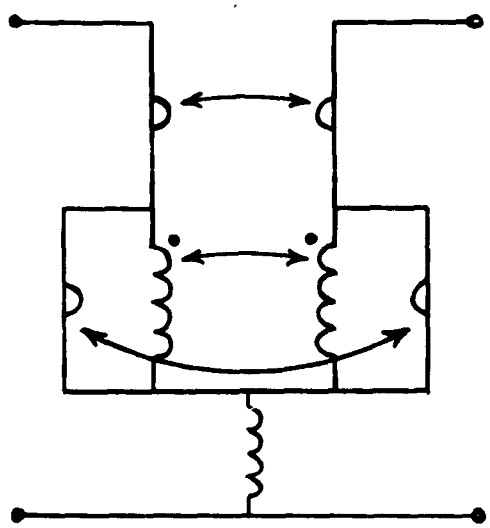
CASCADE SYNTHESIS USING AN IMPEDANCE OPERATOR OF RANK 6

4.1 Introduction

In this chapter, the impedance operator of rank 6 is discussed. Four additional network sections are derived (Fig. 4-1) which may always be removed from an RLC driving point impedance function. Each section contains three constants also contained in the terminating impedance. In Figs. (4-1a) and (4-1b), one constant must be positive real while the others may be complex conjugates with non-negative real parts. In Figs. (4-1c) and (4-1d), all three constants must, in general, be positive real with two of them equal. As in the previous chapter, the constants may be chosen to produce desired characteristics in either the removed sections or the terminating impedance.



(a)



(b)

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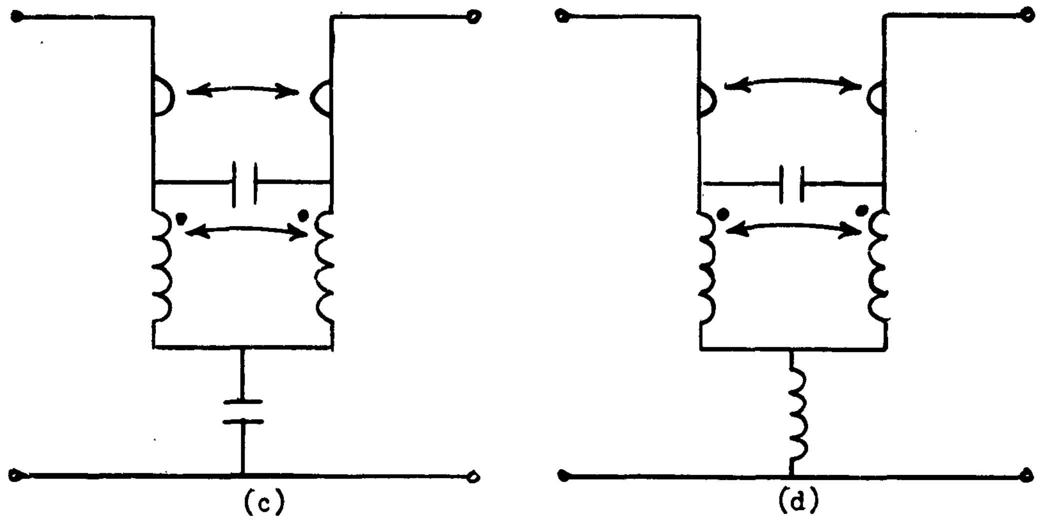

Fig. (4-1) Additional Sections which may Always be Removed from Z

4.2 Rank 6 Operator Formulation

Let Z, ζ_1 and ζ_2 be represented by eqs. (2-18), (2-19) and (3-49), respectively. These equations are repeated below.

$$Z = \frac{a\zeta_1 + sZ_a}{a + \frac{s}{Z_a}\zeta_1} \tag{4-1}$$

$$\zeta_1 = \frac{b\zeta_2 + s\zeta_{1b}}{b + \frac{s}{\zeta_{1b}}\zeta_2} \tag{4-2}$$

$$\zeta_2 = \frac{c\zeta_3 + s\zeta_{2c}}{c + \frac{s}{\zeta_{2c}}\zeta_3} \tag{4-3}$$

By combining these equations, Z may be expressed in terms of ζ_3.

$$Z = \frac{(C_o + C_2 s^2)\zeta_3 + s(C_1 + C_3 s^2)}{(D_o + D_2 s^2) + s(D_1 + D_3 s^2)\zeta_3} \tag{4-4}$$

where

$$C_o = abc = D_o$$

$$C_1 = ab\zeta_{2c} + bcZ_a + ca\zeta_{1b}$$

$$D_1 = \frac{ab}{\zeta_{2c}} + \frac{bc}{Z_a} + \frac{ca}{\zeta_{1b}}$$

$$C_2 = a\,\frac{\zeta_{1b}}{\zeta_{2c}} + b\,\frac{Z_a}{\zeta_{2c}} + c\,\frac{Z_a}{\zeta_{1b}} \tag{4-5}$$

$$D_2 = a\,\frac{\zeta_{2c}}{\zeta_{1b}} + b\,\frac{\zeta_{2c}}{Z_a} + c\,\frac{\zeta_{1b}}{Z_a}$$

$$C_3 = \frac{Z_a\zeta_{2c}}{\zeta_{1b}} = \frac{1}{D_3}$$

The associated V operator is

$$V = \frac{C_o + C_1 s + C_2 s^2 + C_3 s^3}{D_o + D_1 s + D_2 s^2 + D_3 s^3} \tag{4-6}$$

Its even part numerator, from eq. (2-15), is

$$\text{num Ev } V = (a^2 - s^2)(b^2 - s^2)(c^2 - s^2)$$

$$= \left[(a + b + c)s^2 + abc\right]^2 - s^2\left[s^2 + ab + bc + ca\right]^2 \tag{4-7}$$

In each of the equations (4-1), (4-2) and (4-3), the impedance on the right (ζ_1, ζ_2 or ζ_3) is prf if the impedance on the left (Z, ζ_1 or ζ_2) is prf and if a, b and c are real. Also ζ_2 is prf if a and b are complex conjugates with a non-negative real part and Z is prf. It follows that ζ_3 is prf if c is positive real. Similarly, if a is positive real, ζ_1 is prf. Then, if b and c are complex conjugates with a non-negative real part, ζ_3 is prf.

Theorem K

In eq. (4-4), ζ_3 and V are prf if b is positive real and a and c are complex conjugates with a non-negative real part.

To verify Theorem K, it is sufficient to show that the coefficients of the V operator in eq. (4-6) are unchanged by a permutation of the three constants a, b and c.* Then, letting a —> b, b —> c and c —> a does not change V and thus does not change ζ_3. For example, a = 1, b = 1 + j and c = 1 - j is a permissable set of constants to insure that V and ζ_3 are prf if Z is prf. It follows that a = 1 - j, b = 1 and c = 1 + j is also a permissable set.

4.3 Rank 6 Operator Synthesis

The general n - and m-type Darlington syntheses of eq. (4-6) are now derived. The procedure is much the same as that used in

*The invariance of V under a permutation of the three constants is proved in Appendix IV.

connection with the general rank 6 impedance in Section 1.7 except that the termination is no longer one ohm.

For the n-type synthesis,

$$V_{11} = \frac{C_2 s^2 + abc}{s(D_3 s^2 + D_1)} = \frac{abc}{D_1 s} + \frac{\left(\frac{C_2}{D_3} - \frac{abc}{D_1}\right) s}{s^2 + \frac{D_1}{D_3}}$$

$$V_{22} = \frac{D_2 s^2 + abc}{s(D_3 s^2 + D_1)} = \frac{abc}{D_1 s} + \frac{\left(\frac{D_2}{D_3} - \frac{abc}{D_1}\right) s}{s^2 + \frac{D_1}{D_3}} \qquad (4\text{-}8)$$

$$V_{\substack{12\\21}} = \frac{(a + b + c)s^2 + abc \pm s(s^2 + ab + bc + ca)}{s(D_3 s^2 + D_1)}$$

The equation for $V_{\substack{12\\21}}$ may be separated in two ways to give:

$$V_{\substack{12\\21}} = \frac{abc}{D_1 s} + \frac{\left(\frac{a + b + c}{D_3} - \frac{abc}{D_1}\right) s \pm \frac{s^2}{D_1}\left(\frac{D_1}{D_3} - ab - bc - ca\right)}{s^2 + \frac{D_1}{D_3}} \pm \frac{ab + bc + ca}{D_1} \qquad (4\text{-}9)$$

or

$$V_{\substack{12\\21}} = \frac{abc}{D_1 s} + \frac{\left(\frac{a + b + c}{D_3} - \frac{abc}{D_1}\right) s \pm \frac{1}{D_3}\left(ab + bc + ca - \frac{D_1}{D_3}\right)}{s^2 + \frac{D_1}{D_3}} \pm \frac{1}{D_3} \qquad (4\text{-}10)$$

As explained in Section 1.7, the first terms in V_{11}, V_{22} and $V_{\substack{12\\21}}$ form a capacitor. The last term in $V_{\substack{12\\21}}$ is a non-loaded gyrator. The residues of the last term in V_{11} and V_{22} along with those of the middle terms in $V_{\substack{12\\21}}$ satisfy the extended residue conditions of eqs. (1-25) and (1-27) with the equal sign, indicating the use of Figs. (1-2) and (1-3). The complete synthesis of Z, using the inductive form of Fig. (1-2) and eqs. (1-34), (4-8) and (4-9), appears in Fig. (4-2) where

$$
\begin{aligned}
K &= \frac{1}{D_3} - \frac{ab + bc + ca}{D_1} \\
L_{11} &= \frac{1}{D_1}\left(C_2 - abc\,\frac{D_3}{D_1}\right) \\
L_{22} &= \frac{1}{D_1}\left(D_2 - abc\,\frac{D_3}{D_1}\right) \\
M &= \frac{1}{D_1}\left(a + b + c - abc\,\frac{D_3}{D_1}\right)
\end{aligned}
\qquad (4\text{-}11)
$$

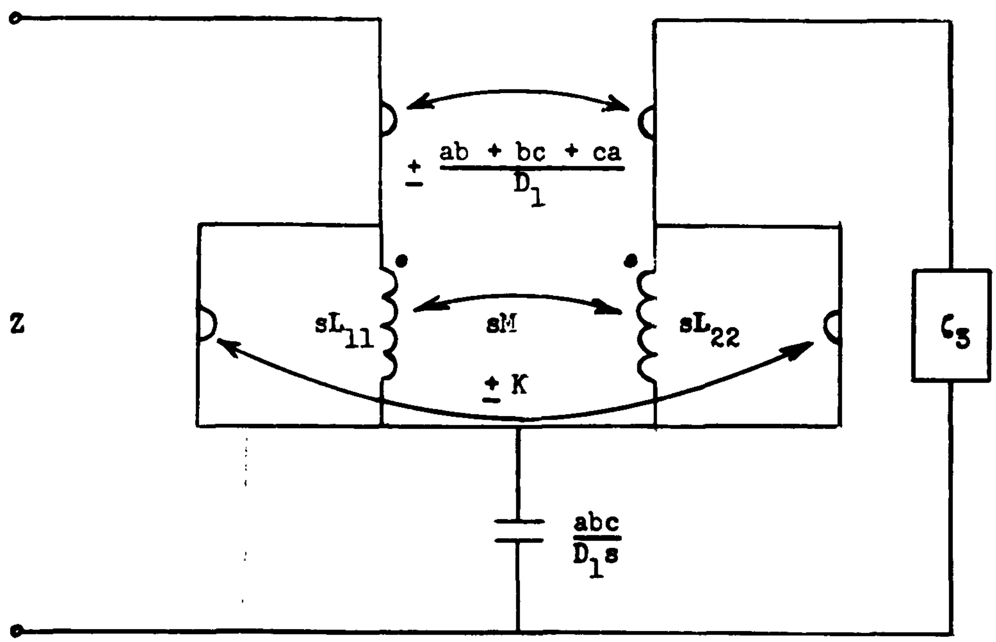

Fig. (4-2) n-Type Rank 6 Inductive

In order to avoid a second transformer in the m-type synthesis of V, the terminating impedance ζ_3 may be scaled. Let the new termination be ζ_3/k^2, where k is an arbitrary positive real constant. Then V in eq. (4-6) becomes

$$V = \frac{C_o k^2 + C_1 s + C_2 k^2 s^2 + C_3 s^3}{D_o + D_1 k^2 s + D_2 s^2 + D_3 k^2 s^3} \qquad (4\text{-}12)$$

and eq. (4-7) becomes

$$\text{num Ev } V = k^2\left[(a + b + c)s^2 + abc\right]^2 - k^2 s^2\left[s^2 + ab + bc + ca\right]^2 \qquad (4\text{-}13)$$

Then

$$V_{11} = \frac{C_3 s}{D_2} + \frac{\frac{1}{D_2}\left(C_1 - \frac{abc\ C_3}{D_2}\right)s}{s^2 + \frac{abc}{D_2}}$$

$$V_{22} = \frac{D_3 k^2 s}{D_2} + \frac{\frac{k^2}{D_2}\left(D_1 - \frac{abc\ D_3}{D_2}\right)s}{s^2 + \frac{abc}{D_2}} \qquad (4\text{-}14)$$

$$V_{\substack{12\\21}} = \frac{ks(s^2 + ab + bc + ca) \pm k\left[(a+b+c)s^2 + abc\right]}{D_2 s^2 + abc}$$

Rearranging the $V_{\substack{12\\21}}$ expression yields

$$V_{\substack{12\\21}} = \frac{ks}{D_2} + \frac{\frac{k}{D_2}\left(ab + bc + ca - \frac{abc}{D_2}\right)s \pm k\left(\frac{a + b + c}{D_2} - 1\right)s^2}{s^2 + \frac{abc}{D_2}} \pm k \qquad (4\text{-}15)$$

or

$$V_{\substack{12\\21}} = \frac{ks}{D_2} + \frac{\frac{k}{D_2}\left(ab + bc + ca - \frac{abc}{D_2}\right) s \pm \frac{abck}{D_2}\left(1 - \frac{a + b + c}{D_2}\right)}{s^2 + \frac{abc}{D_2}} \pm \frac{k(a+b+c)}{D_2}$$

(4-16)

If $k = C_3 = \frac{1}{D_3}$, the first terms in V_{11}, V_{22} and $V_{\substack{12\\21}}$ reduce to an inductor. The complete synthesis of Z, using the inductive form of Fig. (1-2) and eqs. (1-34) and (4-14) appears in Fig. (4-3), where in this case,

$$K = C_3\left(\frac{a + b + c}{D_2} - 1\right)$$

$$L_{11} = \frac{C_1}{abc} - \frac{C_3}{D_2}$$

$$L_{22} = C_3^2 \frac{D_1}{abc} - \frac{C_3}{D_2} \qquad (4\text{-}17)$$

$$M = C_3 \frac{ab + bc + ca}{abc} - \frac{C_3}{D_2}$$

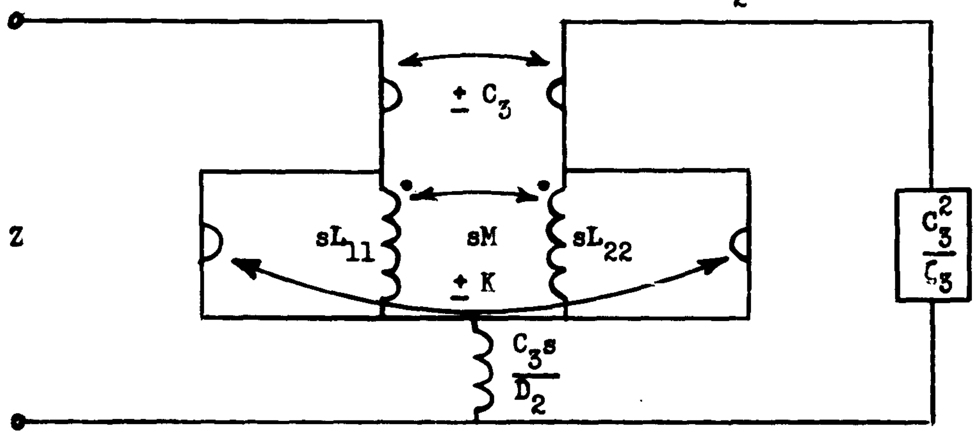

Fig. (4-3) m-Type Rank 6 Inductive

The syntheses appearing in Figs. (4-2) and (4-3) are perfectly general and the constants a, b and c are still arbitrary in both the removed network section and the termination ζ_3. Each of the sections of Figs. (4-2) and (4-3) can always be removed from Z and therefore they are included as the first two sections in Fig. (4-1). Also, each section realizes six even part zeros, two of which are real and the remaining four complex.

For impedances of rank 6, 10, 14 ...., the even part numerator of Z always contains at least one positive real root in addition to its pairs of complex conjugate roots. Hence, for such impedances, it is always possible to choose a, b and c such that Ev Z_a = Ev Z_b = Ev Z_c = 0. This synthesis procedure reduces the rank of ζ_3 by six. For impedances of rank 4, 8, 12 ...., there is no guarantee of a positive real even part zero and thus the reduction in rank of ζ_3 by six is not always possible. In these latter cases, the rank 4 V operator discussed in Chapter III may be employed and a reduction in rank by four obtained.

The arbitrary constants may also be utilized to simplify the networks of Figs. (4-2) and (4-3). This choice of the arbitrary constants is now investigated.

4.4 <u>Eliminating a Gyrator from Figures (4-2) and (4-3)</u>

In the discussion of the m-type Darlington synthesis of the rank 4 V operator, the terminating impedance was scaled to eliminate an element from the removed sections. In the case of the rank 6 V operator, scaling has already been employed to obtain the single capacitor in Fig. (4-2)* and the single inductor in Fig. (4-3). Thus further simplification of these networks requires a proper choice of one or more of the arbitrary constants.

