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# SURFACE-TO-AIR GUIDED MISSILE SYSTEMS METHODS OF TACTICAL ANALYSIS

by
M. C. Waddell

APPLIED PHYSICS LABORATORY

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# SURFACE-TO-AIR GUIDED MISSILE SYSTEMS
# METHODS OF TACTICAL ANALYSIS

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M. C. Waddell



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# PREFACE

The present volume deals with the tactical analysis of surface-to-air missile systems; it is intended to provide a general introduction to the basic problems arising in such analyses, and to suggest techniques useful in their solution. Regretfully, all problems concerned with the possible defeat of surface-to-air missile systems by means of electronic countermeasures have intentionally been omitted from the discussion; appropriate treatment of countermeasures is not possible in an unclassified document.

The material here summarized has resulted from work carried out by the Assessment Division of the Applied Physics Laboratory and by many . : . organizations, over a period of years, with frequent interchange of ideas both in personal discussions and by written report. Consequently it is usually difficult and frequently impossible to determine individual priorities and ascribe credits for particular parts of the material. For this reason, no attempt is made here to do so.

In preparing this volume, the author has drawn heavily upon unclassified parts of an earlier classified work, Tactical Analysis of Surface-to-Air Guided Missile Systems, compiled by the Assessment Division of the Laboratory. Indeed, the present volume may be thought of as an unclassified revision of the earlier work. The author is indebted to those who compiled the earlier work and to many in the Division who reviewed and commented upon the present volume.

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# CHAPTER I

## Tactical Analysis

### A. Introduction

Before proceeding with the primary purpose of this book, the discussion of methods of tactical analysis of guided missile systems, it will be well to consider briefly the needs for such analysis and what is to be analyzed.

### B. Need for Tactical Analysis

A major need for tactical analysis is in the development of new weapon systems. The role tactical analysis plays in such a development can perhaps best be brought out by tracing a typical development process. In general, this process is iterative and can be conveniently represented by a multi-loop feed-back diagram as shown in Figure 1.

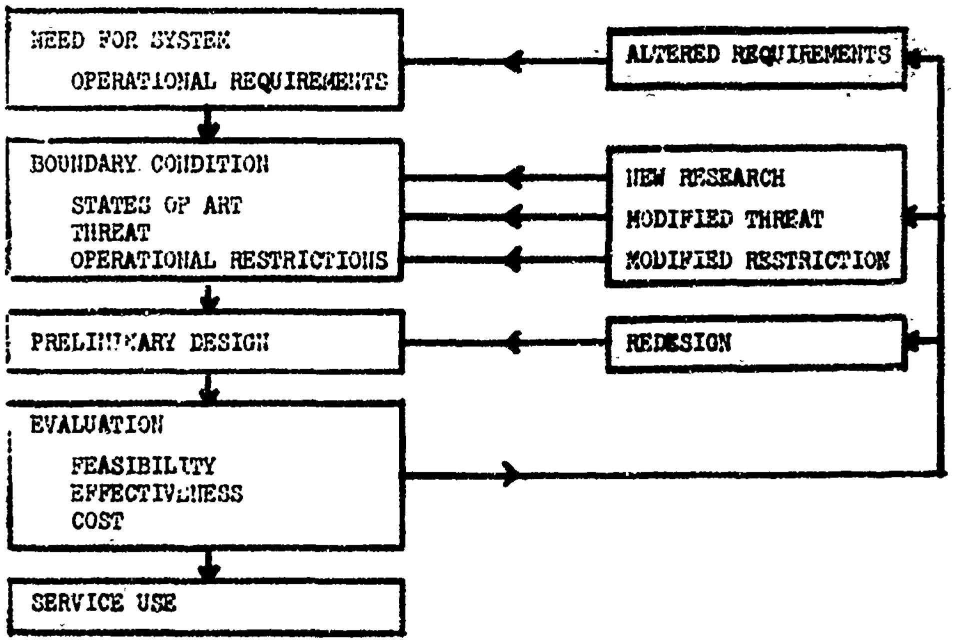



Figure 1

The loops are traversed several times in the process, as a consequence of the many interactions of all the factors involved in the design. The process may start in any of several different places, for example, with an operational requirement, or in an advance in the state of a technical art, or with a threat analysis, or perhaps with some existing system design.

In order to describe the development process in greater detail, suppose that it starts with an (initially broad) statement of the need for a system, answering the question: What do we wish to do? Next must come a listing of the boundary conditions existing; that is: Where are we now? The boundary conditions include the states of the various technical arts involved, the characteristics of the threat to be met and the various operational restrictions that may obtain. With answers to these two questions: Where do we go? and Where are we? the next step is to devise one or more preliminary system designs which are presumed to fill the need. Each of these designs must then be evaluated to determine whether it is feasible from all points of view and whether it indeed fills the stated need effectively and at reasonable cost in resources. The results of the evaluation are then fed back as indicated in the figure, leading, on the next circuit, to refined system designs and to their evaluation.

It is in the evaluation phase of the development process that tactical analysis enters. In particular, a measure of the effectiveness of each proposed system is a product of tactical analysis of the system as a whole, operating in the specified environment against the specified threat. The results of the tactical analyses may be put to several uses. First, they form the basis for a comparison of alternative system designs. They may suggest changes or improvements in the system leading to redesign of the system, and perhaps to further research in order to improve the state of an applicable art. They may suggest changes in the expected threat that would result from an enemy's reaction to the introduction of the proposed system in service use. They may suggest changes in the operating restrictions assumed. Finally they may suggest changes in the operational requirement; that is, the need for or purpose of the system.

In the early stages of the development process the system design will usually be simple, with little detail specified. The corresponding tactical analysis must of necessity be relatively crude and simple, owing to the lack of sufficiently complete system definition. As the system design is refined on successive circuits of the loop, the tactical analysis must be refined to deal with the greater detail of the design. Gross approximations must be replaced by more precise analyses.

A system, in very general terms, is a complex of people and equipment organized to perform one or more functions. It is made up of many subsystems which interact with each other and with the environment. A system design must specify not only what the system will consist of but also how it will operate, what the interactions of the subsystems and their reactions to external environmental influences will be. Early in the development process, tactical analysis deals primarily with predictions of what the system will be. Suggestions for change, coming out of the analysis, will most often involve the equipment itself. As the development process progresses, many parts of the system will have taken physical form. Consequently in the later stages tactical analyses will more often affect the procedures laid down for operation of the system.

Tactical analyses can be of great help to military planners as well as to weapon system designers. The strategic planner, in deciding how best to achieve a military objective, can learn much from the results of tactical analysis. What weapons will be most effective? What forces will be required? Tactical analyses can assist the procurement planner in determining what quantities of weapons to purchase. The tactical commander, in deploying his forces and selecting tactics to be used, can derive much useful information from analyses.

In all places where tactical analysis can be applied, decisions still must ultimately be based upon sound judgment. But in rendering sound judgment, it is preferable to rely on rational analysis than on personal intuition. The former, if indeed rational, will by definition, demonstrate that certain basic assumptions must lead to certain conclusions. Thus, the problem of passing judgment can be reduced, by rational analysis, to that of judging the validity of basic assumptions.

C. Typical Air Battle

The forerunner to any tactical analysis must quite naturally be a clear understanding of what is to be analyzed. In the present case, this is the air battle. Its principal elements are: surface targets which are subject to air attack, active defense installations to counter the attack, and the attack itself. These elements interact, in an environment generated in part by them, in a sequence of events which make up the air battle. Figure 2 depicts these air battle elements schematically.

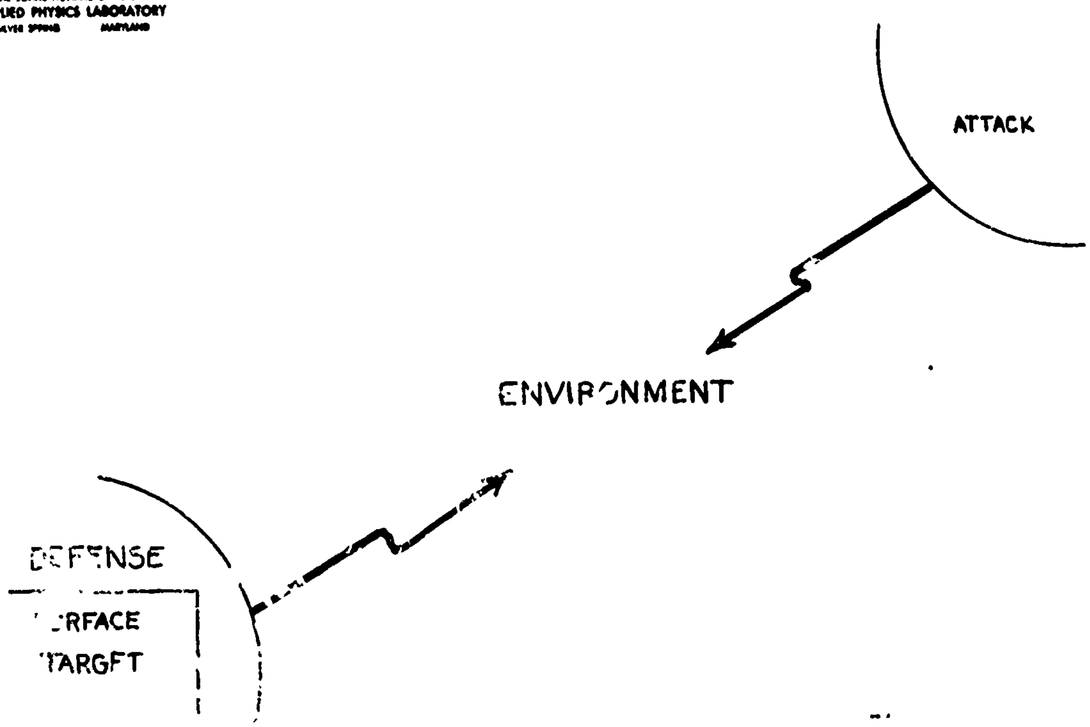



Figure 2

The present section will describe the major characteristics of these elements.

The first necessary ingredient for an air battle is, of course, the existence of one or more surface targets which an enemy wishes to destroy. These targets might be cities, industrial installations, air bases, groups of ships, or the like. Although very different, they have certain common characteristics such as the area of each, their geographic distribution, the value of each to the defender and to the attacker, the hardness of each (i.e., ability to withstand damaging forces). These characteristics will influence both the defense and offense Deployment of defensive weapons will depend in part on surface target geometry. Commitment of offensive weapons to individual targets may be affected by the hardness and also the worth of the targets. Most of the surface target characteristics of major importance in tactical analyses are common to most surface targets. Thus general methods of analysis, that are independent of the identity of the surface target, can often be devised.

A second essential ingredient for an air battle is the active air defense system provided for the defense of the ground targets. The principal functions of the defense are detection of targets and their surveillance, decision making, and employment of weapons. Detection and surveillance are performed by any of a variety of sensing instruments, such as radars. Their performance in general is characterized by their coverage in range and altitude, by the accuracy with which they measure target position and by their ability to resolve individual targets. A supporting part of the decision-making function is the processing and storing of data. The completeness, speed, and accuracy with which these tasks are done characterize the performance of this support. The decision making itself is done either manually, with the aid of suitable data displays, or automatically, usually by electronic computers. The decisions themselves are tactical in nature. The weapons are characterized by such things as coverage, range and altitude, by the number of weapons that can be brought to bear and by the effectiveness of the weapon in destroying its target. Communications facilities for the transmission of pertinent target and other data and of orders are essential to all functions. The important features of these facilities are their reliability and their accuracy. The many pieces of equipment necessary for surveillance, for data processing, and for the weapons systems themselves must be deployed in some manner about the surface targets being defended. A further necessary provision in the active defense is that of doctrines of use, insofar as these can be prescribed in advance.

The third ingredient in the air battle is, of course, the attack. A full description of an attack will include the numbers and kinds of aircraft, of warheads to be delivered and of delivery techniques used, as well as any other equipment the enemy may employ, such as jammers. The performance capabilities of these must all be included as well as the tactics the enemy will use. Included in this last are such things as the coordination of his attack and the extent to which he uses formations or evasive maneuvers.

The environment within which the air battle takes place can be an all important factor in determining the outcome of the battle. The environment is in part comprised of conditions which would obtain independent of the battle; notable among these is the weather, including winds, temperature, cloud cover, and numerous other features of the weather. Other characteristics of the environment are determined by the attacking force or the defense. If, for example, the attacker employs electronic countermeasures of some sort, the environment within which the radars of the defense operate is, of course, significantly altered.

Given these three essential ingredients and the environment within which they must function, the air battle then takes place. Initially the attacking force will proceed toward the surface targets, searching for targets. Certain tactical decisions by the offense are necessary during the course of battle; these include identifications of surface targets following detection, choice of tactics to be used and in some cases, in later stages, assessment of damage inflicted. On the part of the defense, the events are as follows; the attack is detected and tracks established on individual aircraft or groups of aircraft. As data are gathered from different sources, they must be processed, correlated, and the necessary displays generated. Various tactical decisions must then be made such as identification, evaluation and assignment of targets to weapon units. Finally, targets are engaged, weapons are launched and fly out to intercept, warheads detonated, and if possible damage is assessed. The battle thus proceeds until all attacking aircraft are either killed or deliver their bombs and depart, or until the defense becomes inactive, through damage suffered or through exhausting its supply of weapons.

Many of the events which occur during the battle are random events, e.g. detecting a target, or inflicting lethal damage on a target. Consequently the outcome of a particular battle is a chance event; and so one speaks of the probability of a particular outcome occurring, e.g. all attacking aircraft are killed, or a specified number survive. This element of uncertainty is a major contributor to the difficulty encountered in carrying out a tactical analysis.

D. Problems of Tactical Analysis

The purpose of a tactical analysis is to provide a suitable quantitative measure of the effectiveness of a guided missile system in defending surface targets. Two problems arise at the outset, 1) the choice of a proper measure of effectiveness, and 2) the choice of a proper method of analysis, i.e., means for getting from the characteristics of the air battle elements to the measure. In making both choices, care must be exercised to insure that effects of all significant characteristics are included, so that the calculated measure will be meaningful. The measure must reflect the true purpose of the defense, and the method must not require excessive computational labor.

The remaining chapters of this book are devoted to the two problems mentioned above. The choice of a measure of effectiveness is considered in some detail in Chapter VII. Whatever the choice of measure, it must depend upon kill probability, firepower, and coordination of fire. By kill probability is meant the probability that, once undertaken, an engagement of a target will result in damage to that target. By firepower is meant the number of target engagements the defense is capable of during the attack (the particular number is subject to statistical fluctuation and hence we refer to the expected number of engagements). By coordination of fire is meant the degree to which overkilling of targets, with consequent waste of missiles and firing time, is avoided. Each of these three is treated at length, in turn, in Chapters III, IV, V, VI. As a prelude to the treatment, the characteristics of a surface-to-air guided missile system are described in Chapter II.

# CHAPTER II

## SAM System Characteristics

### A. Introduction

A surface-to-air missile system (hereafter referred to as SAM system) may range in complexity from a single installation, entirely self-contained, to a continent-wide net of radars, communication links, evaluation centers, and firing units. Accomplishment of the air defense mission requires proper functioning and coordination of many diverse system components other than the missiles themselves. It is, therefore, impossible to speak of the tactical effectiveness of a missile; one can only speak of the tactical effectiveness of a missile system. It is, of course, possible, and in most cases necessary, to consider the functioning of various parts of the system on the basis of assumed inputs from the rest of the system.

The characteristics of a SAM system which are important for tactical analysis do not entirely coincide with those which are important from the points of view of development, engineering, production, logistics, or combat use, although there are obviously close interdependences among these various sets of characteristics. The first task of tactical analysis is to find out which characteristics determine tactical effectiveness and to explore the connection between these and the characteristics obtainable from or to be furnished to the design engineer. The present chapter attempts to point out certain system characteristics which are most important from the tactical point of view. It seems advisable to give a discussion, necessarily very brief, on the relation between these tactical characteristics and the physical nature of the missile system.

### B. System Functions

The immediate mission of a SAM system is to damage attacking enemy aircraft. In order to carry out this mission, the system must in general be able to carry out the following functions:

Detection: The system must be able to discover the presence of potentially hostile aircraft at long enough range to allow time for the rest of the functions to be carried out.

Surveillance: The position of each aircraft must be repeatedly observed with sufficient accuracy to permit proper performance of remaining functions; i.e., each aircraft must be tracked.

Identification: It must be determined whether the aircraft is friendly or hostile.

Evaluation: If the target is hostile, the system must decide whether and with how much force to engage it, in view of the missile supply available, other threats which must be countered, etc.

Assignment: If the target is to be engaged, it must be assigned to the firing unit which is to carry out the engagement.

Designation: The firing unit must be informed of the target assignment and be given the current location of the target and any instructions that may be pertinent to the forthcoming engagement, e.g., size of salvo to be fired.

Acquisition: The responsible firing unit must acquire the target with its own fire-data net (e.g., tracking radar).

Loading: The missiles must be on the launcher in condition to fire at the proper time.

Launching. The missiles must be launched at the proper time and in the proper direction (for the system, of course; the launcher may be fixed).

Flight: The missile must be brought near the target.

Fuzing: The proper time for warhead detonation must be determined.

Detonation and Burst: The warhead must detonate at the proper time and must be lethal enough to attain a satisfactory probability of kill.

Damage Assessment: Damage inflicted on the target must be assessed as quickly and accurately as possible.

Re-evaluation: If the verdict of the assessment is that lethal damage was not inflicted, the target must then be considered for further assignment to a firing unit for reengagement.

In order to carry out these direct functions the system must carry out certain supporting operations, such as supply of missiles and spare parts, maintenance and repair of equipment, and intercommunication.

A SAM system must be able to engage several aircraft during an attack. Since in general different parts of the system perform the different functions, several engagements can be in progress simultaneously. For example, while a missile is in flight to intercept one target, the launcher may be reloading in preparation for a second target, which is being assigned. Meanwhile a third target may be undergoing identification. Thus considerable parallelism is to be found in most systems. Moreover, since some functions take considerably longer to perform than others, the faster working parts of the system may often be idle, waiting for their slower companions to complete their tasks. In order to avoid, or at least reduce such idleness, the slower parts of a system are often duplicated. For example, immediately following launching of a missile, the launcher can be reloaded in preparation for the next target. Meanwhile, the guidance channel, e.g., tracking radar, is tied up with the guidance of the first missile, and will remain tied up throughout the missile flight.

Suppose the missile time-of-flight exceeds the loading cycle time; then the launcher, loaded and ready for the next target, must remain idle until the guidance channel is free. If, however, a second guidance channel were provided, the next missile could be fired as soon as the launcher was loaded and ready.

It is convenient to divide the above listed functions into two groups; those that are predominantly decision-making in nature and those that involve launching and fire control. The former can be thought of as being performed in a decision or control center, and the latter in or by a firing unit, even though in some SAM systems no physical separation exists between the two. This conceptual distinction will be followed in the sequel, whether the decision functions of a particular SAM system are carried out locally at each individual SAM site or centrally at one control center serving several sites. The purpose in adhering to this distinction is to emphasize the fact that a particular target is not paired to a particular SAM unit until the designation function has been performed.

C. Significant Times

Certain instants of time and the time consumed in the performance of certain of the above listed functions occur so often in tactical discussions that they are given names. These instants and times consumed are depicted schematically in Figure 3.

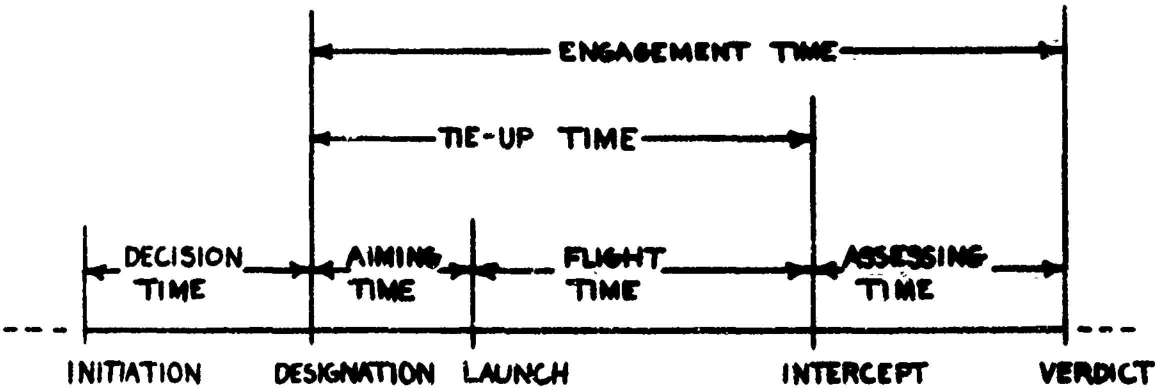



Figure 3

The significant instants of time are:

Time of initiation--the time at which the sequence of events leading to the engagement of a target starts. In case of a first engagement, the sequence of events of course starts with detection, and in the case of subsequent engagements the time of initiation is when re-evaluation starts.

Time of designation--the time at which a target is designated to a firing unit following the various decision functions which led to its assignment.

Time of launch--the time at which a missile or salvo of missiles is launched at a target. This time is sometimes referred to as present time.

Time of intercept--the time at which a missile intercepts the target. This time is on occasion referred to as future time.

Time of verdict--the time at which the verdict as to whether lethal damage was inflicted reaches the decision center.

Associated with each time above is a corresponding range from the SAM site to the target; e.g., designation range, launch range, intercept range. Present and future times and the corresponding present and future ranges are terms held over from anti-aircraft artillery discussions.

The important times consumed in the performance of various functions are:

Decision time--the time from initiation to designation, consumed in the performance of decision functions

Aiming time--the time, from designation to launch, consumed in final preparations for launch. It may include such functions as target acquisition, launcher slewing, etc.

Flight time--the time consumed by the flight of the missile to intercept.

Assessing time--the time following intercept before the verdict of the assessing process is received at the decision center.

Tie-up time--the time from designation to intercept. A guidance channel is occupied with the designated target during this time.

Engagement time--the time from designation of a target to a firing unit to receipt of the verdict at the decision center. During this time the decision center looks upon the target as being under engagement by the firing unit.

It should be pointed out that some of the above times can on occasion be zero; for example, a verdict of no kill could lead to immediate re-designation to the same firing unit, and, further, to immediate launching, since the aiming functions may not have to be performed again. A verdict regarding damage may be possible almost immediately following intercept so that the assessing time will be zero for all practical purposes. In contrast, one or another of the above times could be effectively infinite. If a target is detected and entered in the surveillance system, but is never designated to a firing unit, the decision time will be infinite. Similarly if at any other point a function is neither completed nor abandoned, the corresponding time will be infinite. If the SAM system is unable to assess damage, no verdict can be rendered. In this instance, engagement time could terminate at intercept; i.e., could coincide with the tie-up time while the assessing time could be thought of as infinite. Clearly the flight time and so the tie-up time and the engagement time will be dependent upon the range of intercept. Certain of the other times may also be range dependent according to how the SAM system performs the functions covered by the times.

Many of the variables upon which the above times depend must properly be regarded as random. The spread of their distributions arise from errors in estimation and/or the statistical nature of the quantities themselves. Thus, even for targets having specific characteristics, the times will vary randomly from engagement to engagement; i.e., are themselves random variables. Consequently we define the above times as the mean values of these random variables. In the sequel only the mean values will enter the discussion unless specific statement is made to the contrary.

D. Time Intervals

Of considerable interest to the tactical analyst are the time intervals between successive events of the same kind. Important among these are:

Initiation interval--extending from the initiation of a sequence of events leading to one target engagement to the sequence leading to the next engagement.

Designation interval--extending from one target designation to a firing unit to the next designation to the same firing unit.

Launch interval--extending from launch of one missile or salvo of missiles to the launch of the next missile or salvo by the same firing unit.

Intercept interval--extending from intercept of one target to the intercept of the next target by the same firing unit (perhaps employing different guidance channels).

Verdict interval--extending from one verdict to the next by the same firing unit.

Figure 4 illustrates these intervals in their relations to the times and instants of time discussed in Section C.

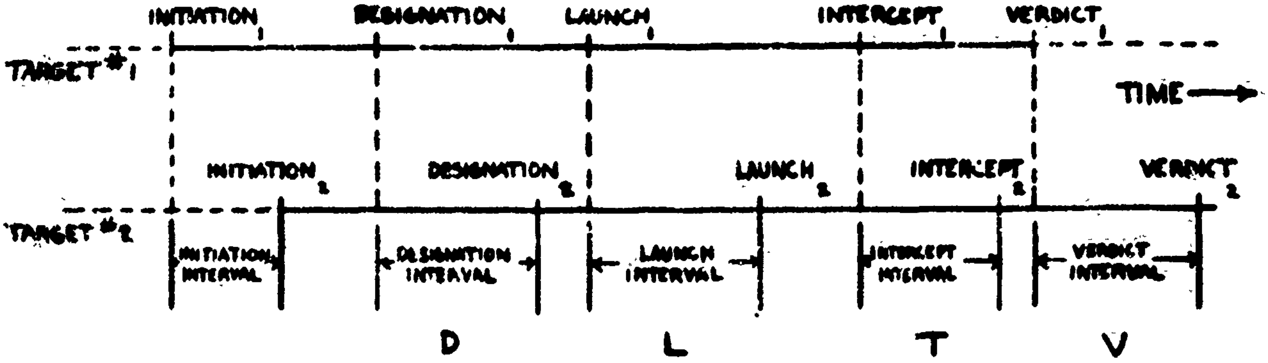



Figure 4

The importance of the above intervals stems from the fact that one or another of them will determine the number of target engagements undertaken, i.e., firepower. If the firing rate is limited by the number of guidance channels available, the successive intercept intervals will determine the firepower. On the other hand, if the firing rate is limited by the launcher cycle time (the minimum time needed to load and slew the launcher), the launch interval will govern firepower.

Any of the above intervals may on occasion be zero. Indeed, if a SAM unit having two launchers and two guidance channels engages targets in a wave attack, then two engagements could take place coincidentally, step by step; detection, designation, launch, intercept, and verdict could all occur simultaneously for both targets. On the other hand, the composition and characteristics of a SAM system may impose a minimum time for one or another of the intervals. For a system with a single launcher, the launch interval must be at least as great as the launcher cycle time. For a system with a single guidance channel, the intercept interval will be at least as great as the time of flight to the latter of the two successive intercepts and in most circumstances must be even greater to include that part (frequently all) of the aiming time during which the guidance channel is required.

The relations between certain of the above intervals and certain of the various times consumed in the performance of functions, as described in Section C, are of interest to the tactical analyst. We let (see Figure 5)

D = designation interval

L = launch interval

T = intercept level

V = verdict interval

and $t_f(\rho_i)$, $\phi(\rho_i)$, $E(\rho_i)$ = flight time, tie-up time, and engagement time for intercept at range $\rho_i$, $i=1,2$, respectively, where $\rho_1$ and $\rho_2$ are successive intercept ranges.

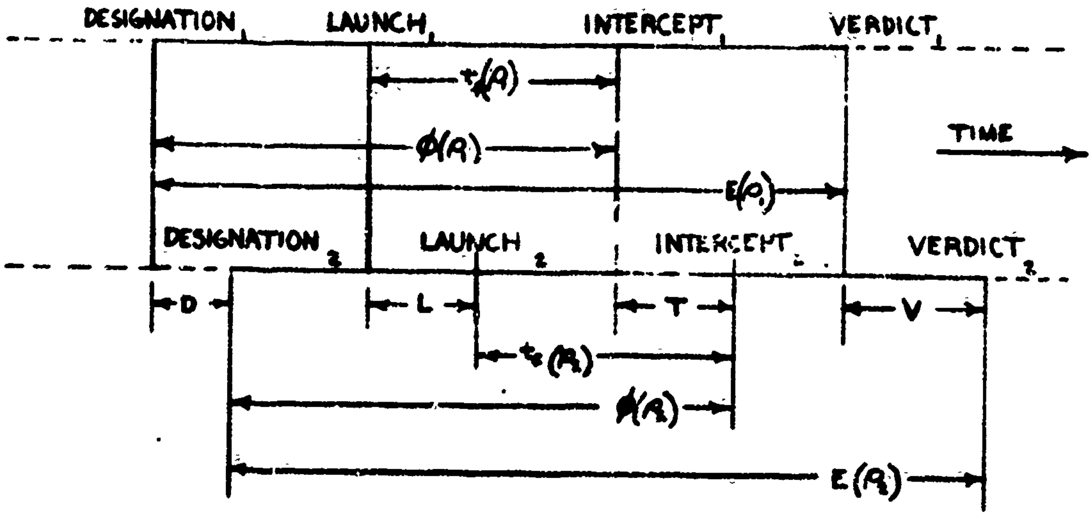



Figure 5

We have

$$t_f(\rho_1) + T = L + t_f(\rho_2)$$

$$\phi(\rho_1) + T = D + \phi(\rho_2)$$

$$E(\rho_1) + V = D + E(\rho_2)$$

or

$$L - T = t_f(\rho_1) - t_f(\rho_2)$$

$$D - T = \phi(\rho_1) - \phi(\rho_2)$$

$$D - V = E(\rho_1) - E(\rho_2) \quad .$$

If the aiming time is constant,

$$D = L$$

and if the assessing time is constant

$$T = V,$$

and so the differences in times of flight, intercept times and engagement times are equal.

In general, either D or T will be determined by the characteristics of the particular SAM system and the other will be derived from it by means of the above relation. In many cases of interest, data gathering and decision making functions (detection to designation) can be performed at a rate sufficient to keep the firing unit occupied. Also the interval from designation to launch is often independent of range and so the interval between successive launches is the same as the interval between designations. If the above hold, the number of missiles fired is determined by the rate at which missiles can be loaded and launched or by the number of guidance channels available for simultaneous control of missiles in flight. In the former case, the governing delay is for launcher readiness; the launch interval is then the launcher cycle time of the SAM system, and the intercept interval is derived from it. In the latter case, the governing delay is for guidance channel availability; this delay dictates the intercept interval, and the launch interval is then derived from it.

If the governing delay is for launcher readiness, and there are $M_L$ launchers, each with a cycle time $T_L$, then there will be $M_L$ launchings in time $T_L$, so that on the average the launch interval for the SAM system is $T_L/M_L$. Similarly if the governing delay is for guidance channel availability, and there are $M_G$ guidance channels, each having an interval $T_G$ between successive intercepts, then there will be $M_G$ intercepts in time $T_G$, and so on the average, the intercept interval is $T_G/M_G$. It will frequently suffice to use these averages for launch interval and intercept interval (See Chapter IV). However, some caution should be exercised, particularly since $T_G$ is usually range dependent.

Details of an engagement may be discussed in either time of designation, time of launch, or time of intercept, and each is most convenient for certain parts of the analysis. Certain precautions are necessary when switching from one time to another; such a change may involve a Doppler correction, as illustrated by the following example.

If missiles with an average horizontal speed v are launched t seconds apart against a straight incoming target with an average horizontal speed u, let $r_i$ be the launch range and $\rho_i$ the intercept range for the i-th missile (i=1,2). While the missile travels a distance $\rho_i$ at speed v, the target travels a distance $(r_i-\rho_i)$ at speed u, so

$$\frac{\rho_1}{v} = \frac{r_1 - \rho_1}{u}$$

or

$$r_1 = (1 + \frac{u}{v})\rho_1 \quad .$$

The interval between times of launch is

$$t = (r_1 - r_2)/u.$$

The interval between times of intercept is, however,

$$t' = (\rho_1 - \rho_2)/u = \frac{1}{1 + \frac{u}{v}} t < t \quad .$$

The conversion factor, $1/(1 + u/v)$, is, as usual, called the Doppler correction.

Both slant and horizontal ranges are in common use. The horizontal range is the distance from the fire unit to a point on the ground directly beneath the target; the slant range is the distance from the fire unit to the target itself. Neglecting the curvature of the earth,

$$r_s^2 = r_h^2 + H^2 ,$$

where

$r_s$ = slant range,

$r_h$ = horizontal range,

$H$ = target altitude.

If H is small compared to $r_h$, the difference between slant and horizontal range is small.

For most of the studies here described, horizontal range has been found more convenient to use than slant range, and the word range means horizontal range unless the contrary is explicitly stated.

## E. Zone of Fire

The zone of effective fire for a SAM firing unit is the region of space within which targets can be successfully engaged. The zone of fire is specified in terms of the point in space where intercept will occur. To each intercept point there corresponds a launch point, the position of the target at the time of launch. Thus corresponding to the zone of fire there is a zone of launch; to be successfully intercepted, a target must be within this zone at the time of launch. In like manner the zone of assignment is defined; a target must be in this zone at the time of assignment. It is this last zone that is of interest to the target assigner.

The boundaries of the zone of effective fire are determined by a number of considerations, the more common of which are mentioned below; not all of these considerations are relevant to every system, but every system has limits set by some of the considerations listed or by similar ones.

It is emphasized that some of the boundaries (in the state of the defender's ignorance) are statistical in nature: they are regions within which the performance of the system deteriorates more or less rapidly, so the zone of fire may be slightly extended in these regions at the cost of decreased kill probability or reliability.

The boundaries of the zone of effective fire are indicated in Figure 6 for a target passing overhead; the boundaries are discussed briefly below.