A considerable simplification results if the non-reciprocal portions of the middle terms in the $V_{\substack{12\\21}}$ expressions of eqs. (4-9), (4-10), (4-15) and (4-16) can be made to vanish. To obtain this simplification requires that

$$\frac{D_1}{D_3} = \frac{Z_a \zeta_{2c}}{\zeta_{1b}}\left(\frac{ab}{\zeta_{2c}} + \frac{bc}{Z_a} + \frac{ca}{\zeta_{1b}}\right) = ab + bc + ca \qquad (4\text{-}18)$$

in eqs. (4-9) and (4-10) and that

$$D_2 = a\,\frac{\zeta_{2c}}{\zeta_{1b}} + b\,\frac{\zeta_{2c}}{Z_a} + c\,\frac{\zeta_{1b}}{Z_a} = a + b + c \qquad (4\text{-}19)$$

in eqs. (4-15) and (4-16). Eq. (4-18) applies to the n-type and eq. (4-19) to the m-type Darlington synthesis. Using eqs. (4-1) and (4-2), it is possible to express eqs. (4-18) and (4-19) in terms of a, b, c, Z_a, Z_b and Z_c. The result for eq. (4-18) is

$$ab + bc + ca = -\,\frac{(b^2 - c^2)a^3 Z_a + (c^2 - a^2)b^3 Z_b + (a^2 - b^2)c^3 Z_c}{(b^2 - c^2)aZ_a + (c^2 - a^2)bZ_b + (a^2 - b^2)cZ_c} \qquad (4\text{-}20)$$

*The scaling factor is unity for the n-type synthesis.

which reduces to

$$aZ_a(b - c) + bZ_b(c - a) + cZ_c(a - b) = 0 \qquad (4\text{-}21)$$

while the result for eq. (4-19) is

$$a + b + c = -\frac{(b^2 - c^2)bcZ_a + (c^2 - a^2)caZ_b + (a^2 - b^2)abZ_c}{(b^2 - c^2)aZ_a + (c^2 - a^2)bZ_b + (a^2 - b^2)cZ_c} \qquad (4\text{-}22)$$

which reduces to

$$Z_a(b - c) + Z_b(c - a) + Z_c(a - b) = 0 \qquad (4\text{-}23)$$

If any two of the arbitrary constants are chosen equal, eqs. (4-21) and (4-23) are satisfied. But if, for example, b is chosen equal to a, then a and b must be positive real and therefore c must be positive real, if V and ζ_3 are to be prf. Thus the arbitrariness of the constants is reduced.

The question arises as to whether a positive real c can be found such that eqs. (4-21) and (4-23) are satisfied with a and b remaining arbitrary. These constraints are investigated in detail in Appendix V, where it is shown that it is often, but not always, possible to choose such a value of c.

With the conditions of eqs. (4-21) and (4-23) satisfied, the $V_{\substack{12\\21}}$ expressions in eqs. (4-9) and (4-10) become

$$V_{\substack{12\\21}} = \frac{abc}{D_1 s} + \frac{\left(\frac{a+b+c}{D_3} - \frac{abc}{D_1}\right)s}{s^2 + \frac{D_1}{D_3}} \pm \frac{1}{D_3} \qquad (4\text{-}24)$$

while those in eqs. (4-15) and (4-16) become (with $k = C_3 = \frac{1}{D_3}$)

$$V_{\substack{12\\21}} = \frac{C_3 s}{D_2} + \frac{\frac{C_3}{D_2}\left(ab + bc + ca - \frac{abc}{D_2}\right)s}{s^2 + \frac{abc}{D_2}} \pm \frac{1}{D_3} \qquad (4\text{-}25)$$

The n- and m-type syntheses appear in Figs. (4-4) and (4-5), respectively, where in Fig. (4-4),*

$$\begin{aligned} L_{11} &= \frac{1}{D_1}\left(C_2 - \frac{abc}{ab + bc + ca}\right) \\ L_{22} &= \frac{1}{D_1}\left(D_2 - \frac{abc}{ab + bc + ca}\right) \\ M &= \frac{1}{D_1}\left(a + b + c - \frac{abc}{ab + bc + ca}\right) \\ \frac{1}{C} &= (ab + bc + ca)(L_{11} + L_{22} + 2M) \end{aligned} \qquad (4\text{-}26)$$

and in Fig. (4-5),*

$$\begin{aligned} L_{11} &= \frac{C_1}{abc} - \frac{C_3}{a + b + c} \\ L_{22} &= \frac{C_3^2 D_1}{abc} - \frac{C_3}{a + b + c} \end{aligned} \qquad (4\text{-}27)$$

*The values for L_{11}, L_{22}, M and C are justified by eq. (3-60).

$$M = \frac{ab + bc + ca}{abc} - \frac{C_3}{a + b + c} \qquad (4\text{-}27)$$

$$\frac{1}{C} = \frac{abc}{a + b + c} (L_{11} + L_{22} + 2M)$$

These two network sections can always be removed from Z through proper choice of the arbitrary constants and thus are included in Fig. (4-1).

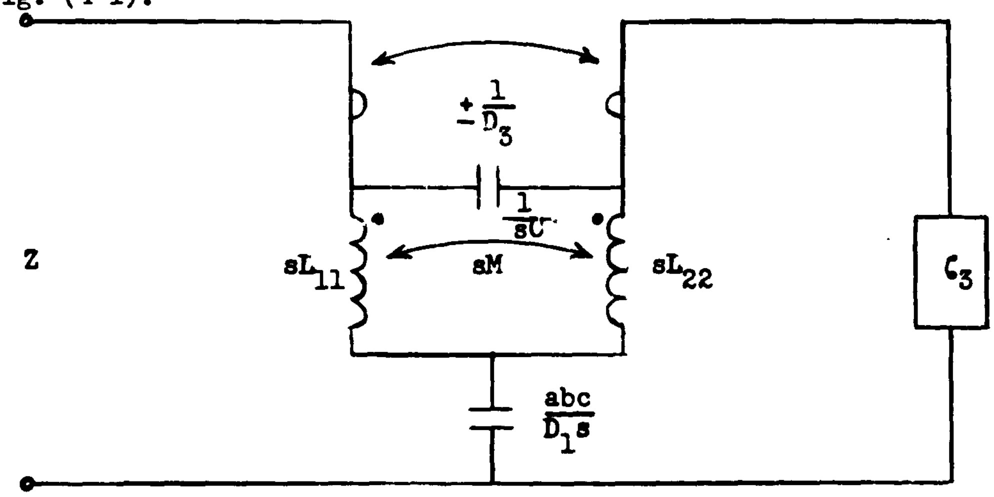

Fig. (4-4) n-Type Rank 6 Simplification

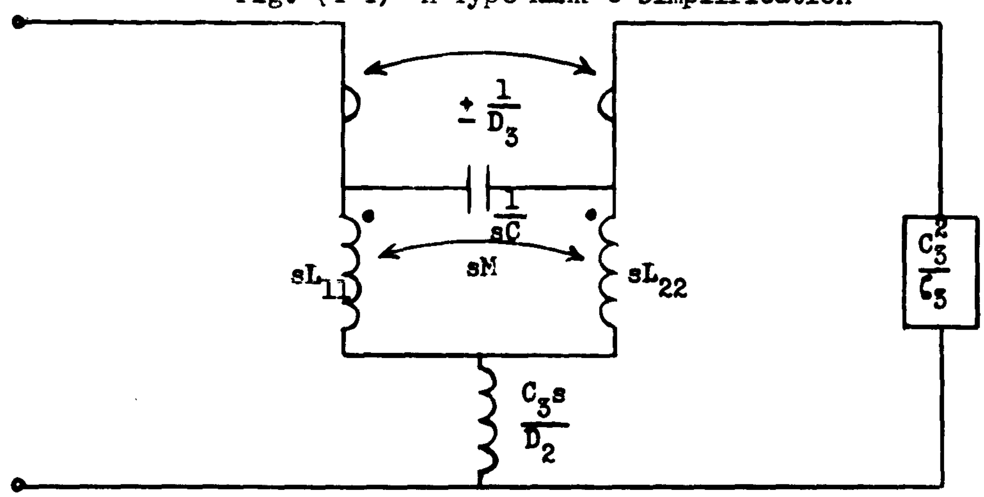

Fig. (4-5) m-Type Rank 6 Simplification

A second simplification in the network sections of Figs. (4-2) and (4-3) results if the arbitrary constants are chosen to make an element vanish in the T equivalent circuits of the transformers in these sections. The components of the T equivalent circuits are given by eq. (3-42). To make L_2 vanish in Figs. (4-2) and (4-3) requires that $L_{22} - M = 0$ for each case. For Fig. (4-2), $L_{22} - M = 0$ means that $D_2 = a + b + c$, while in Fig. (4-3) the requirement is that $D_1/D_3 = ab + bc + ca$. But these are eqs. (4-19) and (4-18), respectively, which can always be satisfied by choosing two of the arbitrary constants equal. With $L_2 = 0$, L_1 is positive by an argument similar to that presented in eqs. (3-45) and (3-46). Thus, when eq. (4-18) is satisfied, the n-type synthesis in Fig. (4-2) is simplified to that in Fig. (4-4) and L_2 vanishes in the m-type synthesis of Fig. (4-3). Eq. (4-19) performs a similar double function.

To make L_1 vanish in Figs. (4-2) and (4-3) requires that $L_{11} - M = 0$ for each case. For Fig. (4-2), this requirement is met if

$$C_2 = a\frac{\zeta_{1b}}{\zeta_{2c}} + b\frac{Z_a}{\zeta_{2c}} + c\frac{Z_a}{\zeta_{1b}} = a + b + c \qquad (4\text{-}28)$$

while for Fig. (4-3), it is necessary that

$$\frac{C_1}{C_3} = \frac{\zeta_{1b}}{Z_a\zeta_{2c}}(ab\zeta_{2c} + bcZ_a + ca\zeta_{1b}) = ab + bc + ca \qquad (4\text{-}29)$$

It may be shown that each of these equations is also satisfied if any two of the arbitrary constants are chosen equal. It should also be noted that if $\frac{1}{Z}$ in eq. (4-4) is synthesized using $\frac{1}{V}$ in eq. (4-6), eqs. (4-28) and (4-29) would have to be satisfied to obtain the network sections of Figs. (4-4) and (4-5).

4.5 Rank 6 Operator Example

The principles developed in this chapter are illustrated by considering the following driving point impedance function

$$Z = \frac{s^3 + \frac{14}{3}s^2 + 2s + 4}{s^3 + 4s^2 + \frac{44}{3}s + 2}$$

where

$$\text{num Ev } Z = (2 - s^2)(s^2 + 2s + 2)(s^2 - 2s + 2)$$

Zero cancellation synthesis may be used in connection with the rank 6 V operator to synthesize Z. Since Z is of rank 6, a, b and c may be chosen so that Ev Z_a = Ev Z_b = Ev Z_c = 0, which guarantees that ζ_3 will be six less in rank than Z. The calculations yield

$$a = 1 + j \quad b = 1 - j \quad c = \sqrt{2}$$

$$Z_a = \frac{8 + 2j}{17}, Z_b = \frac{8 - 2j}{17}, Z_c = \frac{2\sqrt{2}}{5}$$

$$\frac{Z_a}{\zeta_{1b}} = \frac{5}{3}, \zeta_{2c} = \frac{3\sqrt{2}}{5}, \zeta_3 = 2$$

From the above values, V in eq. (4-6) is

$$V = \frac{2\sqrt{2} + 2\sqrt{2}\,s + \frac{7}{3}\sqrt{2}\,s^2 + \sqrt{2}\,s^3}{2\sqrt{2} + \frac{22}{3}\sqrt{2}\,s + 4\sqrt{2}\,s^2 + \frac{1}{\sqrt{2}}\,s^3}$$

and

$$\text{num Ev } V = (2 - s^2)(s^2 + 2s + 2)(s^2 - 2s + 2)$$

$$= \left[(2+\sqrt{2})s^2 + 2\sqrt{2}\right]^2 - s^2\left[s^2 + 2 + 2\sqrt{2}\right]^2$$

The n-type synthesis of V follows directly from eq. (4-11) and Fig. (4-2). The result appears in Fig. (4-6).

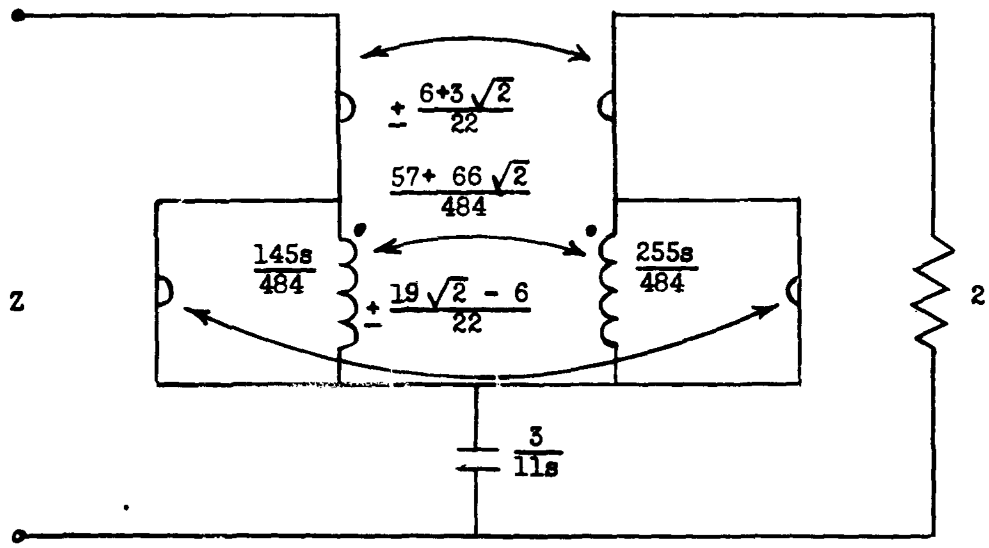

Fig. (4-6) n-Type Rank 6 Inductive

For the m-type synthesis of V, a scaling factor of $k = C_3 = \sqrt{2}$ is used. Then the synthesis follows directly from eq. (4-17) and Fig. (4-3). The result appears in Fig. (4-7).

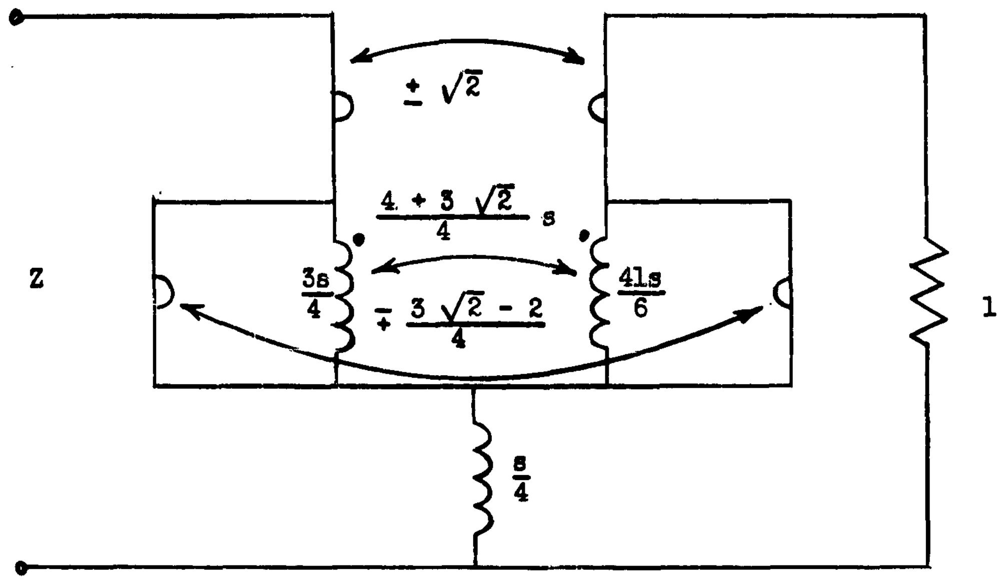

Fig. (4-7) m-Type Rank 6 Inductive

Note that the lossless sections in Figs. (4-6) and (4-7) each realize two real and four complex zeros of the even part of Z.

CHAPTER V

CASCADED AND DISTRIBUTED V OPERATOR SYNTHESES

5.1 Introduction

In Chapter IV the rank 6 V operator was introduced and syntheses were developed which treated the operator in its entirety. In the first part of this chapter, the rank 6 V operator is split into two operators, of rank 2 and rank 4, and the synthesis procedures of Chapter III are applied to these two operators in cascade.

In Chapter III considerable emphasis was given to eliminating the gyrator which appears in the synthesis of rank 2 and rank 4 operators. In each case, elimination of the gyrator resulted in the inclusion of a perfect transformer in the V operator synthesis. In this chapter, the emphasis is changed and it is shown that any prf driving point impedance may be realized by a series of cascaded network sections terminated in a realizable impedance of reduced rank, where each network section is reactive and contains one gyrator but no transformer.

In the second part of this chapter, the Darlington split even part and the Miyata synthesis procedures are reviewed. These procedures are then discussed in terms of the V operator and it is shown that each procedure may be considered as the synthesis of the V operator distributed in a prescribed way, with Foster-type expansions necessary in the Darlington split even part procedure and Cauer-type expansions required in the Miyata procedure. Lastly, the Bott-Duffin network is shown to result from a synthesis of the rank 2 distributed V operator.

5.2 Synthesis of Impedance Operators in Cascade

Consider the three impedance operators defined by eqs. (4-1), (4-2) and (4-3). These equations are repeated below.

$$V_1 = \frac{a + sZ_a}{a + \frac{s}{Z_a}} \tag{5-1}$$

$$V_2 = \frac{b + s\zeta_{1b}}{b + \frac{s}{\zeta_{1b}}} \tag{5-2}$$

$$V_3 = \frac{c + s\zeta_{2c}}{c + \frac{s}{\zeta_{2c}}} \tag{5-3}$$

Z may be represented in terms of these operators as

$$Z = V_1 V_2 V_3 \zeta_3 \tag{5-4}$$

It is desired to synthesize Z in terms of a rank 2 and a rank 4 V operator terminated in ζ_3. Such a synthesis may be accomplished in two ways in eq. (5-4); either V_1 and V_2 or V_2 and V_3 may be combined into a rank 4 V operator.* The combinations are

*The combination of V_1 and V_3 is prohibited since neither V_1 and V_2 nor V_2 and V_3 are commutative in general.

$$V_1V_2 = \frac{ab + s(bZ_a + a\zeta_{1b}) + s^2 \dfrac{Z_a}{\zeta_{1b}}}{ab + s\left(\dfrac{b}{Z_a} + \dfrac{a}{\zeta_{1b}}\right) + s^2 \dfrac{\zeta_{1b}}{Z_a}} \qquad (5\text{-}5)$$

$$V_2V_3 = \frac{bc + s(c\zeta_{1b} + b\zeta_{2c}) + s^2 \dfrac{\zeta_{1b}}{\zeta_{2c}}}{bc + s\left(\dfrac{c}{\zeta_{1b}} + \dfrac{b}{\zeta_{2c}}\right) + s^2 \dfrac{\zeta_{2c}}{\zeta_{1b}}} \qquad (5\text{-}6)$$

Z may now be synthesized using either eqs. (5-1) and (5-6) or eqs. (5-3) and (5-5). Using the former two equations requires that a be positive real while b and c may be complex conjugates with non-negative real parts. The latter two equations require c to be positive real while a and b may be complex conjugates with non-negative real parts. As pointed out in Section 4.3, if Z is of rank 6, 10, 14 ...., then num Ev Z contains at least one positive real root. For such impedances, the arbitrary constants may be chosen so that ζ_3 is reduced in rank by six while still insuring that both V operators are prf. For impedances of rank 4, 8, 12 ...., the arbitrary constants may be chosen to reduce the rank of ζ_3 by four with both V operators prf.

The n-type syntheses of the rank 2 and rank 4 V operators have been developed in Sections 3.2 and 3.7, respectively and will be used here. The synthesis of Z using eqs. (5-3) and (5-5) appears in Fig. (5-1) while that for eqs. (5-1) and (5-6) appears in Fig. (5-2). In Figs. (5-1) and (5-2),

$$D = \frac{b}{Z_a} + \frac{a}{\zeta_{1b}} \qquad (5\text{-}7)$$

$$E = \frac{c}{\zeta_{1b}} + \frac{b}{\zeta_{2c}} \qquad (5\text{-}8)$$

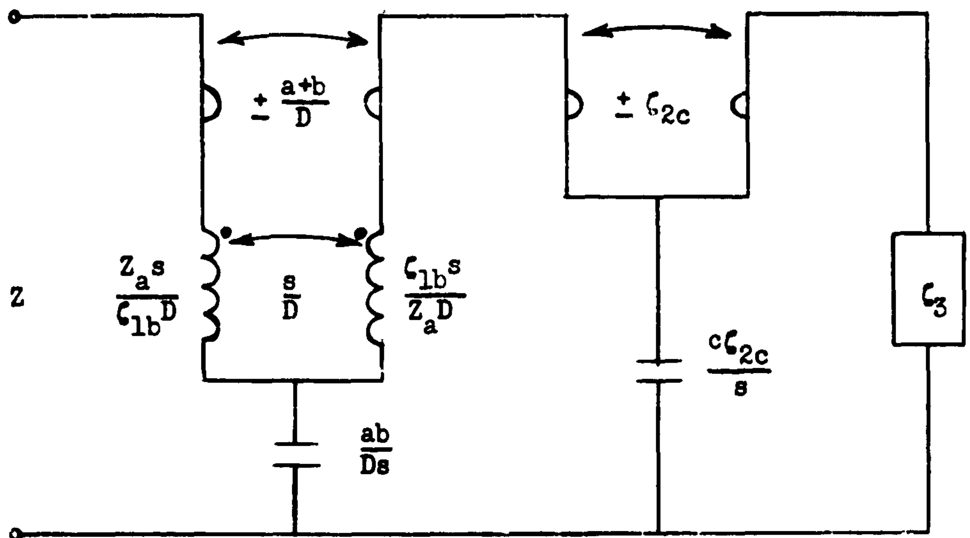

Fig. (5-1) Cascade Operator Synthesis Using Eqs. (5-3) & (5-5)

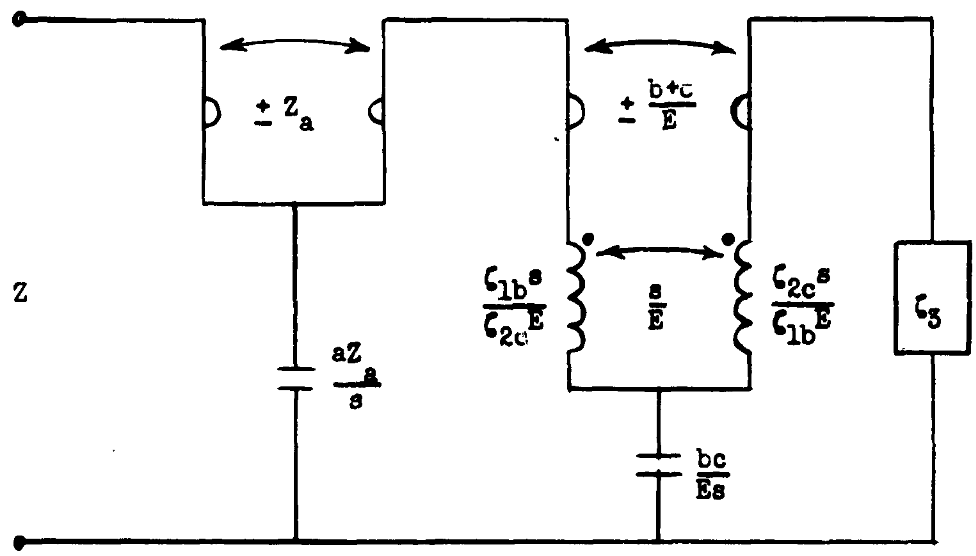

Fig. (5-2) Cascade Operator Synthesis Using Eqs. (5-1) & (5-6)

The first section in Fig. (5-1) realizes a quadruplet of complex even part zeros whereas the second section realizes a pair of real even part zeros. These realizations are reversed in Fig. (5-2).

5.3 A General Cascade Synthesis Procedure not Requiring Transformers

Theorem L

Any prf driving point impedance function may be realized by a series of cascaded network sections terminated in a realizable impedance of reduced rank without the use of transformers.

The networks of Figs. (5-1) and (5-2) each contain one transformer. Consider the transformer in Fig. (5-1). It becomes an inductor if $Z_a = \zeta_{1b}$, ie if V_1 and V_2 are commutative. This constraint was investigated in Appendix III, where it was shown that it is often, but not always, possible to choose the arbitrary constants such that the condition $Z_a = \zeta_{1b}$ is satisfied.

Consider next the transformer in Fig. (5-2). In order that it reduce to an inductor it is necessary that

$$\zeta_{2c} = \zeta_{1b} \tag{5-9}$$

It is shown in the following development that it is always possible to choose a positive real value of a such that eq. (5-9) is satisfied with b and c remaining arbitrary complex conjugates with nonnegative real parts. Thus, by sacrificing one of the constants, a, it is possible to achieve a cascade synthesis without transformers in which the termination (ζ_3) is four less in rank than Z.

Using eq. (4-2), eq. (5-9) takes the form

$$\frac{\zeta_{2c}}{\zeta_{1b}} = \frac{b\zeta_{1c} - c\zeta_{1b}}{b\zeta_{1b} - c\zeta_{1c}} = 1 \qquad (5\text{-}10)$$

or

$$(b + c)(\zeta_{1b} - \zeta_{1c}) = 0 \qquad (5\text{-}11)$$

Eq. (5-11) gives three alternative conditions similar to those in eq. (A-33) in Appendix III, namely

$$c = + b \quad \text{and } \zeta'_{1b} = 0^{*}$$

$$c = - b \quad \text{and Ev } \zeta_{1b} \neq 0 \qquad (5\text{-}12)$$

$$\zeta_{1b} = \zeta_{1c} \text{ (ie } \zeta_{1b} \text{ and } \zeta_{1c} \text{ real and equal)}$$

The first two conditions are discarded since they restrict b and c to be either both real or both imaginary. The third condition, using eq. (4-1), requires that

$$\frac{aZ_b - bZ_a}{aZ_a - bZ_b} = \frac{aZ_c - cZ_a}{aZ_a - cZ_c} \qquad (5\text{-}13)$$

Solving eq. (5-13) for Z_a gives

$$Z_a^2 + Z_a \frac{(a^2 - bc)(Z_c - Z_b)}{a(b - c)} - Z_b Z_c = 0 \qquad (5\text{-}14)$$

---

*ζ'_{1b} is the derivative of ζ_1 with respect to s evaluated at s = b.

$$Z_a = \frac{(a^2 - bc)(Z_b - Z_c)}{2a(b - c)} \pm \sqrt{\frac{(a^2 - bc)^2(Z_b - Z_c)^2}{4a^2(b - c)^2} + Z_b Z_c} \tag{5-15}$$

Thus the problem reduces to finding whether a positive real value of a always exists such that eq. (5-15) is satisfied with b and c remaining arbitrary.

Define the following,

$$\begin{aligned} b &= x + jy, \quad c = x - jy \\ Z_b &= u + jv, \quad Z_c = u - jv \end{aligned} \tag{5-16}$$

where, as in the discussion in Appendix V, x and u are always positive and y may always be chosen positive. Then v may be either positive or negative.

For large values of a, eq. (5-15) can be put in the form

$$Z_a = \frac{a(Z_b - Z_c)}{(b - c)} = a\,\frac{v}{y} \tag{5-17}$$

For small values of a, eq. (5-15) becomes, using the first two terms of a binomial expansion to represent the radical,

$$Z_a = -\frac{bc}{2a}\left(\frac{Z_b - Z_c}{b - c}\right) + \frac{bc}{2a}\left(\frac{Z_b - Z_c}{b - c}\right) + \frac{2a(b - c)Z_b Z_c}{bc(Z_b - Z_c)} \tag{5-18}$$

which reduces to

$$Z_a = \frac{2a(b - c)Z_b Z_c}{bc(Z_b - Z_c)} = \frac{2ay}{v}\,\frac{u^2 + v^2}{x^2 + y^2} \qquad (5\text{-}19)$$

Let Z_a be expressed in general form by

$$Z_a = \frac{a_o + a_1 a + a_2 a^2 + \ldots.\, a_n a^n}{b_o + b_1 a + b_2 a^2 + \ldots.\, b_n a^n} \qquad (5\text{-}20)$$

and require that none of the coefficients vanish.

For large a, Z_a in eq. (5-20) approaches the value a_n/b_n, while, for small a, it approaches a_o/b_o. Each of these ratios is finite and non-zero if Z has no pole or zero at the origin or infinity. Therefore Z_a in eq. (5-17) is greater than a_n/b_n as $a \rightarrow \infty$ and less than a_o/b_o as $a \rightarrow o$. Since both expressions for Z_a are continuous functions of a, their curves must cross at least once and yield a value of a which satisfies each equation and permits the transformer in Fig. (5-2) to be replaced by an inductor. The resulting network is shown in Fig. (5-3).* By choosing b and c such that Ev Z_b = Ev Z_c = 0, ζ_3 is four less in rank than Z.

*This network, in essence, was derived by Fialkow and Gerst using a different procedure[11] and hence it will be called the Fialkow-Gerst network.

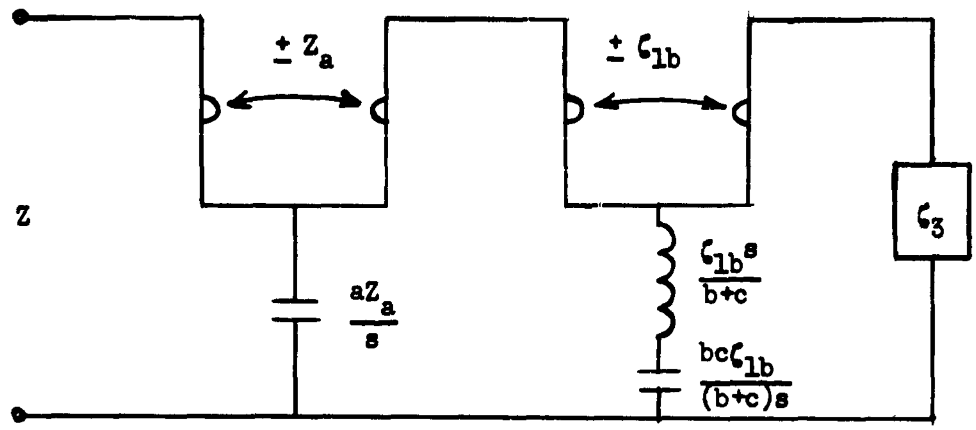

Fig. (5-3) Elimination of the Transformer from Fig. (5-2) (the Fialkow-Gerst Network)

Figs. (5-1), (5-2) and (5-3) all represent n-type syntheses. Consider now the possibility of an m-type synthesis of V_2V_3 in eq. (5-6). Normally this would require loaded gyrator networks similar to those in Figs. (3-14) and (3-15). But if a is chosen such that $\zeta_{2c} = \zeta_{1b}$, the network section of Fig. (A-2) in Appendix III results. Then the complete syntheses of Z is as shown in Fig. (5-4).

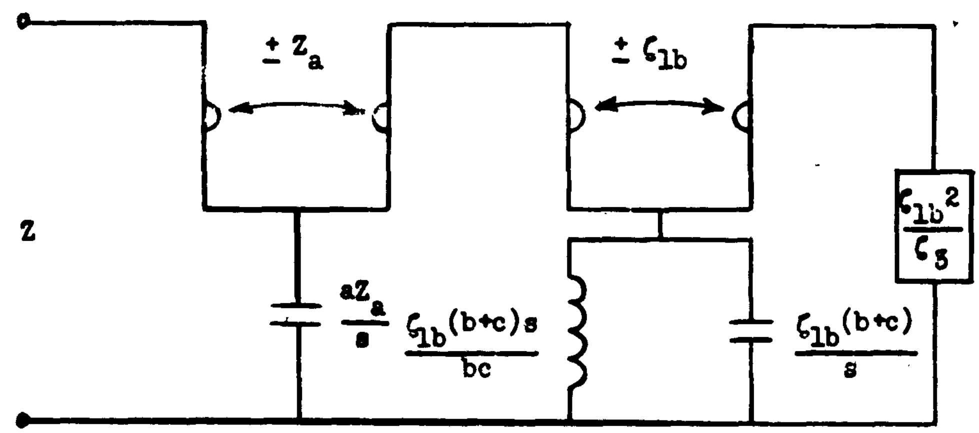

Fig. (5-4) m-Type Transformerless Synthesis of Eq. (5-6)

The syntheses in Figs. (5-3) and (5-4) are perfectly general and the rank of the terminating impedance may always be reduced by four.

5.4 Cascaded Operator Examples

The preceding principles are illustrated by considering the following driving point impedance function.

$$Z = \frac{s^3 + \frac{14}{3}s^2 + 2s + 4}{s^3 + 4s^2 + \frac{44}{3}s + 2}$$

where

$$\text{num Ev } Z = (2 - s^2)(s^4 + 4)$$

Solution A

Since Z is of rank 6, the arbitrary constants may be chosen to reduce the rank of the termination by six. Then, to develop the network of Fig. (5-2) requires that

$$a = \sqrt{2}, \quad b = 1 + j, \quad c = 1 - j, \quad Z_a = \frac{2\sqrt{2}}{5}$$

It follows that

$$V_1 = \frac{\sqrt{2} + \frac{2\sqrt{2}}{5}s}{\sqrt{2} + \frac{5}{2\sqrt{2}}s}, \quad \text{num Ev } V_1 = 2 - s^2$$

$$\zeta_1 = \frac{\sqrt{2}\,Z - \frac{2\sqrt{2}}{5}s}{\sqrt{2} - \frac{5}{2\sqrt{2}}sZ} = \frac{\frac{2}{5}s^2 + \frac{3}{5}s + 2}{\frac{5}{4}s^2 + \frac{29}{6}s + 1}$$

Note that ζ_1 is two less in rank than Z since a was chosen to make Ev $Z_a = 0$. Further calculations yield

$$\zeta_{1b} = \frac{6}{5}\left(\frac{7 - j3}{29}\right), \zeta_{1c} = \frac{6}{5}\left(\frac{7 + j3}{29}\right), \quad \frac{\zeta_{2c}}{\zeta_{1b}} = \frac{5}{2}$$

From these results, V_2V_3 and ζ_3 may be found as

$$V_2V_3 = \frac{\frac{2}{5}s^2 + \frac{6}{5}s + 2}{\frac{5}{2}s^2 + \frac{29}{6}s + 2}, \quad \text{num Ev } V_2V_3 = s^4 + 4$$

$$\zeta_3 = \frac{(\frac{5}{2}s^2 + 2)\zeta_1 - \frac{6}{5}s}{\frac{2}{5}s^2 + 2 - \frac{29}{6}s\zeta_1} = 2$$

ζ_3 is six less in rank than Z since b and c were chosen so that Ev Z_b = Ev $Z_c = 0$. V_2V_3 may now be synthesized directly by the n-type procedure. The entire synthesis of Z appears in Fig. (5-5).