High-Angle Limit: The high-angle limit, which results in an overhead dead zone, is usually inside the bomb release surface (see below) in tactical situations of interest, and hence does not usually limit tactical performance. Occasionally, however, the fire unit may be placed far enough outside the defended area for the high-angle limit to take effect; this limit must also be taken into account when the fire unit is part of the perimeter or area defense and engages both approaching and receding targets. One of the following limitations on the system may set the high-angle limit:

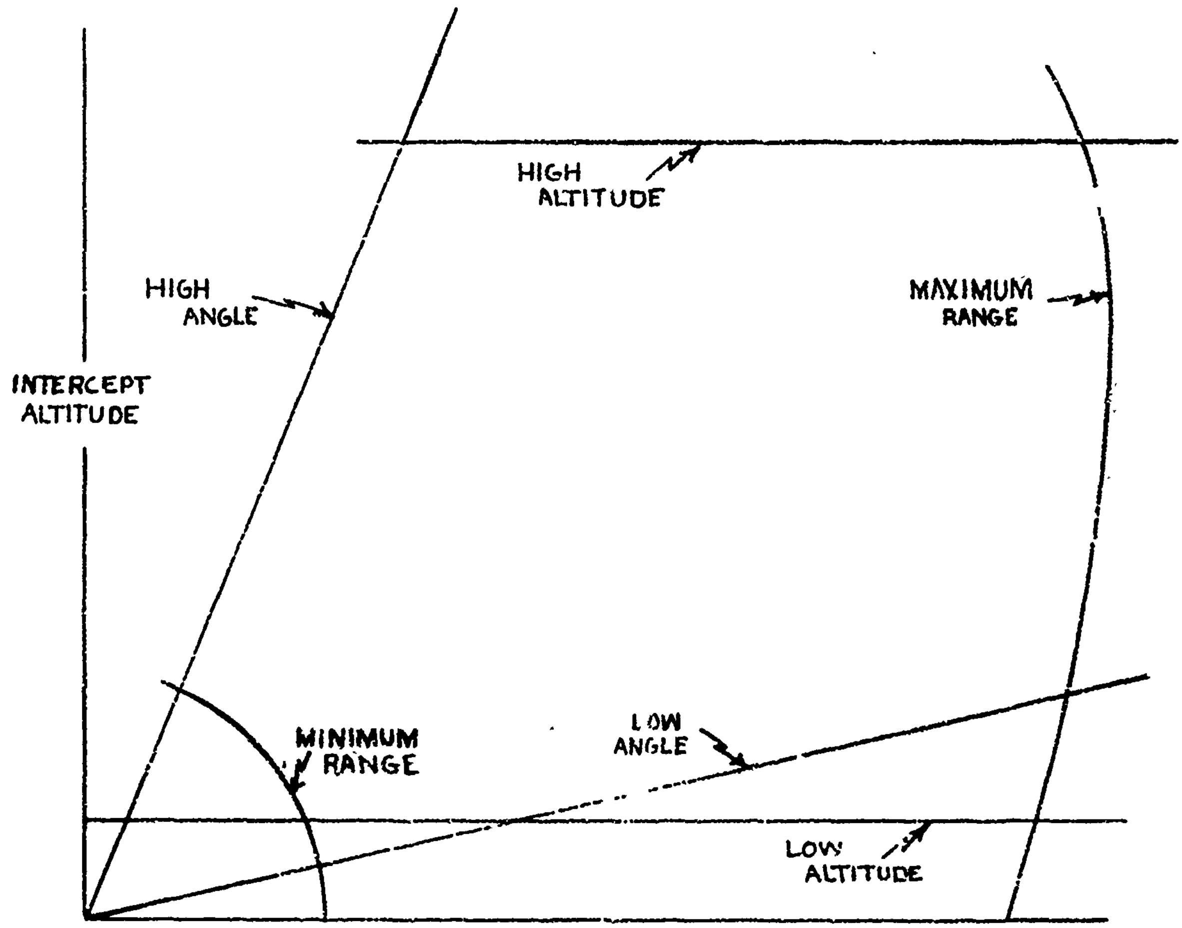
The launcher may have physical limitations on maximum elevation, which for beamriders and certain homers sets an upper bound on the missile trajectories.

A maximum elevation limit on a radar used for guidance would set a high-angle limit on the zone of fire. It is not to be expected that such a limit would exist for tracking or illuminating radars (except perhaps on shipboard), but such an elevation limit is usual on many search radars, and in some systems these furnish at least a portion of the guidance or assignment intelligence.

The missile itself may set a limit on the elevation of engagement; for instance, one or more of the missile gyros may tumble if the missile axis gets too near the vertical.

In a beamriding system it becomes difficult to capture the missile in the capture beam at the end of boost if the beamrate (angular rate of the radar beam) is too high. For incoming targets at fixed speed and altitude this is an increasing function of elevation angle, and may set a limit on the usable elevation angle.

High-Altitude Limit: The high-altitude limit on system performance is usually set by the missile itself, and constitutes an effective limit on system defense effectiveness, since if an effective attack can be brought in above the system high-altitude limit, the system provides no defense at all against this attack. There is, therefore, a real advantage in trying to make the system high-altitude limit greater than the altitude at which attack is profitable, as determined by the difficulty of aircraft flight at high altitudes, the loss of bombing accuracy with increase in altitude, etc.




BOUNDARIES OF ZONE OF EFFECTIVE FIRE

Figure 6

The principal limitation on missile effectiveness at high altitude is the drop in missile maneuverability resulting from reduced effectiveness of aerodynamic surfaces at low atmospheric pressures. The maneuverability of the attacking aircraft is likewise reduced, but since the missile usually needs more maneuverability than its target, this fact merely postpones the limit. Control independent of aerodynamic surfaces (e.g., by jet vanes) could be used to raise the limit imposed by this consideration if necessary.

The variation of aerodynamic characteristics with altitude may impose a high-altitude limit somewhat less directly, since increased effectiveness at high altitude (beyond a certain point) implies decreased effectiveness at lower altitudes. For example, the provision of large aerodynamic surfaces to obtain high maneuverability at high altitude increases drag and reduces range at lower altitudes. A compromise must often be made between high-altitude limit and low-to-medium altitude performance. The compromise becomes easier in systems with greater freedom of trajectory, (e.g., command systems, in which the "cruising altitude" may be chosen at will).

Maximum Intercept Range: The maximum effective range of the system plays an important part in the analysis of system effectiveness; since the effects of range are considered in detail in what follows, no particular comment seems necessary here. The maximum range limit may be set by limitations on any of the following functions.

Detection: The target cannot be engaged until a certain minimum time after target detection; the effective range of the system is therefore not greater than the detection range minus the distance travelled by the target during the time interval required for identification, evaluation, assignment, acquisition, launching, and flight. The detection range depends not only on the missile system, but also upon the target characteristics (i.e., radar echoing area, altitude, and speed). The same argument applies to the system functions of identification, evaluation, assignment, and acquisition when they show a range dependence different from that of detection. At low altitude, the radar horizon may critically limit the detection range.

Guidance: The guidance system has a range limitation whose nature varies from system to system. For prediction guidance (guns and unguided rockets) the limit is set by the time of flight over which prediction is sufficiently accurate. For command and beamrider guidance the limit is set by the fact that the target position is known at the firing unit with an error which is angular and hence corresponds to a miss distance proportional to range at long ranges if the angular is roughly independent of range. For active or semi-active homing guidance systems the range is limited by the range product* obtainable.

*The range product is the product of the range from transmitter to target and the range from target to receiver. For an active homing system, in which the transmitter and receiver are both in the missile, the range product is the square of the missile-to-target range. For a semi-active system, in which the transmitter is at the SAM firing site, the range product is the SAM site-to-target range multiplied by the missile-to-target range.

Flight: The range of the missile itself is limited by its fuel supply and by the trajectory the missile follows. Fuel consumption in the dense lower atmosphere is far greater than at high altitude.

Low-Altitude Limit: A lower limit on the altitude of successful engagement is set by difficulties in successful fuzing near the surface of the earth and/or by difficulties in homing successfully near the surface of the earth.

Low-Angle Limit: A limit on the minimum angle at which targets can be engaged, distinct from the maximum range limitation set by the radar horizon and also from the low-altitude limit is imposed on some systems by the difficulty of radar tracking and/or missile capture after boost at low radar elevation angles.

Minimum Range: The minimum range limitation, like the high-angle limit, is usually not troublesome tactically because it lies inside the bomb release line in many cases, though not in all.

For safety reasons it is customary not to arm the missile warhead until some time after launch; successful engagement is unlikely until the warhead has been armed.

Many missile systems have a sequence of operations (launch, stabilization, capture) which must be completed at the beginning of flight before the missile is under adequate control of the guidance. The distance travelled by the missile during this part of flight is the usual limitation on minimum slant range for effective interception.

Certain systems have a fixed minimum launching angle which results in a dead zone at short ranges and low altitudes, since the missile trajectory cannot curve sharply enough to engage targets in this region.

Clearance Shadows: SAM's effective zone of fire is normally further reduced, in particular installations, by the "shadows" cast by nearby obstructions as seen from either radars or launchers, by safety restrictions on certain angles of fire resulting from the danger from falling boosters or special warheads, etc. These clearance shadows are for the most part ignored in the present general discussion for the sake of simplicity, but obviously play an important role in the analysis of particular installations.

Crossing Distance: The crossing distance of a target is defined as the distance from the firing site to the horizontal projection of the target trajectory, i.e., the shortest distance from site to projected trajectory. Thus, for instance, a target whose trajectory passes directly over the firing unit has a zero crossing distance. Even ignoring clearance shadows, the zone of effective fire is not exactly a volume of revolution; the various maximum and minimum ranges at a given altitude depend to some extent upon crossing distance and upon whether the target is approaching or receding (i.e., whether it has or has not passed the point of closest approach to the firing unit).

The determination of the exact extent of the deformation of the solid of revolution is complicated and depends on the details of the missile system. But this deformation is often relatively slight and is frequently ignored in rough estimates of tactical effectiveness.

Bomb Release Point: The primary interest of the analyst often is defense (prevention of damage to the defended area) rather than attrition (preventing the attackers from returning home to attack again some other time) and hence attention is directed to intercepts that may prevent the attacker from damaging the defended area. The last point at which such intercepts are possible is called the bomb release point, even though the attacker's armament actually may be a torpedo, salvo of rockets, or some other weapon than a bomb. If the attacker's armament is a guided air-to-surface missile, then the bomb release point may not be the actual point of launch of the air-to-surface missile, but rather the last point at which destruction of the mother plane disrupts the guidance enough to cause a miss. Or it may be the last point at which the missile itself can be intercepted to prevent damage from its warhead. Similar modifications of meaning in other cases permit the same terminology to be used throughout. Because the surface target often is not a point, and the lethal radius of the bomb is significantly greater than zero, the bomb release point may not be a point at all. There may be a series of points such that if the bomb is released at any one of them, damage to the surface target may result. It is frequently convenient to define one point of the series as the bomb release point; often the first point of the series is picked.

From the point of view of defense, as opposed to attrition, the effective zone of fire is only that part of the region defined in preceding paragraphs which lies outside the surface of bomb release points.

# CHAPTER III

## Kill Probability

### A. Introduction

The immediate intent of the defense, having undertaken the engagement of a target, is to inflict damage upon that target. The outcome of such an engagement cannot with certainty be predicted in advance. Many of the variables upon which the outcome depends are of necessity random. Consequently the outcome itself is a random event. Thus we can speak of the probability that the target will be damaged; i.e., of the kill probability. In broad terms the kill probability of a missile depends upon the reliability of the missile (does it function as intended), upon the guidance accuracy of the missile system, upon the fuzing (is the warhead detonated at the proper time), and upon the warhead lethality. Before investigating these factors in any detail, it is first necessary to consider the kinds of damage that can be inflicted upon the target, the ways in which damage may be inflicted, and the means for inflicting it.

A target can be damaged in many ways. Most defensive missiles are designed to produce damage in one or more of the following ways:

Destroy vital structural members of the target aircraft.
Incapacitate a sufficient number of components of the control system of the target aircraft to prevent continued control of flight.
Disable the propulsion system of the target aircraft either by damaging enough components of the system to prevent continued operation or by cutting off the fuel supply (severing fuel lines or igniting fuel supply).
Incapacitate a sufficient number of components of the offensive weapon control system to prevent satisfactory delivery of the weapon.
Disable the firing mechanism of the offensive weapon or cause premature detonation of the weapon.
Incapacitate a sufficient number of air crew members to prevent satisfactory control of the target aircraft or delivery of its weapon.

Clearly many of the damaging effects listed above will lead ultimately, if not immediately, to the destruction of the target aircraft. If the aircraft control or propulsion systems are disabled the aircraft may very likely crash. Nor is the aircraft apt to remain intact if its offensive weapon is detonated. On the other hand, other effects listed above may prevent successful delivery of the offensive weapon but leave the target aircraft itself undamaged. The time at which damage takes effect may be important. The damaging effect may in time bring about the destruction of the target aircraft, but not until after the offensive weapon has been successfully delivered. Thus the importance of any one damaging effect may rest upon whether the primary objective of the defense is to exact attrition (destroy target aircraft) or to prevent damage to the surface target (prevent successful delivery of the offensive weapon). Because of the importance of the defense objective and of the time interval following the defensive missile warhead burst before damage takes effect, a standard nomenclature for categories of aircraft damage has been adopted in the U.S., based on the time interval.

The types of damage and their definitions are listed below.

KK damage - immediate catastrophic disintegration of target aircraft.

K damage - damage from which the target aircraft begins to fall within 10 seconds of the defensive missile warhead burst.

A damage - damage from which the target aircraft begins to fall within 5 minutes of the defensive missile warhead burst.

B damage - damage from which the target aircraft begins to fall within 2 hours of the defensive missile warhead burst.

C damage - damage such that the aircraft is unable to successfully deliver its offensive weapon.

E damage - damage such that the aircraft is unable to successfully land at its home base.

These categories obviously are not mutually exclusive. The category most appropriate for the analyst to consider will depend upon the nature of the problem facing him. If the attack is by aircraft on one way missions, e.g., surface-to-surface missiles, then C damage is of particular interest. If the purpose of the defense is to exact attrition, the analyst will devote his attention to E damage. If the problem facing the analyst involves early damage assessment by the defense in order to improve the coordination of fire, then KK and K damage are the categories of primary concern.

A variety of lethal agents may be employed in the design of defensive missile warheads to produce different damaging effects. Typical are high speed metal fragments, blast, metal rods, intense heat, and radiation. A warhead may incorporate one or more lethal agents. The common types of warheads are:

Fragment warhead, designed to emit a large number of small pieces of metal moving at sufficiently high speed to damage vulnerable components of the target.
Internal blast warhead, designed to emit one or more small packages of high explosive, each able to penetrate the target aircraft skin and then detonate, producing blast internal to the aircraft.
External blast warhead, designed to produce impulsive loading on portions of the aircraft structures.
Rod warhead, designed to emit metal rods or bars of sufficient length and mass and moving at sufficient speed to sever structural elements and/or damage vulnerable components of the target aircraft.
Nuclear warhead, producing blast, thermal effects, and radiation.

A defensive missile must be provided with some means for detonating its warhead at the appropriate time. Two widely used means are proximity fuzing, wherein a fuzing device senses when the missile is in the vicinity of its target and then actuates the firing mechanism of the warhead, and command detonation, wherein the SAM site determines when the missile is in the vicinity of the target and then transmits a signal to the missile which actuates the firing mechanism. Another less widely used means for warhead detonation is contact fuzing; on direct contact of the missile with the target aircraft the contact fuze actuates the firing mechanism.

B. Definition of Kill Probability

Suppose that, as an experiment, a single missile were launched at a single target. A kill either would or would not result and the ratio of number of targets killed to number of missiles launched would be 1 or 0. If the identical experiment were repeated, the cumulative ratio could be computed again. With each repetition the ratio could be recomputed and would show a change; as the number of experiments increased, the ratio would show less and less variation. We define the kill probability p of a missile as the limit of the ratio of the number of targets killed $n_K$ to the number of missiles launched $n_L$, as the number of launchings increases without limit.

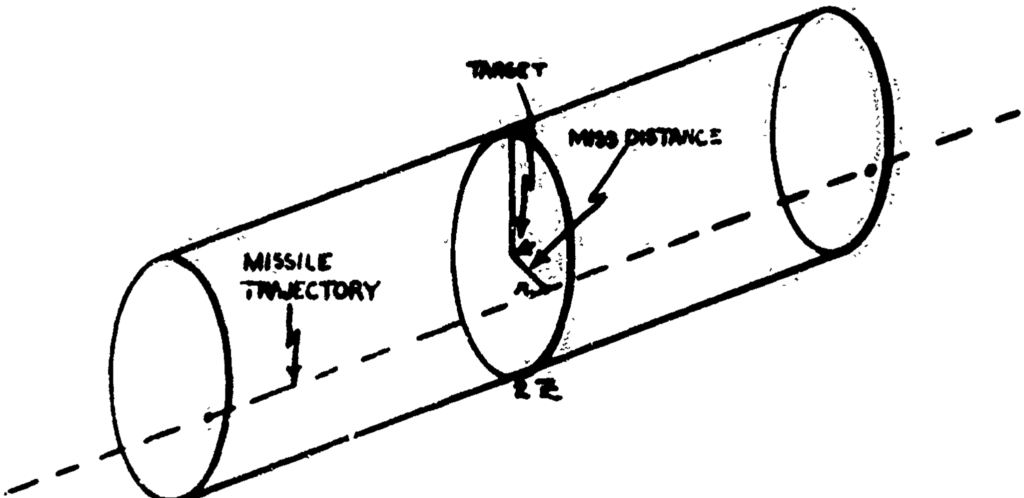
$$(1) \qquad p = \lim_{n_L \to \infty} \frac{n_K}{n_L}$$

In the above definition the category of kill must, of course, be specified as well as what constitutes a launching; e.g., when the missile leaves the launcher, or when the launching button is pressed.

The sequence of events that must occur in order that an engagement of a target by a defensive missile, once undertaken, shall result in a kill can be summarized as follows: the missile must fly along a trajectory that passes through or near to the target, the warhead must be detonated at some point along the trajectory near the target, and the damaging agent or agents of the warhead must inflict the prescribed damage.

The missile may fail to fly a satisfactory trajectory because components of the missile itself or of equipment at the SAM site do not function as intended, or because active countermeasures taken by the enemy so alter the environment that the SAM system as designed cannot perform as desired. In either case, the failure may be complete in that the missile does not fly at all or the flight terminates long before intercept. Alternatively the failure may be one of degree--the missile flies but the trajectory passes too wide of the mark. Thus "satisfactory trajectory" connotes nearness to target. How near the trajectory must be to be satisfactory is of necessity an arbitrary choice. Before making this choice explicitly, we must first state what is meant by nearness. We adopt a widely used notion of miss distance, defined as the distance from the target center of gravity to the point of closest approach of the trajectory (or its extension). It is then convenient to define a satisfactory trajectory as one for which the miss distance is less than some prescribed distance R. Clearly R should be chosen large enough to include all missiles that perform as intended in a normal environment. We shall later see that R should also be sufficiently large so that no missile with miss distance greater than R can inflict damage on the target; otherwise our later development will yield a false value for p.

Even if the missile trajectory is satisfactory, the warhead in the missile may fail to detonate near the target. Failure may be caused by malfunction of the fuzing or firing mechanism, by active countermeasures taken by the enemy, by natural environmental conditions such as weather, or by characteristics intentionally incorporated in the fuze. An example of the last, often found in proximity fuzes, is a rather sharp cut-off range to prevent the fuze from triggering on anything other than the intended target; e.g., the earth's surface in the case of low altitude intercepts. Whatever the cause, the detonation may not occur at all or it may occur too early or too late. "Satisfactory detonation" also connotes nearness to target, and how near a detonation must be to be satisfactory is also a necessarily arbitrary choice. Again we must state what is meant by nearness. We measure the distance along the trajectory from the plane normal to the trajectory and through the target CG to the point of detonation. We then define a satisfactory detonation as one occurring within a prescribed distance Z of this normal plane. Clearly Z should be chosen large enough to include all detonations that occur from normal operation of the fuze, firing mechanism, etc. in a normal environment. It is also important that Z be large enough to insure that no detonation at a distance greater than Z will cause damage to the target. By our definitions a missile will have a satisfactory trajectory and satisfactory detonation if its warhead detonates within a truncated tube (see Figure 7) centered at the target and having length 2Z and circular cross section of radius R.




TUBE OF OPERABILITY

Figure 7

A missile whose warhead detonates within this truncated tube or whose trajectory passes through the tube but detonation fails to occur by design intent is said to be operable. Whether a missile will be operable or not is dictated by many chance occurrences; we can speak only of the probability that it will be operable. This probability is commonly referred to as missile operability or missile reliability. We define missile reliability $p_R$ as the limit of the ratio of the number of operable missiles $n_R$ to the number of missiles launched $n_L$ as the number of launchings increases without limit.

$$p_R = \lim_{n_L \to \infty} \frac{n_R}{n_L} .$$

From equation (1) we then have

$$(2) \qquad p = \lim_{n_L \to \infty} \frac{n_R}{n_L} \frac{n_K}{n_R}$$

$$= \left(\lim_{n_L \to \infty} \frac{n_R}{n_L}\right) \left(\lim_{n_R \to \infty} \frac{n_K}{n_R}\right)$$

$$= p_R \left(\lim_{n_R \to \infty} \frac{n_K}{n_R}\right) .$$

In some contexts, for example when considering quality control in missile production, a missile is said to be operable unless its warhead fails to detonate inside the truncated tube because of missile malfunctions only. In this case the missile reliability $p_R$ as defined above can be broken down into two factors, one being the probability of no missile malfunctions and the other the conditional probability that, if no malfunctions occur, the missile will be operable in our more restrictive sense, i.e., the effects of countermeasures and other influences are separated from those of malfunctions. In like manner the effects of different kinds of malfunctions can be separated, e.g., those involving missile propulsion, guidance, or fuze and warhead. Hereafter we will adhere to our more restrictive definition of reliability, admitting to the select group of operable missiles only those missiles that detonate within the truncated tube or fail to so detonate by design intent.

An operable missile may still fail to inflict the prescribed damage on the target. Because of the discreteness of the projectiles ejected by some warheads no vital portion of the target may be hit, or impulsive loading from an external blast warhead may be insufficient to cause the prescribed damage. We define the kill probability $p_K$ of an operable missile to be the limit of the ratio of number of missiles that produce kills $n_K$ to the number of operable missiles $n_R$ as the latter increases without limit.

(3) $$p_K = \lim_{n_R \to \infty} \frac{n_K}{n_R}$$

The probability $p_K$ is a conditional probability, i.e., is the probability that, if operable, a missile will kill. However, to avoid later confusion, we will call $p_K$ simply the kill probability of an operable missile.

Combining equations (2) and (3) we have

$$p = p_R \, p_K \quad .$$

C. Kill Probability of an Operable Missile

Confining our attention now to operable missiles, the likelihood of kill will in general vary depending on the precise position of detonation. Thus the kill probability $p_K$ of an operable missile is the product of the conditional kill probability that if detonation occurs at a particular point in the tube a kill will result and the probability that the detonation occurs at that point, summed over all points in the truncated tube. Employing an integral representation for $p_K$ we have

(4) $$p_K = \int_{\Sigma} p_D(S) h(S) dS \quad .$$

where dS is an element of volume in the truncated tube

$h(S)dS$ is the probability that detonation occurs in dS

$p_D(S)$ is the conditional probability of kill if detonation occurs at a point S in dS

$\Sigma$ is the volume of the truncated tube.

Although we are here considering only operable missiles, a detonation still may not occur, because of fuse cut-off range, for example. Thus, the integral of h(S) over the truncated tube may be less than one, and so we have

(5) $$\int_{\Sigma} h(S) dS \leq 1 \quad .$$

In most situations of interest the segment of missile trajectory lying within the truncated tube may, for the purposes of this chapter, be adequately approximated by a straight line segment and so the truncated tube may be thought of as a right circular cylinder of radius R and height 2Z. Using a cylindrical coordinate system centered at the target and with z axis parallel to the missile trajectory the element of volume dS may then be written as

$$dS = r\,dr\,d\theta\,dz .$$

Furthermore, the miss distance is independent of the position along the trajectory at which detonation occurs, although converse is not in general true. Hence the probability density function h(S) can be written

$$h(S) = g(r,\theta)f(r,\theta,z)$$

where

$g(r,\theta)rdrd\theta$ = probability that the trajectory or its extension passes through an element of area normal to the trajectory and containing the point $(r,\theta)$,

$f(r,\theta,z)dz$ = conditional probability that if the trajectory is at $(r,\theta)$, the detonation will occur in a linear element $dz$ containing $z$.

Since we are here speaking only of operable missiles, and by definition a missile with trajectory passing outside the tube is not operable, we must have

$$\int_0^R \int_0^{2\pi} g(r,\theta)rdrd\theta = 1 \quad .$$

Hence, from equation (5) we have

$$h(S)dS = \int_0^R \int_0^{2\pi} g(r,\theta) \left( \int_{-Z}^{Z} f(r,\theta,z)dz \right) rdrd\theta \leq 1$$

and so the integral of f with respect to z may be less than one, a consequence of fuze cut-off, for example. So we have

$$(6) \qquad \int_{-Z}^{Z} f(r,\theta,z)dz \leq 1 \quad .$$

Substituting for h(S) and dS in equation (4) we have

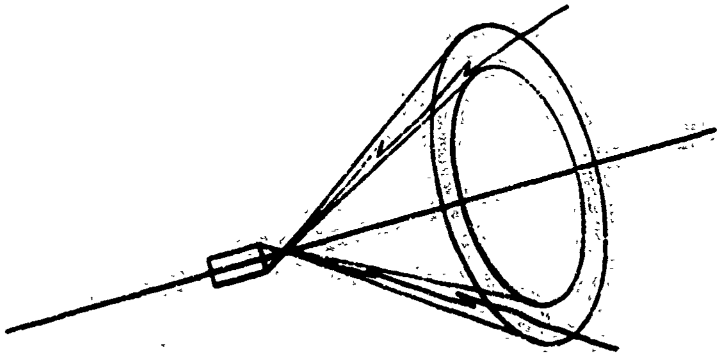
$$(7) \qquad p_K = \iiint_{\Sigma} p_D(r,\theta,z)f(r,\theta,z)g(r,\theta)rdrd\theta dz \quad .$$

D. <u>Guidance Accuracy</u>

The distribution of missile trajectories in the tube can most easily be described by the distribution of the intersections of the trajectories with the plane normal to them and passing through the target. For most guidance systems, including beamriding, command, and homing systems, this distribution can be reasonably approximated by a two-dimensional normal or Gaussian distribution in which the two dimensions, one vertical and the other horizontal, are statistically independent. In many cases, vertical and horizontal standard deviations are equal and the distribution of trajectories reduce to a circular normal or radial normal distribution. However, at times such is not the case; for example, at low angles greater dispersion may occur in the vertical than in the horizontal direction. Often the distribution of trajectories will be centered at the target, though not always. For example, a boresight error in the guidance radar may mean that all missiles are biased in some manner.

E. Fuzing Accuracy

A common type of proximity fuze is the fixed angle fuze which emits electronic radiation symmetrically in a narrow beam at fixed angle to the missile axis as shown in Figure 8.



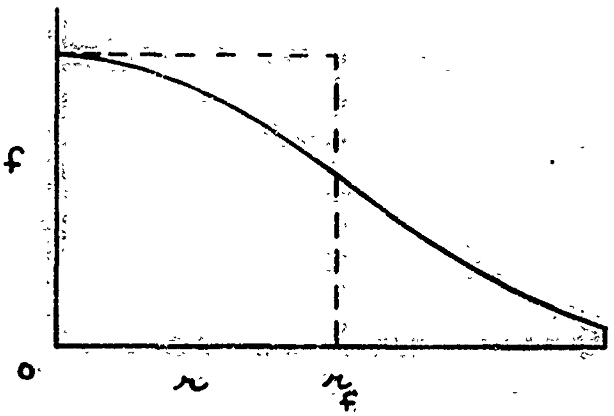
FUZE PATTERN

Figure 8

If the target aircraft intercepts the radiation, the fuze is triggered by the return echo and firing mechanism actuated. A small delay inevitably occurs between fuze triggering, i.e., the instant at which the return echo from target is detected, and detonation of the warhead. Frequently this delay is intentionally lengthened in order to place the detonation at the desired point. As mentioned earlier, a proximity fuze is often designed to have a rather sharp cut-off range to prevent the fuze from triggering on anything other than the intended target.

The position of the warhead detonation along the trajectory will depend upon the size and shape of the target and the aspect of approach of the missile to the target as well as on the fuze radiation angle and time delay. The cut-off range will prevent detonation if the miss distance is sufficiently large. Because of irregular shapes of targets and vagaries in reflection coefficients of targets to electromagnetic radiation, the probability density function f may assume any of several forms. For some purposes it is permissible to assume that the distribution of detonation positions is Gaussian with regards to z. For other purposes it is necessary to treat individual cases separately or to make alternative simplifying assumptions.

In the event a cut-off range is incorporated in the fuze, the dependence of f on miss distance r may take any of several forms. If the cut-off is indeed sharp and the target is small, f can be approximated by a rectangular distribution in r, extending from 0 to the cut-off range $r_f$.




FUZING PROBABILITY

Figure 9

If $r_f$ is sufficiently large, the cut-off will have no significant effect on $p_K$, i.e., either $p_D$ or g will become very small before the cut-off range is reached. Often the cut-off is not so precisely defined. A more suitable approximation to f is then some form of exponential, as depicted in Figure 9; a convenient approximation, borne out in many cases by more detailed analysis is

$$e^{-\frac{r^2}{2a^2}}$$

where a is a constant characteristic of the fuze and the target.

A second type of proximity fuze uses the doppler shift in the return radiation from the target to determine the proper detonation position. Unless the radiation is confined to a sharp beam, considerable dispersion in detonation positions will occur and use of a Gaussian distribution of detonation positions may be permissible. If the radiation beam is sharp, the distribution of detonation positions may be similar to that for the fixed angle fuze.

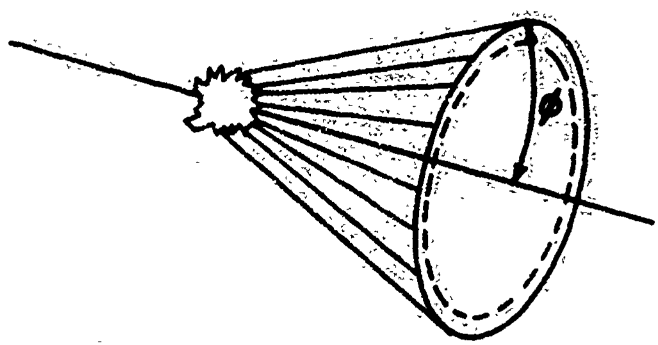
In the most common command detonation scheme the ranges to missile and to target are measured at the SAM site and when these measured ranges become coincident a detonation signal is transmitted to the missile. The distribution of detonation positions is thus similar to the distribution of errors in range measurement and usually can be assumed to be Gaussian.

F. Conditional Kill Probability

The conditional probability $p_D$, that if the detonation occurs at a particular point damage of the prescribed category will occur, is critically dependent on the type of warhead and on the characteristics of the target. For this reason the discussion of $p_D$ will be given separately for each warhead type in turn.

Fragmentation Warhead

A fragment warhead is designed to emit a large number of fragments at high velocity, often confined to a fairly narrow beam near the equatorial plane of the missile and symmetrical about the missile axis. Figure 10 depicts a typical fragment warhead burst.




FRAGMENT WARHEAD
Figure 10

The detonation of a fragment warhead at any point S can be characterized by the fragment density, fragment mass, fragment striking velocity and the angle of approach of fragments to aircraft components. A common measure of fragment distribution is the expected number of fragments per unit solid angle, having vertex at the warhead center. A typical fragment distribution is illustrated in Figure 11.

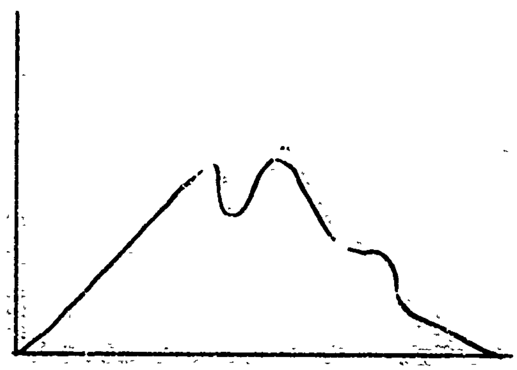



Figure 11

Fragment density at any specified distance from the warhead is defined as the expected number of fragments passing through a unit area normal to the fragment path. The distance from warhead center, expressed in terms of miss distance r, is $r/\sin\phi$, and so fragment density is given by

$$\frac{D(\phi)\ \sin^2\phi}{r^2}$$

where $D(\phi)$ is the expected number of fragments per unit solid angle and $\phi$ is the angle from the forward missile axis. Since $\phi = \text{arc tan } r/z$, angular density $D(\phi)$ can be expressed in terms of r and z, variables of integration in equation (7).

In many fragment warhead designs provision is made to control the mass of individual fragments. Test firings indicate that a high degree of control is possible. The striking velocity of fragments depends, of course, upon the initial fragment velocity, the air density, the distance from warhead center to the impact point, and the missile and target velocities. Experimental firings indicate that the striking velocity falls off exponentially with this distance. The vulnerability of aircraft components to the impact of high velocity fragments is usually determined experimentally. A large number of fragments of given mass and velocity are fired at whole aircraft or parts of aircraft under simulated flight conditions. Experienced warhead analysts then assess the damage to aircraft components using their good judgment to ascertain how many of the fragments that hit a component would inflict damage of the prescribed category. The ratio of killing fragments to hits provides an estimate of the probability $p_c$ that if hit, a component will be killed. Having determined $p_c$ as a function of fragment characteristics and aspect, we then define the vulnerable area $A_v$ of the component as an equivalent area such that if hit by a fragment a kill is certain to result; i.e.,

$$A_v = p_c A_p$$

where $A_p$ is the presented area of the component. Because of air drag on fragments, $p_c$ and so $A_v$ may depend upon miss distance r.

The distribution of fragment hits on an aircraft may be biased because of large fuzing errors or may be highly localized because the detonation occurs close to the aircraft. The latter circumstance is of limited importance since for such near misses, the blast from the warhead detonation may itself be lethal. The former will not be important if fuzing errors are small compared to target size and the fragment beam covers the target. Neglecting fuzing errors and near misses, the distribution of fragment hits is likely to be random; i.e., all parts of the aircraft are likely to be hit. Assuming the target to have just one vulnerable component, the conditional kill probability $p_D$ is the probability of at least one hit occurring on the vulnerable area of the component, since such a hit is by definition lethal. It is reasonable to assume that the distribution of hits on the component follows a Poisson law. The conditional kill probability is then

$$p_D = 1 - e^{\frac{-D(\phi)\sin^2\phi\, A_v}{r^2}} \tag{8}$$

since the average number of hits on the vulnerable area of the component is

$$\frac{D(\phi)\sin^2\phi\, A_v}{r^2} .$$

If the target possesses several vulnerable components, the killing of any one of which will constitute a kill of the target; i.e., the target is singly vulnerable, then the average number of hits on vulnerable areas of components is

$$\sum_i \frac{D(\phi_i)\sin^2\phi_i\, A_{vi}}{r_i^2}$$

where the index i ranges over all components. The conditional kill probability is then

$$p_D = 1 - e^{-\sum_i \frac{D(\phi_i) \sin^2\phi_i A_{vi}}{r_i^2}}$$

If the target is small relative to the distance to detonation point and to the fragment beam width, or if all vulnerable components are clustered so that $r_i$ and $\phi_i$ do not vary significantly with i then

$$p_D = 1 - e^{\frac{-D(\phi) \sin^2\phi}{r^2} \sum_i A_{vi}} .$$

Some targets are so constituted that a combination of vulnerable components must be killed in order to kill the target (e.g., 3 out of 4 engines or both pilot and copilot). Such targets are called multiply vulnerable. The calculation of $p_D$ for multiply vulnerable targets is far more complex. Most missile warheads today are being designed to inflict structural kills and most targets are singly vulnerable to structural damage. Consequently we omit any detailed consideration of multiply vulnerable targets.