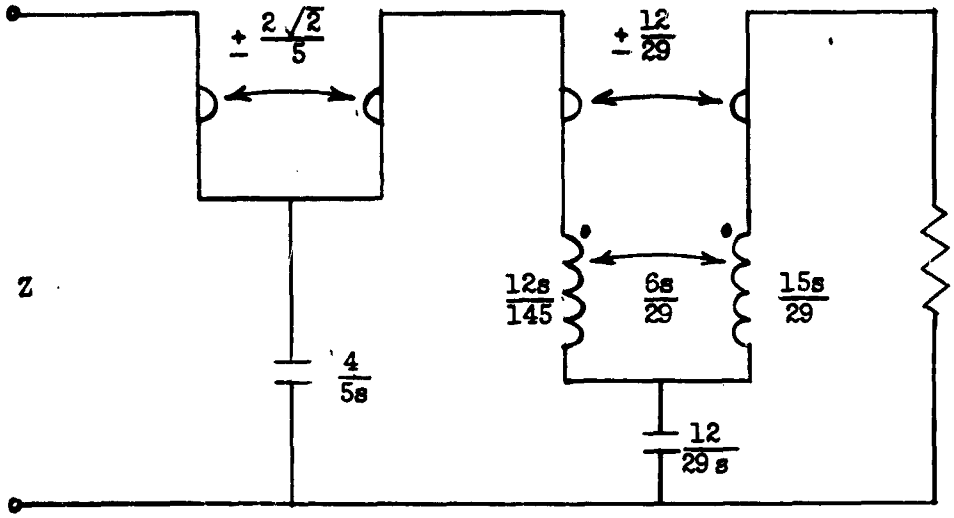

Fig. (5-5) Synthesis which Reduces Rank by 6

<u>Solution B</u>

The constant a is now chosen to eliminate the transformer in Fig. (5-5). The constants b and c remain the same so that ζ_3 is four less in rank than Z. For this case,

$$Z_b = \frac{8 + 2j}{17}, \quad Z_c = \frac{8 - 2j}{17}$$

and thus eq. (5-15) becomes

$$Z_a = \frac{a^2 - 2 \pm \sqrt{a^4 + 64a^2 + 4}}{17a}$$

Also

$$Z_a = \frac{a^3 + \frac{14}{3}a^2 + 2a + 4}{a^3 + 4a^2 + \frac{44}{3}a + 2}$$

The approximate solution to these two equations is

$$a \approx \frac{11}{2}, \; Z_a \approx .872$$

Then V_1 and ζ_1 become

$$V_1 = \frac{5.5 + .872s}{5.5 + \frac{s}{.872}}, \quad \text{num Ev } V_1 = 5.5^2 - s^2$$

$$\zeta_1 = \frac{.872s^3 + 2.78s^2 + 2.43s + 4.13}{1.15s^3 + 6.16s^2 + 14.2s + 1.96}$$

Note that ζ_1 is the same rank as Z since a was not chosen to make Ev $Z_a = 0$. Further calculations yield

$$\zeta_{1b} = \zeta_{1c} = \zeta_{2c} = .338$$

$$V_2V_3 = \frac{s^2 + .676s + 2}{s^2 + 5.92s + 2}, \quad \text{num Ev } V_2V_3 = s^4 + 4$$

$$\zeta_3 = \frac{.872s + 2}{1.15s + 1}$$

The complete synthesis appears in Fig. (5-6), where V_2V_3 has been synthesized using the n-type procedure and ζ_3 is four less in rank than Z.

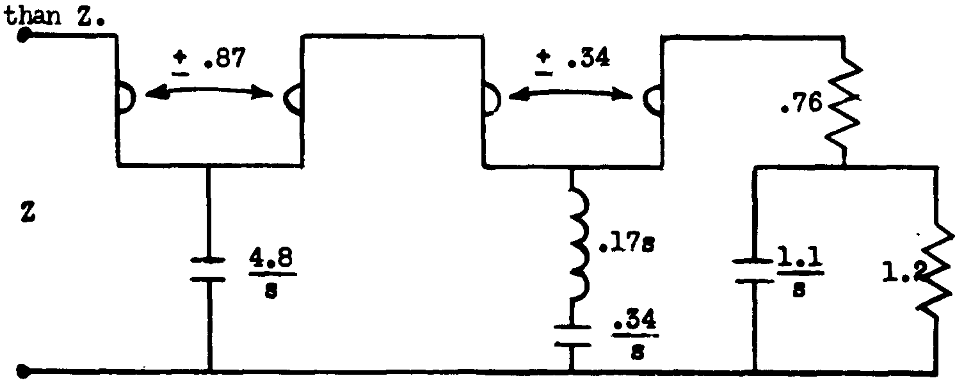

Fig. (5-6) n-Type Transformerless Synthesis

Solution C

The m-type synthesis of V_2V_3 may be obtained directly since the required value of a is the same as that in Solution B. Both ζ_3 and V_2V_3 are scaled so that

$$V_2V_3 = \frac{.114s^2 + .676s + .228}{s^2 + .676s + 2}$$

$$\frac{\zeta_3}{\zeta_{1b}^2} = \frac{.872s + 2}{.131s + .114}$$

The entire synthesis appears in Fig. (5-7).

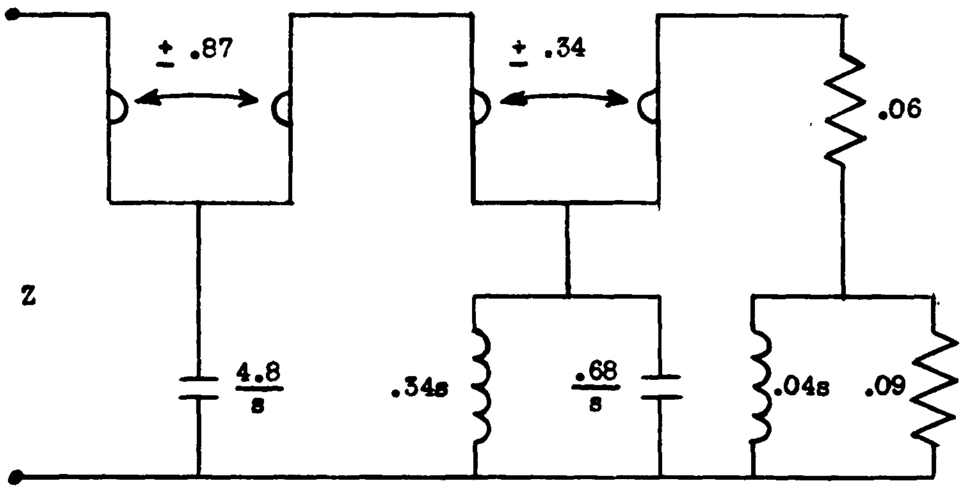

Fig. (5-7) m-Type Transformerless Synthesis

5.5 Even Part Synthesis Procedures

It is well known that a prf driving point impedance is determined by its even part within an arbitrary reactance function. Utilizing this fact, a number of synthesis procedures have been developed in which the even part is split into two or more parts and each part is synthesized separately. The Darlington synthesis of a split even part[14] and the Miyata[13] synthesis are two such procedures. The first method permits a synthesis without gyrators, the second without gyrators or transformers, in return for which the number of elements is increased and the cascade nature of the synthesis is lost.

These two methods are now briefly reviewed, after which it is shown that each may interpreted as the synthesis of the impedance operator V distributed in a prescribed way.

A. Darlington Synthesis of a Split Even Part[14]

Let Z be given by

$$Z = \frac{m_1 + n_1}{m_2 + n_2} = \frac{a_n s^n + a_{n-1} s^{n-1} + \ldots . a_o}{b_n s^n + b_{n-1} s^{n-1} + \ldots . b_o} \tag{5-21}$$

where

$$\text{Ev } Z = \frac{m_1 m_2 - n_1 n_2}{m_2^2 - n_2^2} \tag{5-22}$$

Assume for the moment that num Ev Z is a perfect square so that a reciprocal Darlington synthesis is possible. Also assume that only eq. (5-22) is given. Then, since m_2 and n_2 are known, Z_{12} can be obtained directly. Also Z_{22} is found from m_2/n_2, since the n-type Darlington procedure is applicable in this case.*

*Z does not have a pole or zero at s = o.

Since the residue condition is satisfied with the equal sign at all $j\omega$ axis poles (the degree of n_1 is not greater than n_2), the components of Z_{11} may be obtained from those of Z_{12} and Z_{22}. Thus the entire synthesis of Z is accomplished using only the even part of Z.

Now let the even part of Z be expressed as

$$\text{Ev } Z = \frac{A_n(-s^2)^n + A_{n-2}(-s^2)^{n-2} + \dots\dots A_o}{m_2^2 - n_2^2} \qquad (5\text{-}23)$$

or

$$\text{Ev } Z = \frac{A_n(-s^2)^2}{m_2^2 - n_2^2} + \frac{A_{n-2}(-s^2)^{n-2}}{m_2^2 - n_2^2} + \dots\dots \frac{A_o}{m_2^2 - n_2^2} \qquad (5\text{-}24)$$

If the coefficients in eq. (5-24) are all positive then each term on the right is positive everywhere on the $j\omega$ axis. Furthermore all denominators are identical, equal to that of Z, and hence are Hurwitz polynomials. Thus each term on the right hand side of eq. (5-24) is the even part of a realizable, minimum reactance impedance. The sum of these impedances is Z.

The Darlington synthesis of a split even part may be considered in another manner. Let the driving point impedance function be given by

$$Z = \frac{s^2 + a_1 s + a_o}{s^2 + b_1 s + b_o} \qquad (5\text{-}25)$$

where

$$\text{Ev } Z = \frac{s^4 - (a_1 b_1 - a_o - b_o)s^2 + a_o b_o}{(s^2 + b_o)^2 - b_1^2 s^2} \qquad (5\text{-}26)$$

Assume the numerator coefficient in the parentheses is positive, a necessary condition for the split even part procedure to be applicable. Let Z in eq. (5-25) be rewritten as

$$Z = \frac{s^2 + \frac{b_o}{b_1} s}{s^2 + b_1 s + b_o} + \frac{\left(a_1 - \frac{a_o + b_o}{b_1}\right) s}{s^2 + b_1 s + b_o} + \frac{a_o + \frac{a_o}{b_1} s}{s^2 + b_1 s + b_o}$$

$$= \quad Z_4 \quad + \quad Z_2 \quad + \quad Z_o \qquad (5\text{-}27)$$

Computing the numerators of the even parts of Z_4, Z_2 and Z_o gives values identical with those in the numerator of eq. (5-26).

$$\begin{aligned} \text{num Ev } Z_4 &= s^4 \\ \text{num Ev } Z_2 &= -(a_1 b_1 - a_o - b_o)s^2 \\ \text{num Ev } Z_o &= a_o b_o \end{aligned} \qquad (5\text{-}29)$$

In general, if Ev Z is given by eq. (5-24) with all coefficients positive, then Z may be written as

$$Z = \frac{N_n(s)}{m_2 + n_2} + \frac{N_{n-2}(s)}{m_2 + n_2} + \dots \frac{N_o(s)}{m_2 + n_2}$$

$$= Z_n \quad + Z_{n-2} \quad + \dots Z_o \qquad (5\text{-}30)$$

where each term on the right is prf. Thus, in general, the Darlington synthesis of a split even part may be considered as the synthesis of the parts of Z, distributed in such a way that num Ev $Z_q = A_q(-s^2)^q$ (q = 0,2,4 ... n). This concept of synthesizing the distributed Z is useful in the impedance operator discussion to follow.

B. Miyata Synthesis

The chief disadvantage of the previous procedure is that perfect transformers are often required. The Miyata procedure avoids their use. Again consider Z in eq. (5-21) and its even part given by eq. (5-22). If $a_o = 0$, ie Z has a zero at the origin, then $A_o = 0$ and num Ev Z has the factor $-s^2$. Thus $Y = \frac{1}{Z}$ has a pole at the origin which can be removed as a shunt inductance L. Removing this pole from the reciprocal of eq. (5-21) gives

$$Y_1 = \frac{m_2 + n_2}{m_1 + n_1} - \frac{1}{sL} = \frac{\frac{1}{s}\left(n_2 - \frac{m_1}{sL}\right) + \frac{1}{s}\left(m_2 - \frac{n_1}{sL}\right)}{\frac{1}{s}(m_1 + n_1)} \tag{5-31}$$

$$EvY_1 = \frac{\dfrac{m_1m_2 - n_1n_2}{-s^2}}{\dfrac{m_1^2 - n_1^2}{-s^2}} = Ev\ Y \tag{5-32}$$

The denominator of eq. (5-32) has no zero at the origin. Its numerator may be expressed as

$$\text{num Ev } Y_1 = \frac{\text{num Ev } Z}{- s^2} \qquad (5\text{-}33)$$

If A_2 is also zero, then the constant term in the numerator of Y_1 is zero and Y_1 has a zero at origin. Hence Z_1 has a pole there which may be removed as a series capacitance. If num Ev Z has its last k terms missing, k reactive elements may be removed from Z in this fashion.

In a similar manner, if $a_n = 0$, then $A_n = 0$ and Z has a zero at infinity. This can be removed as a shunt capacitance. If the first ℓ terms of num Ev Z are missing, ℓ reactive elements may be removed. If $k + \ell = n$, which implies that num Ev Z has zeros only at the origin and infinity, Z is synthesized by n reactive elements and a resistive termination.

The Miyata synthesis procedure utilizes the above properties. Consider the individual parts of Ev Z as given in eq. (5-24). For each part, $k + \ell = n$ and thus each part may be realized by n reactive elements and a resistance, assuming all the numerator coefficients are positive. However this process would be computationally laborious were it not for the fact that a prototype impedance can be found and all other impedances computed from it. The prototype is defined by its even part as

$$\text{Ev } Z_p = \frac{1}{m_2^2 - n_2^2} \qquad (5\text{-}34)$$

Z_p is now obtained (using the Gewertz procedure,[3] for example) and the numerator of each even part term is multiplied by Z_p. The resulting expressions are not in general prf, but their even parts are always positive on the $j\omega$ axis. Each expression is "divided out" until the order of its numerator no longer exceeds that of its denominator. The result is the sum of an odd polynomial in s (whose even part is zero) plus a rational function of s. Each rational function so obtained has the same denominator (that of Z) and an even part identical with the portion of the even part of Z from which it was derived. Thus it is the desired impedance. Each such impedance may now be synthesized by n reactive and one resistive element and the resulting networks added in series to give Z.

The Miyata procedure may also be interpreted in terms of a distributed Z rather than a split even part of Z. Consider Z_4 in eq. (5-27). The numerator of its even part is s^4. Thus for Z_4, A_o and A_2 are missing in eq. (5-24). This means that two reactive elements may be removed from Z_4 and Z_4 may be expanded into the following form

$$Z_4 = \cfrac{1}{\cfrac{b_1}{s} + \cfrac{1}{\cfrac{b_o}{b_1 s} + 1}} \qquad (5\text{-}35)$$

Thus a shunt inductor $\left(\frac{1}{b_1}\right)$ and a series capacitor $\left(\frac{b_1}{b_o}\right)$ may be removed from Z_4 leaving a resistive termination (1 ohm). The process may be repeated for Z_2 and Z_0 and the results added in series to give the Miyata synthesis of Z.

Eq. (5-35) is a Cauer-type expansion of Z_4 whereas the previous Darlington syntheses are Foster-type expansions. Once again, this concept of synthesizing the distributed Z is useful in the development of Miyata-type networks using the impedance operator. This is discussed subsequently.

5.6 V Operator Split Even Part Synthesis

In this and the following section, the principles of the previous section are applied to the V operator. It is shown that the Darlington split even part procedure and the Miyata procedure may be interpreted as syntheses of the V operator distributed in a prescribed way.

The impedance operator lends itself readily to even part synthesis largely because of three relations developed in Chapter II. These relations are eqs. (2-17), (2-48) and (2-50), which are repeated below in slightly revised form.

$$\text{num Ev } Z = (\text{num Ev } V)(\text{num Ev } \zeta) \qquad (5\text{-}36)$$

$$V\zeta = (V_1 + V_2 + \dots V_n)\zeta = V_1\zeta + V_2\zeta + \dots V_n\zeta \qquad (5\text{-}37)$$

$$\text{num Ev } V = \text{num Ev } V_1 + \text{num Ev } V_2 + \dots \text{num Ev } V_n \qquad (5\text{-}38)$$

Eq. (5-35) points out that the zeros of num Ev Z are split between num Ev V and num Ev ζ. Thus any of the zeros of num Ev Z which appear in num Ev V are absent from num Ev ζ. This equation also points out that num Ev V can be split without interfering with the relationship between num Ev Z and num Ev ζ. Eqs. (5-37) and (5-38) are valid only if the denominators of V_1, V_2 ... V_n are equal or differ by a positive real constant. The three equations taken together demonstrate that the Darlington split even part procedure may be readily applied to the V operator, in other words num Ev V may be split, the individual parts synthesized, ζ included as the termination in each synthesis, and the resulting networks assembled in series to give the desired Z. Eqs. (5-37) and (5-38) point out that the synthesis may be considered in two ways, either as a synthesis of the parts of Ev V or as a synthesis of the parts of the distributed V operator. To further illustrate this latter concept, let Z in eq. (5-25) represent an impedance operator such that

$$V\zeta = \frac{(s^2 + a_o)\zeta + a_1 s}{s^2 + b_o + b_1 s\zeta} \qquad (5\text{-}39)$$

Eq. (5-39) may be distributed in the following way.

$$V\zeta = \frac{s^2\zeta + \frac{b_o}{b_1}s}{s^2 + b_o + b_1 s\zeta} + \frac{\left(a_1 - \frac{a_o + b_o}{b_1}\right)s}{s^2 + b_o + b_1 s\zeta} + \frac{a_o\zeta + \frac{a_o}{b_1}s}{s^2 + b_o + b_1 s\zeta} \qquad (5\text{-}40)$$

$$= V_4\zeta + V_2\zeta + V_o\zeta$$

V_4, V_2 and V_o may now be synthesized by the Darlington procedure, the proper terminations added, and the resulting networks placed in series, assuming, as in eq. (5-25) that the numerator coefficient of $V_2\zeta$ is positive.

Example 1

Let it be required to synthesize Z given by

$$Z = \frac{(s^2 + 2)\zeta_2 + \frac{7}{3}s}{s^2 + 3 + 3s\zeta_2}$$

by synthesizing the distributed V operator and also by synthesizing the split even part of V. The V operator and its even part are

$$V = \frac{s^2 + \frac{7}{3}s + 2}{s^2 + 3s + 3}, \quad \text{Ev } V = \frac{s^4 - 2s^2 + 6}{(s^2 + 3)^2 - 9s^2}$$

The coefficients in num Ev V satisfy the requirements for a split even part synthesis. Let V be distributed according to eq. (5-27).

$$V = \frac{s^2 + s}{s^2 + 3s + 3} + \frac{\frac{2}{3}s}{s^2 + 3s + 3} + \frac{2 + \frac{2}{3}s}{s^2 + 3s + 3}$$

$$= V_4 + V_2 + V_o$$

$$\text{num Ev } V = s^4 - 2s^2 + 6$$

$$= \text{num Ev } V_4 + \text{num Ev } V_2 + \text{num Ev } V_o$$

V_4 and V_o are synthesized by the n-type procedure while V_2 requires the m-type procedure. In the cases of V_2 and V_o, the terminations are scaled to avoid transformers. The complete synthesis appears in Fig. (5-8).

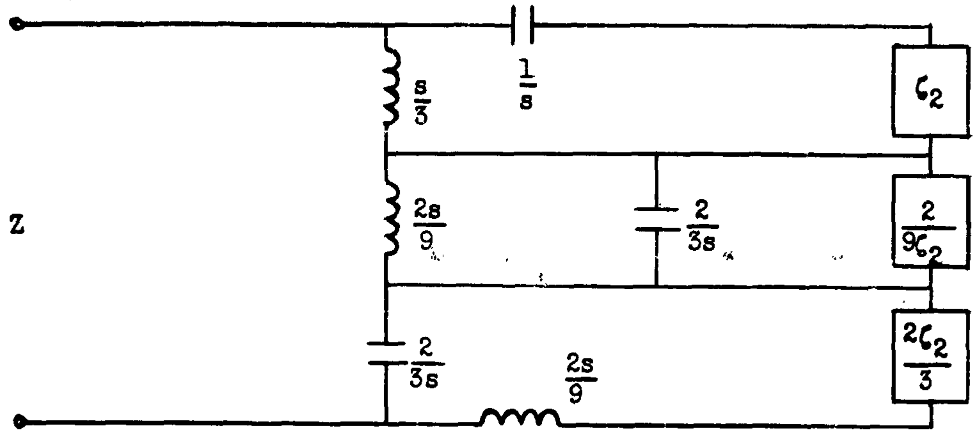

Fig. (5-8) Split Even Part Synthesis of V

The synthesis in Fig. (5-8) may also be achieved by synthesizing the split even part of V given by

$$\mathrm{Ev}\ V = \frac{s^4}{m_2^2 - n_2^2} + \frac{2}{9}\ \frac{-9s^2}{m_2^2 - n_2^2} + \frac{2}{3}\ \frac{9}{m_2^2 - n_2^2}$$

where the $\frac{2}{3}$ and $\frac{2}{9}$ scaling factors again permit the synthesis of Z without transformers.

5.7 V Operator Miyata-Type Synthesis

The Miyata procedure also applies in the case of the V operator, again largely because of the relations in eqs. (5-36) through (5-38). Alternatively, the V operator may be distributed, each part expanded into a Cauer network, the appropriate terminations included, and the resulting networks placed in series. To further illustrate the latter procedure, consider the Cauer expansion of $V_4\zeta$ in eq. (5-40). This is given by

$$V_4\zeta = \cfrac{1}{\cfrac{b_1}{s} + \cfrac{1}{\cfrac{b_o}{b_1 s} + \zeta}} \tag{5-41}$$

Note the similarity between eqs. (5-35) and (5-41). The only difference is that the one-ohm termination in eq. (5-35) is replaced by ζ in eq. (5-41). Stated another way, eq. (5-40) does not specify the manner in which the individual V operators are to be synthesized; it merely states that each is to operate in a prescribed way on ζ and the results summed to give $V\zeta$.

Example 2

Let it be required to synthesize the Z of example 1 through Cauer expansions of each part of the distributed V operator and also by applying the Miyata procedure to the split even part of V.

The Cauer expansions of V_4, V_2 and V_o are

$$V_4 = \cfrac{1}{\cfrac{3}{s} + \cfrac{1}{\cfrac{1}{s} + 1}}$$

$$V_2 = \cfrac{1}{\cfrac{3s}{2} + \cfrac{9}{2s} + \cfrac{9}{2}}$$

$$V_o = \cfrac{1}{\cfrac{3s}{2} + \cfrac{1}{\cfrac{2s}{9} + \cfrac{2}{3}}}$$

When the terminations are included and the syntheses summed, the network of Fig. (5-8) results.

The Miyata prototype in this case has an even part given by

$$\text{Ev } V_p = \frac{1}{(s^2 + 3)^2 - 9s^2}$$

Synthesizing Ev V_p gives

$$V_p = \frac{1}{9} \; \frac{s + 3}{s^2 + 3s + 3}$$

Multiplying the terms in num Ev V from example 1 by V_p gives

$$V_{4+} = \frac{s^4(s + 3)}{9(s^2 + 3s + 3)}$$

$$V_{2+} = \frac{-2s^2(s + 3)}{9(s^2 + 3s + 3)}$$

$$V_{o+} = \frac{6(s + 3)}{9(s^2 + 3s + 3)} = V_o$$

V_{4+} and V_{2+} are non-prf. The even parts of V_{4+} V_{2+} and V_{o+} are equal to those of V_4, V_2 and V_o, respectively.

Dividing out V_{4+} and V_{2+} until the numerator degrees do not exceed those of the denominator yields

$$V_{4+} = \frac{s^3}{9} - \frac{s}{3} + \frac{s^2 + s}{s^2 + 3s + 3} = \frac{s^3}{9} - \frac{s}{3} + V_4$$

$$V_{2+} = -\frac{2s}{9} + \frac{\frac{2}{3}s}{s^2 + 3s + 3} = -\frac{2s}{9} + V_2$$

The reactive elements may now be removed from V_4, V_2 and V_o, again yielding the network of Fig. (5-8).*

5.8 The Bott-Duffin Network from the Distributed V Operator

The Bott-Duffin network may be shown to result from a particular distribution of the rank 2 V operator developed in Chapter III. To show this relationship, let Z, as given by eq. (3-1), be rewritten as

$$Z = \frac{aZ_a \frac{\zeta_1}{Z_a} + sZ_a}{a + s\frac{\zeta_1}{Z_a}} \tag{5-42}$$

*The networks resulting from the split even part Darlington and Miyata procedures are not always identical as in Examples 1 and 2, since it is not always possible to achieve a split even part Darlington synthesis without transformers, especially for impedances of rank greater than four.

or

$$\frac{Z}{Z_a} = \frac{a \frac{\zeta_1}{Z_a} + s}{a + s \frac{\zeta_1}{Z_a}} \tag{5-43}$$

If the pertinent driving point impedance is considered to be Z/Z_a and the termination is defined as $Z_1 = \zeta_1/Z_a$, the Bott-Duffin V operator is

$$V_{BD} = \frac{a + s}{a + s} \tag{5-44}$$

Let eq. (5-44) be distributed to give

$$V_{BD} = \frac{a}{a + s} + \frac{s}{a + s} \tag{5-45}$$

where

$$\text{Ev } V_{BD} = \frac{a^2}{a^2 - s^2} + \frac{- s^2}{a^2 - s^2} \tag{5-46}$$

The requirement of positive coefficients is satisfied and thus the first and second parts of V may be synthesized by the n - and m-type Darlington procedures, respectively. The result is the network of Fig. (5-9), which is the non-cascade representation of the Bott-Duffin network.[6] The positive real constant a is arbitrary and may be chosen so that Z_1, given by

$$Z_1 = \frac{aZ - sZ_a}{aZ_a - sZ} \tag{5-47}$$

has a $j\omega$ axis zero or pole at the point where the even part of Z is zero, according to the Bott-Duffin procedure.

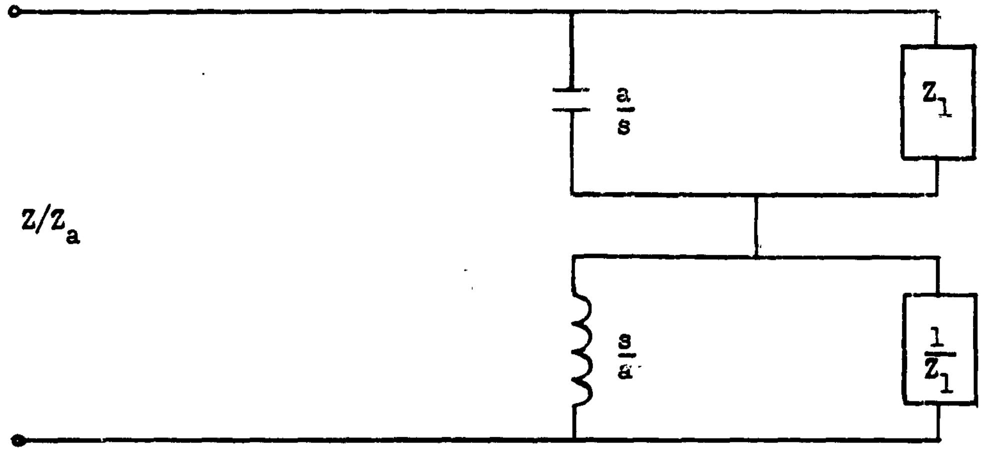

Fig. (5-9) Bott-Duffin Network

Guillemin[7] also has an alternate method of obtaining the Bott-Duffin result* which is now reviewed so as to compare it with the distributed operator procedure. Let Z, given by eq. (1-1) and here assumed to be a minimum resistance function, be augmented by the polynomial s + a to yield

$$Z = \frac{(m_1 + n_1)(s + a)}{(m_2 + n_2)(s + a)} = \frac{(am_1 + sn_1) + (an_1 + sm_1)}{(am_2 + sn_2) + (an_2 + sm_2)} = \frac{M_1 + N_1}{M_2 + N_2} \tag{5-47}$$

The even part of Z is given by

$$\text{Ev } Z = \frac{(m_1 m_2 - n_1 n_2)(a^2 - s^2)}{M_2^2 - N_2^2} \tag{5-48}$$

*The method is actually a special application of a general procedure for extending the Miyata synthesis procedure. In the general procedure the polynomial s + a is replaced by $m_o + n_o$.

Eq. (5-48) may be separated to give

$$\text{Ev } Z = \frac{a^2(m_1m_2 - n_1n_2)}{M_2^2 - N_2^2} + \frac{-s^2(m_1m_2 - n_1n_2)}{M_2^2 - N_2^2} \quad (5\text{-}49)$$

The polynomial s + a is now specified to fulfill the condition that M_2 vanish at the point $s = j\omega_0$ where the even part of Z is zero.* It follows that N_1 also vanishes at $s = j\omega_0$ since $m_1m_2 - n_1n_2$ vanishes there.

Each term in eq. (5-49) represents the even part of a prf impedance. These impedances may be found from their even parts. Since $(m_1m_2 - n_1n_2)^{1/2}$ is a factor of both M_2 and N_1, the first impedance in eq. (5-49) has $j\omega$ axis zeros at $s = \infty$ and at $s = j\omega_0$ and the second impedance has $j\omega$ axis zeros at $s = 0$ and at $s = j\omega_0$. These zeros may be removed from each impedance.

The impedance operator approach utilizes the synthesis of the distributed V operator (with its built-in surplus factor) to achieve the Bott-Duffin network whereas, in the Guillemin approach, Z is augmented by the auxiliary polynomial at the outset and a split-even part synthesis is used. Both methods employ the constant a to create a finite $j\omega$ axis zero or pole in the two resulting impedances.

*If this requirement yields a negative a, then M_1 can be required to vanish at $s = j\omega_0$.

CHAPTER VI

SUMMARY, CONCLUSIONS, FUTURE INVESTIGATION

6.1 Introduction

The final chapter consists of two parts. In the first portion, a summary of the results of the previous chapters is presented and conclusions are drawn therefrom. The summary is two-fold. First the overall contribution of the thesis in terms of the development of the impedance operator concept and its application to network synthesis is discussed. Secondly, a summary of the specific contributions resulting from a consideration of the properties of the various impedance operators and their network realizations is presented.

The second portion of the chapter deals with possible further applications of the impedance operator approach to network analysis and synthesis. Special emphasis is given to the distributed impedance operator and the possibility of achieving syntheses which do not include gyrators or transformers by relaxing the requirement of a single termination.

6.2 Overall Contribution

In Section 1.1 it was stated that the purpose of this thesis was to develop a general, systematic, flexible and easily applied approach to driving point impedance synthesis using the concept of the impedance operator. This purpose has been achieved in the previous five chapters. The impedance operator approach is general in that it is applicable for any configuration of even part zeros and therefore permits the synthesis of any prf driving point impedance through its use. It is systematic in that the same basic operations are required no matter what the rank of the driving point impedance or the type of cascade realization desired. The method is flexible because of the arbitrary constants incorporated in each impedance operator and the terminating impedances. These constants permit considerable latitude in the network structures which realize a given driving point impedance. The method is easy to apply, in that the required computations for a given driving point impedance are straightforward and involve only a reasonable amount of algebra.

It has been shown that the synthesis procedures of Brune, Darlington, Miyata and Bott-Duffin readily lend themselves to the impedance operator approach. The Brune procedure results from a specific synthesis of the rank 4 impedance operator in Section 3.4. In Sections 5.6 and 5.7, it is shown that the split even part Darlington and Miyata procedures may be considered as syntheses of the distributed impedance operator in conjunction with Foster and Cauer expansions, respectively. The Bott-Duffin network is shown to result from a particular distribution of the rank 2 impedance operator in Section 5.8.

From the impedance operator approach, three cascade synthesis procedures have been developed. These are the procedures of Sections 3.5, 3.10 and 5.3. The procedure of Section 3.5 is an extension of the Bott-Duffin procedure and is a specific contribution of this thesis. It differs from the usual Bott-Duffin procedure in that two applications of Richards' Theorem are employed rather than one. The extended procedure permits realizations of the form of Fig. (3-11) which do not occur in the usual Bott-Duffin synthesis procedures. The procedure of Section 3.10 is a general cascade reciprocal synthesis procedure applicable to any prf driving point impedance. As pointed out in the footnote to Section 3.10, this synthesis procedure is new but the philosophy behind it is that of Guillemin as described in Section 1.9. The procedure of Section 5.3 permits any prf driving point impedance to be realized by a non-reciprocal cascade synthesis procedure which does not require mutual coupling. Again the procedure is new but one of the resulting networks has been obtained by Fialkow and Gerst by a different method, as mentioned in Section 5.3.

6.3 Specific Contributions

In addition to the overall contribution summarized above, there are a number of specific contributions which are dispersed throughout the thesis. The more important of these will now be summarized in the order of their occurrence.

A. A general non-reciprocal Darlington synthesis procedure applicable to any prf driving point impedance (Theorem B) has been developed in Section 1.3 and applied to impedances of rank 2, 4 and 6 in Sections 1.6 and 1.7 using the network sections of Figs. (1-2) and (1-3). The results in Section 1.6 are not original[4,14]; whereas those in Section 1.7 are. The procedure is a non-cascade type since all of the even part zeros are realized in one "box" rather than in cascaded "boxes" as in Fig. (1-11). The realization is always in the form of a lossless (generally non-reciprocal) network terminated in a pure resistance.

B. The extended residue condition of eq. (1-25) and the fact that the two extended residue conditions of eqs. (1-25) and (1-27) are equivalent (Theorem C in Section 1.4) are specific contributions of this thesis. The extended residue condition of eq. (1-27) is not.[2,14] Eq. (1-27) permits realizations in terms of the capacitive structure of Fig. (1-3) whereas eq. (1-25) applies when the inductive structure of Fig. 1-2 is desired.

C. The associative, commutative and distributed laws have been applied to the impedance operator in Chapter II. The operator is shown to always obey the associative law through Theorem D in Section 2.3; it is shown to obey the commutative law only if the conditions of Theorem G in Section 2.7 are satisfied; and it is shown to obey the distributive law only if the conditions of Theorem H in Section 2.8 are satisfied.