Carlton Approximation

An alternative approximate expression for the conditional kill probability $p_D$, known as the Carlton approximation, is of interest in that it permits ready evaluation of the integral of equation (7) for several situations of interest. The approximation assumes that $p_D$ is independent of θ. We write

$$(9) \qquad p_D = e^{-\frac{r^2}{2c^2}}$$

where c is a constant, characteristic of the warhead and target. Unfortunately no complete relationship is known between the constant c and the many parameters describing the warhead and target. In consequence the Carlton approximation is useful primarily in investigating the effects of variations in fuzing and guidance, after a value for c has been chosen to fit more careful calculations for one set of assumptions regarding fuzing and guidance.

Internal Blast Warhead

The detonation of an internal blast warhead is in many ways similar to that of a fragment warhead, in that, on detonation, the internal blast warhead emits one or more subprojectiles, each containing a high explosive charge which is designed to detonate after penetrating the target aircraft skin. Two critical problems facing the internal blast warhead designer are provision of a sufficient number of subprojectiles, each containing a lethal high explosive charge, and provision of a subprojectile fuzing mechanism that will be insensitive to the forces it experiences at ejection but sensitive to impact with the target, in order to cause detonation of the high explosive charge at or just inside the target skin. However, such a detonation, if it does occur, will usually be lethal.

Characteristics of an internal blast warhead that differ significantly from those of a fragment warhead are:

The density of subprojectiles is much lower.
A considerably greater portion of the target is vulnerable to subprojectiles.
The subprojectile beam angle $\phi$, measured from the forward missile axis, is usually smaller.
In some cases more than one beam of subprojectiles is ejected.

The conditional kill probability expression for fragment warheads (equation (8)) is often suitable for internal blast warheads, although density, angle, and vulnerable area will differ.

External Blast Warhead

The detonation of an external blast warhead produces a shock wave which may be characterized at any nearby point by its peak overpressure and its positive impulse (the integral of pressure over the interval of time during which the pressure exceeds the ambient atmospheric pressure). Damage to an aircraft from such a detonation appears to be a consequence of a combination of the effects of peak overpressure and positive impulse.

The vulnerability of an aircraft to blast is customarily determined experimentally, in a manner similar to that used for fragment warheads. Static warhead firings are made against whole aircraft or parts of aircraft, and the resulting deformations are assessed by experienced warhead analysts to ascertain whether the deformations constitute damage of the prescribed category. Pressure measurements are made during the firings, from which peak overpressure and positive impulse are obtained. Some combinations of pressure and impulse produce lethal damage, and others do not. On a pressure-impulse plot, these combinations fall in two regions, a damage region and a no-damage region, having a fairly well defined boundary, called a damage curve, as depicted in Figure 12.

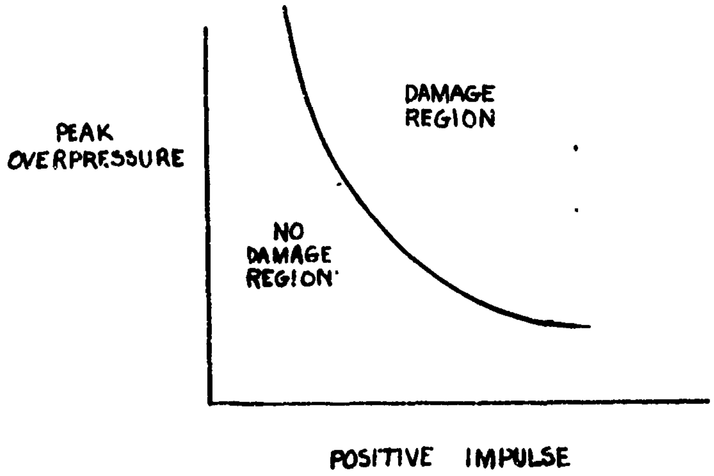
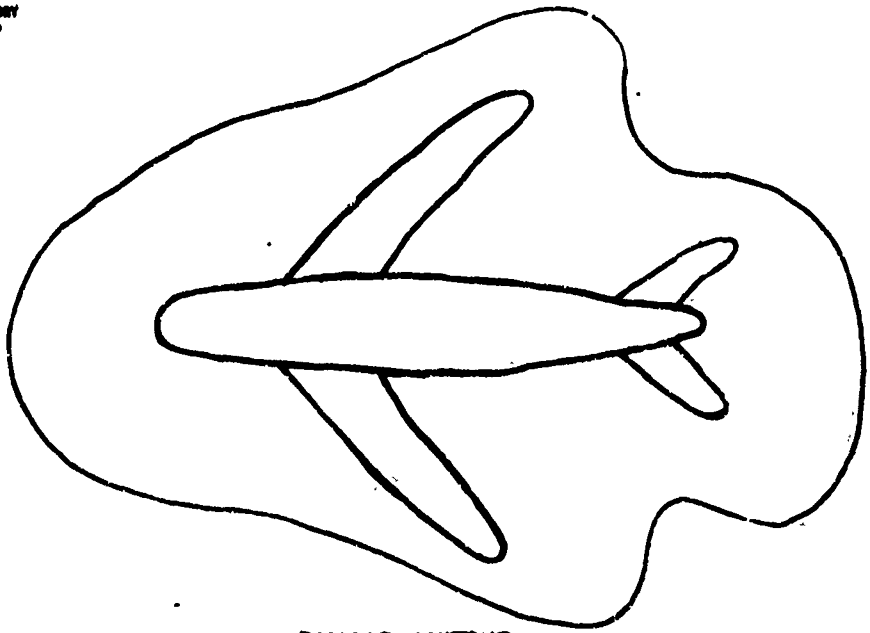



Figure 12

The location of the damage curve on such a pressure-impulse plot will depend upon the toughness of the type of target considered, i.e., how well it can withstand the effects of blast.

Since both pressure and impulse fall off with increasing distance from the detonation point of the warhead, a crossover from the damage region to the no-damage region of the pressure-impulse plot will occur at some distance out from the detonation. Thus a damage contour about the aircraft can be defined such that detonations within the contour produce damage, and detonations outside the contour do not. In fact, damage contours are customarily derived directly from the results of test firings. The distances and aspects between the aircraft or aircraft parts and the detonation point are recorded, points for which static firings do and also do not produce lethal damage are plotted about an outline of the target, and the boundary dividing the damage and no-damage regions is drawn. Typically this boundary is rather sharply defined. The damage contour thus drawn will depend on the size, shape, and toughness of the target and on the high explosive charge in the warhead. Because of the irregular shape and varying toughness of most aircraft, the contour will usually itself be irregular; Figure 13 shows a cross-section of a typical contour.



DAMAGE CONTOUR
Figure 13

To each point on the damage contour there corresponds a combination of peak overpressure and positive impulse. Each combination corresponds in turn to a point of the damage curve on the pressure-impulse plot.

The discussion thus far has dealt with static firings at or near sea level. However, in an air battle many intercepts will occur at high altitude, where variations in ambient atmospheric pressure and in temperature will significantly alter the behavior of a shock wave. Thus a means for scaling the sea level results to altitude is required. A convenient and satisfactory means is to scale both peak overpressure and positive impulse to altitude, and then define a new damage contour to correspond to the damage curve derived earlier for the target in question. It should be noted that a simple increase in distance between target and detonation at higher altitude will not alone reproduce any particular combination of pressure and impulse. However, an appropriate increase in distance at higher altitude will produce another point on the damage curve, and so the damage contour can be scaled.

Both peak overpressure and positive impulse of a shock wave impinging on a surface may be reinforced by reflection from the surface; the extent to which reinforcement occurs will depend critically on the angle of impingement. The extreme cases are face-on (the surface is normal to the direction of motion of the shock wave) and side-on (the surface is parallel to the direction of motion). But face-on and side-on pressure and impulse scale differently, and so in scaling results to altitude some consideration must be given to the angle of impingement. The irregular shape of most targets greatly complicates this problem; however, some evidence exists to support the use of face-on scaling.

Because of the sharp definition of the damage contour it is permissible in most instances to assume that the conditional kill probability $p_D$ is 1 if detonation occurs inside the contour, and is 0 if the detonation is outside.

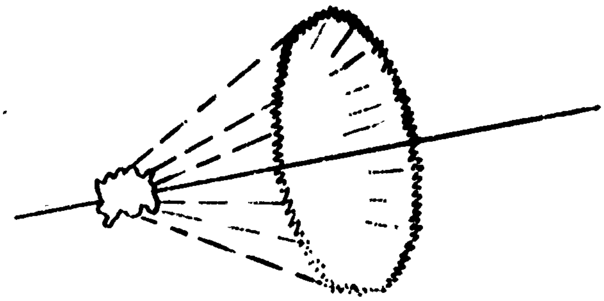
A simplification of considerable practical value has been used extensively. For purposes of estimating blast damage, the target is assumed to be spherical and of uniform toughness. The damage contour for the spherical target will then also be spherical. The difference in radii of the two spheres is, of course, the greatest distance from warhead burst at which the target will suffer lethal damage. A convenient estimate of the radius of the spherical damage contour, which yields satisfactory accuracy, is the radius of a circle having the same area as that enclosed by a cross section of the damage contour for the real target, taken through the target center. Since the latter area will depend on the orientation of the cross section, an average of these areas for several orientations can be used. To simplify scaling to altitude, it is useful to specify the radius of the spherical target as well. A satisfactory estimate of this radius can be obtained in the same manner as above, but here using the projected area of the real target rather than the cross sectional area of the real damage contour. For the spherical model we assign the value 1 to the conditional kill probability $p_D$ if detonation occurs inside the sphere, and the value 0 if detonation occurs outside.

Rod Warheads

Rod warheads of two types will be considered. One of these, the discrete rod warhead, on detonation, emits a number of discrete metal rods, much as a fragment warhead emits fragments. The pattern of emitted rods is similar to the fragment beam, although the rod beam is usually narrower and the number of rods in the beam is less. Damage to aircraft from a discrete rod warhead is primarily a consequence of rods striking vulnerable components of the aircraft, as in the case of a fragment warhead. To a lesser extent damage may be caused by rods severing structural members of the aircraft. Vulnerable areas of targets to rods are considerably greater than to fragments, since individual rods are larger and heavier than fragments.

The conditional kill probability expression for fragment warheads (equation (8)) will sometimes provide sufficient numerical accuracy when applied to discrete rod warheads to warrant its use.

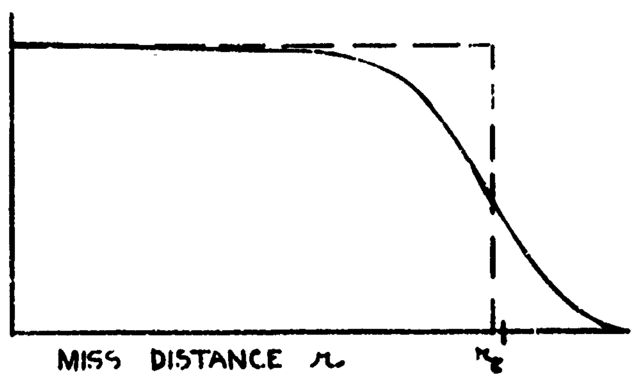
The other type of rod warhead is the continuous rod warhead. A single long continuous metal rod is folded onto the warhead over a high explosive charge. On detonation the rod expands in a circle approximately normal to and centered at the missile axis, as shown in Figure 14.



CONTINUOUS ROD WARHEAD

Figure 14

The circle of metal expands as a virtually continuous rod until the circumference of the circle is approximately equal to the length of the unfolded rod; the rod then breaks randomly into discrete lengths. The speed, thickness, and mass of the rod are such that if the rod strikes the target aircraft, it will usually sever the aircraft structure. Such structural damage will usually cause flight failure, unless the hit is on one of certain extremities, such as a wing-tip. Thus if the miss distance is within the maximum radius of continuity of the rod, a catastrophic kill (K-kill) will usually be produced unless the rod fails to strike the target because of fusing errors. The conditional kill probability falls off rather rapidly with miss distance beyond the maximum radius of continuity, primarily because of rod break-up. With proper fusing, the conditional kill probability $p_D$ typically varies with miss distance as shown in Figure 15.




CONTINUOUS ROD CONDITIONAL KILL PROBABILITY

Figure 15

A convenient and (as is evident from Figure 15) reasonable assumption is to assign to $p_D$ the value 1 for miss distances inside the maximum radius of continuity $r_c$, and the value 0 for miss distances outside.

If fuzing errors cause the rod to fail to strike vulnerable structure of the target, a kill may still be produced by blast from the warhead detonation. Thus within blast lethal radius of the target the conditional kill probability will also have the value 1.

Nuclear Warhead

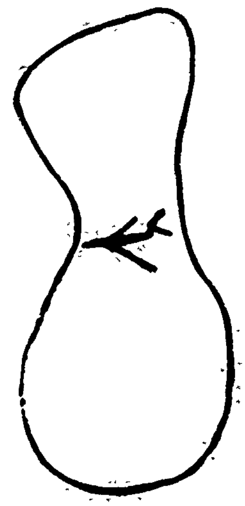
The detonation of a nuclear warhead in the atmosphere produces a devastating shock wave, intense heat, and a high level of radiation. From the point of view of damage to aircraft, the shock wave is of primary concern, since in most cases it will inflict lethal damage to the aircraft itself at considerably greater distance than the others.

One form of damage inflicted by the shock wave is deformation of the target, caused by the overpressure incident on the target, and characterised by peak overpressure and positive impulse as in the case of high explosive external blast warheads discussed earlier.

A second form of damage inflicted by the shock wave is structural failure of aerodynamic surfaces of the target (wings or other surfaces which provide aerodynamic lift) caused by gust loading. A characteristic of a shock wave is a rapid forward motion of the atmosphere immediately behind the shock front. This gust of wind exerts an added force on the aerodynamic surfaces well in excess of the loading experienced in normal flight. If the combined loading exceeds the ultimate strength of the aircraft structure, the structure will fail. Targets having no aerodynamic surfaces will, of course, not be vulnerable to effects of gust loading.

The gust produced by detonation of a high explosive external blast warhead is of too short duration to have lethal effects; to be lethal, the gust must be of significant duration, requiring a much higher level of energy release.

The distance from burst point of the nuclear warhead at which gust loading will be lethal depends on the orientation of the target relative to the burst. The distance will be greatest if the aerodynamic surfaces are approximately normal to the direction of motion of the shock wave. A typical lethal envelope for gust loading is shown in Figure 16.



LETHAL ENVELOPE

Figure 16

The size of the lethal envelope will be determined not only by the level of energy released by the burst, but also by the speed and shape of the target aircraft. Both high speed and large wing area tend to increase the lethal envelope.

Adoption of a spherical damage contour, with conditional kill probability set equal to 1 inside the sphere and 0 outside, has proved to be less satisfactory for nuclear warheads than for high explosive external blast warheads. However, under some circumstances, especially when target deformation caused directly by blast is the principle form of damage, the assumption may be sufficiently accurate to warrant its use.

An alternative assumption in many instances gives reasonably close agreement with the results of detailed calculations. The conditional kill probability $p_D$ is assumed to decrease exponentially as the square of the distance from the point of detonation, much as in the Carlton approximation for fragment warheads (equation (9)). The distinction is that in the present instance the distance is measured between target and point of detonation rather than point of closest approach of the trajectory (miss distance).

## C. Calculation of the Kill Probability of an Operable Missile

We consider the evaluation of the integral of equation (7) for various combinations of guidance, fuzing, and warhead. Unless suitable simplifications are made the precise forms of one or more of the functions $p_D$, $f$, and $g$ are usually such as to preclude expression of the integral in closed form; the integral must then be evaluated by some method of numerical integration. Such methods are usually lengthy and tedious, though they can often be carried out expeditiously with the aid of a high speed computer.

## H. Lotto Method

Where simplifying assumptions are not appropriate, the value of the integral can be estimated by the so-called Lotto method, a sampling process. In this process a sample of detonation points is drawn randomly from the assumed distributions f and g. For example, if g is assumed to be radial normal, a value of r can be selected by drawing a random number between 0 and 1 and determining the value of r that will make

$$\int_0^r g\,dr = \text{random number.}$$

Similarly, under the same assumption for g, all values of $\theta$ are equally likely, so a value of $\theta$ can be selected by drawing a second random number and determining $\theta$ to make

$$\frac{1}{2\pi}\int_0^\theta d\theta = \text{random number.}$$

In like manner, a value of z can be selected by drawing a third random number and, using in f the values of r and $\theta$ just selected, determining z to make

$$\int_{-Z}^{z} f\,dz = \text{random number.}$$

Since (see equation (6)) the integral of f may be less than 1, we may not obtain a value of z for every random number drawn. If no z is obtained, implying that no detonation occurs, it is convenient to count this trial as a detonation in our sample anyway, but as one for which $p_D = 0$.

By repeating the above steps many times, we obtain the coordinates $(r,\theta,z)$ of the detonation points in our sample. The size of the sample will dictate the accuracy of the estimated value of the integral, the larger the sample the greater the accuracy.

For each point selected, we calculate the conditional kill probability $p_D(r,\theta,z)$, and draw another random number between 0 and 1 to determine if a kill occurs. If $p_D$ is less than or equal to the random number, a kill is scored, and if $p_D$ is greater than the random number, no kill is scored. Clearly, if $p_D = 0$, no kill will occur. We then count the number of kills scored. The ratio of number of kills to number of detonation points selected provides an estimate of $p_K$.

Once the sample of detonation points has been selected, an alternative approximation to $p_K$ is given by

$$p_K = \frac{\sum p_D(r_i,\theta_i,z_i)f(r_i,\theta_i,z_i)g(r_i,\theta_i)}{\sum f(r_i,\theta_i,z_i)g(r_i,\theta_i)}$$

where the summations are over all detonation points selected. This approximation arises from the definition of expected value; $p_K$ is the expected value of $p_D$, and the sampling process provides an approximation to this expected value.

## I. Analytic Evaluations of the Integral

It is possible to obtain relatively simple expressions for $p_K$ from the integral of equation (7) by introducing suitable simplifying assumptions for the functions $p_D$, f, and g. Simplifying assumptions can often be made that will provide sufficient accuracy in the resulting expressions for $p_K$ to meet the needs of some, though by no means all, investigations of guidance accuracy, fuzing accuracy, fuze cut-off, and warhead lethality. However, great care should be exercised in employing these simplifications, to insure that they are applicable. Different sets of assumptions are appropriate to different warheads; for this reason expressions for $p_K$ will be derived separately for each warhead type in turn.

A few assumptions will be common to the treatment of all warhead types. A circular or radial normal distribution of trajectories, having standard deviation σ in any one dimension, will be assumed. One of three alternative assumptions will be made concerning fuze cut-off: 1) no fuze cut-off, 2) a sharp fuze cut-off range $r_f$, or 3) an exponential fuze cut-off, varying with miss distance according to

$$e^{-\frac{r^2}{2a^2}}$$

where a is a constant, characteristic of the fuze and target.

### Fragment Warhead - Ideal Fuze Model

The integral of equation (7) can be greatly simplified if we assume that all vulnerable components are clustered close together, that the vulnerable areas of components are independent of miss distance r and angle θ, and that fuze operation is ideal in the sense that the detonation occurs so as to place the most dense part of the fragment beam at the center of the vulnerable components, i.e., ∅ and so r/z are constant, and hence

$$p_D = 1 - e^{-\frac{k^2}{r^2}}$$

where

$$k^2 = D(\phi)\ \sin^2\phi\ \sum A_{vi}$$

is independent of r and θ. With no fuze cut-off we have

$$p_K = \frac{1}{\sigma^2} \int_0^{\infty} \left(1 - e^{\frac{-k^2}{r^2}}\right) e^{\frac{-r^2}{2\sigma^2}}\ rdr \quad .$$

Since the radius R of the tube of operability should be chosen large enough to preclude kills from detonations outside the tube, no significant error is introduced by taking the above integral from 0 to ∞. The above integral can be written

$$p_K = \frac{1}{\sigma^2} \int_0^{\infty} e^{\frac{-r^2}{2\sigma^2}}\ rdr - \frac{1}{\sigma^2} \int_0^{\infty} e^{\frac{-k^2}{r^2}\ \frac{-r^2}{2\sigma^2}}\ rdr \quad .$$

The first of these integrals has the value one, since it is simply the integral over the entire range of the radial normal distribution. The second integral can be expressed in terms of $K_1(x)$, the modified Bessel function of the second kind and first order. We then have

$$p_K = 1 - \frac{\sqrt{2}k}{\sigma} K_1 \left(\frac{\sqrt{2}k}{\sigma}\right) \quad .$$

Values of the function $K_1(x)$ can be obtained from tables available in the literature (National Bureau of Standards and Government Printing Office, Applied Mathematics Series 25 "Tables of the Bessel functions $Y_o(x)$, $Y_1(x)$, $K_o(x)$, $K_1(x)$, $0 \leq x \leq 1$", Sept. 1952).

If we assume an exponential fuze cut-off but otherwise retain the above assumptions, we can write

$$p_K = \frac{1}{\sigma^2} \int_0^{\infty} \left(1 - e^{\frac{-k^2}{r^2}}\right) e^{\frac{-r^2}{2a^2}}\ e^{\frac{-r^2}{2\sigma^2}}\ rdr$$

which reduces to

$$p_K = \frac{1}{\sigma^2 w^2}\left[1 - \sqrt{2}\, kwK_1\,(\sqrt{2}\, kw)\right]$$

where $$w^2 = \frac{1}{c^2} + \frac{1}{a^2} \quad .$$

## Calculations Using Carlton Approximation

The Carlton approximation to the conditional kill probability $p_D$ (equation (9)) permits simple evaluation of the integral of equation (7) for several variations in guidance and fuzing.

If fuzing dispersion is small, so that the fragment beam covers the target, we have, with no fuze cut-off

$$p_K = \frac{1}{\sigma^2}\int_0^{\infty} e^{\frac{-r^2}{2c^2}}\, e^{\frac{-r^2}{2\sigma^2}}\, r dr$$

which reduces to

$$p_K = \frac{c^2}{c^2 + \sigma^2} \quad .$$

The effect of a sharp fuze cut-off range $r_f$ can readily be introduced by taking $r_f$ as the upper limit of integration. We then have

$$p_K = \frac{1}{\sigma^2}\int_0^{\infty} e^{\frac{-r^2}{2c^2}}\, e^{\frac{-r^2}{2\sigma^2}}\, r dr$$

which reduces to

$$p_K = \frac{c^2}{c^2 + \sigma^2}\left[1 - e^{\frac{-r_f^2}{2}\left(\frac{1}{c^2} + \frac{1}{\sigma^2}\right)}\right]$$

Thus the effect of the sharp cut-off is to introduce the bracketed expression as a correction factor.

Assuming an exponential fuze cut-off, we have

$$p_K = \frac{1}{\sigma^2} \int_0^\infty e^{\frac{-r^2}{2c^2}} e^{\frac{-r^2}{2a^2}} e^{\frac{-r^2}{2\sigma^2}} r dr .$$

Comparing this expression with the corresponding expression for no fuze cut-off, it is evident that the effect of an exponential cut-off is to replace

$$\frac{1}{c^2} \text{ by } \frac{1}{a^2} + \frac{1}{c^2} .$$

It is convenient to write the exponential cut-off result in the form

$$p_K = \frac{c^2}{c^2 + \sigma^2} \left[ \frac{1}{1 + \frac{c^2 \sigma^2}{a^2 (c^2 + \sigma^2)}} \right]$$

The effect of exponential fuze cut-off is to again introduce a correction factor, the quantity in brackets.

The Carlton approximation is useful in examining the effects of biased guidance. It is convenient in this instance to express the distribution in Cartesian coordinates, with origin at the target, as in Figure 17.

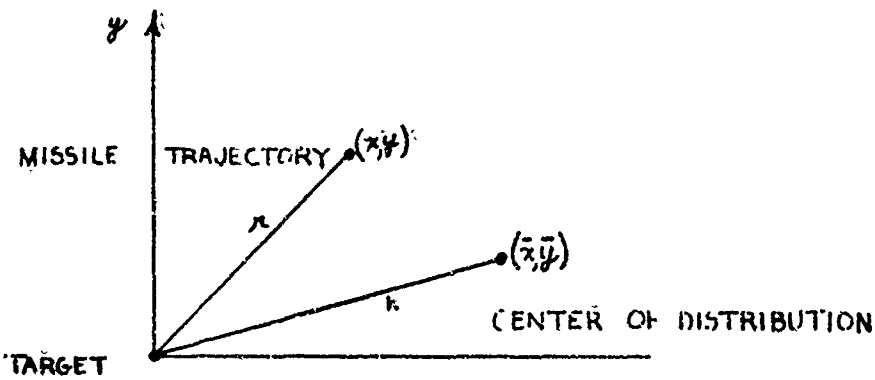



Figure 17

We let $(\overline{x},\overline{y})$ denote the center of the distribution, and h the distance from the target to the center, so

$$h^2 = \overline{x}^2 + \overline{y}^2 \quad .$$

The miss distance r of course satisfies the relation

$$r^2 = x^2 + y^2 \quad .$$

The assumption of a radial normal distribution of trajectories of course implies that the distributions with respect to both x and y are Gaussian having a common standard deviation σ. We may then write

$$g(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{(x-\overline{x})^2+(y-\overline{y})^2}{2\sigma^2}}$$

Employing the Carlton approximation for $p_D$ and assuming no-fuze cut-off, we have

$$p_K = \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{(x-\overline{x})^2+(y-\overline{y})^2}{2\sigma^2}} e^{-\frac{x^2+y^2}{2c^2}} dxdy \quad .$$

The exponentials of the integrand can be combined to give

$$p_K = \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{(c^2+\sigma^2)}{2c^2\sigma^2}\left[\left(x - \frac{\overline{x}c^2}{c^2+\sigma^2}\right)^2 + \left(y - \frac{\overline{y}c^2}{c^2+\sigma^2}\right)^2\right]} e^{-\frac{h^2}{2(c^2+\sigma^2)}} dxdy$$

$$= \frac{1}{2\pi\sigma^2} e^{-\frac{h^2}{2(c^2+\sigma^2)}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{(c^2+\sigma^2)}{2c^2\sigma^2}\left[\left(x - \frac{\overline{x}c^2}{c^2+\sigma^2}\right)^2 + \left(y - \frac{\overline{y}c^2}{c^2+c^2}\right)^2\right]} dxdy$$

The double integral has the value

$$\frac{2\pi c^2\sigma^2}{c^2+\sigma^2}$$

and so

$$p_K = \frac{c^2}{c^2+\sigma^2}\left[e^{-\frac{h^2}{2(c^2+\sigma^2)}}\right] \qquad (10)$$

Thus the effect of biased guidance is also to introduce a correction factor, the bracketed quantity.

The effect of biased guidance together with an exponential fuze cut-off can readily be obtained if, in the preceding derivation, we replace

$$\frac{1}{c^2} \text{ by } \frac{1}{a^2} + \frac{1}{c^2} \quad .$$

Fuzing Dispersion

The assumption of ideal fuze operation set forth above is sometimes not appropriate, and so it is necessary to consider the effects of dispersion in fuzing. For the sake of simplicity we suppose that all fragments are confined to a beam bounded by the angles $\phi_1$ and $\phi_2$, measured from the forward missile axis, and that fragment density is constant within this beam. If we again confine our attention to targets having closely clustered vulnerable components, clearly a kill can occur only if the warhead detonates between $z_1 = -r \cot \phi_1$ and $z_2 = -r \cot \phi_2$ (see Figure 18).

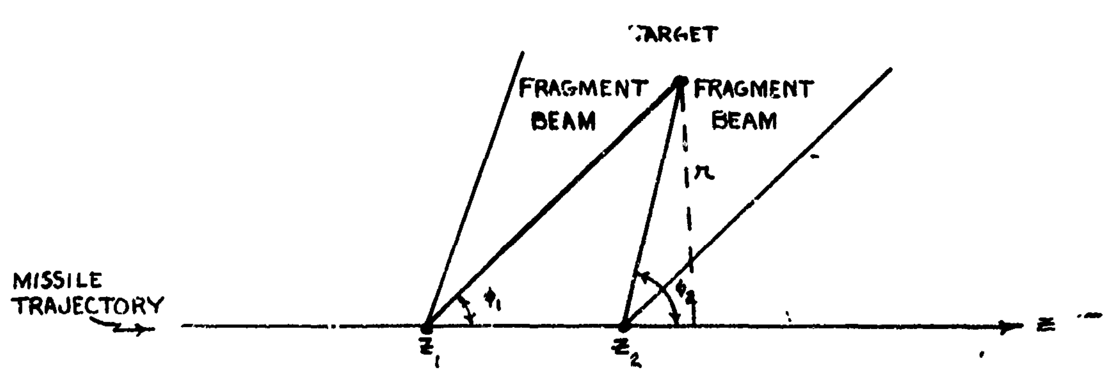



Figure 18

We assume that the distribution of detonation points along the trajectory is Gaussian with mean $\bar{z}$ midway between $z_1$ and $z_2$, and standard deviation $\sigma_z$. Using the Carlton approximation for $p_D$ and no fuze cut-off, we have

$$(11) \qquad p_K = \frac{1}{\sqrt{2\pi}\,\sigma_z \sigma^2} \int_0^{\infty} e^{-\frac{r^2}{2c^2}} e^{-\frac{r^2}{2\sigma^2}} r \int_{z_1}^{z_2} e^{-\frac{(z-\bar{z})^2}{2\sigma_z^2}} dz dr$$

Since $\bar{z}$ is midway between $z_1$ and $z_2$, we have, on change of variable of integration

$$\int_{z_1}^{z_2} e^{-\frac{(z-\bar{z})^2}{2\sigma_z^2}} dz = 2\int_0^{\alpha r} e^{-\frac{z^2}{2\sigma_z^2}} dz$$

where

$$\alpha = \frac{\cot \phi_1 - \cot \phi_2}{2} .$$

Substituting in equation (11) and integrating by parts, we have

$$(12) \qquad p_K = \frac{c^2}{c^2+\sigma^2}\left(\frac{1}{\sqrt{1 + \frac{\sigma_z^2(c^2+\sigma^2)}{\alpha^2 c^2 c^2}}}\right) .$$

We see that the effect of fuzing dispersion is again to introduce a correction factor, the expression in brackets.

For an exponential fuze cut-off we have

$$f = \frac{1}{\sqrt{2\pi}\,\sigma_z} e^{-\frac{(z-\bar{z})^2}{2\sigma_z^2}} e^{-\frac{r^2}{2a^2}}$$

and so we can readily obtain an expression for $p_K$ if we replace

$$\frac{1}{c^2} \text{ by } \frac{1}{a^2} + \frac{1}{c^2}$$

in equation (12).

For a sharp fuze cut-off range $r_f$, the integration by parts that led to equation (12) now gives

$$p_K = \frac{c^2}{c^2+\sigma^2}\left[\frac{1}{\sqrt{1+\frac{\sigma_z^2(c^2+\sigma^2)}{\alpha^2c^2\sigma^2}}}\; I\left(\frac{\alpha r_f}{\sigma_z}\sqrt{1+\frac{\alpha_z^2(c^2+\sigma^2)}{\alpha^2c^2\sigma^2}}\right) - e^{-\frac{c^2+\sigma^2}{2c^2\sigma^2}r_f^2}\; I\left(\frac{\alpha r_f}{\sigma_z}\right)\right]$$

where I(x) is the probability integral,

$$I(x) = \sqrt{\frac{2}{\pi}}\int_0^x e^{-\frac{t^2}{2}}\,dt$$

tables of values of which may be found in the literature.

## Internal Blast Warhead

In some situations and for some purposes the methods for computing $p_K$ which were derived for fragment warheads may be used to obtain estimates of $p_K$ for internal blast warheads. In particular, the beam of subprojectiles must cover the entire target, and the density of subprojectiles must be fairly uniform within the beam. The methods derived for fragment warheads are most useful when the analyst is concerned with the gross effects of guidance accuracy and fuze performance.

## External Blast Warhead

Adoption of a spherical damage contour of radius $r_W$ for an external blast warhead leads to extremely simple evaluation of the integral of equation (7).

Since distance from target is the critical parameter in determining kill, ideally the warhead should detonate at the point of closest approach of the missile to the target. Assuming ideal fuzing in the above sense, and no fuze cut-off, we have

$$p_K = \frac{1}{\sigma^2}\int_0^{r_W} e^{-\frac{r^2}{2\sigma^2}}\, r\,dr$$

$$= 1 - e^{-\frac{r_W^2}{2\sigma^2}} .$$

A sharp fuze cut-off range $r_f$ will have no effect if $R < r_f$. If $r_f$ is the smaller, its effect is to replace R by $r_f$ in the above expression.

Assuming an exponential fuze cut-off

$$p_K = \frac{1}{\sigma^2} \int_0^{r_W} e^{-\frac{r^2}{2\sigma^2}} e^{-\frac{r^2}{2a^2}} rdr$$

which reduces to

$$p_K = \frac{a^2}{a^2+\sigma^2} \left( 1 - e^{-\frac{1}{2}\left(\frac{1}{a^2}+\frac{1}{\sigma^2}\right) r_W^2} \right) .$$

If a fixed angle fuze is used, the warhead detonation may occur somewhat before or somewhat after the missile reaches the point of closest approach. Neglecting fuzing dispersion and assuming either no time delay or a delay proportional to miss distance, expressions for $p_K$ can readily be obtained from those for ideal fuzing if we replace $r_W$ by $r_W \sin \phi$, where $\phi$ is the fuzing angle, including delay.

Some indication of the consequence of fuzing dispersion can be obtained if we assume a Gaussian distribution of burst points along the trajectory, with mean at the point of closest approach (z=0) and standard deviation $\sigma_z$. With no fuze cut-off, we have

$$p_K = \int_0^{r_W} \int_{-z(r)}^{z(r)} \frac{1}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} \frac{1}{\sqrt{2\pi}\,\sigma_z} e^{-\frac{z^2}{2\sigma_z^2}} rdrdz .$$

Changing the order of integration, this expression becomes

$$p_K = \frac{1}{\sqrt{2\pi}\,\sigma_z} \int_{-r_W}^{r_W} e^{-\frac{z^2}{2\sigma_z^2}} \left\{ \frac{1}{\sigma^2} \int_0^{\sqrt{r_W^2-z^2}} e^{-\frac{r^2}{2\sigma^2}} rdr \right\} dz$$

which reduces to

$$p_K = \sqrt{\frac{2}{\pi}}\,\frac{1}{\sigma_z}\int_0^{r_W} e^{-\frac{z^2}{2\sigma_z^2}}\,dz - \sqrt{\frac{2}{\pi}}\,\frac{1}{\sigma_z}\,e^{-\frac{r_W^2}{2\sigma^2}}\int_0^{r_W} e^{-\frac{1}{2}\left(\frac{1}{\sigma_z^2}-\frac{1}{\sigma^2}\right)z^2}\,dz$$

The first term on the right hand side of this equation is the probability integral $I(r_W/\sigma_z)$. Evaluation of the second term depends upon the relative values of $\sigma_z$ and $\sigma$. If $\sigma_z = \sigma$, the integrand in the second term is 1, and we have

$$p_K = I\left(\frac{r_W}{\sigma_z}\right) - \sqrt{\frac{2}{\pi}}\,\frac{1}{\sigma_z}\,r_W\,e^{-\frac{r_W^2}{2\sigma^2}}.$$

If $\sigma_z < \sigma$, we have

$$p_K = I\left(\frac{r_W}{\sigma_z}\right) - \frac{1}{\sqrt{\frac{\sigma_z^2}{\sigma} - 1}}\,e^{-\frac{r_W^2}{2\sigma^2}}\left[\sqrt{\frac{2}{\pi}}\int_0^{r_W\sqrt{\frac{1}{\sigma^2}-\frac{1}{\sigma_z^2}}} e^{+\frac{t^2}{2}}\,dt\right]$$

Tables of values of the integral in brackets are not as readily available as in the case of the probability integral. However, numerical integration is not unduly tedious.

### Rod Warhead

Accurate evaluation of the probability integral of equation (7) for discrete rod warheads in most cases requires detailed computer calculations, or use of the Lotto technique. In the treatment of fragment warheads a variety of simplifying assumptions were introduced which led to closed form evaluations of the integral. These assumptions are less palatable in the consideration of discrete rod warheads, because of the narrower rod beam and smaller number of rods. However for some purposes they may be justified, and the corresponding expressions for $p_K$ may be used.

The almost certain lethality of the continuous rod warhead out to the maximum radius of rod continuity permits simple evaluation of the probability integral of equation (7). If we assume ideal fuzing in the sense that fuzing errors do not prevent the rod from striking the vulnerable portion of the target, we have, with no fuze cut-off

$$p_K = \frac{1}{\sigma^2}\int_0^{r_c} e^{-\frac{r^2}{2\sigma^2}}\, r\,dr$$

$$= 1 - e^{-\frac{r_c^2}{2\sigma^2}}$$

where $r_c$ is the maximum radius of continuity.

A sharp fuze cut-off range $r_f$ has no effect if $r_c < r_f$. If, on the other hand, $r_f$ is smaller, its effect is to replace $r_c$ by $r_f$ in the above expression.

If the fuze cut-off is exponential, we have

$$p_K = \frac{1}{\sigma^2}\int_0^{r_c} e^{-\frac{r^2}{2\sigma^2}}\, e^{-\frac{r^2}{2a^2}}\, r\,dr$$

$$= \frac{a^2}{a^2+\sigma^2}\left(1 - e^{-\frac{1}{2}\left(\frac{1}{a^2}+\frac{1}{\sigma^2}\right) r_c^2}\right) .$$

The above definition of ideal fuzing implies that the warhead detonation will occur within an interval along the trajectory of length L equal to the length of the vulnerable portion of the target, measured in a direction parallel to the trajectory. The continuous rod does not expand precisely in the plan normal to the trajectory and through the point of detonation. Rather, the rod sweeps out a cone which makes an angle $\phi$ with the forward missile axis. Consequently the effective burst interval is displaced along the trajectory as indicated in Figure 19.

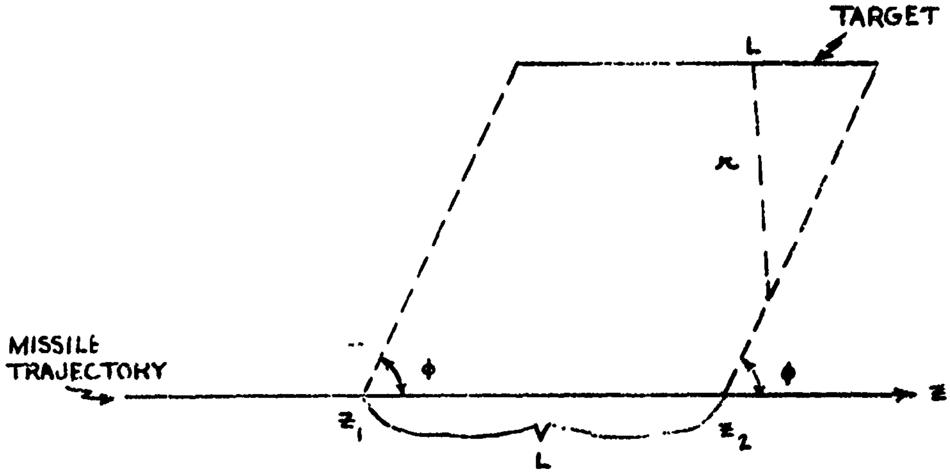



Figure 19

If we assume that the distribution of detonation points along the trajectory is normal with mean at the midpoint of the effective burst interval and standard deviation $\sigma_z$, then, with no fuze cut-off, we have

$$p_K = \frac{1}{\sigma^2} \int_0^{r_c} e^{-\frac{r^2}{2\sigma^2}} r dr \frac{1}{\sqrt{2\pi}\,\sigma_z} \int_{-L/2}^{L/2} e^{-\frac{z^2}{2\sigma_z^2}} dz$$

$$= \left(1 - e^{-\frac{r_c^2}{2\sigma^2}}\right) I\left(L/2\sigma_z\right)$$

where I(x) is the probability integral

$$I(x) = \sqrt{\frac{2}{\pi}} \int_0^x e^{-\frac{t^2}{2}} dt \quad .$$

The last expression for $p_K$ above neglects the contribution made by blast damage when the burst occurs outside the interval of length L but within the blast lethal radius.

## Nuclear Warhead

If a spherical damage contour for a nuclear warhead can appropriately be assumed, the expressions for $p_K$ for an external blast warhead can be used. The alternative assumption that the conditional kill probability behaves in a manner similar to the Carlton approximation leads to two simple expressions for $p_K$. If the detonation occurs at or near the point of closest approach to the target, we may write

$$p_D = e^{-\frac{r^2}{2c^2}}$$

where as usual r denotes miss distance, and c is a constant characteristic of the lethal envelope. Assuming no fuze cut-off, we have

$$p_K = \frac{1}{\sigma^2} \int_0^{\infty} e^{-\frac{r^2}{2c^2}} \, e^{-\frac{r^2}{2\sigma^2}} \, r dr$$

$$= \frac{c^2}{c^2+\sigma^2} .$$

Because the lethal envelope for a nuclear warhead is so large, a fuze cut-off is rarely employed. Indeed, crude command fuzing is often adequate. If we assume a normal distribution of burst points along the trajectory with mean at the point of closest approach to the target and standard deviation equal to that of the guidance accuracy, we can then employ a spherical normal distribution to represent the distribution of burst points in space about the target. We then have

$$p_K = \sqrt{\frac{2}{\pi}} \, \frac{1}{\sigma^3} \int_0^{\infty} e^{-\frac{1}{2}\left(\frac{1}{c^2}+\frac{1}{\sigma^2}\right)\rho^2} \rho^2 d\rho$$

where $\rho$ here denotes distance from target center to burst point. This expression readily reduces to

$$p_K = \frac{c^3}{(c^2+\sigma^2)^{3/2}} .$$

J. Salvo Kill Probability

If the kill probability of a single missile against a target is not sufficiently high, it may, in some circumstances, be desirable to fire several missiles at the target simultaneously. To meet this need, many SAM systems are designed to be capable of salvo fire; a single guidance channel can guide two or more missiles to the same target at the same time. The tactical analyst is then faced with the question of what the probability of killing a target with a salvo may be.

If the vulnerability of the target to the type of warhead in question is such that cumulative damage inflicted by successive warheads can be ignored, the salvo kill probability can readily be derived. If a salvo of $\nu$ missiles is fired at a target, some number j of these will prove to be operable, and the probability $p_R(j)$ that exactly j will be operable is given by

$$p_R(j) = \binom{\nu}{j} p_R^{\,j} (1-p_R)^{\nu-j}$$

where $p_R$ is the single missile reliability. The probability $p_K(j)$ that if j missiles are operable the target will suffer lethal damage is just the probability that not all j missiles will fail to kill, i.e.,

$$p_K(j) = 1 - (1-p_K)^j$$

where $p_K$ is the probability that a single operable missile will kill. The salvo kill probability $p(\nu)$ is given by

$$\begin{aligned} p(\nu) &= \sum_{j=1}^{\nu} p_R(j) p_K(j) \\ &= \sum_{j=1}^{\nu} \binom{\nu}{j} p_R^{\,j} (1-p_R)^{\nu-j} \left[1 - (1-p_K)^j\right] \\ &= 1 - (1-p_R p_K)^{\nu} \\ &= 1 - (1-p)^{\nu} \end{aligned}$$

where p is the single missile kill probability.

Few investigations of the mechanics of cumulative damage to singly vulnerable targets have been carried out. The limited investigations that have been made suggest that cumulative damage to singly vulnerable targets is of only minor importance.

K. Kill Probability Against Formations

The discussion up to this point has been concerned with the kill probabilities of missiles against single isolated target aircraft. However, aircraft may attack in any of a wide variety of formations. If the formation is fairly loose, the spacing between aircraft being large, each target can be engaged individually as an isolated target and the existence of the formation will have no effect on kill probability. If, on the other hand, the aircraft fly sufficiently close together, the guidance equipment of the SAM system may become confused, causing a substantial increase in miss distance and so a degradation in kill probability. Of course, a very tight formation may permit a missile to inflict lethal damage on several aircraft simultaneously, to the detriment of the attacker.

The source of confusion in the guidance equipment of a SAM system is the inability of a radar or missile seeker to single out an individual target in the formation in sufficient time and then remain locked on that target during the remaining flight of the missile. A radar or seeker, unable to discriminate between targets, tends to wander randomly over the formation, skipping erratically from target to target, and even moving outside the formation.

Different guidance systems are influenced quite differently by the wander of the radar or seeker over the formation. A beamrider will follow the radar beam approximately if the beam does not move with too great an angular rate. Thus the distribution of beamrider trajectories through a formation can be expected to approximate the distribution of radar beam positions. In a command system the steering orders transmitted to the missile will be affected not only by the instantaneous position of the radar beam, but also by the rate of change of position, in that the latter will in part determine the predicted course and speed of the target. In consequence a dispersion in missile trajectories larger than in radar beam position can be expected. A homing missile will usually be able to discriminate between targets when it reaches a point close enough to the formation. However, if discrimination occurs too late, the speed and maneuverability of the missile may be such as to prevent it from correcting the error in trajectory obtaining at the time of discrimination.

One further formation effect warrants consideration, that of fuze screening, pertinent to proximity fuzes. A missile may be flying toward one aircraft in a formation along a trajectory having a satisfactorily small miss distance. However, before reaching this target, the proximity fuze may detect another target in the formation, causing early detonation of the warhead at a relatively ineffective position for both targets.

Radars and missile seekers usually employ one or more of four means for discriminating between targets:

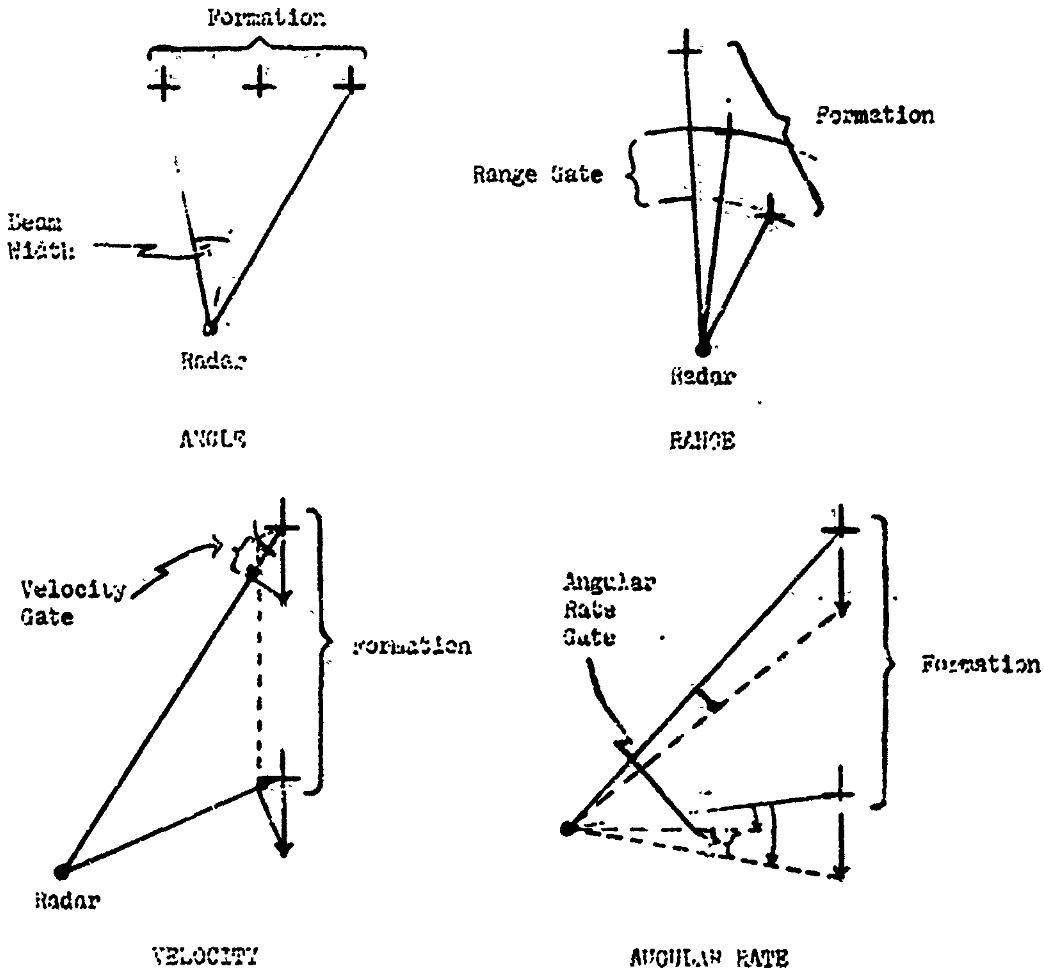
Angular discrimination, determined by the beam width of the radar.

Range discrimination, by means of a range gate.

Range rate discrimination, using a velocity gate.

Angular rate discrimination.

The various methods for discrimination are illustrated schematically in Figure 20.




Discrimination Means

Figure 20

The success any combination of these will have in discriminating between targets in time to avoid a degradation in missile guidance will depend upon the geometry of the formation relative to the radar or the missile seeker. Angular discrimination may be effective against a line abreast formation flying toward the radar, but not against a single file formation. The converse applies to range discrimination. Range rate and angular rate discrimination will be most effective when the formation is flying on a crossing course.

A convenient measure of the effect of a formation on the killing capability of a missile is the ratio of the expected number of kills inflicted by the missile on the formation to the expected number of kills inflicted on an isolated target, i.e., the probability of killing an isolated target. We call this ratio the <u>formation index</u>.

The expected number of kills in a formation can most easily be calculated by attempting to separate guidance errors arising from radar wander over the formation from those normally associated with an isolated target. To this end we assume these two types of error to be independent. Consider now a formation and a particular radar beam position within the formation, as depicted in Figure 21.

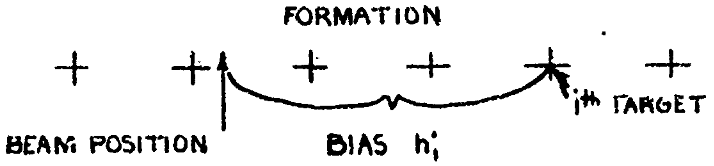



Figure 21

If we assume that the usual errors associated with an isolated target lead to a distribution of trajectories about this beam position, the probability of killing the i-th target can be computed for the particular bias $h_i$, (see Figure 21) using one of the techniques discussed earlier for isolated targets. If we then assume a suitable distribution of radar beam positions over the formation, we can average the above kill probability with respect to this distribution of bias $h_i$ to obtain the kill probability for the i-th target $p_{Ki}$. Summing $p_{Ki}$ over all targets, we obtain the expected number of kills for the missile against the formation. We have

$$K = \sum \int p_{Ki}(h_i)\, B(h_i)\, dh_i$$

where

$K$ = expected number of kills

$p_{Ki}(h_i)$ = probability of killing the i-th target if bias is $h_i$

$B(h_i)$ = distribution of bias $h_i$ .

The above formulation of expected number of kills against a formation is inadequate for a homing missile without some modification. The difficulty lies in the fact that a homing missile can usually discriminate between targets when it reaches a point close enough to the formation but may or may not be able to home on a single target satisfactorily. We must distinguish between the two cases of satisfactory and unsatisfactory homing.

With satisfactory homing multiple kills may occur if the formation spacing is small enough, even though the missile passes close to one target. However, the chance of multiple kills will fall off more rapidly with increased spacing in this case. The probability of killing the i-th target, if the missile is homing on another target, can be computed as above, the guidance bias being simply the lateral distance between the targets. This probability must then be averaged over all targets on which the missile could home satisfactorily to give $p_i$. The sum of all the $p_i$ is the expected number of kills $K_H$ if the missile homes satisfactorily on some target.

If the missile fails to home satisfactorily, it can be thought of as entering the formation randomly, and the expected number of kills $K_U$ can be computed as before.

There is a probability $p_H$ that a missile will home satisfactorily. If the initially random entry into the formation is such as to permit discrimination in time, homing will be satisfactory. Whether or not a particular entry position will permit discrimination is dependent on the geometry of the formation and the method of discrimination employed by the seeker.

The expected number of kills K is given by

$$K = p_H K_H + (1-p_H) K_U$$

It is of interest to consider qualitatively the variation of formation index as a function of spacing between adjacent aircraft. If the spacing is very small, a single missile may be expected to kill several targets; the formation index will in this instance be high, considerably in excess of 1. As the spacing increases, fewer targets will be within lethal distance of the warhead burst, wherever the burst may occur; consequently fewer kills will be inflicted and the index will decrease. At some sufficiently great spacing, all targets will appear resolved and can be intercepted individually as isolated targets, and no multiple kills can occur. At this and all greater spacings, the index will have the value 1. At spacings less than this critical spacing the value of the index will depend upon the lethality of the warhead. If the lethal diameter of the warhead is as great or greater than the critical spacing, multiple kills can be expected at all spacings less than critical, and so the index will never be less than 1. On the other hand, if the lethal diameter is significantly less than the critical spacing, the random wander of the radar or seeker over the formation will so degrade the guidance accuracy that at spacings less than critical the index may fall well below the value 1. Figure 22 depicts typical plots of formation index versus spacing for both large and small lethal diameters.

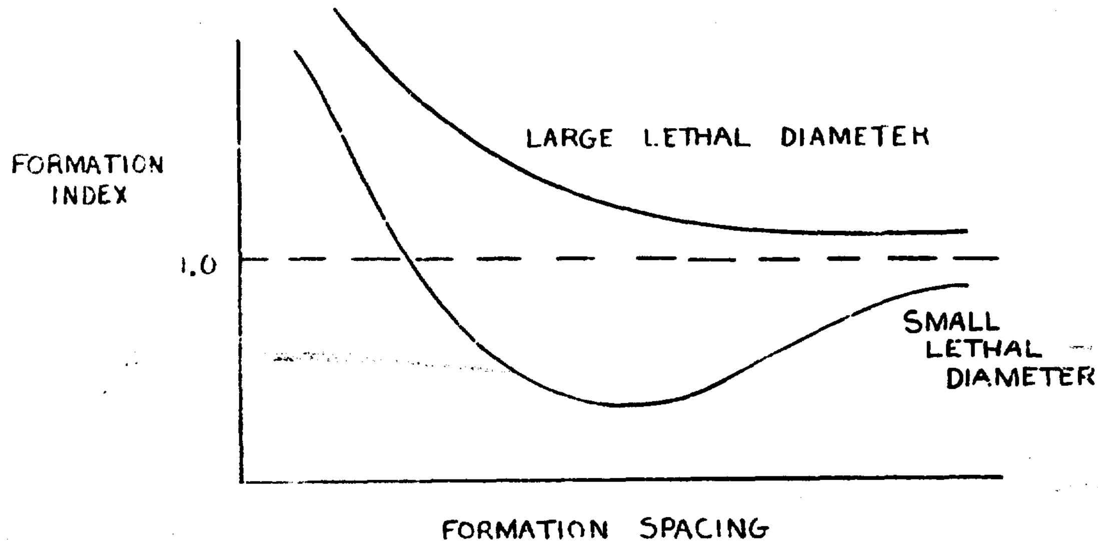



Figure 22

Calculation of Formation Index

Calculation of the formation index usually requires fairly detailed graphical or computer techniques. Intricacies in the means for discrimination coupled with variations in formation geometry, as well as the many difficulties in calculating kill probability for an isolated target, all preclude the use of simple analytic expressions in most cases. However, by way of illustration two simple cases are treated here, one analytically and the other graphically.

Consider b targets flying directly toward an SAM site in line abreast formation with spacing S between adjacent targets, as shown in Figure 23.

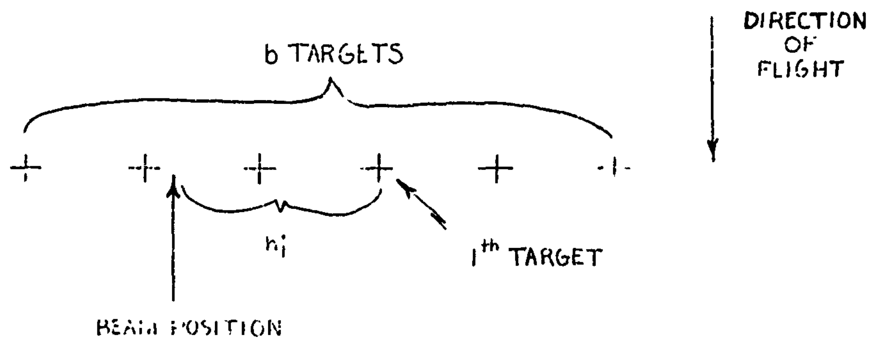



Figure 23

Suppose that the conditional kill probability $p_D$ can be expressed in the form of the Carlton approximation (equation (9)). From equation (10) we then have

$$p_{Ki}(h_i) = \frac{c^2}{c^2+\sigma^2}\, e^{-\frac{h_i^2}{2(c^2+\sigma^2)}} .$$

Assume a uniform distribution of radar beam position extending from one end of the formation to the other. We then have

$$p_{Ki} = \frac{1}{(b-1)S} \int_0^{(b-1)S} p_{Ki}(h_i)\, dh_i$$

$$= \frac{c^2}{c^2+\sigma^2}\, \frac{1}{(b-1)S} \int_0^{(b-1)S} e^{-\frac{h_i^2}{2(c^2+\sigma^2)}}\, dh_i .$$

The expected number of kills is then given by

$$K = \frac{c^2}{c^2+\sigma^2} \frac{1}{(b-1)S} \sum_{i=1}^{b} \int_0^{(b-1)S} e^{-\frac{h_i^2}{2(c^2+\sigma^2)}} dh_i$$

$$= \frac{c^2}{c^2+\sigma^2} \left[ \sqrt{2\pi} \frac{\sqrt{c^2+\sigma^2}}{(b-1)S} \sum_{j=1}^{b-1} I \left( \frac{jS}{\sqrt{c^2+\sigma^2}} \right) \right]$$

where again I(x) is the probability integral

$$I(x) = \sqrt{\frac{2}{\pi}} \int_0^x e^{-\frac{t^2}{2}} dt \qquad .$$

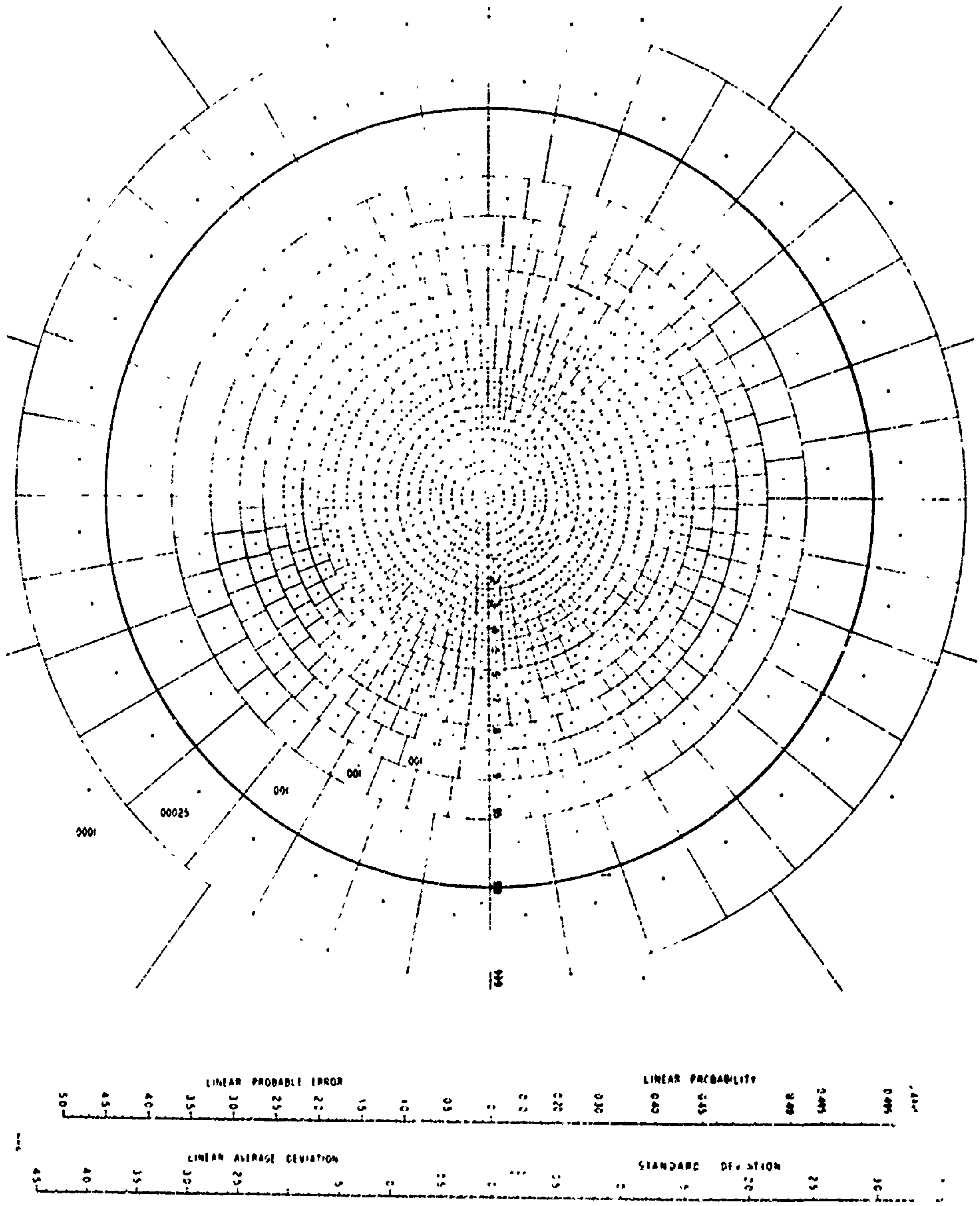
Since the quantity on the right outside the brackets is simply the kill probability $p_K$ for an isolated target, the quantity within the brackets is the formation index.

The second illustration introduces a tool convenient for many purposes, known as probability cell paper for a two dimensional distribution. The plane is divided into n (e.g., n = 1000) cells, each cell containing area corresponding to a cumulative probability of 1/n. In the case of a radial normal distribution the cells are usually arranged in concentric rings, as shown in Figure 24. Cells near the center of the distribution are small, and those farther out are larger. One point in each cell (e.g., the centroid) is clearly marked; its purpose will be explained below.

Consider again the same formation attack as above, but suppose now that the conditional kill probability $p_D$ follows the lethal radius definition, i.e., is 1 inside a spherical damage contour of radius $r_W$, and is 0 outside. Assuming ideal fuzing, only a trajectory passing within a distance R of a target can kill the target. To calculate the expected number of kills, plot on a transparent sheet the target positions, using as unit length the length corresponding to σ on the probability cell paper. Around each target draw a circle of radius $r_W$. Select a suitable representative set of radar beam positions. (The larger the set, the more accurate the resulting computation.) Place the transparent sheet over the probability cell paper such that the first of the selected beam positions coincides with the center of the probability cell paper (see Figure 25).

# CELLS OF EQUAL PROBABILITY

(FOR A CIRCULAR GAUSSIAN DISTRIBUTION)



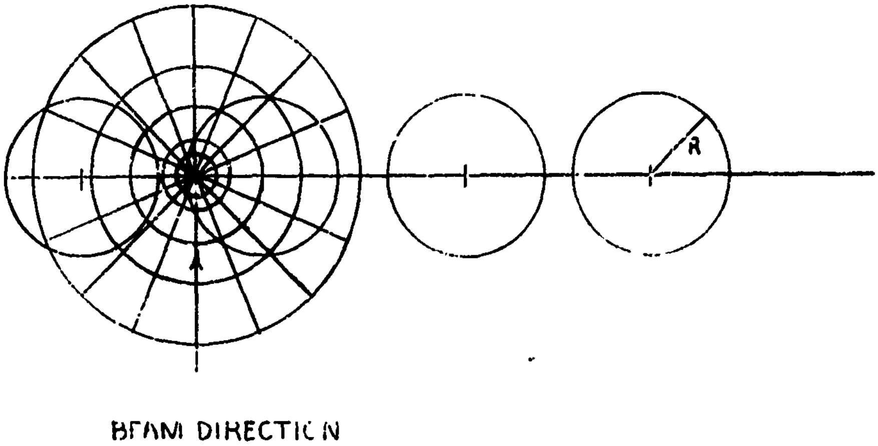



Figure 25

For each target, count the number of cells lying within the lethal radius circle of that target; a cell is counted only if its centroid falls within the circle. The number of such cells multiplied by 1/n is the probability that, if the radar beam position is the one selected, the target in question is killed. By repeating the above process for each selected beam position, and averaging over all the positions, the probability of killing each target is determined.

# CHAPTER IV

## Firepower

### A. Introduction

The number of targets engaged by a SAM firing unit will vary from battle to battle, not only because of distinctive characteristics of the attack but also because of random variations in many of the variables involved for any given set of attack characteristics. Thus the number of engagements itself is a random variable. Firepower is defined as the mean number of target engagements occurring during an attack having given characteristics.

Firepower depends on the time available for fire and on the time consumed for each intercept. The first intercept can occur as soon as the first target arrives in the firing zone and the last as the last target leaves the zone. Thus the time available for fire is at least as long as the time for one target to traverse the zone, which is determined by the intersection of the flight path with the boundaries of the firing zone, including BRL, and by the target speed. The time will further be augmented by any time-spread in the attack. For example, consider a stream attack at constant speed and altitude flying directly toward the defensive site. The time available is then the traverse time of the lead aircraft plus the time interval between arrivals of the first and last aircraft at BRL.

The time consumed for each intercept is appropriately expressed as the intercept interval, being the time from the intercept of one target by a firing unit to the intercept of the next target by the same firing unit, as defined in Chapter II, Section D. A brief review of Sections C and D of Chapter II will provide a useful background for the ensuing discussion. As there indicated, the intercept interval can be and often is range dependent. Hence, the firepower may be smaller or larger as intercepts occur predominantly at long or at short range. The dominant range in turn is determined by the distribution of targets in space and time and by the order in which they are engaged; i.e., the firing doctrine used.

### B. Single Firing Unit - Radial Attack

To introduce certain basic notions without confusing complications, consider first a very simple tactical situation.

A single SAM firing unit is employed in its own defense. All parts of the system are assumed to operate (equipment reliability is ignored). No consideration is given to clearance shadows or to degradations arising from poor data. The specified firing doctrine is assumed to be followed without error.

The attack is assumed to be homogeneous, employing a single aircraft type. All aircraft fly at the same speed and altitude along a single radial path directly toward the SAM firing unit. Consequently the same firing zone and BRL apply to all aircraft. Individual aircraft can be distinguished by the radars. The number of aircraft in the attack is not specified but is assumed to exceed the number of missile salvos fired if no aircraft is engaged more than once.

C. Wave Attack

Assume initially that the aircraft approach in a wave; i.e., with no significant time spread. The aircraft are engaged in arbitrary order, no aircraft being engaged more than once. Let $\rho_{max}$ and $\rho_{min}$ be the maximum and minimum intercept ranges as determined by the zone of fire and BRL; let u denote target speed. Let $\rho_{max} = \rho_1 \geq \rho_2 \geq \cdots \quad \rho_n \geq \rho_{min}$ denote the ranges at which successive targets are intercepted. The time between the (i-1)st and i-th intercepts is on the one hand the intercept interval $T(\rho_i)$ and on the other hand the time $(\rho_{i-1} - \rho_i)/u$ required for the attacking wave to travel from range $\rho_{i-1}$ to range $\rho_i$. Using a linear approximation

$$T(\rho) = \alpha + \beta\rho$$

for the intercept interval, this equality may be written

$$\alpha + \beta\rho_i = \frac{\rho_{i-1} - \rho_i}{u}$$

or

$$\rho_i = \frac{\rho_{i-1} - u\alpha}{1 + u\beta} .$$

But $\rho_1 = \rho_{max}$, so successive intercept ranges can be computed, and n determined as the greatest integer for which $\rho_n \geq \rho_{min}$.