D. Two important theorems concerning the even parts of a series of impedance operators are developed in Chapter II. These are Theorem E in Section 2.4, which relates the even part numerators of a series of cascaded impedance operators, and Theorem I in Section 2.8, which relates the even part numerators of a distributed impedance operator. The former has been used extensively in the development of impedance operator cascade syntheses throughout the thesis while the latter has been applied in the distributed impedance syntheses in Chapter V.

E. The pseudo-commutative property of the impedance operator discussed in Section 2.7 is included here because it permits resistance to be included in the removed network section. This property merits further study and thus is also included in the discussion of proposed future investigations later in the chapter.

F. The representation of the basic impedance operator relations in matrix form has been presented in Section 2.9. No immediate advantage of these matrix forms has been found (other than their conciseness) but it is very possible that they could be of value in the development of two-port synthesis procedures using the impedance operator approach.

G. The non-reciprocal network sections of Figs. (3-7), (3-8) and (3-16) (similar to the Brune section and the Darlington C and D sections) merit inclusion here, not because they are new sections, but rather because they have been arrived at in a new way, contain arbitrary constants and can always be obtained through the elimination of gyrators from other network sections. These points have been discussed in detail in Chapter III.

H. The non-reciprocal network sections of Figs. (4-1a) and (4-1b) are of interest largely because of the fact that they realize six even part zeros (generally two real and four complex).

I. In the discussion of the rank 6 impedance operator in Section 4.2 (Theorem K), it was shown that a permutation of the three arbitrary constants in no way changed the impedance operator or the terminating impedance, ζ_3. Thus Theorem K permits the "middle constant", b, to be positive real while a and c are complex conjugates with a non-negative real part. No application of this result has as yet been found.

6.4 Future Investigation

A. Distributed Operators

In Section 5.8, the rank 2 V operator of eq. (3-2) was changed slightly in form and distributed into two parts. The resulting synthesis yielded the Bott-Duffin network of Fig. (5-9).

No other specific distributed operator syntheses have been developed in this thesis and thus there is a considerable amount of work still to be done in this area.

To illustrate a possible course for future study of the distributed impedance operator, consider the rank 4 V operator given by eq. (3-13), which is repeated below.

$$V = \frac{ab + s(bZ_a + a\zeta_{1b}) + s^2 \dfrac{Z_a}{\zeta_{1b}}}{ab + s\left(\dfrac{b}{Z_a} + \dfrac{a}{\zeta_{1b}}\right) + s^2 \dfrac{\zeta_{1b}}{Z_a}} \qquad (6\text{-}1)$$

Its even part may be split as follows:

$$\text{Ev } V = \frac{a^2b^2}{m_2^2 - n_2^2} - \frac{(a^2 + b^2)s^2}{m_2^2 - n_2^2} + \frac{s^4}{m_2^2 - n_2^2} \qquad (6\text{-}2)$$

In order to achieve a Darlington split even part or a Miyata synthesis of eq. (6-2), it is necessary that all numerator coefficients be positive. The quantity a^2b^2 is always positive and the s^4 coefficient is unity. The quantity $a^2 + b^2$ may or may not be positive, depending on the values of a and b. It is positive if

$$\text{Re } a > |\text{Im } a|^* \qquad (6\text{-}3)$$

It is always possible to choose a (and therefore b) so that eq. (6-3) is satisfied but, in so doing, it may not be possible to make Ev Za = Ev Zb = 0 so as to reduce the rank of the terminating impedance through zero cancellation synthesis. Thus a constraint is imposed on the synthesis of eq. (6-2) by eq. (6-4).

Following the pattern of Section 5.6 and eq. (5-40), the synthesis of eq. (6-2) may also be interpreted as the synthesis of V distributed in the following way.

$$V = \frac{\frac{Z_a}{\zeta_{1b}}\left(s^2 + \frac{ab}{D}s\right)}{m_2 + n_2} + \frac{\left[bZ_a + a\zeta_{1b} - \frac{ab}{D}\left(\frac{\zeta_{1b}}{Z_a} + \frac{Z_a}{\zeta_{1b}}\right)\right]s}{m_2 + n_2} + \frac{ab\left(1 + \frac{\zeta_{1b}s}{Z_a D}\right)}{m_2 + n_2}$$

$$= V_4 + V_2 + V_o \qquad (6\text{-}4)$$

*A similar constraint is given by Guillemin[7] for driving point impedances of all ranks to the effect that all even part numerator coefficients are positive if the even part zeros of the given impedance do not lie within 45° of the jω axis.

where $D = \frac{b}{Z_a} + \frac{a}{Z_{1b}}$ as in eq. (3-33) and

$$\text{num Ev } V_4 = s^4$$
$$\text{num Ev } V_2 = -(a^2 + b^2)s^2 \qquad (6\text{-}5)$$
$$\text{num Ev } V_o = a^2b^2$$

Eqs. (6-2), (6-3) and (6-4) raise several questions regarding distributed operator synthesis procedures which should be investigated.

1) Is it desirable to distribute the impedance operator in ways other than that of eq. (6-4) (where only one power of s appears in Ev V_4, Ev V_2 and Ev V_o). Are there other distributions of V which will always permit syntheses without transformers and/or gyrators or will allow a reduction in the number of elements required in a given synthesis?*

This latter question has been discussed by Kuh[18] with regard to splitting the even part of a driving point impedance function. Kuh's procedure splits the even part of Z into only two parts. Then reactive elements are removed from each of these parts leaving terminating impedances of reduced rank. The process is repeated on these terminations.

*It has been suggested by Darlington[20] that a useful approach would be to try to develop a synthesis without transformers and with no more than k redundant elements.

Possible application of Kuh's procedure to the impedance operator should be investigated. Furthermore, other even part separations, in which more than one power of s appears in each of the parts of the even part numerator, should be studied.

2) Is there any advantage to be gained by distributing the rank 6 impedance operator of eq. (4-6) with its three arbitrary constants? Is it possible to choose one of the arbitrary constants to simplify the distributed operator synthesis while retaining the other two constants for rank reduction of the terminating impedance? Is it possible to choose one or more of the arbitrary constants so as to significantly reduce the number of elements required in the synthesis of the distributed operator?

3) In eqs. (5-1), (5-2) and (5-3), three specific impedance operators were presented. These operators were then cascaded in various ways to achieve several cascaded operator syntheses, two of which did not require transformers. The question arises as to whether it might not be possible and desirable to develop syntheses in which one or more of these three operators is distributed. Could a synthesis procedure be developed which requires neither gyrators nor transformers, perhaps again through proper choice of one or more of the arbitrary constants?

4) All of the distributed operator syntheses suggested by items 1, 2 and 3 above should also be investigated from the Miyata synthesis point of view, in which Cauer, rather than Foster, expansions are used. It is again entirely possible that syntheses which do not require

transformers or gyrators may be achieved, although in this case undoubtedly at the expense of an additional number of elements.

B. Driving Point Impedances of Odd Rank

Throughout this thesis, driving point impedances of even rank have been considered. It has been assumed that, if impedances of odd rank were encountered, the poles and zeros at the origin and infinity could be removed prior to the application of the impedance operator approach. The question arises as to whether it is always desirable to remove such poles and zeros at the outset from a driving point impedance function. Consider, as a first example, the discussion in Appendix V. There it was shown that it is often, but not always, possible to choose the constant, c, with a and b remaining arbitrary, so as to achieve the syntheses of Figs. (4-4) and (4-5). The lack of generality in Appendix V results from the fact that Z was assumed finite and non-zero at both the origin and infinity. Suppose, for example, that Z has a zero at both the origin and infinity. Then it is always possible to find a positive real value for c such that the synthesis of Fig. (4-4) is guaranteed when v is positive and the synthesis of Fig. (4-5) when v is negative.

As a second example, consider the discussion in Appendix III. There it was shown also that it is often, but not always possible, to choose the constants a and b so as to achieve the transformerless syntheses of Figs. (A-1) and (A-2). Suppose that Z in eq. (A-35) has a pole at $s = \infty$. Then the numerator of Z will contain a term $a_{n+1}\, s^{n+1}$ and therefore eq. (A-36) will also contain this term (and its coefficient is always positive). Thus, employing the reasoning following eq. (A-36), it is always possible to find two suitable roots of eq. (A-36) except in the case where the first coefficient yields the minimum value of α. Thus the presence of the pole at infinity eliminates one-half of the exceptional cases of eq. (A-36).

As a third example, consider the network section of Fig. (6-1). Its Z parameters are

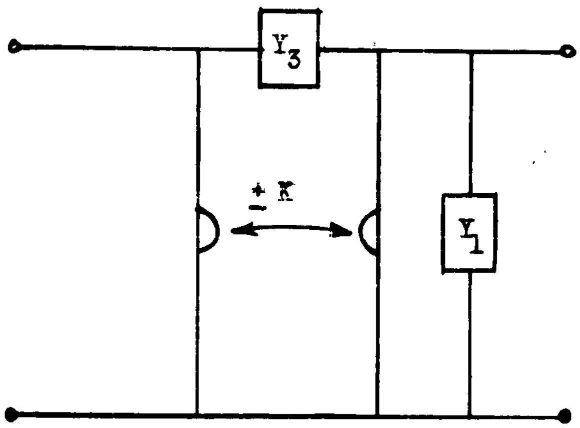

Fig. (6-1)

$$Z_{11} = \frac{Y_3 + Y_1}{Y_1 Y_3 + \frac{1}{K^2}}$$

$$Z_{22} = \frac{Y_3}{Y_1 Y_3 + \frac{1}{K^2}} \qquad (6\text{-}6)$$

$$Z_{\substack{12\\21}} = \frac{Y_3 \pm \frac{1}{K}}{Y_1 Y_3 + \frac{1}{K^2}}$$

Let $Y_1 = sC$ and $Y_3 = \frac{1}{sL}$. Then eq. (6-6) becomes

$$Z_{11} = \frac{K^2}{s(K^2C + L)} + \frac{sK^2CL}{K^2C + L}$$

$$Z_{22} = \frac{K^2}{s(K^2C + L)} \qquad (6\text{-}7)$$

$$Z_{\substack{12\\21}} = \frac{K^2}{s(K^2C + L)} \pm \frac{KL}{K^2C + L}$$

The resulting network appears in Fig. (6-2a). Now consider the network of Fig. (6-2b). By inspection, its Z parameters are given by eq. (6-7) and thus the networks of Figs. (6-2a) and (6-2b) are equivalent.

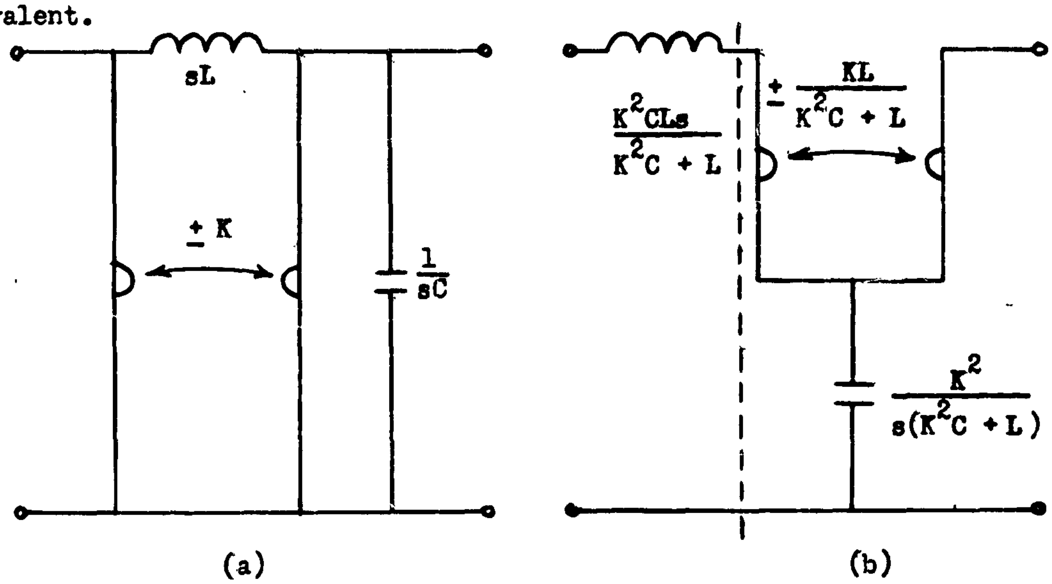

Fig. (6-2) Equivalent Networks

The network section to the right of the dotted line in Fig. (6-2b) is a familiar one, being identical in form to that of Fig. (3-2), which was obtained from the n-type synthesis of the rank 2 V operator of eq. (3-2). Thus the two parts of Fig. (6-2) suggest two options in the synthesis of a driving point impedance of odd rank which has a pole at infinity. Either the pole can be removed first and the synthesis leading to Fig. (3-2) employed or the impedance can be synthesized directly without removing the pole leading to the network of Fig. (6-2a). To handle other types of impedances of odd rank, Y_1 and Y_3 can be selected differently than in eq. (6-7).

The three above examples show a need for a further investigation of the impedance operator approach as applied to driving point impedances of odd rank.

C. Equivalent Networks

The impedance operator approach, because of the fact that the basic operation of eq. (2-1) may be applied as many times as desired in a given synthesis and because an additional arbitrary constant (s) is included with each basic operation, can yield many equivalent realizations for the same driving point impedance function. This has been illustrated throughout the thesis, notably in Chapters III and IV and in item B of Section 6.4. It has also been pointed out in Section 1.9 that equivalent realizations can sometimes be obtained in the form of bridged T, twin T, or lattices structures.[7] These results indicate the desirability of making a study of the use of the impedance operator approach in obtaining equivalent networks.

D. Four-Terminal Network Synthesis

This thesis has almost entirely concerned itself with the problem of driving point impedance synthesis. The use of the impedance operator in transfer impedance synthesis and in the synthesis of two terminal pair networks has not been discussed. Thus, the following questions may be raised.

1) Can the impedance operator approach be used with profit in synthesizing a prescribed Z_{12} in, say, a filter design problem? Does the impedance operator, with its arbitrary constants, provide a flexi-

bility in such design problems not possessed by existing synthesis procedures?

2) Can a general method be devised, using the impedance operator approach, for the realization of a network whose four-terminal immittance parameters are specified.* Do the matrix forms developed in Section 2.9 offer any advantage in the solution of this problem?

E. Pseudo-Commutative Property

The pseudo-commutative property of the V operator is discussed in Section 2.7. This property is especially interesting because it permits resistance to be included in the impedance operator. As pointed out in that section, no study has been made of methods by which a given driving point impedance might be separated to obtain V_x and γ_2 or V_y and γ_1. Such a study should be undertaken.

One example in which the separation is straightforward is afforded by refering to eqs. (3-1) and (3-2) and comparing the network of Fig. (3-2) with those of Fig. (2-6). In Fig. (2-6), let

$$\gamma_1 = \frac{aZ_a}{s}, \quad \gamma_2 = \zeta_1, \quad K = Z_a \qquad (6\text{-}8)$$

Then the networks of Fig. (2-6) become those of Fig. (6-3). Fig. (6-3a) is identical with Fig. (3-2) and Fig. (6-2b) illustrates the pseudo-commutative property in that the termination, ζ_1, which can

*Some significant results have been obtained in regard to this problem by H. J. Nain and D. Hazony in unpublished work.

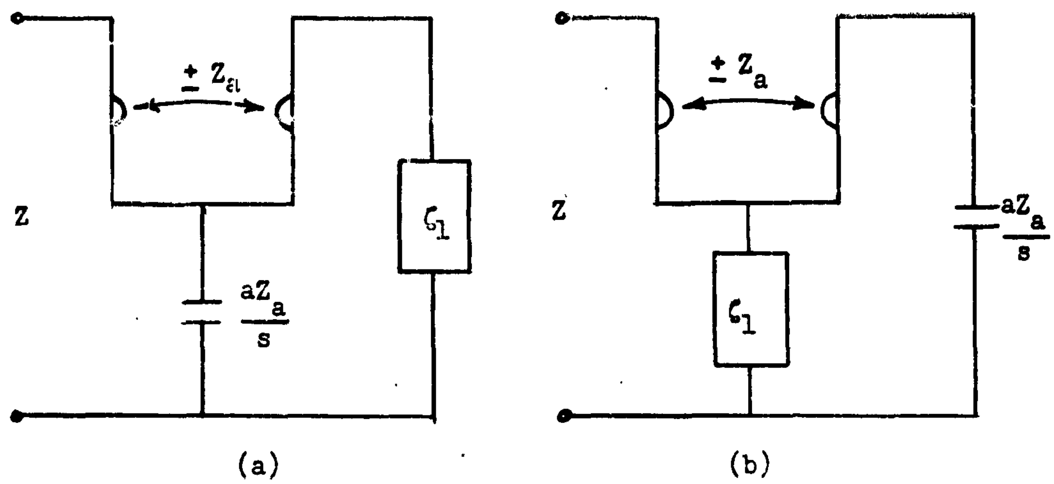

Fig. (6-3) Pseudo-Commutative Networks

contain resistance, is now included as a part of the impedance operator. The equivalence of Figs. (6-3a) and (6.3b) suggests that it may be possible to develop a Darlington-type synthesis procedure in which resistance can be included in the removed network sections but which may require a prescribed (perhaps reactive) terminating impedance. This matter merits further investigation.