Alternatively, it is easily seen by induction that

$$\text{(13)} \qquad \rho_n = \frac{1}{(1 + u\beta)^{n-1}} \rho_1 - \frac{u\alpha}{1 + u\beta} \left(1 + \frac{1}{1 + u\beta} + \ldots + \frac{1}{(1 + u\beta)^{n-2}}\right)$$

If $\beta \neq 0$

$$\rho_n = \frac{1}{(1 + u\beta)^{n-1}} \rho_1 - \frac{\alpha}{\beta} \left(1 - \frac{1}{(1 + u\beta)^{n-1}}\right)$$

so

$$(1 + u\beta)^{n-1} = \frac{\alpha + \beta\rho_1}{\alpha + \beta\rho_n}$$

and

$$n = 1 + \frac{\log \dfrac{\alpha + \beta\rho_1}{\alpha + \beta\rho_n}}{\log (1 + u\beta)} .$$

But since $\rho_1 = \rho_{max}$ and $\rho_n \geq \rho_{min} > \rho_{n+1}$, one may write

$$n = \left[ 1 + \frac{\log \dfrac{\alpha + \beta\rho_{max}}{\alpha + \beta\rho_{min}}}{\log (1 + u\beta)} \right]$$

where $[x]$ denotes the greatest integer not exceeding x.

If $\beta = 0$, then equation (13) reduces to

$$\rho_n = \rho_1 - u\alpha(n-1)$$

and

$$n = 1 + \frac{\rho_1 - \rho_n}{u\alpha} .$$

Again, replacing $\rho_1$ by $\rho_{max}$, $\rho_n$ by $\rho_{min}$, and taking the integral part of the resulting expression

$$(14) \qquad n = \left[ 1 + \frac{\rho_{max} - \rho_{min}}{u\alpha} \right] .$$

It should here be emphasized that $T(\rho)$ is the time between intercepts and the preceding equation applies to an intercept interval that is independent of range. Such independence will arise when the rate of fire is launcher governed, provided the attack path passes directly over the SAM firing site. In this event the time between the (i-1)st and i-th intercepts is the launcher cycle time $T_L$ plus the time of flight to $\rho_i$ less the time of flight to $\rho_{i-1}$. The time between intercepts, for approaching targets, is

$\dfrac{\rho_{i-1} - \rho_i}{u}$ . Hence, letting v denote mean missile speed,

$$T_L + \frac{\rho_i}{v} - \frac{\rho_{i-1}}{v} = \frac{\rho_{i-1} - \rho_i}{u}$$

or

$$\rho_i = \rho_{i-1} - \frac{uT_L}{1 + \frac{u}{v}} \quad .$$

By induction

$$\rho_n = \rho_1 - (n-1) \frac{uT_L}{1 + \frac{u}{v}}$$

and

$$n = 1 + (1 + \frac{u}{v}) \frac{\rho_1 - \rho_n}{uT_L}$$

or, as before

$$n = \left[1 + (1 + \frac{u}{v}) \frac{\rho_{max} - \rho_{min}}{uT_L}\right] \quad .$$

Comparing this equation with equation (14), clearly

$$\alpha = \frac{1}{1 + \frac{u}{v}} T_L$$

(See Chapter II, Section C).

If the targets are receding (outbound), the time between intercepts is $\frac{\rho_i - \rho_{i-1}}{u}$, and

$$\rho_i = \rho_{i-1} - \frac{uT_L}{1 - \frac{u}{v}} \quad .$$

In this case

$$\alpha = \frac{1}{1 - \frac{u}{v}} T_L \quad .$$

D. Continuous Approximation

The expressions for n given above provide estimates of firepower during a wave attack. Each is a step function over the range interval $\rho_{max}$ to $\rho_{min}$, whose values jump from one integral value to the next, as indicated in Figure 26.

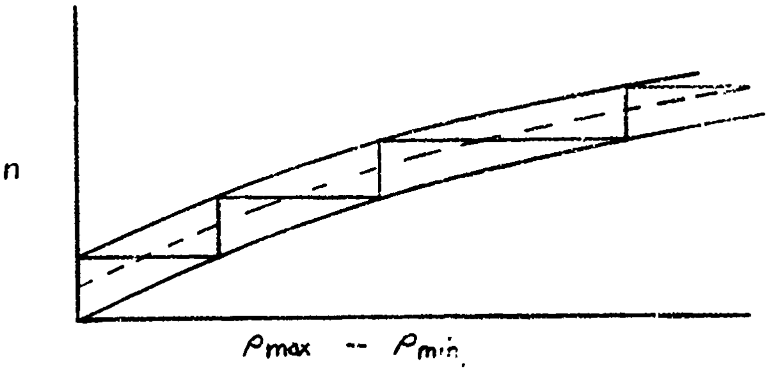



Figure 26

Situations in which it is desirable to approximate a step function by a continuous function arise so often in this and similar calculations that a few remarks on this point are in order. These remarks, while specific to the situation at hand, are typical of arguments used in a variety of similar situations.

It should first be pointed out that three continuous approximations are of particular interest:

The major approximation, which touches the tops of the steps

The minor approximation, which touches the bottoms of the steps

The mean approximation, which cuts each riser at its midpoint.

Since the step function lies between its major and minor approximations, the relative error in using any of these approximations is less than

$$\frac{(\text{major approx}) - (\text{minor approx})}{\text{function}} = \frac{1}{n}$$

and hence is small when n is large. The relative error in using the mean approximation is $\leq 1/2\, n$, which is still smaller. Hence, when n is large, any of the three approximations is acceptable without further argument.

The independent variables ($\rho_{max}$, $\rho_{min}$, u, $\alpha$, $\beta$) from which the values of the function in question are computed are, in an actual tactical analysis, not known exactly but estimated from available data and hence properly regarded as random variables, the spread of whose distribution arises from errors of estimate and/or the statistical nature of the quantity being estimated. The number of targets engaged as a function of these random variables is itself a random variable with a distribution function which in principle can be computed, but in practice is difficult to determine. This view of the expected number of targets engaged as arising (in the usual way) from a random variable permits a straightforward treatment of the discontinuities in n.

At those values of the independent variable corresponding to a discontinuity in n, what actually happens statistically is that in about half the actual cases one gets the higher value, and in about half the actual cases the lower value. When these favorable and unfavorable cases are averaged together, the expected value so obtained is simply the average of the higher and lower values. This argument leads one to prefer the mean approximation, not only to other continuous approximations but even to the step function itself, as an estimate of the expected number of targets engaged.

E. Continuous Approximation Formulas

On the basis of the last two sections, the following estimates are suggested for the firepower during a wave attack:

$$N = \frac{1}{2} + \frac{\log \dfrac{\alpha + \beta\rho_{max}}{\alpha + \beta\rho_{min}}}{\log (1 + u\beta)} \qquad \text{if } \beta \neq 0$$

$$N = \frac{1}{2} + \frac{\rho_{max} - \rho_{min}}{u\alpha} \qquad \text{if } \beta = 0 \quad .$$

In the first formula above, the base to which the logarithms are taken is immaterial, since only ratios of logarithms are involved. If $\alpha$ is determined by the launcher cycle time $T_L$, the second formula above becomes

$$N = \begin{cases} \dfrac{1}{2} + \left(1 + \dfrac{u}{v}\right) \dfrac{\rho_{max} - \rho_{min}}{uT_L} & \text{for approaching targets} \\[2ex] \dfrac{1}{2} + \left(1 - \dfrac{u}{v}\right) \dfrac{\rho_{max} - \rho_{min}}{uT_L} & \text{for receding targets} \end{cases}$$

F. Extensions of Formulas

In some instances a single linear approximation to the intercept interval $T(\rho)$ is not satisfactory, but a piece-wise linear approximation is. To illustrate, suppose

$$T(\rho) = \begin{cases} \alpha + \beta\rho & \rho' \le \rho \le \rho_{max} \\ \alpha' + \beta'\rho & \rho'' \le \rho \le \rho' \\ \alpha'' & \rho_{min} \le \rho \le \rho'' \end{cases}$$

(See Figure 27)

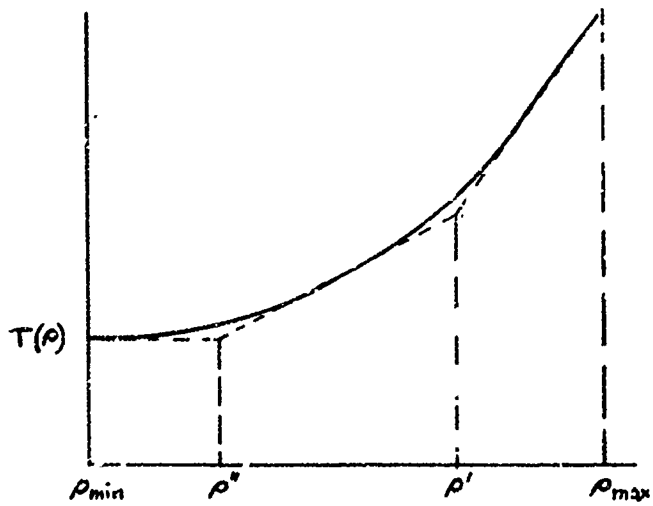



Figure 27

Using the methods derived earlier, one can calculate the firepower achieved in each range interval, and add these to obtain the total firepower. However, the 1/2 should be added only once, since it is included to allow only for the first intercept, at $\rho_{max}$. One may then write

$$N = \frac{1}{2} + \frac{\log \frac{T(\rho_{max})}{T(\rho')}}{\log (1 + u\beta)} + \frac{\log \frac{T(\rho')}{T(\rho'')}}{\log (1 + u\beta')} + \frac{\rho'' - \rho_{min}}{u\alpha''} .$$

Consider now a SAM system having one launcher with cycle time $T_L$, and two guidance channels, each with intercept interval $T(\rho) = \alpha + \beta\rho$. Let the first intercept occur at maximum range $\rho_{max}$. Since another guidance channel is available, the next engagement will be delayed only until the launcher is again loaded and ready. Hence the second intercept range is

$$\rho_2 = \rho_1 - \frac{uT_L}{1 + \frac{u}{v}} .$$

The next engagement will be delayed until either the launcher is loaded and ready, or until the first guidance channel becomes free and available, whichever delay is longer. The third intercept range is thus given by

$$\rho_3 = \min \begin{cases} \rho_2 - \dfrac{uT_L}{1 + \frac{u}{v}} \\ \dfrac{\rho_1 - u\alpha}{1 + u\beta} \end{cases} .$$

In similar manner successive intercept ranges can be computed, and the firepower determined. An analogous calculation can, of course, be made for any number of launchers and guidance channels.

If an estimate of the firepower is all that is wanted, the individual intercept ranges being of no interest, a simpler procedure can be followed. Instead of the step-by-step procedure used above, in which each piece of equipment must be kept track of, one can use the average launcher interval and the average intercept interval described in Chapter II, Section D, to define an intercept interval for the entire SAM unit, and then compute firepower by any of the methods derived earlier. The intercept interval to be used is

$$T(\rho) = \max \begin{cases} \dfrac{T_L/M_L}{1 + u/v} \\ \dfrac{\alpha + \beta\rho}{M_G} \end{cases} .$$

where $M_L$ and $M_G$ are the numbers of launchers and guidance channels respectively.

If the step-by-step procedure is followed, the intercept points will usually be rather irregularly spaced along the attack path. Use of the average intercept interval just defined will lead to about the same number of intercept points, but spaced at more nearly regular intervals.

## G. Means and Dispersions

The computations above are typical of many in this field in that they use mean values of the independent variables throughout, rather than performing calculations with random variables and then averaging in the final step. The reason for doing this is that, generally speaking, the earlier the averaging is performed, the simpler the calculations become, and, furthermore, the distributions of the random variables are rarely known. Such an inversion of the correct order of operations does not generally yield correct answers, as a very simple example shows. The inversion does often provide an adequate approximation, however.

Let x be a random variable which has the three equally probable values 1, 2, 3. Let $y = 1/x$. Then $\overline{x} = 2$ and $1/\overline{x} = 1/2$; but

$$\overline{y} = \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3}\right) = \frac{11}{18} \neq 1/\overline{x} \quad .$$

Hence, inverting the averaging operation and the (non-linear) operation of taking reciprocals yields a wrong answer by some 1/9 = 11 percent.

However, the following argument partially justifies the inversion in some cases of present interest:

Let $f(x_1, x_2, \ldots, x_n)$ be a function of the random variables $x_1$, $x_2, \ldots, x_n$. Let $\overline{x}_i$ be the mean of $x_i$ and let $h_i = x_i - \overline{x}_i$. If $f(x_i)$ has continuous second partial derivatives, then its Taylor expansion can be written about the mean point:

$$f(x_i) = f(\overline{x}_i) + \sum_i f_i h_i + R,$$

where $f_i = \frac{\partial}{\partial x_i} f(\overline{x}_i)$ and R, the remainder, is a sum of higher order terms. The mean, $E(f)$, and variance, $\sigma^2(f)$, can then be written

$$E(f) = f(\overline{x}_i) + E(R)$$

$$\sigma^2(f) = \sum f_i^{\,2} \sigma^2(h_i) + \sigma^2(R),$$

since $E(h_i) = 0$, $\sigma^2(f(\overline{x}_i)) = 0$, and the operations of taking mean and variance are linear.

When the remainder term, R, is so small (in probability) that its mean and variance can be neglected in the approximate computation at hand, the approximations

$$E(f) \doteq f(\bar{x}_i)$$

$$\sigma^2(f) \doteq \sum f_i^2 \sigma^2(h_i)$$

can be used. The first of these justifies averaging at the beginning rather than at the end of the calculation and the second permits a comparatively simple estimation of the variance when the calculation is completed.

The remainder term is of the form

$$R = \frac{1}{2} \sum_i \sum_j f_{ij} h_i h_j$$

where

$$f_{ij} = \frac{\partial^2}{\partial x_i \partial x_j} f(\bar{x}_i + h_k')$$

with $\left|h_k'\right| < \left|h_k\right|$ .

This remainder is small, therefore, when the second order derivatives are small (i.e., the function f is almost linear) or when the deviations $h_i$ about the means are small in probability (i.e., when $\sigma(h_i)$ is small compared to $\bar{x}_i$). One or the other of these two conditions holds in many cases of interest. In particular, the intercept interval, $I(\rho)$, typically has a dispersion which is small compared to its mean value.

It should be noted that the argument above makes no assumptions about the independence of the random variables, $x_i$, or about the forms of their distribution functions, except existence of first and second moments.

H. Integral Formulas

In many instances of interest, the quantity uβ is small relative to unity (e.g., as the ratio of target speed to missile speed). If this is so and natural logarithms are used, the expansion

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

gives the approximation

$$N = \frac{1}{2} + \frac{1}{u\beta} \ln \frac{T(\rho_{max})}{T(\rho_{min})} \qquad (u\beta \ll 1)$$

I. Alternate Computation

It is interesting to notice the last equation can be derived in a somewhat different fashion:

$$N = \frac{1}{2} + 1 + \ldots 1,$$

where 1/2 is allowed, as in Section E, for the target engaged at $\rho_1 = \rho_{max}$, and 1 for each of the targets engaged at $\rho_2, \ldots, \rho_n \geq \rho_{min}$. As in Section C, the i-th of these 1's may be written

$$1 = \frac{1}{u} \frac{\rho_{i-1} - \rho_i}{\alpha + \beta\rho_i} = \frac{1}{u} \frac{\Delta \rho_i}{\alpha + \beta\rho_i}$$

whence

$$N = \frac{1}{2} + \frac{1}{u} \sum_i \frac{\Delta \rho_i}{\alpha + \beta\rho_i} .$$

Approximating the sum by the corresponding integral,

$$N = \frac{1}{2} + \frac{1}{u} \int_{\rho_{min}}^{\rho_{max}} \frac{d\rho}{\alpha + \beta\rho}$$

$$= \frac{1}{2} + \frac{1}{u\beta} \ln \frac{\alpha + \beta\rho_{max}}{\alpha + \beta\rho_{min}} .$$

J. Graphical Integration

The expression for N in terms of an integral, given above, suggests a graphical means for computing N that is particularly useful when $T(\rho)$ does not admit a simple linear approximation. The reciprocal of $uT(\rho)$, is plotted against $\rho$ on suitably ruled graph paper (Figure 28).

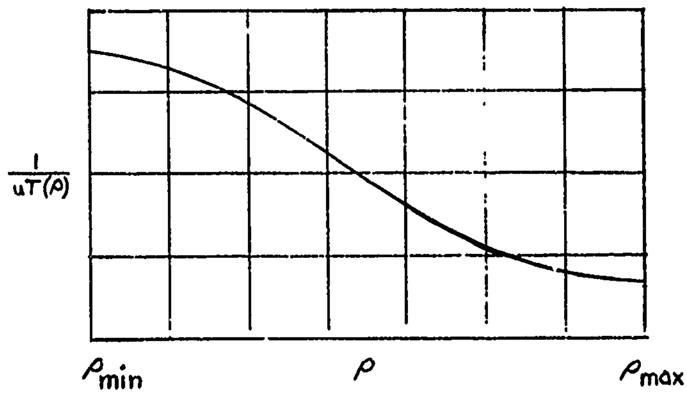



Figure 28

The desired integral, which is the area under the curve, can then readily be evaluated by counting squares and multiplying the count by the value per square:

$$\Delta\left(\frac{1}{uT(\rho)}\right)\cdot \quad \Delta\rho \quad .$$

K. Stream Attack

The discussion thus far has been concerned with a wave attack; consider now the effect of a time spread in the attack. To be specific, suppose that the attacking aircraft approach in a stream, with a time spacing of s seconds between arrivals of successive aircraft. Aircraft are engaged in the order of their arrival, no target being engaged more than once. As in Section C, equate the intercept interval $T(\rho_i)$ to the time it takes for the i-th target to fly from range $\rho_{i-1}$ + us, its range when the preceding intercept occurred, to range $\rho_i$.

$$T(\rho_i) = \frac{\rho_{i-1} + us - \rho_i}{u}$$

or

$$T(\rho_i) - s = \frac{\rho_{i-1} - \rho_i}{u}$$

Comparison with the corresponding equation of Section C shows that the effect of a time spacing s between successive targets is to substitute $T(\rho) - s$ for $T(\rho)$. Otherwise stated, a time spacing s between targets has the same effect as a decrease of s in the intercept interval.

The discussion of firepower against a wave attack, the derivations of formulas, and the description of the graphical integration computation are applicable to the stream attack with spacing s as well, the only change being to replace $T(\rho)$ by $T(\rho) - s$. Indeed the wave attack as treated is just a special case of the stream attack, with s set equal to zero. The more general continuous approximation formulas, for arbitrary spacing s, are

$$N = \frac{1}{2} + \frac{\log \frac{T(\rho_{max}) - s}{T(\rho_{min}) - s}}{\log (1 + u\beta)} \qquad \text{if } \beta \neq 0$$

$$N = \frac{1}{2} + \frac{\rho_{max} - \rho_{min}}{u(\alpha - s)} \qquad \text{if } \beta = 0$$

In particular, for launcher governed fire against approaching targets

$$N = \frac{1}{2} + \left(1 + \frac{u}{v}\right) \frac{\rho_{max} - \rho_{min}}{u(T_L - s)}$$

## L. Two Graphical Computing Methods

A graphical means for computing successive intercept ranges $\rho_i$, applicable to arbitrary spacing, is the familiar strip-matching technique; it is particularly useful when a linear approximation to $T(\rho)$ is not satisfactory. The method has wider applications which will be discussed later; it is explained here to avoid unnecessary complications.

Suppose, first, that fire is guidance channel limited. After selecting a convenient distance unit (e.g., 10 kiloyards per inch), prepare an intercept interval strip, on which conveniently chosen values of $T(\rho)$, covering the interval from $T(\rho_{min})$ to $T(\rho_{max})$ are entered at the corresponding ranges $\rho$, measured in the selected distance unit. A convenient means for doing this is to plot $T(\rho)$ versus $\rho$, the latter in the selected unit, and mark off the chosen values $T_0$, $T_1$, $T_2$, ... of $T(\rho)$ on the T-axis. From this graph, read off and mark the corresponding ranges $\rho_0$, $\rho_1$, $\rho_2$, ... . Lay a strip of paper along the $\rho$-axis, mark off the ranges on the strip, and enter the corresponding times (see Figure 29).

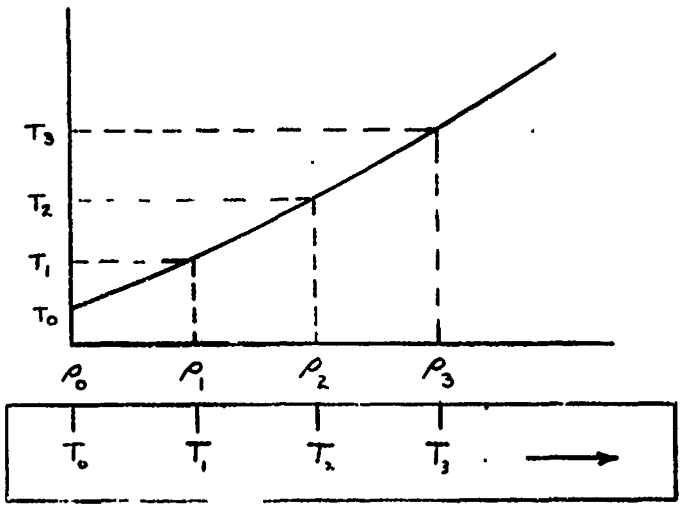



Figure 29

Next prepare a target strip showing the distance of target travel, in the selected unit, versus time (Figure 30).

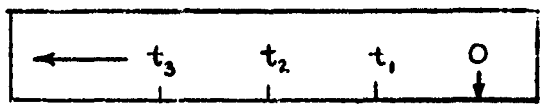



Figure 30

On a separate sheet draw a straight line attack path and mark off the firing unit position and the maximum and minimum intercept ranges. Place the two strips along the line as shown in Figure 31. The index of the intercept interval strip should be at the firing unit position. The appropriate position for the target strip will depend upon the target spacing; for spacing s the point t = s on the strip should coincide with the maximum intercept range $\rho_{max}$. With the strips properly placed, determine the point at which the times on the two scales coincide (i.e., T = t). Mark this point on the attack path line; it is the second intercept range $\rho_2$ (recall that $\rho_1 = \rho_{max}$). (The two strips are no more than a simple analog computer for the solution of the equation $T(\rho_2) = (\rho_1 - \rho_2)/u$). Next slide the target strip along until the point t = s coincides with the second intercept point, and mark the point at which the two scales coincide (i.e., T = t) to obtain $\rho_3$.

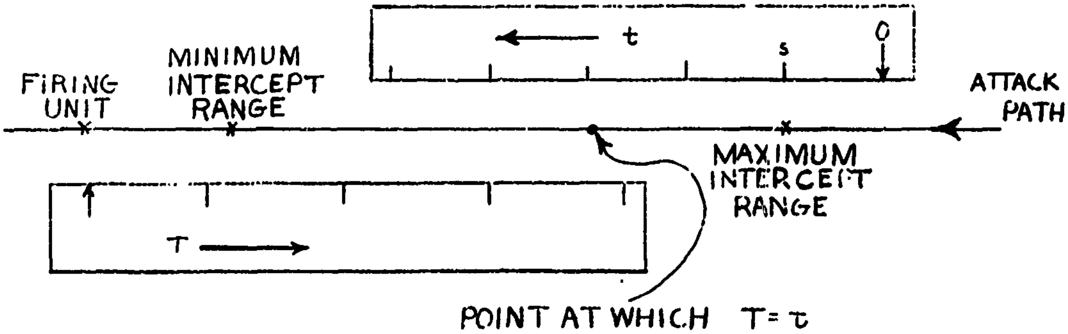



Figure 31

Proceed to mark the successive intercept points in this manner until some range $\rho_{n+1}$ falls inside the minimum range $\rho_{min}$. If a continuous approximation to the firepower is desired, one can interpolate between $\rho_n$ and $\rho_{n+1}$. Further, if the mean approximation is desired, n should be reduced by 1/2 as before. Thus

$$N = n - \frac{1}{2} + \frac{\rho_n - \rho_{min}}{\rho_n - \rho_{n+1}} .$$

If the intercept interval T is independent of range over a part of the interval from $\rho_{max}$ to $\rho_{min}$, the intercept interval strip should show the constant time T corresponding to all of that part of the interval. (See Figure 32)

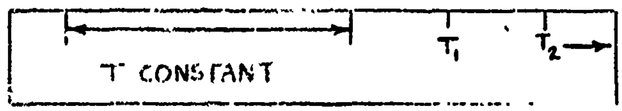



Figure 32

Thus in matching strips to find intercept points, one need only look for the point on the target strip at which t = T constant. If the intercept interval is constant over the entire range interval, no intercept interval strip is needed.

An alternate graphical method for computing successive intercept ranges follows. It, too, has wider applications, which will be discussed later.

First plot intercept interval $T(\rho)$ versus range $\rho$ on a translucent sheet (Chart A). On a separate sheet plot target range versus time (Chart B) for a single target, using the same time and range scales as in Chart A. (See Figure 33)

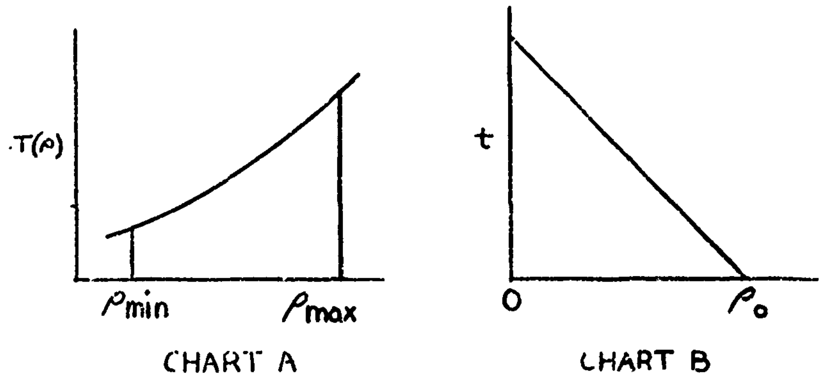



Figure 33

The first intercept range $\rho_1 = \rho_{max}$. To find the next intercept range $\rho_2$, place Chart A on Chart B with $\rho = \rho_o$ of B at $\rho = \rho_{max} (=\rho_1)$ of A and $t = 0$ of B at $T = s$ of A, where s denotes spacing between targets. Read off the range $\rho = \rho_2$, on A, at which the target line of B intersects the intercept interval curve. To find $\rho_3$, move Chart A so that $\rho = \rho_o$ of B is at $\rho_2$ of A and $t = 0$ of B is at $T = s$ of A. Read off the range $\rho = \rho_3$ on A at which the target line of B intersects the intercept interval. Proceed in this manner until some $\rho_{n+1}$ falls to the left of $\rho_{min}$. As before a continuous approximation to firepower can be obtained by interpolation between $\rho_n$ and $\rho_{n+1}$. (See Figure 34)

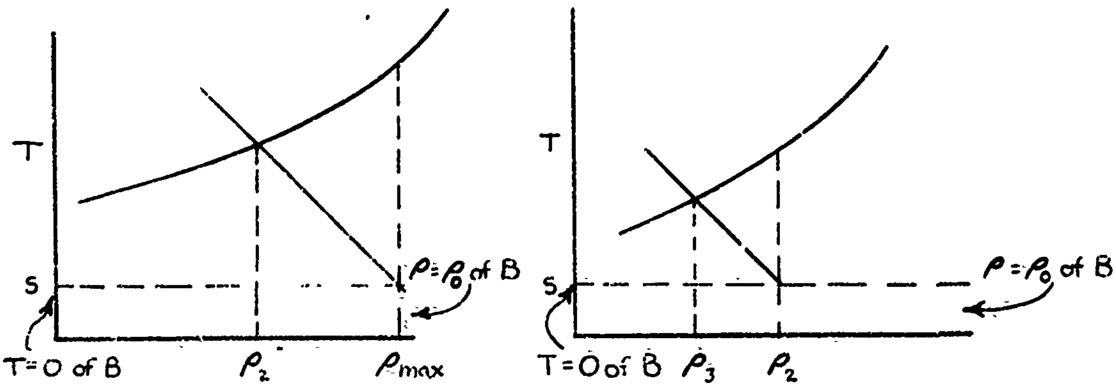



Figure 34

M. Single Firing Unit - Crossing Attack

Thus far we have considered a special tactical situation of a single SAM firing unit engaged in its own defense. The attack followed a radial path directly toward the SAM unit. The present section extends the results to the situation in which the SAM unit is engaged in the defense of a surface target other than itself and the attack follows a single straight path toward the surface target. In general, the path will not pass directly over the SAM unit; i.e., will be a crossing attack path. The effect of a crossing path is to introduce certain geometrical complications into the calculations. All simplifying assumptions previously made are here retained except for the position of the SAM unit relative to the attack path.

N. Wave Attack

Initially we will assume a wave attack. The aircraft are engaged in arbitrary order, but no aircraft is engaged more than once. To account for the geometry, it is convenient to introduce a Cartesian coordinate system with origin at the surface target center, x-axis along the attack path so that the attack approaches from the positive direction and y-axis normal to it. Let $(x_0, y_0)$ be the coordinates of the SAM firing unit position (the distance $y_0$ is often called the crossing distance or crossing range); again let $\rho$ denote intercept range to the target (see Figure 35).

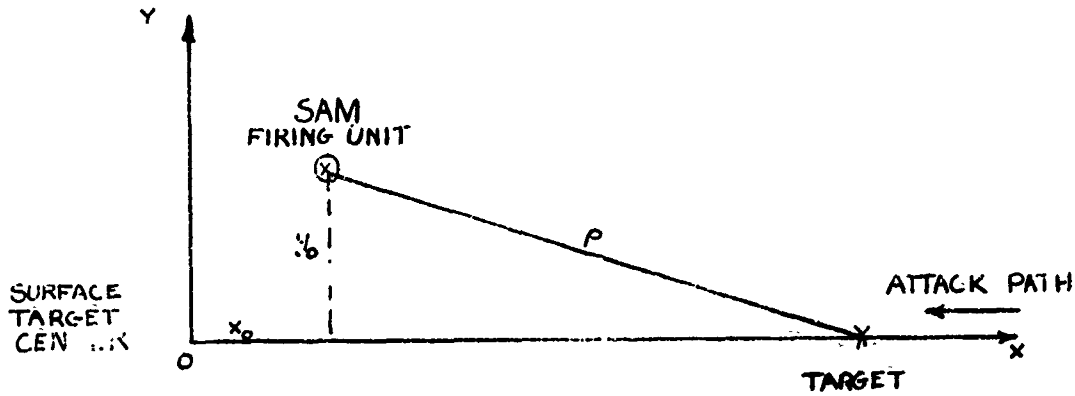



Figure 35

The relation between intercept range ρ and the corresponding target position x may be written as

$$x = x_o + \sqrt{\rho^2 - y_o^2}$$

or

$$\rho = \sqrt{(x-x_o)^2 + y_o^2}$$

Let $\rho_{max}$, $\rho_{min}$ be the maximum and minimum intercept ranges as determined by the zone of fire and weapon release point, i.e., the first possible and last possible intercept ranges; let $x_{max}$ and $x_{min}$ be the corresponding target positions along the attack path; let u denote target speed. Let $x_{max} = x_1 > x_2 > \ldots\ x_n > x_{min}$ denote positions at which successive targets are intercepted, and $\rho_1, \rho_2 \ldots \rho_n$ the corresponding intercept ranges. The time between the (i-1)st and i-th intercepts is on the one hand the intercept interval $T(\rho_i)$ and on the other hand the time $(x_{i-1} - x_i)/u$ required for the attacking wave to travel from $x_{i-1}$ to $x_i$. Using the linear approximation

$$T(\rho) = \alpha + \beta\rho$$

for the intercept interval, this equality may be written

$$\alpha + \beta\rho_i = \frac{x_{i-1} - x_i}{u} .$$

But

$$\rho_i = \sqrt{(x_i - x_o)^2 + y_o^2} . \tag{15}$$

Substituting this expression for $\rho_i$ in the preceding equation and solving for $x_i$, a recursion formula giving $x_i$ in terms of $x_{i-1}$ is obtained. Setting $x_1 = x_{max}$, successive intercept positions can be computed and n determined as the greatest integer for which $x_n \geq x_{min}$.

A recursion formula, more convenient for computing purposes, can be obtained by assuming the last intercept at $x_{min}$ and then calculating the next-to-last intercept position, the second-from-last, etc. To this end, let $x_{min} = x'_1 \leq x'_2 \leq \ldots \; x'_n \leq x_{max}$ denote the intercept positions, numbered in the reverse order of occurrence, and let $\rho'_0, \rho'_2, \ldots, \rho'_n$ be the corresponding intercept ranges. Then, as above,

$$T(\rho'_i) = \frac{x'_{i+1} - x'_i}{u}$$

or

$$x'_{i+1} = x'_i + uT(\rho'_i) \quad .$$

But, again,

$$\rho'_i = \sqrt{(x'_i - x'_o)^2 + y_o^{\,2}}$$

Thus, setting $x'_1 = x_{min}$, one may compute $\rho'_1$, then $x'_2$, and so $\rho'_2$, then $x'_3$, etc.

Because of the quadratic relation between $x_i$ and $\rho_i$, an inductive argument does not lead directly to a convenient formula for n, as was the case with a radial attack. However, if one writes the intercept interval as a function of x,

$$T = T\left(\sqrt{(x - x_o)^2 + y_o^{\,2}}\right)$$

and then chooses a suitable approximation <u>which is linear</u> in x, i.e.,

$$T = \alpha' + \beta' x$$

the earlier inductive argument may then be followed and one obtains

$$n = 1 + \frac{\log \dfrac{\alpha' + \beta' x_1}{\alpha' + \beta' x_n}}{\log\,(1 + u\beta')} \qquad \text{if } \beta' \neq 0$$

$$n = 1 + \frac{x_1 - x_n}{u\alpha'} \qquad \text{if } \beta' = 0$$

The mean continuous approximation can, as before, be written as

$$N = \frac{1}{2} + \frac{\log \dfrac{\alpha' + \beta' x_{max}}{\alpha' + \beta' x_{min}}}{\log (1 + u\beta')} \qquad \text{if } \beta' \neq 0$$

$$N = \frac{1}{2} + \frac{x_{max} - x_{min}}{u\alpha'} \qquad \text{if } \beta' = 0$$

When the rate of fire is launcher governed, the intercept interval will not be independent of intercept position, as in Se--'on C. For, recall that the intercept interval is equal to the launch -cle time $T_L$ plus the difference in times of flight of the i-th and (i-1)s vos. But this difference in times of flight is not independent of range when the attack path is a crossing path. For example, if the intercept range $\rho_i$ is long compared to the crossing distance $y_o$, then the difference in intercept ranges is approximately equal to the difference in intercept positions, and so the difference in times of flight is approximately $(x_{i-1} - x_i)/v$ (see Figure 36). If, on the other hand $\rho_i$ is about equal to $y_o$, then so may $\rho_{i-1}$ be about equal to $y_o$, then so may $\rho_{i-1}$ be about equal to $y_o$, and the difference in times of flight is small.