APPENDIX I

Proofs of V Operator Properties

1.A <u>Proof of Theorem D</u>

Let $$V_1 = \frac{m_1 + n_1}{m_2 + n_2}, \quad V_2 = \frac{m_3 + n_3}{m_4 + n_4}, \quad V_3 = \frac{m_5 + n_5}{m_6 + n_6} \tag{A-1}$$

Then

$$V_2V_3 = \frac{m_3 \dfrac{m_5 + n_5}{m_6 + n_6} + n_3}{m_4 + n_4 \dfrac{m_5 + n_5}{m_6 + n_6}} = \frac{m_3(m_5 + n_5) + n_3(m_6 + n_6)}{m_4(m_6 + n_6) + n_4(m_5 + n_5)} \tag{A-2}$$

and

$$V_1(V_2V_3) = \frac{m_1 \left[\dfrac{m_3(m_5 + n_5) + n_3(m_6 + n_6)}{m_4(m_6 + n_6) + n_4(m_5 + n_5)}\right] + n_1}{m_2 + n_2 \left[\dfrac{m_3(m_5 + n_5) + n_3(m_6 + n_6)}{m_4(m_6 + n_6) + n_4(m_5 + n_5)}\right]}$$

$$= \frac{(m_1m_3 + n_1n_4)(m_5 + n_5) + (m_1n_3 + n_1m_4)(m_6 + n_6)}{(m_2n_4 + n_2m_3)(m_5 + n_5) + (m_2m_4 + n_2n_3)(m_6 + n_6)} \tag{A-3}$$

In a similar manner,

$$V_1V_2 = \frac{m_1 \dfrac{m_3 + n_3}{m_4 + n_4} + n_1}{m_2 + n_2 \dfrac{m_3 + n_3}{m_4 + n_4}} = \frac{m_1(m_3 + n_3) + n_1(m_4 + n_4)}{m_2(m_4 + n_4) + n_2(m_3 + n_3)} \tag{A-4}$$

and

$$(V_1V_2)V_3 = \frac{(m_1m_3 + n_1n_4) \dfrac{(m_5 + n_5)}{(m_6 + n_6)} + (m_1n_3 + n_1m_4)}{(m_2m_4 + n_2n_3) + (m_2n_4 + n_2m_3) \dfrac{(m_5 + n_5)}{(m_6 + n_6)}}$$

$$= \frac{(m_1m_3 + n_1n_4)(m_5 + n_5) + (m_1n_3 + n_1m_4)(m_6 + n_6)}{(m_2m_4 + n_2n_3)(m_6 + n_6) + (m_2n_4 + n_2m_3)(m_5 + n_5)} \tag{A-5}$$

Therefore $(V_1V_2)V_3 = V_1(V_2V_3) = V_1V_2V_3$ (A-6)

The result in eq. (A-6) may be extended by mathematical induction to include any number of prf V operators.

1.B Proof of Theorem E

Let V_1, V_2, and V_3 be given by eq. (A-1) and let V be defined by eq. (A-6). Let V also be given by

$$V = \frac{M_1 + N_1}{M_2 + N_2} \tag{A-7}$$

where M_1 and N_1 are the even and odd parts, respectively, of the numerator of V and M_2 and N_2 are likewise for the denominator of V. The numerator of the even part of V may be expressed as

$$\text{num } EvV = M_1M_2 - N_1N_2 \tag{A-8}$$

and, by either eq. (A-3) or eq. (A-5),

$$\begin{aligned} \text{num } EvV &= (m_1m_3 + n_1n_4)(m_2m_4 + n_2n_3)(m_5m_6 - n_5n_6) \\ &\quad - (m_1n_3 + n_1m_4)(m_2n_4 + n_2m_3)(m_5m_6 - n_5n_6) \\ &= (m_1m_2 - n_1n_2)(m_3m_4 - n_3n_4)(m_5m_6 - n_5n_6) \\ &= (\text{num } EvV_1)(\text{num } EvV_2)(\text{num } EvV_3) \end{aligned} \tag{A-9}$$

The proof can be extended by mathematical induction to include any number of prf V operators.

1.C Restrictions on V_2 in Equation (2-31)

$$V_2 = \frac{m_1F(s) + G(s) + n_1F(s)}{m_2F(s) + G(s) + n_2F(s)}, \quad V_1 = \frac{m_1 + n_1}{m_2 + n_2} \tag{A-10}$$

Checking V_1 and V_2 in eq. (2-30) gives:

$$\frac{n_1}{n_1F(s)} = \frac{n_2}{n_2F(s)} = \frac{m_2 - m_1}{m_2F(s) + G(s) - m_1F(s) - G(s)} \quad \text{(A-11)}$$

Therefore V_1 and V_2 are commutative.

The numerator and denominator in eq. (2-31) must each be Hurwitz for V_2 to be prf. If they are to be Hurwitz, then

$$\frac{m_1F(s) + G(s)}{n_1F(s)} \text{ and } \frac{m_2F(s) + G(s)}{n_2F(s)} \quad \text{(A-12)}$$

must be reactance functions. By expanding each term it follows that

$$\frac{G(s)}{n_1F(s)} \text{ and } \frac{G(s)}{n_2F(s)} \quad \text{(A-13)}$$

must be reactance functions. In addition,

$$\text{num } EvV_2 \geq 0 \quad \text{(A-14)}$$

everywhere on the $j\omega$ axis. For eq. (A-14) to be satisfied requires that

$$(m_1m_2 - n_1n_2)\ F(s) + G(s)\ F(s)\ (m_1 + m_2) - G^2(s) \geq 0 \quad \text{(A-15)}$$

everywhere on the $j\omega$ axis (and hence in the right half plane).

1.D <u>PRF Nature of V_x and V_y</u>

Consider the operator V_x as given by eq. (2-35).

$$V_x = \frac{(m_1 + K^2m_2) + (n_1 + K^2n_2)}{(m_1 + m_2) + (n_1 + n_2)} \quad \text{(A-16)}$$

For V_x to be prf, its numerator and denominator must be Hurwitz and num $EvV_x \geq 0$ everywhere on the $j\omega$ axis.

The ratio of the even to odd parts of a Hurwitz polynomial is a reactance function. Applying this test to the numerator of V_x gives

$$\frac{m_1 + K^2 m_2}{n_1 + K^2 n_2} = \frac{1}{\dfrac{n_1}{m_1} + \dfrac{K^2 n_2}{m_1}} + \frac{1}{\dfrac{n_1}{K^2 m_2} + \dfrac{n_2}{m_2}} \tag{A-17}$$

Each of the four terms on the right hand side is a reactance function. Therefore the numerator of V_x is Hurwitz and similarly for the denominator. Checking the even part relationship gives

$$\begin{aligned} \text{num } EvV_x &= (m_1 + K^2 m_2)(m_1 + m_2) - (n_1 + K^2 n_2)(n_1 + n_2) \\ &= m_1{}^2 - n_1{}^2 + K^2(m_2{}^2 - n_2{}^2) + (K^2 + 1)(m_1 m_2 - n_1 n_2). \end{aligned} \tag{A-18}$$

Each of the terms on the right hand side of eq. (A-18) is always positive on the $j\omega$ axis (and therefore in the right half plane). Therefore V_x is prf.

A similar proof shows that V_y is prf.

1.E <u>Syntheses of the Networks of Figure (2-6)</u>

Let Z_1 in eq. (2-38) be rewritten as

$$Z_1 = \frac{m_1 + n_1}{m_2 + n_2} \, \frac{\gamma_2 + K^2 \dfrac{m_2 + n_2}{m_1 + n_1}}{\dfrac{m_1 + n_1}{m_2 + n_2} + \gamma_2} \tag{A-19}$$

The form of Z_1 in eq. (A-19) can be matched to that in eq. (2-5) to yield

$$Z_{11} = Z_{22} = \frac{m_1 + n_1}{m_2 + n_2} = \gamma_1 \qquad \text{(A-20)}$$

$$K^2 \frac{m_2 + n_2}{m_1 + n_1} = \frac{\left(\frac{m_1 + n_1}{m_2 + n_2}\right)^2 - Z_{12}Z_{21}}{\frac{m_1 + n_1}{m_2 + n_2}}$$

$$Z_{\substack{12\\21}} = \frac{m_1 + n_1}{m_2 + n_2} \pm K = \gamma_1 \pm K \qquad \text{(A-21)}$$

The network of Fig. (2-6a) follows directly.

To obtain the network of Fig. (2-6b), eq. (2-39) is rewritten in the form of eq. (A-19) and the procedure repeated.

APPENDIX II

Use of Surplus Factors in the Synthesis of V Operators

Let Z be given by

$$Z = \frac{s + 1}{s + 4} \qquad \text{(A-23)}$$

$$\text{num Ev}Z = 4 - s^2 \qquad \text{(A-24)}$$

Num EvZ is not a perfect square but can be made so if Z is multiplied by $\frac{s + 2}{s + 2}$.

$$Z = \frac{(s + 1)(s + 2)}{(s + 4)(s + 2)} \qquad \text{(A-25)}$$

$$\text{num Ev}Z = (4 - s^2)^2 \qquad \text{(A-26)}$$

Z may now be synthesized by the customary reciprocal Darlington procedure with a one-ohm resistive termination.

Now let Z by given by

$$Z = \frac{m_1\zeta_1 + n_1}{m_2 + n_2\zeta_1} = \frac{\zeta_1 + s}{4 + s\zeta_1} \qquad \text{(A-27)}$$

$$V_1 = \frac{s + 1}{s + 4} \qquad \text{(A.28)}$$

Following the above pattern, V is multiplied by $\frac{s + 2}{s + 2}$ to yield

$$V_1 = \frac{(s + 1)(s + 2)}{(s + 4)(s + 2)} = \frac{s^2 + 3s + 2}{s^2 + 6s + 8} \qquad \text{(A-29)}$$

and

$$V_1\zeta_1 = \frac{(s^2 + 2)\,\zeta_1 + 3s}{s^2 + 8 + 6s\,\zeta_1} \neq Z \qquad \text{(A-30)}$$

Eq. (A-30) does not give the same value of Z as eq. (A-27) and therefore, in general,* it is not permissible to utilize surplus factors in the case of the V operator.

- -

* V may always be multiplied by the ratio of two equal even polynomials in s without changing Z.

APPENDIX III

Simplifications in the Syntheses of Rank 4 Operators

As mentioned in Sections 3.7 and 3.8, it is often, but not always, possible to simplify the networks of Figs. (3-13), (3-14), and (3-15) by proper choice of the arbitrary constants. These simplifications require that $Z_a = \zeta_{1b}$. The ratio ζ_{1b}/Z_a may be obtained from eq. (3-23) and equated to unity, namely

$$\frac{\zeta_{1b}}{Z_a} = \frac{aZ_b - bZ_a}{aZ_a - bZ_b} = 1 \qquad \text{(A-31)}$$

or

$$(a + b)(Z_a - Z_b) = 0 \qquad \text{(A-32)}$$

Eq. (A-32) gives three alternative conditions.

$$\begin{aligned} b &= + a \text{ and } Z_a' = 0 \\ b &= - a \text{ and } EvZ_a \neq 0 \\ Z_a &= Z_b \text{ (i.e. } Z_a \text{ and } Z_b \text{ real and equal)} \end{aligned} \qquad \text{(A-33)}$$

The condition b = + a is not sufficient by itself to make $Z_a = \zeta_{1b}$. From the foonote to eq. (3-16), it is also necessary that $Z_a' = 0$. Similarly, the condition b = - a is insufficient. From the footnote to eq. (3-18), it is also necessary that $EvZ_a \neq 0$.

The first condition in eq. (A-33) requires that a and b be positive real, while the second condition requires them to be imaginary. The first condition cannot always be satisfied (consider the simple case Z = s). Assuming proper choice of a, the second condition can always be satisfied but leads to the trivial result of eq. (3-18). The third condition requires that the equation

$$Z - \alpha = 0 \tag{A-34}$$

yield two roots of s ($Z - Z_a = 0$ and $Z - Z_b = 0$) which are positive real or complex conjugates with non-negative real part. To investigate when eq. (A-34) is valid, let Z in its most general form be substituted therein.

$$\frac{a_o + a_1 s + \ldots\ldots a_n s^n}{b_o + b_1 s + \ldots\ldots b_n s^n} - \alpha = 0 \tag{A-35}$$

$$(a_o - b_o\alpha) + s(a_1 - b_1\alpha) + \ldots\ldots\ldots s^n(a_n - b_n\alpha) = 0 \tag{A-36}$$

One of the coefficients in eq. (A-36) will yield a minimum value of α for which that term and all others are positive. Let α be chosen to make that term zero. If the term chosen is an interior term, the resulting polynomial is non-Hurwitz (an interior term is missing) and must have at least one zero in the right half plane. But all the coefficients in eq. (A-32) are positive for the particular choice of α and thus the equation can have no positive real roots. Therefore any right half plane zeros must be complex conjugates and there must be at least one such pair. The same argument applies if α is chosen to make all coefficients negative except the one which becomes zero.

Thus, except for the case where the first coefficient yields the minimum α and the last coefficient the maximum α (or vice versa), it is possible to find at least two roots, s = a and s = b, which are complex conjugates with a non-negative real part such that $Z_a = Z_b$.

When eq. (A-31) holds, the network of Fig. (3-13) reduces to that of Fig. (A-1) but the a and b constants are no longer completely arbitrary.* The components of Fig. (A-1) are derived from eq. (3-33) with $\zeta_{1b} = Z_a$ and therefore $D = \frac{1}{Z_a}(a + b)$.

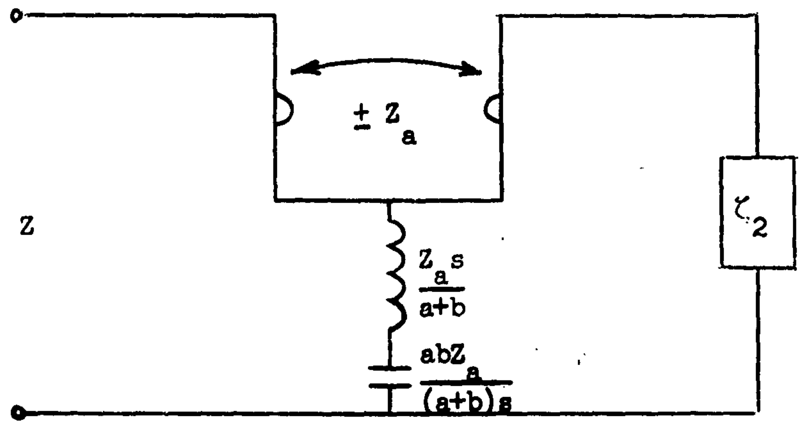

Fig. (A-1) Eliminating the Transformer from Fig. (3-13).

It is interesting to note that at $s = \pm j\sqrt{ab}$, the network of Fig. (A-1) reduces to a gyrator terminated in ζ_2 $(Z = Z_a^2/\zeta_2)$, which illustrates the inverting property of the gyrator.

Similarly, when eq. (A-31) holds, the networks of Figs. (3-14) and (3-15) are simplified since the non-reciprocal term in eq. (3-37) reduces to a single gyrator. Then, choosing $k = Z_a$, V in eq. (3-35) becomes

$$V = \frac{s^2 Z_a^2 + Z_a(a + b) + Z_a^2 ab}{s^2 + sZ_a(a + b) + ab} \tag{A-37}$$

$$\text{num Ev}V = Z_a^2 (s^2 + ab)^2 - Z_a^2 s^2(a + b)^2 \tag{A-38}$$

The components of V are

---

* Generally, there is a range of a and b values which satisfies eq. (A-34).

$$V_{11} = V_{22} = \frac{sZ_a(a+b)}{s^2+ab}$$

$$V_{\substack{12\\21}} = \frac{sZ_a(a+b)}{s^2+ab} \pm Z_a \qquad \text{(A-39)}$$

The resulting network appears in Fig. (A-2). Again at $s = j\sqrt{ab}$, Z becomes $Z_a^{\;2}/\zeta_2$ as in Fig. (A-1).

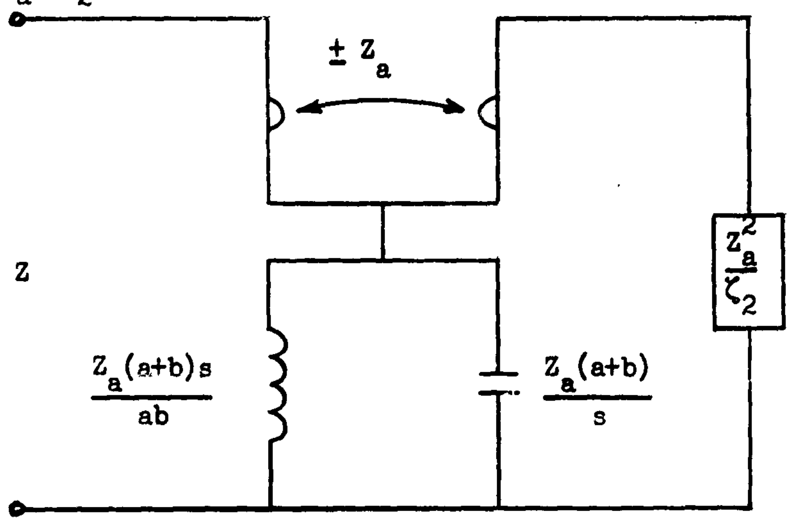

Fig. (A-2) Eliminating the Loaded Gyrators from Figs. (3-14) and (3-15).

APPENDIX IV

Proof of Theorem K

Theorem K: If a and c are complex conjugates with a non-negative real part and b is positive real, then ζ_3 as given in eq. (4-4) and V as given by equation (4-6) are prf.