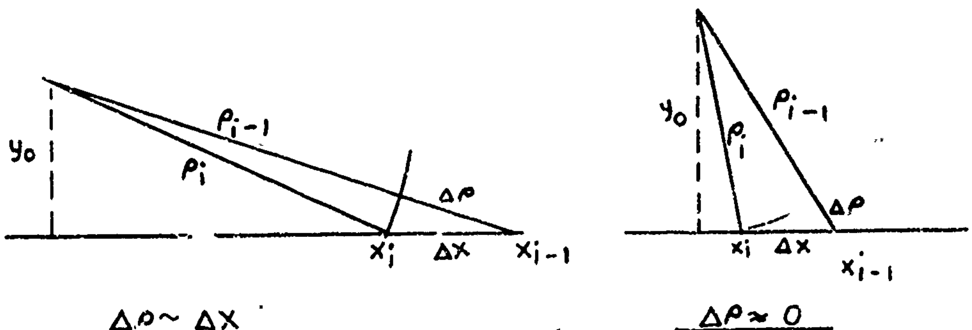



Figure 36

If fire is launcher governed, a recursion formula giving successive intercept positions is obtained by substituting equation (15) into

$$T_L + \frac{\rho_i}{v} - \frac{\rho_{i-1}}{v} = \frac{x_{i-1} - x_i}{u}$$

and solving for $x_i$.

O. Integral Representation

For a crossing attack path the integral representation of firepower N is

$$N = \frac{1}{2} + \frac{1}{u} \int_{x_{min}}^{x_{max}} \frac{dx}{T(\rho)}$$

where ρ is expressed in terms of target position x. Because of the quadratic relation between x and ρ, evaluation of this integral does not lead to a simple closed form even if T is expressible as $T = \alpha + \beta\rho$. If, however, a suitable approximation to T, which is linear in x, or at least is piece-wise linear in x, can be found, the integral can then be evaluated as before. Even without such approximation, graphical or numerical evaluation of the integral can be made.

P. Stream Attack

Extension of the earlier results for a crossing attack path to an arbitrary spacing s is very simple and straightforward. In Section K, it was shown that a time spacing s has the same effect as a decrease in tie-up time. The same argument applies in the case of a crossing attack path, and so, with T replaced by T-s, the results of the preceding section apply to arbitrary spacing s.

Q. Graphical Computing Methods

The method of strip-matching described in Section L is directly applicable to the crossing attack path situation. If fire is guidance channel limited, an intercept interval strip and a target strip are prepared exactly as before. On a separate sheet the attack path is drawn, the SAM firing unit position is indicated, and the first ($x_{max}$) and last ($x_{min}$) intercept positions are marked off. The target strip is placed along the attack path, with the point t = s on the strip coinciding with $x_{max}$, for target spacing s (see Figure 37).

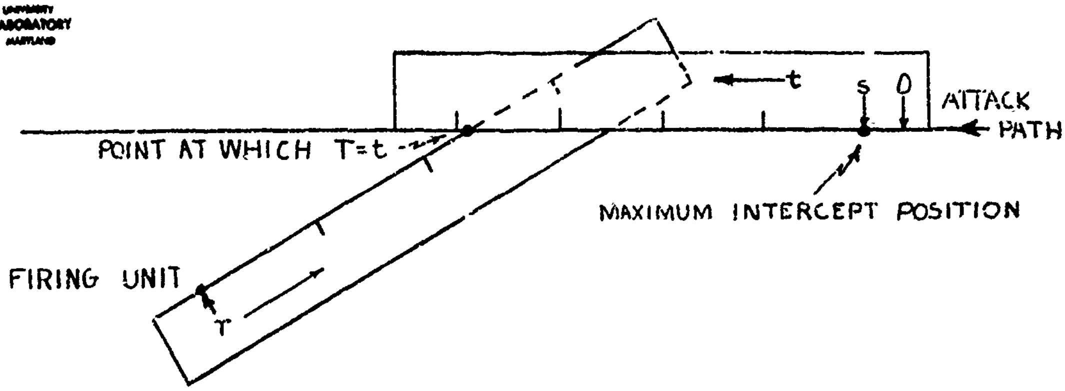



Figure 37

The intercept interval strip is placed with index at the SAM firing unit; it is convenient to insert a pin at this point, to permit the intercept interval strip to rotate about the SAM position. The latter strip is then rotated until the two strips intersect at a point such that $T = t$. This point is marked on the attack path line; it is the second intercept position $x_2$ (recall that $x_1 = x_{max}$). The target strip is next moved along the path until the point $t = s$ of the strip coincides with the position $x_2$. The intercept interval strip is rotated until the two strips intersect at a point such that $T = t$; the point is marked off as position $x_3$. Successive intercept positions are marked off in this manner.

If fire is launcher governed, the intercept interval is given by

$$T = T_L + t_f(\rho_i) - t_f(\rho_{i-1})$$

and the difference in times of flight is not constant (see Section N). The problem to be solved by the use of strips is the solution of the equation

$$T_L + t_f(\rho_i) - t_f(\rho_{i-1}) = \frac{\rho_{i-1} - \rho_i}{u}$$

or

$$T_L + t_f(\rho_i) = \frac{\rho_{i-1} - \rho_i}{u} + t_f(\rho_{i-1}) \quad .$$

The target strip should be prepared as before. The intercept interval strip, however, requires two scales, one giving $T_L + t_f(\rho)$ versus $\rho$, and the other $t_f(\rho)$ versus $\rho$. In preparing the strip, first choose several convenient values $T'_0$, $T'_1$, ... of $T' = T_L + t_f(\rho)$, covering the interval from $T_L + t_f(\rho_{min})$ to $T_L + t_f(\rho_{max})$, and enter these times at the corresponding ranges $\rho$. The method of Section L can be used here. At each of the above ranges, also enter the time-of-flight $t_f(\rho)$ (see Figure 38).

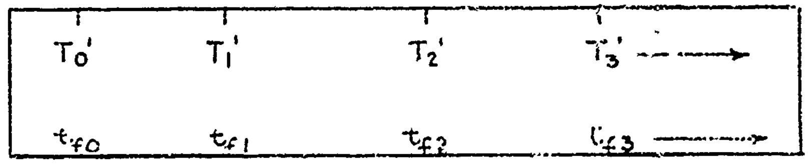



Figure 38

The procedure for finding successive intercept points is as follows. Place the intercept interval strip with index at the SAM firing unit, again inserting a pin to permit rotation of the strip about the SAM site. Rotate the strip so that its ruled edge passes through the first intercept position $x_{max}$, and read off the time-of-flight to this position. Next place the target strip along the attack path so that the first intercept position is opposite the time on the target strip scale which is equal to the time of flight just read off. Holding the target strip fixed, rotate the intercept interval strip until the two strips intersect at a point such that $T' = t$. This point is marked on the attack path line; it is the second intercept position $x_2$ (see Figure 39).

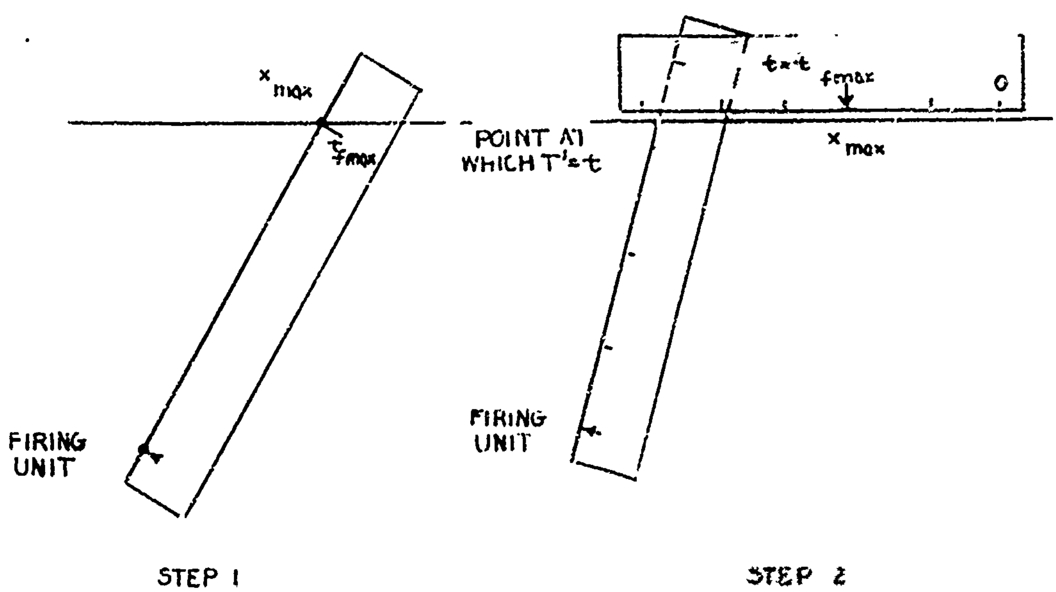



Figure 39

Proceed in this manner to find the successive intercept positions.

If fire is guidance channel limited at long range and launcher limited at short range, proceed as described above for guidance limited fire, so long as this limit is governing. Then switch to the launcher governed procedure, replacing $x_{max}$ by the last intercept position obtained under guidance limited fire.

The alternative graphical method described in Section L is also applicable, with one modification, when fire is guidance channel limited. The intercept interval must be expressed in terms of target position x, by means of the quadratic relation between ρ and x, and be plotted against x to give the counterpart of Chart A (Figure 35).

A similar method can be employed when fire is launcher governed. On a translucent sheet plot flight time $t_f$ and also flight time plus launcher cycle time $T_L$ versus intercept position x, using the quadratic relation between ρ and x to express flight time in terms of x. (See Figure 40) This sheet takes the place of Chart A.

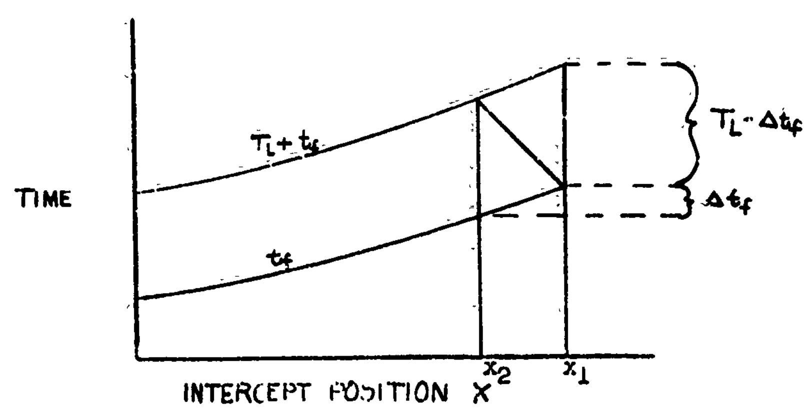



Figure 40

Mark off the first intercept position $x_1$ corresponding to $\rho_{max}$. Place this new Chart A on Chart B so that the target line of B intersects the $t_f$ curve of A at $x_1$, keeping the corresponding axes of the two charts parallel. The second intercept position $x_2$ is the x at which the target line of B intersects the $T_L + t_f$ curve of A. In like manner successive intercept positions can be obtained.

R. Fire Analyzers

A fire analyzer is a device useful in calculating firepower of a single SAM firing unit against a variety of crossing attacks; e.g., in a map exercise one may wish to consider several attack paths and several SAM sites. The fire analyzer consists of a family of parallel attack paths with successive intercept points marked off on each. The loci of intercept points form a family of successive intercept curves.

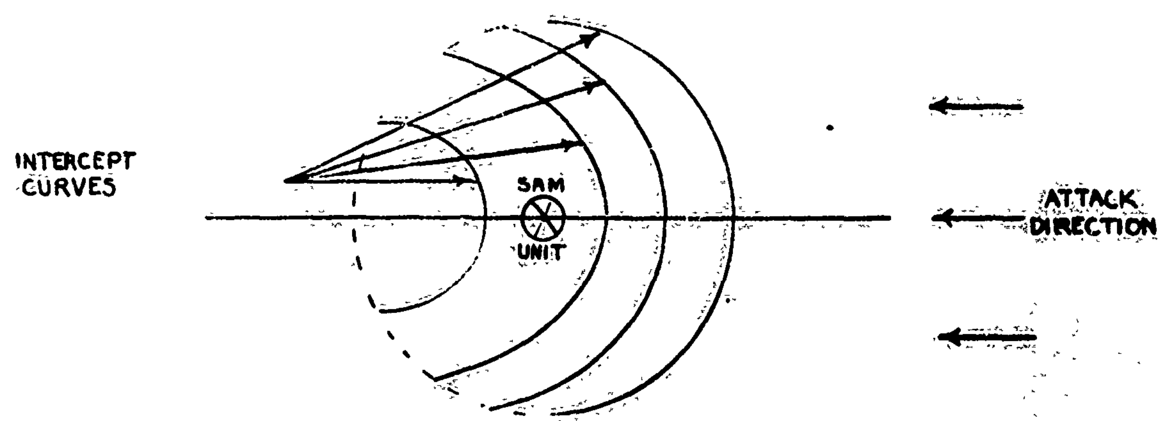



Figure 41

The SAM firing unit location, parallel attack paths, and the successive intercept curves are drawn on transparent material. To use the analyzer, the attack path or paths, bomb release point for each, and the SAM sites are drawn on the map or sheet of paper. The analyzer is then placed with its center at the SAM site and its family of attack paths parallel to the attack path on the map. The number of intercepts that occur before bomb release can then be counted to obtain firepower. If desired, linear interpolation between intercept curves can be employed to give fractional engagements. The fire analyzer is particularly useful when a large number of attack paths and/or SAM sites are to be considered as in the case of a map exercise. Variations in maximum intercept range can be accounted for by constructing the analyzer for the greatest range to be considered, then counting the number of intercepts, including fractions, occurring before the desired maximum intercept range is reached and subtracting this number from the firepower obtained for greatest range. Similarly, the number of intercepts occurring inside a minimum intercept range can be subtracted, to account for the effect of a minimum range.

To construct a fire analyzer, first lay out a family of parallel, evenly spaced attack paths on a sheet of paper, draw a circle with radius equal to the maximum intercept range and center at the SAM firing unit. This is the first intercept curve. The recursion formula of Section N, or the strip matching method of the preceding section can then be used to determine successive intercept points along each attack path in turn. To obtain the successive intercept curves, join the sets of 2nd, of 3rd, etc. intercept points with a smooth curve. The fire analyzer to be used in a map exercise must be drawn to the scale of the map in use. The scale of the analyzer can be changed by photographic enlarging or contracting.

The labor involved in construction of a fire analyzer often is not justified when variations in attack speed, target spacing, or SAM system intercept interval are to be considered, since a new analyzer must be made for each set of these parameter values. However, in some investigations, such as the deployment of units of a given SAM system, fire analyzers can be very helpful. Even though it may be desirable to vary target speed and spacing, the investigation often can be carried out, using analyzers, for one set of attack parameters, and adequate extrapolation of the results to other attacks can then be made.

S. Circular Deployment

A tactical situation of particular interest is that in which several SAM firing units are deployed about a surface target and engaged in its defense. The attack is here assumed to follow a single radial path toward the surface target. All simplifying assumptions stated in Section B other than the positions of the SAM firing units, are here retained. The assumption that the attack approaches along a single radial path; i.e., within a narrow azimuth sector may be justified on several grounds. By so attacking, the enemy minimizes the firepower of the SAM units deployed on the far side of the surface target. He improves his own coordination in that he can minimize spacing between arrivals with greater ease. Also, he may enhance his ECM effectiveness by permitting mutual screening of targets.

To simplify the discussion, the surface target will be approximated by a circle called the Weapon Release Circle, with center at 0 and radius $R_B$. (See Figure 42) The SAM firing units are equally spaced on an emplacement circle with center at 0 and radius $R_E$. The SAM maximum intercept range is assumed to be a constant $\rho_{max}$ independent of the azimuth of attack.

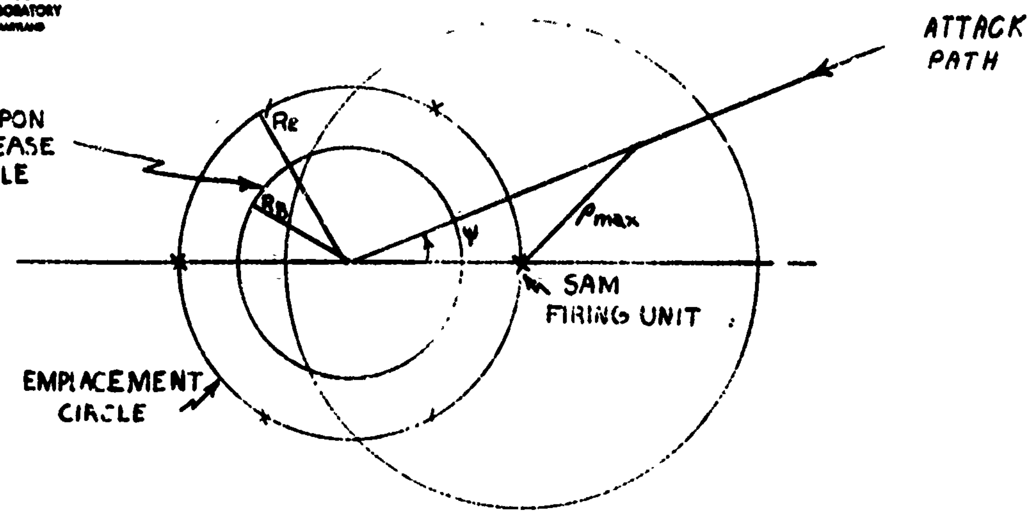



Figure 42

The total firepower of the defense is the sum of the firepowers achieved by each of the SAM units. Hence, we consider first the firepower of one SAM unit and how this firepower varies with the azimuth of attack. For a particular azimuth $\psi$, if the bomb release circle is reached before the maximum intercept range circle, the firepower is of course 0. Otherwise the zone of fire extends from the point of entry into the maximum intercept range circle to either the point of entry into the bomb release circle or the point of exit from the maximum intercept circle, whichever is reached first. Once the zone of fire is determined, the firepower for one and so for every SAM unit can then be calculated by any of the methods described earlier. Thus the variation of firepower with attack azimuth $\psi$ can be obtained.

If the enemy knows those azimuths for which the total firepower $N(\psi)$ is a minimum and if he can indeed attack at one such azimuth, the expected firepower is this minimum. If he has no such knowledge, and if he can attack from any azimuth and has no preference, then all azimuths of attack are equally likely and the expected firepower $N$ is the mean of $N(\psi)$ with respect to $\psi$; i.e.,

$$N = \frac{1}{2\pi} \int_0^{2\pi} N(\psi) d \quad .$$

Certain computational short cuts are available for determining N. Indeed, because of the assumed symmetry of deployment, N is simply the product of the number of sites and the average with respect to $\psi$ of the firepower achieved by any one site. To obtain this average N', it is convenient to assume that the SAM site is at $\psi = 0$ and to calculate the firepower for each of several evenly spaced (in azimuth) attack paths by any of the methods described earlier. The number of paths chosen will be dictated by the accuracy desired. The average N' can readily be obtained by plotting successive intercept points along each path and drawing smooth curves through the sets of successive points, terminating at the edges of the zone of fire. Next, measure the angles $\Delta\psi_1$, $\Delta\psi_2$, ---, subtended by the successive intercept curves.

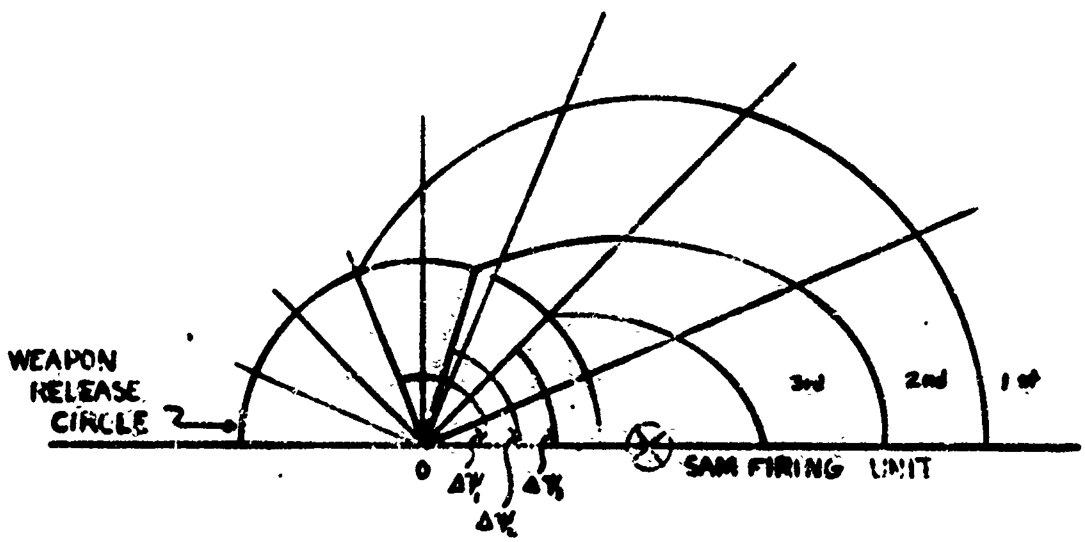



Figure 43

The azimuthal average N' is then

$$N' = \frac{1}{180}\sum \Delta\psi_i$$

where $\Delta\psi_i$ is in degrees.

The labor involved in the above calculations often will not be justified, particularly since they must be repeated for each variation of certain parameters. However, extrapolation from the results of a few sets of calculations can often be advantageous. A quantity useful in extrapolation is the <u>siting factor</u>, defined as the ratio of the average firepower per site $N'$ to the maximum firepower per site $N$ at $\psi = 0$. The siting factor can be computed for a single SAM system and a single attack speed, but for variations in the geometric parameters $R_B$, $R_E$, and $\rho_{max}$. Since the effects of SAM system intercept interval and attack speed will to a large extent cancel out in the ratio, firepower for other intercept intervals and speeds can be estimated by computing $N$ at $\psi = 0$ (using the methods described in Section K and earlier - all relatively simple), and then multiplying by the siting factor computed for the single selected case.

# CHAPTER V

## Coordination of Fire

### A. Introduction

Of major importance to the effectiveness of a SAM defense is the coordination achieved in assigning individual missiles to individual targets during a battle. The purposes of coordination are to avoid overkilling by avoiding assignment of missiles to targets already destined to die from previous engagements and to give highest priority to the most threatening targets. If the attacking force is sufficiently large to completely saturate the defense, the degree of coordination of fire is of little consequence; even with ideal coordination the defense is overwhelmed. On the other hand, if the attacking force is too small, the defense will annihilate it even with poor coordination. Between these limits the coordination achieved may make the difference between defeat and victory. It thus becomes important to assess how much factors contribute to degradation in coordination and how much, in order to determine the best means for achieving a high level of coordination.

### B. Limitations

The ultimate to be achieved in coordination would be to engage each target once, observe damage, re-engage surviving targets, and so on until all are killed. This happy state of affairs can seldom be realized. The number of missiles that can be fired may not suffice. The time required for damage assessment following each intercept may be excessive. Poor target data, poor communications, and errors in damage assessment may lead to confusion in target assignments. The number of times any one target can be engaged and re-engaged following assessment of damage from the preceding engagement may not be sufficient. Deviations from the near ultimate may be made intentionally; e.g., fire may be concentrated on a particularly threatening target to the neglect of certain other targets.

Since no single missile can with certainty be relied on to kill its target (i.e., the kill probability, however high, is less than unity), the ultimate in coordination as defined above could in theory lead to a battle of infinite duration. But missile storage and duration of a battle are always finite, however great, and so in theory and often in practice a limiting factor will be the number of missiles that can be fired. The customary definition of <u>perfect coordination</u> includes this limitation. The definition is as follows:

Suppose that, if need be, N missiles can be launched against an attack of b targets. Perfect coordination of fire is said to obtain if targets continue to be engaged until either all b targets are killed or all N missiles are fired, there being <u>no</u> overkilling.

Strict adherence to the principle of no overkilling may result in a reduction in firepower. The reduction, if any, will depend on the system intercept interval, on the engagement time, i.e., the interval from assignment to completion of assessment, for each engagement, and on the number of targets entering the zone of fire during the battle. To illustrate, suppose that initially the available targets are each engaged once, in turn. If these engagements are completed before the damage resulting from any of them has been assessed, then the system must await assessment and so waste firing time and hence lose some firepower. Any shortening of the assessment process would, of course, reduce the time wasted and so regain some at least of the lost firepower. Even if no waste of firing time were to result, strict avoidance of overkilling could lead to decreased firepower; for the assessment process itself may tie up the guidance channel for a time following intercept and so increase the engagement time by the interval required for assessment, with consequent decrease in firepower. Thus, some compromise between no overkilling and maximum firepower must often be made.

In the extreme case, if the time required for assessment exceeds the duration of the battle, no assessment is possible during the battle. Thus, no overkilling must imply at most one engagement of each target. Often added firing time will remain, permitting added potential firepower. This added firepower will enhance effectiveness even though overkilling may result, provided that every target is engaged at least once. The additional missiles may kill some targets that would survive first engagements. If firing continues without regard to overkilling, then, ideally, the best that can be achieved in the way of coordination of fire is to engage all targets the same number of times. Since the maximum firepower will in general not be an integral multiple of the number of targets, the closest approach to this ideal is for some targets to be engaged n times and the remaining, n+1 times. The number n is the largest integer not exceeding the ratio N/b, where N is the firepower and b is the number of targets; i.e., $n = [N/b]$. When this distribution of missiles among targets obtains, the coordination is said to be <u>discrete uniform</u>.

It is at times convenient for purposes of computation to ignore the fact that only integral numbers of missiles can be fired at targets and so to define n by $n = N/b$ and to assume that all targets are engaged n times. This fictitious case, wherein n is not restricted to integral values, is referred to as <u>continuous uniform coordination</u>.

When overkilling cannot be prevented without a reduction in firepower and yet some effective damage assessment can be made, the best compromise is not always obvious. Spacing of arrival times between targets may be such as to permit excessive overkilling of early targets thereby depleting the store of ready missiles too rapidly, to the detriment of defense effectiveness against later targets. However, except for this storage limitation, overkilling is preferable to a loss of firepower.

One means for coordinating fire that in some situations will achieve maximum firepower and still hold overkilling to a minimum is cyclic coordination. Each target is engaged once, in turn. Upon completion of this first cycle, targets are immediately re-engaged in the same order, any target that is known to be dead (i.e., has been so assessed) being omitted in the second cycle. Targets are thus re-engaged cyclically, those targets assessed as dead being omitted from subsequent cycles. In principle, cyclic coordination is applicable even when there is spacing between successive target arrivals so that not all targets are in the firing zone throughout the duration of the battle. In this case, targets are engaged in the order of their arrival, each cycle running through only those targets in the zone of fire at the time.

Another means for coordinating fire is multiple battle uniform coordination. In many ways it resembles cyclic coordination. The air battle evolves as a series of sub-battles, each of duration too brief to permit any useful damage assessment before its completion. In the first sub-battle $N_1$ missiles are fired, the coordination being uniform (discrete or continuous). The survivors of this sub-battle then enter the second sub-battle in which $N_2$ missiles are fired, the coordination again being uniform. The battle thus proceeds through the full series of sub-battles. It is assumed that the outcome of each sub-battle is fully assessed before the next sub-battle takes place and that all targets entering a sub-battle remain within the zone of fire throughout that sub-battle. These suppositions are in contrast to cyclic coordination wherein cycles fall one upon another without delay for assessment, and spacing between target arrivals is permissible. Another distinction between a sub-battle and a cycle is that in the former the number of missiles fired is not limited to one per available target, as is the case in a cycle.

A serious obstacle in achieving a high level of coordination is the misassignment of targets to SAM units. Because of confusion in the assignment process some targets may unintentionally be simultaneously assigned to two or more SAM units with consequent overkilling, while other targets remain unengaged. Misassignments may occur whether the kill assessment function is rapid or slow. One can in principle specify a best assignment pattern and then, through misassignment, fail to realize this goal. Causes of misassignment are many. Lack of target resolution may prevent positive assignment of a particular target to a SAM unit for engagement. Surveillance of the attack may indicate only that a group of targets are in a region of the sky, individual targets in the group remaining unresolved. Thus, only the group can be assigned to a SAM unit. Selection of a single target is by a SAM guidance channel, e.g., the tracking radar, in the acquisition phase. The selection may be a chance event, and which target is selected may not be known at the SAM site. Thus a subsequent assignment of the group to the same or another SAM unit may result in the same target being selected again.

Inaccuracies in target position data obtained from the surveillance may result in the wrong target being acquired by a tracking radar following assignment. The tracking radar looks at the designated point in space but finds no target there because the position data are in error. The tracking radar then scans a volume of space around this point and may wind up acquiring a target other than the intended one. Repetition of this sequence of events following several assignments could lead to considerable confusion and misassignment of targets.

Confusion between SAM sites as to which target is assigned to which site may arise from data inaccuracies or communication errors. For example, suppose a direction center is making assignments to two SAM sites, A and B. The direction center orders the targets according to range or in some other manner. A is assigned targets 1 and 3, and B is assigned targets 2 and 4. But A fails to see target 1, perhaps due to radar fading, and so according to A targets 1 and 3 are precisely the targets B sees as 2 and 4. (See Figure 44)

| | | | | |
|---|---|---|---|---|
| Targets |  | | | |
| Target numbers according to A | | (1) | 2 | (3) |
| Target numbers according to B | 1 | (2) | 3 | (4) |

Figure 44

As a result two of the targets remain unengaged and the other two are each engaged twice.

Communication links between SAM sites may break down so that each site must select its own targets without knowledge of what targets are being engaged by other sites. Previously agreed upon standard operating procedures can often reduce the number of misassignments that occur in these circumstances; e.g., site A will engage targets on the left, site B on the right. But lack of information concerning what other sites are doing must almost inevitably lead to some misassignments.

A target may be unnecessarily reassigned to a SAM unit because an earlier engagement of it was erroneously assessed as no kill, when in fact the target was killed, or a live target could escape reengagement if erroneously assessed as dead after an earlier engagement.

Confusion in target assignments can sometimes be reduced by one means or another. The defense can hold fire until targets approach close enough to be resolved or to permit more accurate position data to be obtained. Tracking radars usually provide more accurate position data. Hence, following target assignment and acquisition by a tracking radar, but preceding missile launch, the more accurate data from the tracking radar can be fed back to the assignment center and compared with corresponding data on other targets currently under engagement. If the assignment is seen to be a poor one. It can then be cancelled and a new one made. Communication errors can sometimes be eliminated by repeating messages. Any efforts at reducing confusion are very apt to be time consuming, often prohibitively so, and certainly with consequent reduction in firepower. Thus lack of confusion must be weighed against loss of firepower.

The missiles fired during an attack may be distributed among the targets in any of a great many ways depending both on the intended assignment pattern and on the type and degree of confusion. Little is known about these distributions [illegible] confusion exists, as it so often does. This ignorance has led to extensive use of a coordination scheme that clearly [illegible] in a way appropriate [illegible]. This scheme is referred to as random coordination. [illegible] missile [illegible] chance, all targets being [illegible] for each missile. Implicit is the assumption [illegible] in the rate of fire throughout the battle, i.e., [illegible] is constant during the battle. Another scheme for [illegible] the preceding assumption is relaxed is multiple battle random coordination. It is defined in the same way that [illegible] random coordination is defined, except that in each sub-battle random coordination [illegible].

A final [illegible] on the level of coordination achieved is the depth of the missile [illegible] here defined as the number of successive [illegible] engagements of a target that are possible. Assuming the target is engaged and damage is assessed, the target is reengaged if necessary, damage is again assessed, and so on. It is clear from the definition of depth that if kill assessment is not possible, the system depth is never greater than one. This concept of depth is distinct from the customary notion wherein depth [illegible] the distance over which the battle is fought. Conventionally the argument for depth is that with greater combat distance the defense has greater opportunity to strike repeatedly at the attack, gradually whittling it down as the battle progresses. This argument suggests the definition of depth given above; that is, the number of times a target can be engaged successively on a shoot-observe-shoot basis.

Clearly depth depends not only on characteristics of the SAM system (missile range and speed, [illegible] delays, and assessment time) but also on attack characteristics (target speed and [illegible]). Thus one must speak of depth of a SAM system against a specific attack. Lack of depth can of course be compensated for by simultaneously firing enough missiles at each target, accepting the [illegible]. This [illegible] can be very costly in missiles expended, as the following [illegible] example shows. Consider two SAM systems, [illegible] missiles having a kill probability of .5. Suppose that SAM system A has a depth of [illegible] and SAM system B has a depth of 1 against the attack [illegible]. Against A the first and each succeeding target will [illegible], a second, and, if it survives that, a third [illegible] that any one target survives to [illegible] and [illegible] that it is killed before bomb release is .97. The number of missiles fired per target is on the average $1+.5+.5^2+.5^3+.5^4 \approx 1.94$. System B, to achieve the same effectiveness per target, i.e., .97 probability of kill before bomb release, must fire 5 missiles at each target, thus expending about two and one-half times as many missiles as System A.

Since all SAM systems are, in principle at least, limited in their depth, it is of interest to consider a means of coordination known as constrained perfect coordination. Constrained perfect coordination is defined precisely as perfect coordination with the one added restriction that no target can be engaged more than a specified number of times m (m being the depth of the SAM system against the attack in question).

## C. Derivations of Formulas

The various levels of coordination defined above derive their importance in part from the fact that they admit the derivation of relatively simple formulas for certain useful measures of effectiveness. In the following sections these formulas will be derived. The measures of effectiveness considered are 1) the probability that exactly k targets are killed, 2) the probability that all targets are killed, i.e., annihilation probability, and 3) the expected number of targets killed. The merits and uses of these measures of effectiveness will be discussed in Chapter VII; their consideration here will be limited to the derivation of formulas.

Before proceeding with the derivations, it will be helpful to list here certain symbols to be used.

- $b$ number of attacking targets
- $N$ maximum firepower - for entire battle
- $p$ single engagement kill probability
- $q$ single engagement survival probability ($q = 1-p$)
- $P_k$ probability that exactly k targets are killed ($0 \leq k \leq b$)
- $Q_j$ probability that exactly j targets survive ($0 \leq j \leq b$)
- $E$ expected number of kills

In a single engagement of a target by a SAM unit, a single missile or a salvo of two or more missiles may be fired. In the following derivations p denotes the probability that the target will be killed as a consequence of the engagement, whatever the salvo size may be.

### Perfect Coordination

Under the assumption of perfect coordination, if k targets are killed, where $k < b$, then all N missiles must have been fired. Since no overkilling is permitted, k of the missiles must have produced kills, and the remaining N-k must have failed. For any one selection of k missiles, the probability that they all produce kills is $p^k$, and similarly the probability that the remaining N-k missiles all fail is $q^{N-k}$. Hence the probability that the two independent events occur simultaneously is $p^k q^{N-k}$. Since there are $\binom{N}{k}$ ways of selecting the k missiles, the probability of exactly k kills is

$$P_k = \binom{N}{k} p^k q^{N-k} \quad , \quad 0 \leq k \leq b-1 \quad .$$

If on the other hand, all b targets are killed, the number of missiles fired may be any number between b and N, inclusive, i.e., the battle may end before all N missiles are fired. Thus setting k equal to b in the above formula will not suffice. But since the only possible outcomes of the battle are k kills, where k=0, 1, ..., b, it follows that

$$\sum_{k=0}^{b} P_k = 1$$

and so the probability of killing all b targets is

$$P_b = 1 - \sum_{k=0}^{b-1} P_k$$

$$= 1 - \sum_{k=0}^{b-1} \binom{N}{k} p^k q^{N-k}$$

$$= \sum_{k=b}^{N} \binom{N}{k} p^k q^{N-k} \quad .$$

The expected number of kills is defined as

$$E = \sum_{k=0}^{b} k P_k \quad .$$

Substituting for $P_k$ from above

$$E = \sum_{k=0}^{b-1} k \binom{N}{k} p^k q^{N-k} + b \sum_{k=b}^{N} \binom{N}{k} p^k q^{N-k} \quad .$$

But

$$k \binom{N}{k} p^k = Np \binom{N-1}{k-1} p^{k-1}$$

and so, letting $k = i + 1$, we have

$$E = Np \sum_{i=0}^{b-2} \binom{N-1}{i} p^i q^{N-1-i} + b \sum_{k=b}^{N} \binom{N}{k} p^k q^{N-k} \quad .$$

The two summations are partial binomial sums, tabulated values of which are available in the literature. To facilitate use of tables, it is convenient to write

$$E = Np - Np \left( \sum_{i=b-1}^{N-1} \binom{N-1}{i} p^i q^{N-1-i} \right) + b \left( \sum_{k=b}^{N} \binom{N}{k} p^k q^{N-k} \right) \quad .$$

Note that by definition the expected number of survivors is b-E, whatever the coordination may be.

Discrete Uniform Coordination

By definition of discrete uniform coordination a number b' of the targets will be engaged by $n = [N/b]$ missiles, and the remaining $b'' = b - b'$ targets will be engaged by n+1 missiles. Assuming the outcomes of individual engagements are independent, the probability that a particular one of the b' targets will survive is $q^n$, and the probability that it will be killed is $1 - q^n$. The survival and kill probabilities for any one of the b" targets are $q^{n+1}$ and $1 - q^{n+1}$ respectively.

The battle may end with exactly k targets being killed, the kills being divided between the two groups of targets in any of several ways; i.e., k' kills out of the first group, and $k'' = k-k'$ kills out of the second.

The probability that a particular selection of k' out of the b' targets are killed is $(1 - q^n)^{k'}$, and that the remaining b' - k' targets survive is $(q^n)^{b'-k'}$. There being $\binom{b'}{k'}$ ways of selecting the k' targets, the probability of exactly k' kills from among the b' targets is then

$$\binom{b'}{k'} (1-q^n)^{k'} (q^n)^{b'-k'} \quad .$$

In like manner, the probability of exactly k" kills from among the remaining b" targets is

$$\binom{b''}{k''} (1-q^{n+1})^{k''} (q^{n+1})^{b''-k''} .$$

The probability of these two independent events (exactly k' kills out of b' targets and k" kills out of b" targets) occurring simultaneously is the product of the two expressions given above. To obtain the probability of exactly k kills in all, the above product must be summed over all admissible combinations of k' and k"; giving

$$P_k = \sum_{\substack{k'+k''=k \\ 0 \le k' \le b' \\ 0 \le k'' \le b''}} \binom{b'}{k'} (1-q^n)^{k'} (q^n)^{b'-k'} \binom{b''}{k''} (1-q^{n+1})^{k''} (q^{n+1})^{b''-k''}$$

The restrictions $k' \le b'$, $k'' \le b''$ can be ignored, since the corresponding binomial coefficients vanish when the restrictions are violated, and so contribute nothing to the sum.

The probability of killing all b targets is readily obtained from the preceding formula by setting k equal to b, and so $k' = b'$, $k'' = b''$, giving,

$$P_b = (1-q^n)^{b'} (1-q^{n+1})^{b''} .$$

To obtain an expression for expected number of kills E, consider b random variables,

$$x_1, x_2, \ldots, x_{b'}, x_{b'+1}, x_{b'+2}, \ldots, x_{b'+b''}$$

corresponding to the b targets. Each of the first b' variables has the value 1 (corresponding to the target kill) with probability $1-q^n$, and has the value 0 (corresponding to no target kill) with probability $q^n$. Similarly each of the next b" variables has the value 1 with probability $1-q^{n+1}$, and value 0 with probability $q^{n+1}$. The expected value (i.e., the expected number of kills per target) of each of the first b' random variables is

$$1 \cdot (1-q^n) + 0 \cdot q^n = 1 - q^n$$

and of each of the next b" random variables is

$$1 \cdot (1-q^{n+1}) + 0 \cdot q^{n+1} = 1-q^{n+1} .$$

Since the expected value of the sum of the random variables is the sum of their individual expected values, we have

$$E = \sum_{0}^{b'} (1-q^{n}) + \sum_{b'+1}^{b'+b''} (1-q^{n+1})$$

$$= b' (1-q^{n}) + b'' (1-q^{n+1}) \quad .$$

Continuous Uniform Coordination

By definition, every target is engaged by $N/b$ missiles, so the probability that any one target will survive is $q^{N/b}$, and the probability that it will be killed is $1-q^{N/b}$. Hence the probability that exactly k targets are killed is readily obtained by setting $b'=b$, $b''=0$ in the corresponding derivation for discrete uniform coordination. We then have ..

$$P_k = \binom{b}{k} (1-q^{N/b})^{k} (q^{N/b})^{b-k} \quad .$$

Similarly the probability of killing all b targets is

$$P_b = (1-q^{N/b})^{b}$$

and the expected number of kills is

$$E = b(1-q^{N/b}) \quad .$$

Random Coordination

Under the assumption of random coordination, the probability that a specified target is among the survivors of a single missile is the probability that the target is not selected plus the probability that it is selected but is not killed, i.e., is

$$\frac{b-1}{b} + \frac{1}{b} q = 1 - \frac{p}{b} \quad .$$

The probability that the target is among the survivors of N missiles is then

$$\left(1 - \frac{p}{b}\right)^{N} \quad .$$

Since all target-missile pairings are assumed equally likely, the above probability applies to any specified target.

Similarly the probability that j specified targets are among the survivors of a single missile is the probability that none of the j targets is selected plus the probability that the first is selected but not killed, plus the probability that the second is selected but not killed, plus the corresponding probability for each of the others, i.e., is

$$\frac{b-j}{b} + \underbrace{\frac{q}{b} + \frac{q}{b} + \ldots + \frac{q}{b}}_{j} = 1 - \frac{jp}{b} \quad .$$

The probability that the j targets are among the survivors of N missiles is then

$$\left(1 - \frac{jp}{b}\right)^N \quad .$$

Again, this probability applies to any specified set of j targets.

For each set of targets that includes j specified targets, there is a probability that precisely the targets of that entire set, and no others, survive. The sum of these probabilities is the probability that the j specified targets are among the survivors. Furthermore, for all sets of the same size j+i, there being $\binom{b-j}{i}$ such sets, the probabilities are all equal to the probability $\pi_{j+i}$ that any j+i specified targets will survive, since all target-missile pairings are equally likely. Hence the above sum can be written as

$$\sum_{i=0}^{b-j} \binom{b-j}{i} \pi_{j+i} \quad .$$

Equating the two expressions for the probability that the j targets are among the survivors, we have, for successive values of j:

$$\left(1 - \frac{p}{b}\right)^N = \pi_1 + \binom{b-1}{1}\pi_2 + \ldots + \binom{b-1}{j-1}\pi_j + \ldots + \binom{b-1}{j+k-1}\pi_{j+k} + \ldots + \pi_b$$

$$\left(1 - \frac{2p}{b}\right)^N = \pi_2 + \ldots + \binom{b-2}{j-2}\pi_j + \ldots + \binom{b-2}{j+k-2}\pi_{j+k} + \ldots + \pi_b$$

$$\left(1 - \frac{jp}{b}\right)^N = \pi_j + \ldots + \binom{b-j}{k}\pi_{j+k} + \ldots + \pi_b$$

$$(1 - p)^N = \pi_b$$

To solve for $\pi_j$, multiply the (j+i)th equation by $(-1)^i\binom{b-j}{i}$ for each i=0, 1, ..., b-j. The multiplier of the jth equation is then $\binom{b-j}{0}=1$, the multiplier of the (j+1)st equation is $-\binom{b-j}{1}$, the multiplier of the (j+2)nd equation is $\binom{b-j}{2}$, and so on. Add together these modified equations, starting with the jth equation. The left hand side of this sum is

$$\sum_{i=0}^{b-j} (-1)^i \binom{b-j}{i} \left(1 - \frac{(j+i)p}{b}\right)^N \quad .$$

The right hand side of this sum consists of $\pi_j$, plus the sum of terms involving $\pi_{j+1}$, plus the sum of terms involving $\pi_{j+2}$, and so on. The sum of terms involving $\pi_{j+k}$ for each $k \geq 1$ is simply $\pi_{j+k}$ multiplied by a coefficient, the coefficient being

$$\sum_{i=0}^{k} (-1)^i \binom{b-j}{i}\binom{b-j-i}{k-i} = \sum_{i=0}^{k} (-1)^i \frac{(b-j)!}{i!(b-j-i)!} \frac{(b-j-i)!}{(k-i)!\,(b-j-k)!}$$

$$= \frac{(b-j)!}{(b-j-k)!} \sum_{i=0}^{k} (-1)^i \frac{1}{i!\,(k-i)!}$$

$$= \frac{(b-j)!}{k!\,(b-j-k)} \sum_{i=0}^{k} \binom{k}{i}(-1)^i$$

$$= \frac{(b-j)!}{k!\,(b-j-k)!} (1-1)^k$$

$$= 0$$

Hence the right hand side reduces to $\pi_j$. We then have

$$\pi_j = \sum_{i=0}^{b-j} (-1)^i \binom{b-j}{i} \left(1 - \frac{(j+i)p}{b}\right)^N$$

as the probability that a specified set of j targets survive.

There being $\binom{b}{j}$ distinct ways of selecting j targets from a total of b, the probability $Q_j$ of exactly j survivors is

$$Q_j = \binom{b}{j}\pi_j$$

$$= \binom{b}{j} \sum_{i=0}^{b-j} (-1)^i \binom{b-j}{i} \left(1 - \frac{(j+i)p}{b}\right)^N$$

The probability that exactly k targets are killed is of course the probability that exactly b-k targets survive. Thus, substituting b-k for j in the preceding equations, we have

$$P_k = Q_{b-k}$$

$$= \binom{b}{k} \sum_{i=0}^{k} (-1)^i \binom{k}{i} \left(1 - \frac{(b-k+i)p}{b}\right)^N$$

The probability that all b targets are killed is readily obtained from the above expression for $P_k$. Setting k equal to b, we have

$$P_b = \sum_{i=0}^{b} (-1)^i \binom{b}{i} \left(1 - \frac{ip}{b}\right)^N$$

An expression for the expected number of kills is easily derived by the procedure used in the case of discrete uniform coordination. Consider b random variables $x_1, x_2, \ldots, x_b$, each having value 1 (target kill) with probability $1 - (1-p/b)^N$ and value 0 (no target kill) with probability $(1-p/b)^N$. The expected value of each random variable is then

$$1 \cdot \left[1 - (1-p/b)^N\right] + 0 \cdot (1-p/b)^N = 1 - (1-p/b)^N$$

and their sum, the expected number of kills, is

$$E = b\left[1 - (1-p/b)^N\right]$$

The expressions for $P_k$ and $P_b$ derived above are indeed ungainly for hand computation. It is therefore helpful to replace them with approximations that are more manageable. Since

$$\left(1 - \frac{\alpha p}{b}\right) \approx \left(1 - \frac{p}{b}\right)^\alpha$$

if p/b is small, we can write

$$P_k \approx \binom{b}{k} \sum_{i=0}^{k} (-1)^i \binom{k}{i} \left(1 - \frac{p}{b}\right)^{(b-k+i)N}$$

$$= \binom{b}{k}\left(1 - \frac{p}{b}\right)^{N(b-k)} \sum_{i=0}^{k} \binom{k}{i} \left[-\left(1 - \frac{p}{b}\right)^N\right]^i$$

$$= \binom{b}{k} \left[\left(1 - \frac{p}{b}\right)^N\right]^{(b-k)} \left[1 - \left(1 - \frac{p}{b}\right)^N\right]^k$$

and similarly

$$P_b \approx \left[1 - \left(1 - \frac{p}{b}\right)^N\right]^b$$

Another approximation that is often useful follows from

$$1 - \frac{\alpha p}{b} \approx e^{\frac{-\alpha p}{b}} \quad .$$

Making this substitution, we can write

$$P_k \approx \binom{b}{k} \left(e^{\frac{-Np}{b}}\right)^{b-k} \left(1 - e^{\frac{-Np}{b}}\right)^k$$

and

$$P_b \approx \left(1 - e^{\frac{-Np}{b}}\right)^b$$

and also

$$E = b\left(1 - e^{\frac{-Np}{b}}\right) \quad .$$

Multiple Battle Coordination

Under the definition of multiple battle coordination, whether uniform or random, each sub-battle is concluded and its outcome known (at least to the "referee", if not to the defending commander) before the next battle commences. In particular the number of targets that survive one sub-battle and so enter the next is known to the "referee". Therefore general formulas can be derived, independent of the coordination obtaining in each sub-battle. Indeed mixtures are admissible, random in one sub-battle and uniform in another.

To simplify the discussion, consider first the case of two sub-battles. If $j$ targets are to survive the second of these, at least $j$ targets must survive the first and the difference must be killed off in the second. Therefore the probability that exactly $j$ targets survive the entire battle is the probability that exactly $m \geq j$ targets survive the first sub-battle, multiplied by the probability that if $m$ targets enter the second sub-battle exactly $j$ will survive, and summed over all $m$ from $j$ to $b$.

Let $Q_n(m,i)$ denote the probability that if $m$ targets enter the $i$-th sub-battle, exactly $n$ will survive. The appropriate formula for $Q_n(m,i)$ is determined by the coordination obtaining in the $i$-th battle. The probability that exactly $j$ targets survive the entire battle can be written as

$$Q_j = \sum_{m=j}^{b} Q_m(b,1)\, Q_j(m,2) \quad .$$

If the multiplicity of sub-battles is three, the probability $Q_j$ is given by

$$Q_j = \sum_{m=j}^{b} Q_m(b,1) \sum_{n=j}^{m} Q_n(m,2)\, Q_j(n,3)$$

The extension to higher multiplicities is clear.

The probability that exactly k targets are killed in the entire battle is given by

$$P_k = Q_{b-k}$$

and is obtained by setting $j = b-k$ in the appropriate formula above.

The probability that all b bombers are killed is derived directly by placing $k=b$ in the expression for $P_k$.

The expected number of kills is, by definition

$$E = \sum_{k=0}^{b} k\, P_k \quad .$$

Even for two battle continuous uniform coordination the above formulas are not particularly attractive for hand computation; when the multiplicity is higher or when random or discrete uniform coordination occur in a sub-battle, the calculations become even more laborious. An approximate calculation which is sometimes satisfactory is to compute the expected number of survivors of the first sub-battle, $b-E_1$, using the formula appropriate to the coordination obtaining. Then, assuming that precisely $b_1 = b - E_1$ targets enter the second sub-battle, again compute the expected number of survivors $b_2 = b_1 - E_2$. Proceed in this manner up to the last sub-battle. Here compute the desired measure of effectiveness (e.g., $P_k$, $P_b$, or $E$). Great caution should be exercised when using this procedure, especially when $P_k$ or $P_b$ are the desired measures, since the approximation may introduce serious errors.

Constrained Perfect Coordination

In this section a formula for annihilation probability will be derived. If, under the restrictions of constrained perfect coordination, all b targets are killed, the number of missiles fired may be any number between b and N, inclusive. Furthermore, since the number of engagements per target cannot exceed the depth m, the maximum firepower N cannot exceed mb. For any particular distribution of k missiles among b targets, if b kills result, exactly b missiles must produce kills, and the remaining k-b missiles must fail. The probability of such occurrences is $p^b q^{k-b}$. Let $C_{k,b,m}$ denote the number of possible distributions (as permitted by the depth constraint) of k missiles among b targets. The probability that all b targets are killed is then

$$P_b = \sum_{k=b}^{N} C_{k,b,m}\, p^b q^{k-b} \quad .$$

There remains the task of deriving an expression for the coefficient $C_{k,b,m}$. To this end, for a particular distribution, let $n_i$ denote the number of missiles fired at the i-th target. Then $C_{k,b,m}$ is the number of integral solutions to the equation

$$n_1 + n_2 + \ldots + n_b = k$$

subject to the constraints

$$1 \leq n_i \leq m \qquad , \; i=1,\, 2,\, \ldots,\, b \qquad .$$

Consider the polynomial

$$F(x) = (x + x^2 + x^3 + \ldots + x^m)^b$$

$$= \underbrace{(x + x^2 + \ldots + x^m)(x + x^2 + \ldots + x^m)\ldots(x + x^2 + \ldots + x^m)}_{\text{b factors in all}}$$

The expansion of this product is a sum of terms of the form $x^{n_1} x^{n_2} \ldots x^{n_b}$, where $x^{n_1}$ is one of the terms in the first polynomial factor above, $x^{n_2}$ is one of the terms in the second polynomial, etc. Thus $1 \leq n_i \leq m$, $i=1,2,\ldots b$, and the sum is over all terms satisfying these conditions. We may write

$$F(x) = \sum_{1 \leq n_i \leq m} x^{n_1} x^{n_2} \ldots x^{n_b}$$

$$= \sum_{1 \leq n_i \leq m} x^{n_1 + n_2 + \ldots + n_b} \quad .$$

These terms can be divided into groups, those terms for which

$$n_1 + n_2 + \ldots + n_b = k$$

being grouped together. The sum of terms in any one group is then $C_{k,b,m}\, x^k$. Summing over all groups,

(16) $$F(x) = \sum_{k=b}^{mb} C_{k,b,m}\, x^k \quad .$$

Returning to the definition of $F(x)$, we have

$$F(x) = \left[x(1 + x + x^2 + \dots + x^{m-1})\right]^b$$

$$= \left[\frac{x(1-x^m)}{1-x}\right]^b$$

$$= x^b(1-x)^{-b}(1-x^m)^b$$

$$= x^b\left[\sum_{i=0}^{\infty} \binom{-b}{i}(-1)^i x^i\right] \quad \left[\sum_{j=0}^{b} \binom{b}{j}(-1)^j x^{mj}\right] \quad .$$

But

$$\binom{-b}{i} = (-1)^i\binom{b+i-1}{i}$$

and so

$$F(x) = x^b \sum_{i=0}^{\infty} \binom{b+i-1}{i} x^i \sum_{j=0}^{b} \binom{b}{j}(-1)^j x^{mj}$$

$$= \sum_{\substack{0 \le i \\ 0 \le j \le b}} \binom{b+i-1}{i}\binom{b}{j}(-1)^j x^{b+i+mj} \quad .$$

Grouping terms for which $b+i+mj = k$, we have

(17) $$F(x) = \sum_{k=b}^{mb} \left(\sum_{\substack{b+i+mj=k \\ 0 \le i,j}} \binom{b+i-1}{i}\binom{b}{j}(-1)^j\right) x^k \quad .$$

Equating like coefficients in equations (16) and (17), we have

$$C_{k,b,m} = \sum_{\substack{b+i+mj=k \\ 0 \le i,j}} \binom{b+i-1}{i}\binom{b}{j}(-1)^j$$

Clearly the coefficients $C_{k,b,m}$ and so the formula for $P_b$, are not well suited for rapid calculation. In the special cases for depths m=1 and 2, much simpler expressions can be derived for $P_b$. Since the effect of the depth constraint is most stringent for small m, these special cases are cases are of particular interest. Indeed, if the kill probability is reasonably high, the depth constraint is negligible for depth of 3 or more.

Clearly if the depth m=1, then constrained perfect coordination is identical to discrete uniform coordination, with N targets each receiving one shot and the remaining b-N receiving none. Thus

$$P_b = \begin{cases} p^b & \text{if } N \geq b \\ 0 & \text{if } N < b \end{cases} .$$

If the depth m=2, and N is not less than b, so annihilation of all targets is possible, the firing can be thought of as occurring in two rounds. In the first round each target is engaged once, and in the second round each first round survivor is engaged once again. Of course, if the first round survivors outnumber the remaining available firepower, not all targets can be killed.

Let j denote the number of first round survivors. For any one set of j targets, the probability that they all survive is $q^j$, and that the remaining b-j targets all are killed is $p^{b-j}$. There being $\binom{b}{j}$ distinct sets of j targets, the probability that exactly j targets survive the first round fire is

$$\binom{b}{j} p^{b-j} q^j .$$

In order that all targets be killed, each of the j first round survivors must be killed in the second round. The probability of this event is $p^j$, and the total number of engagements in the two rounds of fire is b+j, which cannot exceed N. Given b+j engagements, and so j first round survivors, the probability that all b targets are killed is

$$\binom{b}{j} p^b q^j .$$

Thus the annihilation probability is

$$P_b = \sum_{j=0}^{N-b} \binom{b}{j} p^b q^j$$

which, to facilitate use of binomial sum tables, can be written

$$P_b = (p+pq)^b \left[1 - \sum_{j=N-b+1}^{b} \binom{b}{j} \left(\frac{q}{1+q}\right)^j \left(\frac{1}{1+q}\right)^{b-j}\right] .$$

Clearly, $P_b = 0$ if $N < b$.

# CHAPTER VI

## Simulation

### A. Firing Doctrine

In the course of an air battle, targets are assigned to SAM units according to some definable procedure, formal or otherwise, called a firing doctrine. A firing doctrine may consist of a well thought out set of rules intended to insure a high level of coordination. Or a firing doctrine may be a poorly conceived set of rules, giving rise to unnecessary waste of missiles in overkilling. Or it may dictate that, in the end, assignment becomes a chance event not under the defense commander's control, perhaps because he may lack sufficient information to permit him to make a precise assignment. Whether good or bad, fully determined or random, a firing doctrine is a definite process that admits detailed description.

The choice of firing doctrine for a particular defense, operating in a particular environment against a particular attack, is limited by the capabilities of equipment and of operating personnel. Can the sensing equipment provide accurate up-to-date target data? Can the communications links carry the required load without error? Are operating personnel trained to perform as required? Within the limits imposed by the answers to these questions, the choice of firing doctrine is an exercise of the command function; the doctrine may be laid down in all detail in advance or it may be devised as the battle progresses.

The essential features of the assignment process can be brought out by considering two illustrative firing doctrines. The first of these consists of assigning to a free SAM unit the nearest currently least engaged target not known to be dead. If at the moment there are unengaged live targets, the nearest of these is assigned. If all live targets are currently being engaged, those undergoing but one engagement are singled out and the nearest of these is assigned, and so on. The other doctrine consists of making a random selection of a target from among those currently in the firing zone and still airborne. Both of these firing doctrines, and most other doctrines, are characterized by the fact that only certain targets are eligible for assignment (those not known dead, within range, and least engaged in the first doctrine, those airborne and within range in the second); one target is then selected from among the eligibles (the nearest in the first doctrine, a random choice in the second). Thus a firing doctrine may be defined by specifying the criteria for eligibility and for selection. The criteria must, of course, be specified in unambiguous detail; for example, in the first of the two doctrines stated above, "nearest" must be defined precisely.

### B. Firing Doctrine in Relation to Coordination

Many of the well defined coordination schemes of Chapter V can arise from any of a variety of firing doctrines. Perfect coordination can be effected by engaging and re-engaging the first target until it is killed, then engaging and re-engaging a second target until it is killed, and so on. Alternatively, perfect coordination will result if each target is engaged once, then each, necessarily observed, survivor is engaged a second time, and so on. The coordination of fire will be uniform if the first target is engaged n times, then the second target is engaged n times, and so on, or equally well, every target is engaged once, then every target is engaged a second time, and so on. Many other possible doctrines can be laid down which will also result in perfect or uniform coordination.

Often none of the firing doctrines leading to any of the coordination schemes of Chapter V will be applicable to the SAM system being considered. Limitations imposed by the capabilities of equipment and/or of personnel will preclude the use of such doctrines, or even if such a doctrine could be employed, another doctrine not leading to a neatly definable coordination scheme might appear preferable.

Even if a firing doctrine which would give rise to a coordination scheme of Chapter V were applicable and appeared desirable, the use of any of the corresponding formulas for measures of effectiveness, derived in Chapter V, presents serious difficulties. The formulas require estimates of both firepower and kill probability. Firepower may depend, among other things, on the distribution of intercept ranges, which in turn may be governed by the order in which targets are engaged, i.e., by the firing doctrine used. Thus the average kill probability for all engagements may depend on the distribution of intercept ranges or on the extent to which formations are broken up early in the battle, or both. For reasons such as these the proper values of firepower N and the kill probability p to be used in a formula may not be readily calculable. Indeed the firepower calculations of Chapter IV were based on specific firing doctrines in which no target was engaged more than once; such doctrines are quite different from any that are applicable in Chapter V.

One or another of the formulas of Chapter V has been used in each of a good many studies in the past. In using any of the formulas the implied assumption is not that the firing doctrine presumed to be in force would give rise to that particular coordination scheme. Rather it would lead to a value of the measure of effectiveness about equal to that which the formula provides. Many of the difficulties in estimating suitable values of firepower and kill probability to be used in the formula are usually ignored; e.g., firepower may be calculated by a method of Chapter IV, based on a firing doctrine inconsistent with the coordination scheme assumed. In many cases, where one or another of the formulas is used, the choice of coordination scheme does not rest on any very solid foundation, rather it is primarily an intuitive guess or a choice of convenience.

C. Monte Carlo Method

It is evident from the discussion thus far that the tactical analyst needs an alternative means for computing SAM system effectiveness, a means that is applicable to any firing doctrine. An alternative method for computing effectiveness does exist, employing simulation of the air battle. A simplified model of the air battle is designed, to include all essential features of the air battle. The air battle model is then used to calculate a measure of the effectiveness of the defense. Such use of a simplified model of a part of the real world to investigate phenomena of the real world is a common practice in many fields of science.

But how can an air battle model, incorporating firing doctrine, be used to calculate a measure of effectiveness? In the course of an air battle a wide variety of random events occur, each with its associated probability of occurrence. A variety of different outcomes of the battle are possible, each having its associated probability of occurrence. The probability of a particular outcome is defined as the limit of the ratio of number of battles having this outcome to the total number of battles, as the total number of battles becomes infinite. An estimate of this limit is given by the ratio of number of particular outcomes to total number of battles in a finite sample. As the size of the sample is increased, so the estimate is improved. Using the model, the outcome of a single battle that includes a sequence of random events can be determined by ascertaining the outcome of each random event from a throw of a die. For example, suppose that the probability of kill for a particular missile fired at a particular target is two-thirds. Let a die be thrown; if a 1, 2, 3, or 4 turns up, let the target be killed, and if a 5 or 6 turns up, let the target survive. By successive application of this process to each random event in turn the outcome of the model battle can be determined, and by repeatedly fighting the model battle in this way for a sufficient number of times, a satisfactory estimate of the probability of a particular outcome is obtained. This procedure for calculating the probability of a particular outcome is commonly referred to as the Monte Carlo method.

D. Characteristics of an Air Battle Model for Simulation

An air battle model should include all features of the air battle that will significantly affect one or more of the kill probability, firepower, and firing doctrine. The model must provide whatever information is called for by the eligibility and selection criteria that make up the firing doctrine. It must provide means for ascertaining the time and location of each intercept, and whether or not a kill occurs. In addition, the model should include mechanisms for starting the battle at an appropriate time, for advancing the clock in order that the battle may progress, and for terminating the battle. Finally, provision should be made for recording pertinent information about what has transpired in the course of the battle.

Information needed for operation of the firing doctrine will, of course, be dictated by the doctrine being used. Many eligibility and selection criteria are based in some manner on one or more of present position, course, and speed, predicted position, and current engagement status of each target. The time and location of intercept can be calculated if the position of the target at time of assignment, target speed and path, location of SAM site and time from assignment to intercept as a function of intercept position are known. Occurrence of kill can be ascertained by the throw of a die or some similiar random drawing, provided the kill probability is known as a function of intercept position and the nature of the target. The battle may start when the first target becomes eligible for assignment. The battle, or at least the calculation, may stop when a specified number of attacking aircraft have penetrated to weapon release, when the defense is rendered ineffective by exhausting its ammunition or by being destroyed, or when all aircraft have been killed or have otherwise become permanently ineligible. The choice may be dictated by the measure of effectiveness being computed; e.g., if the probability of at least one penetrator is the measure, the calculation need not continue after one penetration has occurred.

Two means for advancing the clock during the battle are in common use. One is to advance the clock a fixed short interval, examine all targets and all SAM units to see if anything significant has taken place during that interval, carry out whatever calculations or other operations that may be called for, then advance the clock another fixed interval and again examine all targets and all SAM units.

The second means makes use of what we shall call an event store. The battle is thought of as a sequence of events such as:

A new target is assigned to a SAM unit.
Engagement of a target by a SAM unit has been completed.
A target has crossed the weapon release point.

Initially the time at which certain events will occur can be calculated; for example, the first target assignment event for each SAM unit. These are entered in a store in the order of their occurrence. Each event will generate other later events; for example, a target assignment event would generate a later completion of engagement event as well as a later target assignment event for that SAM unit. As each new event is generated, the time of its occurrence is calculated and the event is entered in the store in the appropriate order, behind those occurring earlier and before those occurring later. On completion of the calculations and other operations appropriate to an event, including the generation of new events, the clock is advanced to the time of the next event in the store. Calculations and operations appropriate to this event are then carried out and the clock is again advanced.

## E. A Particular Air Battle Simulation

The discussion thus far has indicated in a general way how air battle simulation can be of use to the tactical analyst and what characteristics are required of an air battle model to be used in the Monte Carlo method. The advantages and limitations of air battle simulation as an analytic tool can be better understood by describing in detail the design and operation of a specific simulation. Such a description follows.

In the tactical situation assumed, a number of air targets are proceeding at the same constant speed and altitude along the same straight path toward a surface target which is defended by a number of SAM units deployed about it. The geometry of the battle can conveniently be described in terms of a cartesian coordinate system having its origin at or behind the surface target and its positive x-axis along the attack path in the direction from which the attack approaches, as shown in Figure 45.

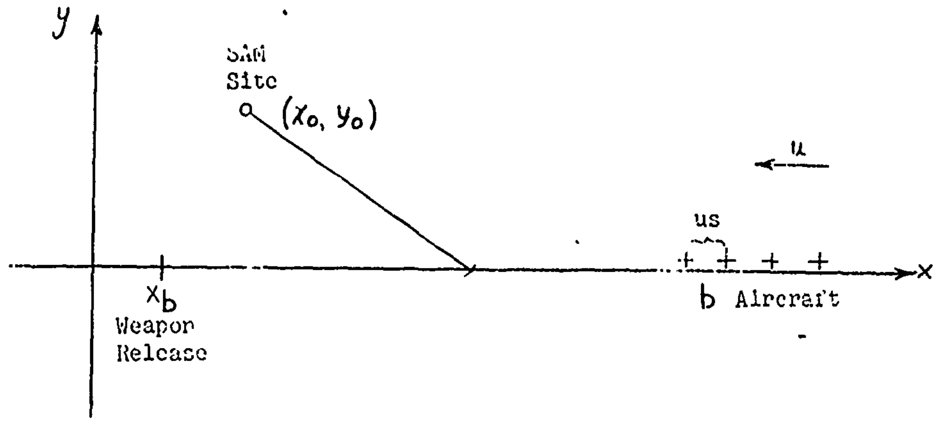



Figure 45

The attack is characterized by the number of aircraft b, their speed u, their weapon release point $x_b$, and the spacing between successive target arrivals. It is often convenient to use a constant mean spacing s between arrival times, although arbitrary spacings do not introduce serious complication; the model admits either choice. Any of the attacking aircraft may be a decoy, if desired. With the specification of weapon release point, the surface target as such need no longer be considered, since the model is concer with intercepts occurring before weapon release only.

Target altitude as such does not enter directly into any of the calculations carried out during the air battle simulation. Rather, its effect is felt in the initial selection of certain parameter values that do enter the calculations. In particular it may influence or govern the choice of weapon release point and maximum intercept ange.

The defense is characterized by the location ($x_o$, $y_o$) of each SAM unit, by the number m of missiles stored at each site, by the maximum intercept range $\rho_{max}$ of the missile as defined by the intersection of the attack path and the zone of fire about the SAM unit, by the kill probability $p(\rho)$ as a function of intercept range $\rho$, and by certain times associated with the SAM unit as described in Sections C and D of Chapter II. A brief review of those sections will provide a useful background for the ensuing discussion. The times of particular interest are (1) tie-up time $\emptyset(\rho)$, the time between designation and intercept of a target, (2) assessing time $t_a$, the time between intercept and receipt of verdict of the assessing process at the decision center, and (3) a time we shall call designation cycle time $T_D$, defined as the greatest designation interval (the interval of time between successive designations to the same SAM unit) that will not cause loss of firepower.