Proof: If it can be shown that the coefficients of the V operator in eq. (4-6) are unchanged by a permutation of the constants a, b, and c, then V and ζ_3 remain prf as explained in Section 4.2. The coefficient $C_3 = (Z_a\zeta_2 c)/\zeta_{1b} = 1/D_3$ is examined as follows:

$$\frac{Z_a\zeta_{2c}}{\zeta_{1b}} = Z_a \frac{b\zeta_{1c} - c\zeta_{1b}}{b\zeta_{1b} - c\zeta_{1c}} \tag{A-40}$$

$$\frac{\zeta_{1c}}{Z_a} = \frac{aZ_c - cZ_a}{aZ_a - cZ_c} \tag{A-41}$$

$$\frac{\zeta_{1b}}{Z_a} = \frac{aZ_b - bZ_a}{aZ_a - bZ_b} \tag{A-42}$$

Combining these three equations yields

$$\frac{Z_a\zeta_{2c}}{\zeta_{1b}} = \frac{bZ_aZ_c(a^2 - c^2) + aZ_bZ_c(c^2 - b^2) + cZ_aZ_b(b^2 - a^2)}{bZ_b\,(a^2 - c^2) + aZ_a\,(c^2 - b^2) + cZ_c\,(b^2 - a^2)} \tag{A-43}$$

Letting a = b, b = c and c = a in eq. (A-43) produces no change in this V coefficient. The remaining V coefficients can be shown to obey the same rule. Thus a permutation of a, b and c produce no change in V and hence no change in ζ_3.

APPENDIX V

Simplifications in the Synthesis of a Rank 6 Operator

Eqs. (4-21) and (4-23), which must be satisfied if the networks of Figs. (4-4) and (4-5) are to be valid, are repeated below.

$$aZ_a(b - c) + bZ_b(c - a) + cZ_c(a - b) = 0 \qquad (4\text{-}21)$$

$$Z_a(b - c) + Z_b(c - a) + Z_c(a - b) = 0 \qquad (4\text{-}23)$$

Let Z_{cn} be the value of Z_c which makes eq. (4-21) valid (n-type synthesis) and likewise let Z_{cm} be the value of Z_c which makes eq. (4-23) valid (m-type synthesis). Then, solving eqs. (4-21) and (4-23) for Z_c gives

$$Z_{cn} = -\frac{ab(Z_a - Z_b)}{c(a - b)} + \frac{aZ_a - bZ_b}{a - b} \qquad (A\text{-}44)$$

$$Z_{cm} = \frac{c(Z_a - Z_b)}{a - b} + \frac{aZ_b - bZ_a}{a - b} \qquad (A\text{-}45)$$

To investigate whether a positive real c can be found such that eqs. (A-44) and (A-45) are satisfied with a and b arbitrary, define the following.

$$a = x + jy \qquad b = x - jy$$
$$Z_a = u + jv \qquad Z_b = u - jv \qquad (A\text{-}46)$$

Then eqs. (A-44) and (A-45) become

$$Z_{cn} = -\frac{v}{cy}(x^2 + y^2) + \frac{uy + vx}{y}\ ^* \qquad (A\text{-}47)$$

---

* $|\arg Z(s)| < |\arg s|$ for $0 < |\arg s| < \frac{\pi}{2}$ [3]. Therefore $uy - vx = ux\left(\frac{y}{x} - \frac{v}{u}\right)$ is always positive. It follows that $uy + vx$ is also always positive.

$$Z_{cm} = c\,\frac{v}{y} + \frac{uy - vx}{y}^{*} \qquad \text{(A-48)}$$

In eq. (A-46), x and u are always positive and y may always be chosen positive. Then v may be either positive or negative. Eqs. (A-47) and (A-48) are plotted in Fig. (A-3) for v positive and in Fig. (A-4) for v negative. The curves are shown as never crossing which may be verified by noting that the difference $Z_{cm} - Z_{cn}$ does not change sign for all positive real c.

Let Z_c be expressed in the general form

$$Z_c = \frac{a_o + a_1 c + \ldots.\ a_n c^n}{b_o + b_1 c + \ldots.\ b_n c^n} \qquad \text{(A-49)}$$

and require that none of the coefficients vanish (Z has no pole or zero at the origin or infinity). Intersections of eq. (A-49) with eqs. (A-47) and (A-48) in the first quadrants of Figs. (A-3) and (A-4) are the desired solutions.

For small values of c, Z_c in eq. (A-49) approaches

$$Z_c = \frac{a_o}{b_o} \qquad \text{(A-50)}$$

while for large values of c, Z_c approaches

$$Z_c = \frac{a_3}{b_3} \qquad \text{(A-51)}$$

---

* $|\arg Z(s)| < |\arg s|$ for $0 < |\arg s| < \frac{\pi}{2}$ [3]. Therefore $uy - vx = ux\left(\frac{y}{x} - \frac{v}{u}\right)$ is always positive. It follows that $uy + vx$ is also always positive.

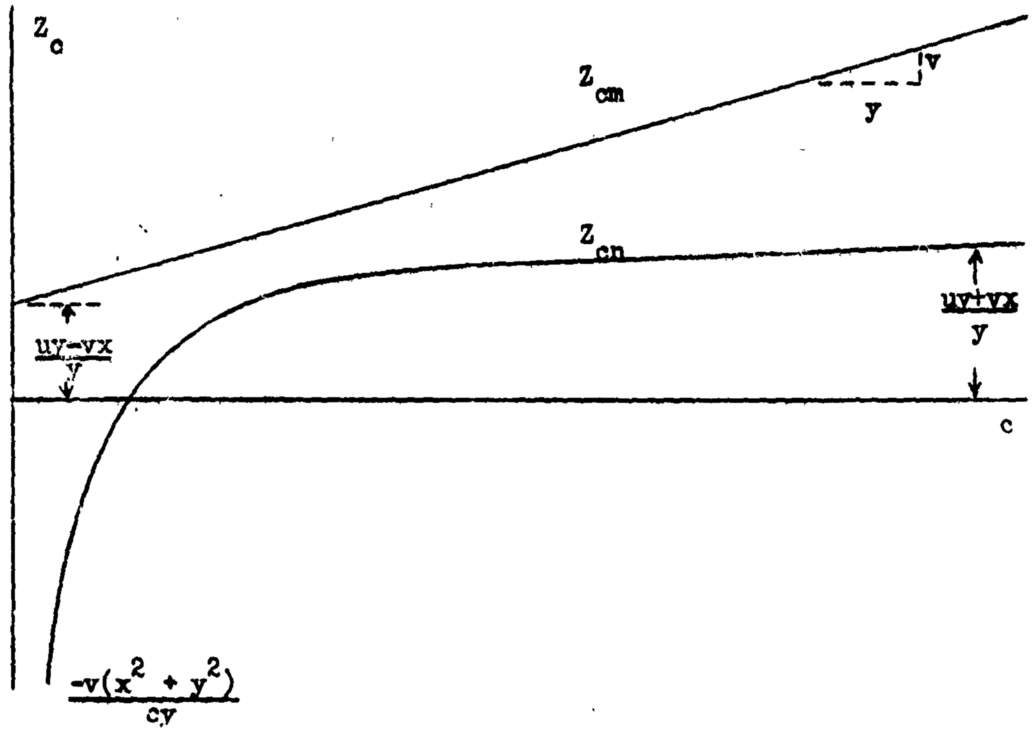

Fig. (A-3) Z_{cn} and Z_{cm} vs c for Positive v

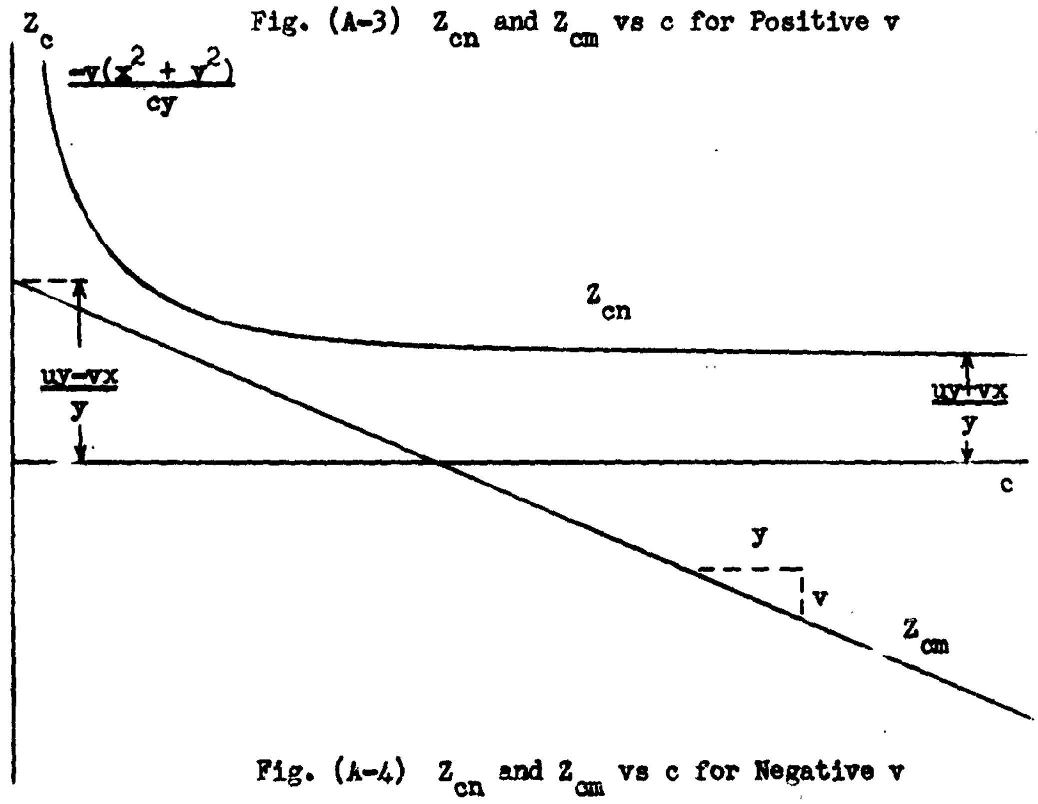

Fig. (A-4) Z_{cn} and Z_{cm} vs c for Negative v

Let

$$\frac{uy - vx}{y} = F \quad , \quad \frac{uy + vx}{y} = G \tag{A-52}$$

Four cases must be considered in each figure. These are tabulated as follows:

		Fig. (A-3)	Fig. (A-4)
1.	$\frac{a_o}{b_o} > F$, $\frac{a_n}{b_n} > G$	Z_c must intersect Z_{cm}	Z_c must intersect Z_{cn}
2.	$\frac{a_o}{b_o} > F$, $\frac{a_n}{b_n} < G$	Z_c must intersect both Z_{cm} and Z_{cn}	no intersection guaranteed
3.	$\frac{a_o}{b_o} < F$, $\frac{a_n}{b_n} > G$	no intersection guaranteed	Z_c must intersect both Z_{cm} and Z_{cn}
4.	$\frac{a_o}{b_o} < F$, $\frac{a_n}{b_n} < G$	Z_c must intersect Z_{cn}	Z_c must intersect Z_{cm}

In every instance except Case 3 for a positive v and Case 2 for a negative v, an intersection is guaranteed and thus a positive real value of c exists with a and b arbitrary such that either the n-type section of Fig. (4-4) or the m-type section of Fig. (4-5) may be removed from Z with a and b arbitrary. In these two cases, intersections may exist but are not guaranteed. This last statement is illustrated in Example 3.

Example 1

Synthesize Z given by

$$Z = \frac{s^3 + \frac{14}{3}s^2 + 2s + 4}{s^3 + 4s^2 + \frac{44}{3}s + 4}$$

where

$$\text{num } EvZ = (2 - s^2)(s^2 + 2s + 2)(s^2 - 2s + 2)$$

Choosing a and b so that $EvZ_a = EvZ_b = 0$ makes ζ_3 four less in rank than Z.

$$a = 1 + j \quad , \quad b = 1 - j$$

Then

$$Z_a = \frac{8 + j2}{17} \qquad Z_b = \frac{8 - j2}{17}$$

Thus

$$x = 1,\ y = 1,\ u = \frac{8}{17},\ v = \frac{2}{17}$$

This is a positive v, Case 1 example and therefore an m-type synthesis is guaranteed.* Solving eq. (4-23) for c gives

$$c = \frac{aZ_b - bZ_a + (b - a)\ Z_c}{Z_b - Z_a} = \frac{-6 + 17Z_c}{2}$$

which yields

$$(2c - 7)(c + 4)(c^2 - 2c + 2) = 0$$

The positive real root is $c = \frac{7}{2}$.

---

*No n-type synthesis is possible in this case.

The calculations then yield

$$\zeta_{1b} = \frac{24 - j6}{85} \quad , \quad \frac{Z_a}{\zeta_{1b}} = \frac{5}{3}$$

$$\zeta_2 = \frac{(2 + \frac{3}{5} s^2)Z - \frac{4}{5} s}{(2 + \frac{5}{3} s^2) - \frac{17}{3} sZ} = \frac{9s + 30}{25s + 15}$$

and from the above relations and eqs. (4-3), (4-5) and (4-6)

$$\zeta_{2c} = \frac{3}{5}, \; \zeta_3 = \frac{s + 2}{s + 1}, \; k = 1$$

$$V = \frac{s^3 + \frac{43}{6} s^2 + 4s + 7}{s^3 + \frac{11}{2} s^2 + \frac{139}{6} s + 7}$$

where

$$\text{num EvV} = (\frac{11}{2} s^2 + 7)^2 - s^2(s^2 + 9)^2$$

The synthesis of V can now be obtained directly through substitution in eqs. (4-14), (4-25) and (4-27). The result appears in Fig. (A-5).

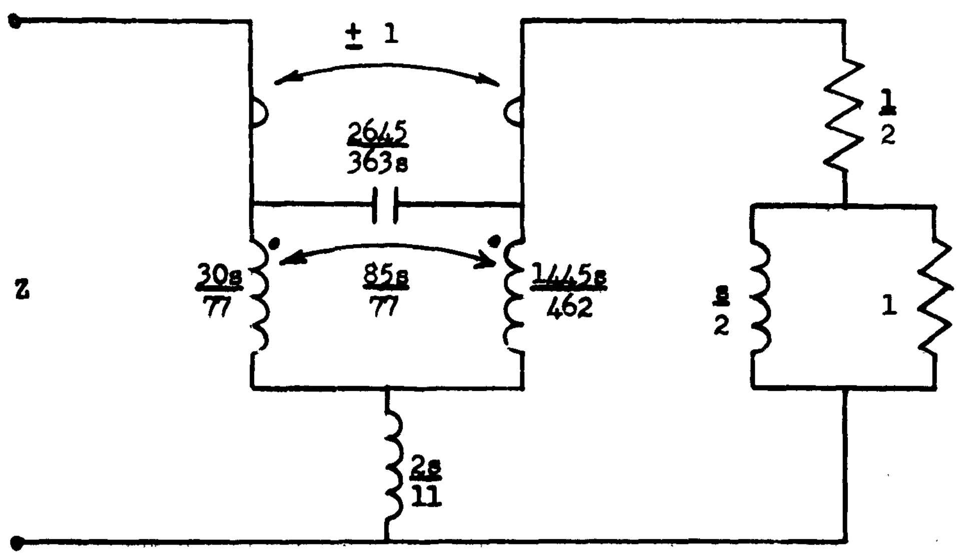

Fig. (A-5) m-Type Rank 6, c = 7/2.

Example 2

Synthesize Z given by

$$Z = \frac{s^3 + 2s^2 + 6s + 4}{s^3 + 7s^2 + 6s + 2}$$

where

$$\text{num EvZ} = (2 - s^2)(s^2 + 2s + 2)(s^2 - 2s + 2)$$

Choosing a and b as in example 1,

$$a = 1 + j \quad , \quad b = 1 - j$$

Then

$$Z_a = \frac{3 - j}{5} \quad , \quad Z_b = \frac{3 + j}{5}$$

Thus

$$x = 1,\ y = 1,\ u = \frac{3}{5},\ v = -\frac{1}{5}$$

This is a -v Case 1 example and thus an n-type synthesis leading to Fig. (4-4) is guaranteed. The required value of c is $\sqrt{6}/3$, which is obtained from eq. (4-21). No m-type synthesis is possible.

Example 3

Synthesize Z given by

$$Z = \frac{s^3 + 2s^2 + 9s + 1}{s^3 + 9s^2 + 5s + 16}$$

where

$$\text{num EvZ} = (4 - s^2)(s^2 + 2s + 2)(s^2 - 2s + 2)$$

Again choosing a and b as in examples 1 and 2

$$Z_a = \frac{31 + 5j}{58} \quad , \quad Z_b = \frac{31 - 5j}{58}$$

This is a +v Case 3 example and thus neither type of synthesis is guaranteed and a further check is necessary.

For an n-type synthesis, from eq. (4-21)

$$c = \frac{ab(Z_b - Z_a)}{bZ_b - aZ_a + (a - b)Z_c} = \frac{5}{18 - 29Z_c}$$

which reduces to

$$(c^2 - 2c + 2)(11c^2 - 77c + 40) = 0$$

The roots are

$$c = 1 + j,\ 1 - j,\ .56,\ 6.43$$

There are two positive real roots and therefore an n-type synthesis leading to Fig. (4-4) is guaranteed.

For an m-type synthesis, using eq. (4-23)

$$c = \frac{aZ_b - bZ_a + (b - a)Z_c}{Z_b - Z_a} = \frac{-26 + 58Z_c}{5}$$

which reduces to

$$(c^2 - 2c + 2)(- 5c^2 - 23c - 179) = 0$$

The roots are

$$c = 1 + j,\ 1 - j,\ 2.3 + j5.5,\ 2.3 - j5.5$$

There are no positive real roots and thus an m-type synthesis leading to Fig. (4-5) is impossible for this choice of a and b.

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