The significance of the designation cycle time is that if the designation interval is equal to this time, then on the one hand the firing rate will be governed by the availability of a launcher or of a guidance channel (i.e., the firing unit will not be idle for lack of a target) and on the other hand the next target designation will be made at the last possible moment that will not cause delay. From this comment we see that if the firing rate is launcher governed, the designation cycle time $T_D$ is equal to the launcher cycle time $T_L$, and if the firing rate is limited by the availability of a guidance channel, the designation cycle time is equal to the intercept interval $T(\rho_2)$ plus the difference in missile times of flight to successive intercept ranges $\rho_1$ and $\rho_2$. Thus we have

$$(18) \qquad T_D = \max \begin{cases} T_L \\ T(\rho_2) + t_f(\rho_1) - t_f(\rho_2) \end{cases} .$$

The simulation, with minor modifications, permits the use of a variety of firing doctrines. To avoid unnecessary confusion, a single doctrine will be employed here; discussion of modifications required for other doctrines will be deferred. The doctrine we here consider is the "nearest currently least engaged" doctrine described earlier. A target is eligible for selection if:

It has not been assessed as killed in consequence of an earlier engagement.
Its present position will permit intercept before weapon release and within maximum intercept range (with one exception noted below).
It is among the currently least engaged of all such targets.

Because of the assumption of constant attack speed and course, the nearest target of any group can be defined as that target closest to but not yet past weapon release point.

The simulation embodies an event store as a means for advancing the clock. In addition, a target store and a SAM unit store are incorporated to facilitate the recording of information needed later in the battle or desired as a permanent record of what transpired during the battle. Events in the simulation are target assignment event, engagement completion event, delayed launching event and weapon release passage event. These events together with the calculations and operations associated with each are discussed in turn.

Target Assignment Event

At the time of the event, when a SAM unit is ready for a new target assignment, the target store is scanned to determine the set of eligible targets, and the nearest of these is selected. The scan can be carried out in a variety of ways. One way is to examine each target in turn in order of nearness, rejecting it if it is labelled killed, if its current position fails to satisfy the elegibility criteria, or if its current engagement status equals or exceeds that of an earlier eligible target. If its engagement status is less than that of an earlier target, then the earlier target is rejected. Clearly, at any stage of the scan, at most one target will be retained. On completion of the scan the one (if any) remaining target is assigned, since it will automatically be the nearest of those eligible. If no target is retained, none are eligible and no assignment is made.

Strict adherence to the stated eligibility criterion that a target's current position be such as to permit intercept within maximum range requires that at the time of designation the target be inside a position

$$x_c = x_{max} + u\phi(\rho_{max}) \quad ,$$

where $x_{max}$ is the intercept position corresponding to maximum range $\rho_{max}$. (see Fig. 46.)

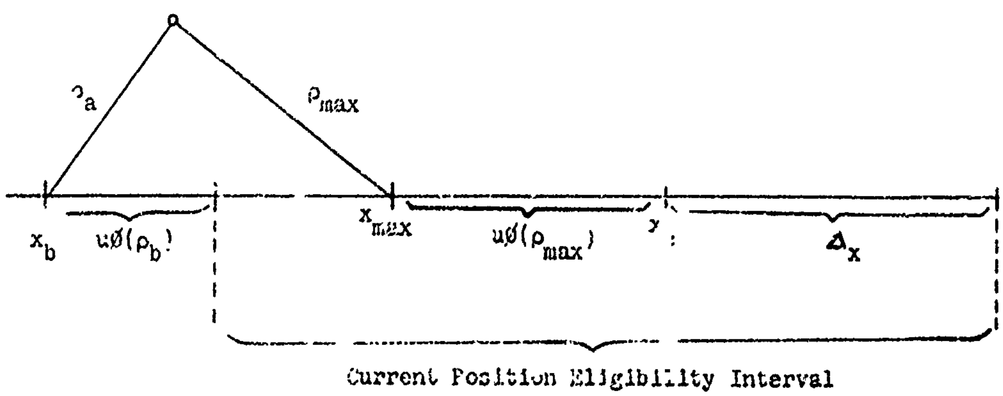



Figure 46

Observance of this requirement could in some instances lead to several SAM units simultaneously engaging the lead target at the beginning of the battle, leaving the next few targets unengaged, unless spacing between targets is very small. To alleviate this undesirable situation we grant eligibility to targets whose current position is in an interval of some specified length $\Delta x$ lying just beyond $x_c$. (See Fig. 46.) Choice of the length $\Delta x$ is a question of doctrine. If too small, it will fail to fulfill its purpose and if too long it will lead to undue delay.

If the designated target is currently within the interval $\Delta x$, a delayed launching event for the SAM unit is generated, including a record of the identity of the designated target; the time of the event is the time when the designated target will pass $x_c$. If the designated target has already passed $x_c$, the intercept position and range can be calculated. We have

(19) $$x_I = x_D - u\phi(\rho)$$

$$\rho = \sqrt{(x_I - x_o)^2 + y_o^2}$$

where (see Fig. 47)

$x_I$ = intercept position

$x_D$ = designation position

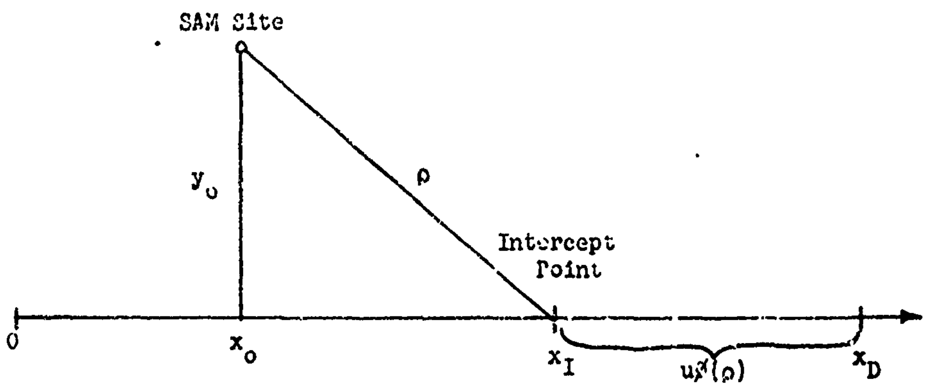



Figure 47

This pair of simultaneous equations can be solved for $\rho$ (and $x_I$), since the target's current position $x_D$ is known.

Having determined $\rho$, we can then obtain the value of the kill probability applicable to this engagement from $p(\rho)$. To ascertain whether the engagement results in a kill, we draw one or more random decimal digits, dependin upon the accuracy to which the kill probability is known, and compare with the kill probability. The use of decimal digits is more convenient than dice, since we customarily work in the decimal system of numbers.

We next record a decrease of one in the SAM unit's missile supply, in the SAM unit store, and an increase of one in the selected target's current engagement status, in the target store. Finally, before proceeding to the next event in the event store, we must generate an engagement completion event for the target in question, including a record of the outcome of the current engagement, and a new target assignment event for the SAM unit involved, unless its missile supply is now exhausted. The time of the engagement completed event will be the engagement time $\emptyset(\rho) + t_a$ later, and the time of the new target assignment event will be the designation cycle time $T_D$ later.

From equation (13) we see that $T_D$ may depend on the range of intercept of the next target to be designated, if the firing rate is limited by availability of a guidance channel. But in the simulation, at the time of one target designation the next target to be designated will not as yet have been selected, so intercept range for the next target cannot be known and $T_D$ may not yet be determinable. However, the simulation calls for an estimate of $T_D$ at this time; and so we must adopt an approximation which is either independent of range or is dependent only on the range of intercept of the target about to be engaged. Such an approximation requires that some estimate of the expected order in which targets will be engaged be beforehand. A convenient and simple assumption is to suppose, <u>for the purpose of defining $T_D$</u>, that the next intercept will occur at the same range as the current one. Equation (13) then reduces to

$$T_D(\rho_1) = \max \begin{cases} T_L \\ T(\rho_1) \end{cases}$$

Other choices can be made to reflect the fact that under the "nearest currently least engaged" doctrine, target engagements tend to move back through the attack stream as the battle progresses. We can assume that the next target to be designated is the next target in the stream. If there are B SAM units, the next target for a particular SAM unit is the B-th target farther back, implying that the SAM units take the targets in turn. Or we can assume that the next target is at an expected arrival spacing ps behind, i.e., at a spacing s if the earlier intercept produces a kill (with probability p), and at spacing 0 if no kill (with probability 1-p). If there are B SAM units, we take instead an expected spacing Bps. These choices all imply that the effect of occasionally moving forward in the stream to pick up a survivor of an earlier engagement is about balanced by the effect of moving backward following this picking up.

If no target was designated to the SAM unit because no target was eligible at the time of the target assignment event, the target store is scanned to determine if some target will become eligible at a later time; i.e., if there is a target located beyond the interval $\Delta x$ associated with the SAM unit in question. If one or more such targets can be found, a new target assignment event is generated for the SAM unit at the time the first of these targets may become eligible. If no such target can be found, no further target assignment event is generated for the SAM unit; rather the unit is retired from action.

At the start of the battle, a target assignment event for each SAM unit is placed in the event store; the time ascribed to each of these events is the time when the lead target reaches the position $x_c$ associated with that SAM unit, i.e., each SAM unit actively enters the battle in time to intercept the lead target at maximum intercept range, should the firing doctrine dictate such a target assignment. Since each target assignment event gives rise to another target assignment event for the same SAM unit, unless the unit's missile supply has been exhausted, each SAM unit then remains actively participating in the battle so long as it is able.

Engagement Completion Event

If the engagement just completed resulted in a kill, this fact is now recorded in the target store, making the target ineligible for further assignment. If the verdict was no kill, the engagement status of the target is reduced by one in the target store. Determination of whether the engagement would or would not result in a kill could have been delayed until this time. However, the range of intercept would then have had to be recorded to allow us to obtain the appropriate value of $p(\rho)$.

Delayed Launching Event

A delayed launching event is generated only when a target, currently in the interval $\Delta x$, is designated to a SAM unit. At the time of designation the target may have been under engagement by another SAM unit, located farther down range or employing longer range missiles and so able to engage the target earlier. Thus the target may have been killed and the verdict to this effect rendered during the time between designation to the SAM unit in question and the delayed launching event. Because of this fact the first operation called for by the delayed launching event is to check the target store to see if the target is now known to be dead. If the target is not found to be dead, we follow the sequence of operations described under target assignment event, starting with calculation of the intercept range. If the target is found to be dead, we attempt to select another target in its stead, again following the sequence of operations under target assignment event, but now starting with the scan of the target store.

Weapon Release Passage Event

The one reason for including weapon release passage events in the simulation is to provide a means for terminating the battle when the number of live penetrators meets the demands of the measure of effectiveness being computed, e.g., one penetration if the measure is the probability of at least one penetrator. The first operation associated with each weapon release passage event is to check the target store to see if the target in question is live and not a decoy (if decoys are included in the attack). If the target is live, a check is then made to see if the total number of live penetrators is sufficient to meet the demands of the measure being computed; if so, the battle is halted. If the target is dead, is a decoy, or if it is live but the number of penetrators is not sufficient, a new weapon release passage event is generated for the next target in the stream at the time it will reach weapon release. At the outset of the battle, a weapon release passage event for the lead target is placed in the event store.

F. Modification to the Simulation

The simulation as described permits a wide variety of applications without modification. The locations and number of SAM units comprising the defense, the initial ready missile supply, the maximum intercept range, the kill probabilities and descriptive times of each are open parameters. The speed, number, arrival spacing, and weapon release position of the attacking aircraft are open parameters. Thus a wide variety of defensive missile systems, even mixtures made up of different systems can be simulated, as can many different attacks.

The simulation limits all attacking aircraft to the same constant speed and straight line attack path. These limitations can be relaxed in any of several ways, usually at the expense of some added complexity in the simulation. A curved or dogleg attack path, the admission of two or more distinct paths, or a variation in target speed along a path will necessitate some change in the calculation of intercept position. If different targets are permitted to fly at different speeds, an alternative definition of "nearest" is required; two common choices are (1) "target with least time to go to intercept", and (2) "target with least time to go to weapon release". Aircraft at different altitudes may have different maximum intercept ranges and different weapon release points.

The structure of the simulation is such as to readily permit changes, particularly in the firing doctrine used. As an example, suppose the attacking aircraft approach in a group, flying sufficiently close together to appear as a single unresolved track to the defensive search radar. The "nearest currently least engaged" doctrine will not provide pairings of SAM units with individual targets; a different doctrine must be used. Let us assume that selection of an individual target is a random process, all live targets having equal chance of being chosen, and no choice being made unless all targets are within range. We replace the target assignment event described above with a random target assignment event. The first operation called for by this new event is to draw a random number (an integer) between one and the number of live targets in the group at the time of the event. We count through the live targets in the target store until we reach the random integer; the corresponding target is selected. Following selection of a target the remaining operations are carried out as before; the intercept range is calculated, occurrence of a kill is determined, the ready missile supply of the SAM unit is decreased by one, and an engagement completion event and a new random target assignment event are generated for the target and SAM unit respectively. Of course no consideration is given to the interval $\Delta x$, nor to the current engagement status of any target. Furthermore, the delayed launching event is not needed. No other changes in the simulation are needed.

Another doctrinal change of interest can easily be incorporated in the simulation. In some situations it may be desirable to regard targets that have penetrated to rather short range as being especially threatening and so warranting heavier fire than the "nearest currently least engaged" doctrine would provide. To provide heavier fire at short range, we define a priority zone extending from weapon release position $x_b$ to an outer boundary $x_p$. We give priority to a target within this zone if the number of engagements it is currently undergoing is less than a specified maximum $K_p$. Our firing doctrine becomes a sequential application of the "nearest currently least engaged" doctrine, first to targets in the priority zone whose current engagement status is less than $K_p$, and then, if no target has been selected, to all targets whether in the priority zone or in the non-priority zone lying beyond $x_p$, as before without regard for the upperbound $K_p$. This change in the target assignment event scan is all we require to have a simulation capable of heavier fire at short range. How much heavier the fire shall be is controlled by the choice of $K_p$, and what constitutes short range is controlled by the choice of $x_p$.

A defense made up of both long and short range SAM units may prove more effective if, early in the battle, a limited number of targets are permitted to penetrate to the firing zone of the short range SAM units, and the long range SAM units meanwhile engage targets farther back in the stream. In this way the short range units actively enter the battle at the earliest possible time, rather than remain idle while the long range units bear the burden of the defense. Clearly the number of targets permitted to penetrate must be carefully limited to insure that the short range units will be able to handle them satisfactorily. One way to achieve this objective is to modify the "nearest currently least engaged" firing doctrine by declaring certain targets (e.g., every third target) ineligible for assignment to long range SAM units. Or the long range units could be limited to a single engagement per target for any target not yet within range of a short range unit.

## G. Critical Examination of Monte Carlo Method

A discussion of the Monte Carlo method would not be complete without at least a brief critical review of its strength and weaknesses; such an examination follows. The particular air battle simulation described above will serve to illustrate several of the points to be made.

The validity of a measure of effectiveness, calculated by the Monte Carlo method, is governed by the appropriateness of the air battle model used in the simulation, the validity of the parameter values selected to describe the attack, the defense, and their interactions with each other and with their environment, and the sample size, i.e., the number of times the air battle simulation is repeated with the same set of parameter values. Of course the need for an appropriate model and for valid parameter values obtains whatever method may be employed in calculating the measure. The suitability of the model and the accuracy of parameter values must ultimately depend on the analyst's knowledge of the defensive missile system, of the attack, of the environment, and of the interactions of these. The use of an overly complex model or of parameter values unjustifiably quoted to many digits cannot compensate for a lack of knowledge.

An air battle model appropriate to the Monte Carlo method should include the simulation of all features of the air battle pertinent to the measure of effectiveness being computed, and should exclude all features of the air battle irrelevant to the problem being investigated. It should be pointed out that the popular desire for realism in models of various sorts of battles is not well founded. A model by definition is not reality; it is a model of reality. No amount of trimming in the form of simulation of irrelevant features can make a model real; nor will it enhance the model in any constructive way. The oft expressed concern for realism is in truth a desire to incorporate in a model all that is pertinent. However, in attempting to fulfill this desire, a clear notion of what various features should be pertinent or relevant <u>to</u>, is essential.

Use of the Monte Carlo method demands that the size of the sample be sufficient to yield statistically meaningful results. Errors from statistical fluctuation can be reduced to any desired level by increasing the sample size. However, no advantage can accrue from making the statistical errors very small compared to uncertainties concerning features of the model or values of the parameters used. Indeed, such action may, and sometimes does, lead to false accuracy. A small statistical error may suggest a high level of accuracy in the results that does not in fact exist. Should uncertainties concerning the model or parameter values be too great, application of the Monte Carlo method will seldom be warranted. Some simpler form of analysis will usually suffice.

The need for an adequate sample size in the Monte Carlo method argues for an air battle model so designed that its running time will be short. The time required to obtain a sample of given size is of course proportional to the running time per simulated battle, and in consequence the cost of calculation is very nearly proportional to running time. Whether calculations are made by hand or on a high speed computer, their cost is usually a matter of major concern. Thus any economy that can be effected by shortening the running time is important.

The structure of an air battle model should be simple enough that the simulation can be thoroughly understood as a whole. Without this complete understanding, the problems of detecting errors in design or operation of the simulator and in resolving the apparent anomalies in results that almost invariably arise are extremely difficult.

The above requirements for an air battle model are often in conflict. Even after due heed has been paid to the question of relevance, so many features of the battle may appear to be essential that the model becomes too complex to make a satisfactory sample size practicable or to permit a simultaneous grasp of all details of the simulation. This dilemma can be resolved by resorting to the sound expedient of subdividing the model. A model is made up of many blocks joined together by prescribed functions or operations. each block in turn can be constructed of many smaller blocks similarly joined. For example, the particular simulation described earlier consists of an initial set of conditions and a sequence of events that operate on these initial conditions. One operation occurring repeatedly is that of determining whether an engaged target is killed or not The scheme used for carrying out this operation calls for a random number to be drawn as each engagement takes place; this number is then compared with a previously selected and stored kill probability to determine kill. In an actual firing of a missile against a target a kill would or would not occur depending upon the outcome of each of an extensive sequence of events described in Chapter III. These outcomes include miss distance, fuze triggering position, fuze delay, and burst position. Since kill probability is pertinent to the measure of effectiveness (penetration probability) being calculated with the aid of the simulation model, the many variables upon which it depends are also pertinent. But even though they are pertinent, explicit treatment of them need not be incorporated in the simulation model. They can be more satisfactorily dealt with separately by one of the methods discussed in Chapter III to provide kill probability as a function of intercept position and target type. More generally, every operation included in the model described earlier could be expanded into a sequence of sub-operations. In turn, each sub-operation could be expanded until, if all ultimate sub-operations were incorporated in a single model, the model would include even the detailed simulation of each radar pulse. Clearly a model incorporating this great detail would become so complex as to be completely unmanageable. But if the model is subdivided, the expansion of each operation being treated separately and only the results of the separate treatments being incorporated in the model, excessive complexity can be avoided.

At best, calculation of a measure of effectiveness by means of the Monte Carlo method is expensive and time-consuming when compared to calculation by any of the formulas of Chapter V. Justification of the Monte Carlo method rests on the inability of the formulas to properly account for many firing doctrines of interest. However, the formulas can often be of considerable value even when firing doctrine is a matter of concern. Once a program of simulation has been carried out, the results obtained can be compared with those provided by the formulas. A formula giving approximately the same results can sometimes be found, and the sets of conditions which constitute bounds within which the approximation is valid can be defined. Within these bounds the formula can be used on later occasions with reasonable confidence.

The need for an adequate sample size will usually dictate the use of a high speed digital computer in carrying out air battle simulation. A computer simulation commonly involves a time-consuming programming task at the outset, but can later pay off handsomely in the form of rapid computation. In contrast, manual simulation requires little initial preparation, but usually is very slow and tedious in operation, and is apt to introduce computational errors; the human operator is not a particularly reliable computer, especially when he is tired.

For all the shortcomings of a manual simulator, it has an important role to play in the simulation of an air battle. In the process of designing a simulation model many features of the design must be tested to determine if the logic of the model is self-consistent, to detect those features and assumptions that are in conflict with the characteristics and interactions of the air battle being simulated, and to discover important omissions in the model. A manual simulator is well suited to this preliminary testing; since it can be quickly and easily prepared for operation, desirable and even necessary changes in the design of the model can then often be discovered in the course of a very few simulator runs, and these changes can readily be incorporated and tested in turn.

The air battle simulation described above was originally designed for a manually operated desk top simulator, and later was programmed for a high speed digital computer. In both forms it has been used extensively in the tactical analysis of SAM systems; during the period of its use many changes and innovations have been introduced. The manual simulator has been a valuable tool not only in working out the details of these changes, but in a number of other ways as well. Experience with the manual simulator has shown that, in the course of a single run, inconsistencies and invalid features in the model not otherwise easily recognized could frequently be observed. A single run often served to demonstrate a previously unsuspected gross weakness in a proposed firing doctrine, and suggested modifications to the doctrine which would strengthen it. Indeed, as a consequence of a single run, an important interaction between elements of the air battle - one that was not evident before - was uncovered in each of several instances. The information gained from a single run was not derived from recorded results of the run - since a sample of one run is completely inadequate from a statistical point of view. The value of the single run stemmed rather from the fact that the operator could observe the sequence of events as they occurred, and often could note why things turned out as they did.

The ability of the operator of a manual simulator to watch the entire battle as it evolves and in so doing to discover why things happen as they do, makes the manual simulator a useful tool in interpreting the results of a Monte Carlo calculation. In many instances the plot of a measure of effectiveness, obtained by the Monte Carlo method, against some parameter of interest (e.g., spacing between successive target arrivals) will behave in a most unexpected manner. A few runs on a manual simulator may quickly provide an explanation for their behavior.

In some instances a measure of effectiveness may be expected to change abruptly as some parameter of interest is varied. If an investigation of such change is to be made, the manual simulator can be useful for calculating purposes. As an example, the plot of penetration probability (the probability that at least one attacker will survive) versus the number of attackers frequently is quite steep, as illustrated in Figure 43.

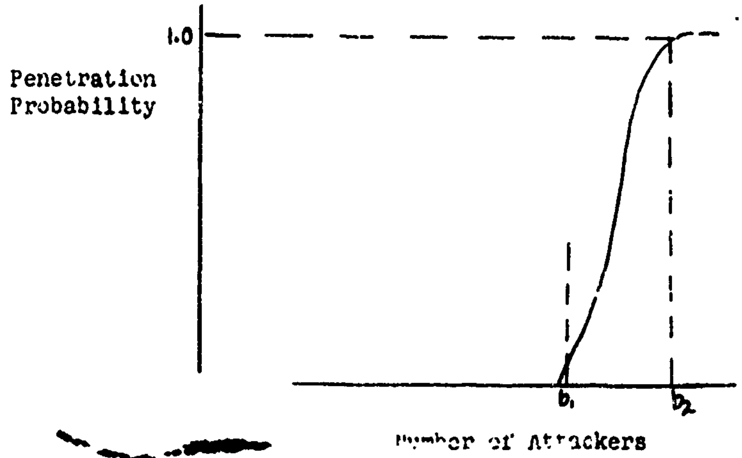


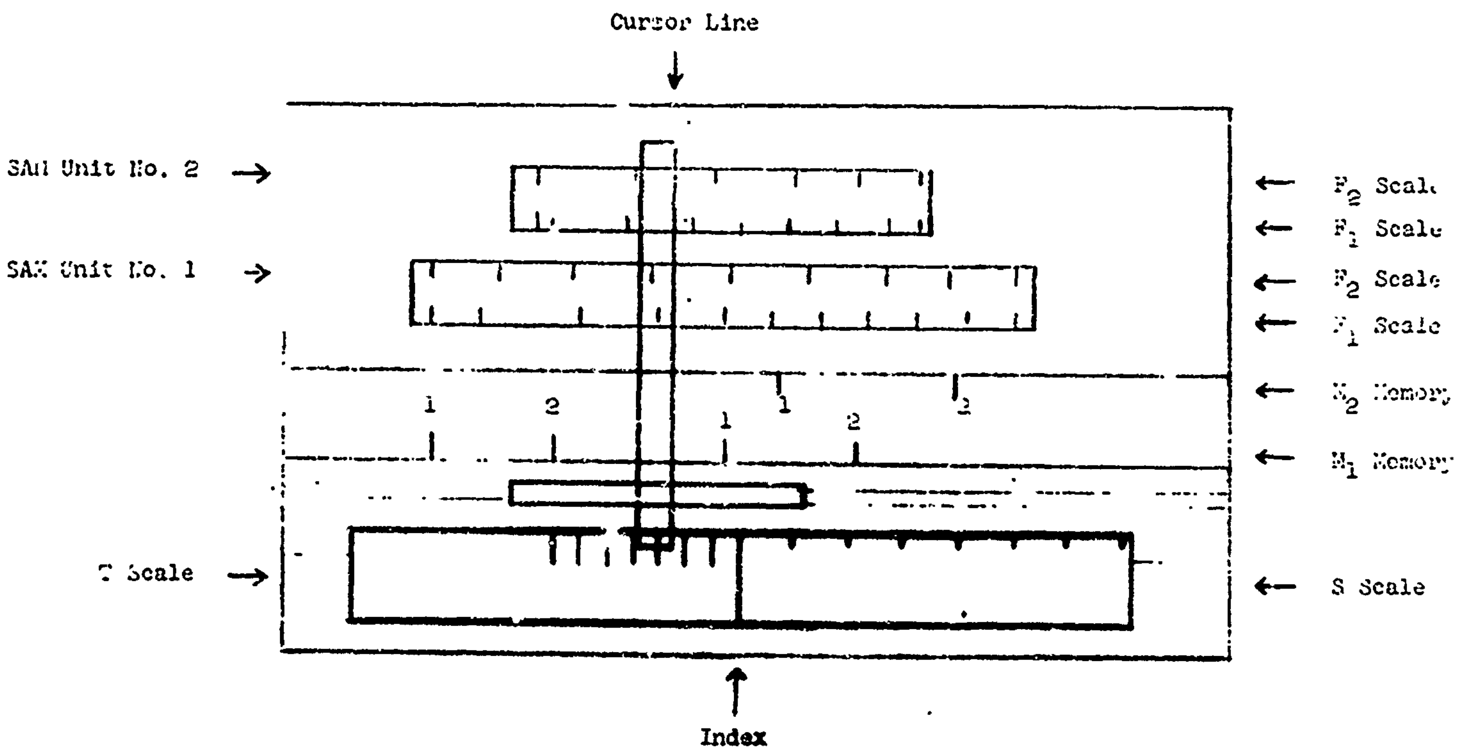

Figure 48

The information of primary interest may be merely the approximate number of attackers at which the curve rises abruptly, rather than the precise value of the penetration probability for any particular number of attackers. The problem then is to locate the "region of interest" ($b_1$ to $b_2$ in the figure), and not to accurately compute points of the curve. In such case relatively few runs on a manual simulator can suffice. For any number of attackers less than $b_1$, a single run will show that only with great good fortune will an attacker penetrate, and for any number of attackers greater than $b_2$, a single run will show that the defense is simply unable to engage all targets, thereby insuring a penetration.

A final important consideration in the application of the Monte Carlo method is the design of the simulator itself, and the choice of specific procedures to be followed in carrying out calculations and other operations, and in recording information of interest. Desirable features for any simulation include efficiency and reliability in carrying out calculations, computing accuracy commensurate with the accuracy of data used, facility in the detection of errors, and speed of initial preparation. Often these features will not be wholly compatible; some compromise must be made. However, general statements on this subject are not apt to be instructive; greater insight into the design problem can be obtained from a description of a particular simulator and of its operation. The manual simulator for which the air battle simulation described above was originally conceived will serve as a useful illustration.

## H. A Manual Simulator

The manual simulator is a simple analog computer of the slide rule variety, made up of two scales on a slide, several fixed scales, a cursor and a memory. Figure 49 shows a sketch of the simulator; the letters in the following description refer to the figure.




MANUAL SIMULATOR

**Figure 49**

On the left hand scale T of the slide, the relative target positions are marked off according to any desired distribution (a uniform distribution with constant target spacing is depicted in the figure). The right hand scale S of the slide gives the distance a target advances as a function of time of advance, the index or origin of the S scale is located at the lead target. The fixed scales $F_1$ give engagement time ($\phi + t_a$) versus target intercept position for each SAM site and the fixed scales $F_2$ give the designation cycle time $T_D$ versus target intercept position for each SAM site. A fixed memory scale $M_1$ serves both as a store of future target assignment events and as a record of all engagements previously undertaken by each SAM unit. The fixed memory scale $M_2$ serves as a store of future engagement completion events and as a record of all engagements of each target. The scale T of the slide, where relative target positions are marked off, is used to record the current engagement status of each target and also the demise of each target killed.

Preparation of the simulator for a particular set of runs consists of drawing up appropriate fixed scales $F_1$ and $F_2$ for each SAM site and scales S and T of the slide. The first step is to select a suitable distance scaling factor. The choice should minimize the real distance per unit simulator distance (e.g., miles per inch), while still including the distance intervals required for operation of the simulator. These adjacent but not overlapping intervals are

a) The distance between the first possible intercept position of a target by any SAM unit and the corresponding target position at designation.

b) The distance between the first possible intercept position and the last possible intercept position, the latter often being the weapon release point.

c) The distance between first and last targets in the attack stream.

The fixed scales $F_1$ and $F_2$ for each SAM unit can readily be laid out as follows. We have (repeating equation (19))

$$x_I = x_D - u\phi(\rho)$$

$$\rho = \sqrt{(x_I - x_o)^2 + y_o^2}$$

where $x_I$, $x_D$ are intercept and designation positions respectively, $(x_o, y_o)$ are SAM site coordinates, $\rho$ is range to intercept from SAM site, $\phi(\rho)$ is tie-up time, and u is target speed. Eliminating $x_I$, we have

$$x_D = x_o + u\phi(\rho) \pm \sqrt{\rho^2 - y_o^2} .$$

For each of several convenient values of $\rho$ covering the interval between $\rho_{max}$ and $\rho_{min}$, we calculate $x_D$; both or just one value of $x_D$ will apply depending upon where the position of last intercept falls relative to the SAM site. For each of the same chosen values of $\rho$ we obtain the corresponding values of engagement time $\emptyset(\rho) + t_a$ and designation cycle time $T_D$ and then plot $\emptyset + t_a$ and $T_D$ versus $x_D$, using the distance scaling factor selected above. We select several values of $\emptyset + t_a$ and of $T_D$ covering the intervals between $(\emptyset + t_a)_{max}$ and $(\emptyset + t_a)_{min}$ and chosen to permit ready interpolation by eye. We mark off these chosen values on the $(\emptyset + t_a)$ and $T_D$ axis, and for each value then mark off the corresponding one or two values of $x_D$. Finally we place a paper strip along the x axis, transfer the markings of the $x_D$ values to the strip, and record the corresponding $(\emptyset + t_a)$ and $T_D$ values opposite each mark to give us the completed scales $F_1$ and $F_2$ for the SAM site in question. In the event that $T_D$ is constant over some $x_D$ interval, as in the case of launcher governed fire, the fact that the same value of $T_D$ applies to the whole interval should be indicated on the scale $F_2$.

When the strips bearing the scales $F_1$ and $F_2$ for different SAM sites are mounted on the simulator, they should be placed so as to coincide in position $x_D$ rather than in time $\emptyset$ or $T_D$. Indeed two SAM units whose locations differ will have different values of $\emptyset$ and usually of $T_D$ corresponding to the same designation position $x_D$. Furthermore, positions of first and last designation for two such SAM sites will not in general coincide, even though positions of first and last intercept may coincide.

The scale T of the slide is readily prepared by marking off positions of successive targets as desired, using the selected distance scaling factor. Note that the relative positions of targets remain unchanged during the battle, since all targets are assumed to be flying at the same constant speed. The scale S of the slide is obtained by choosing a convenient time interval $\Delta t$, and marking off a series of points a distance $u\,\Delta t$ apart, labeling them 0, $\Delta t$, $2\Delta t$, ... up to the maximum value of $\emptyset$ or $T_D$ occurring on any of the scales $F_1$ and $F_2$.

The operation of the simulator starts with entry on the memory $M_1$ of an initial target assignment event for each SAM unit, and a weapon release passage event for the lead target. The target assignment event for each SAM unit is entered by placing the cursor at the left-hand end point of the scale $F_1$ of that SAM unit, and then, holding the cursor fixed, making a mark on the memory $M_1$ and labeling the mark with the number of the SAM unit. The weapon release passage event for the lead target is entered by the following sequence of steps. The cursor is placed at the right-hand end point of the scale $F_1$ of any SAM unit (it is immaterial which SAM unit is chosen) and the value of $(\emptyset + t_a)$ is read off. Holding the cursor fixed, the index of the scale S is moved to coincide with the cursor. Then, holding the slide fixed, the cursor is moved to the right to the point $\emptyset + t_a$ on the scale S, a mark is made on the memory at this point and is labeled WR (for "weapon release").

Following entry of the above events, the cursor is placed at the left most mark on the memory $M_1$ (the first target assignment event to come up). The slide is then moved so that the first target to be engaged, e.g., the lead target, falls at the cursor line. The target-SAM unit pairing for the first engagement has now been fixed. To complete the engagement, the designation cycle time $T_D$ and the engagement time $(\emptyset + t_a)$ for the engagement are read from the points where the cursor line cuts the scales $F_1$ and $F_2$ associated with the SAM unit in question. Holding the slide fixed, the cursor is moved to the right to the point $T_D$ on the scale S, a mark is made on the memory $M_1$, and the SAM unit number is recorded above the mark. This mark constitutes entry of a new target assignment event for the SAM unit. Next the cursor is moved to the point $(\emptyset + t_a)$ on the scale S, a mark is made on the memory $M_2$, and the target number is recorded below this mark. This mark constitutes an engagement completion event for the engagement being undertaken.

Next, a random digit is drawn to determine if the target has been killed. If a kill results, the target number entered on the $M_2$ memory is underlined or encircled. Finally, on the scale T a check mark is placed over the target being engaged to indicate that the target is now undergoing an engagement, and the recorded number of ready missiles available at the SAM unit is reduced by one. The operations called for by the first target assignment event have now been completed.

The next event to occur is that corresponding to the left most mark on either the $M_1$ or $M_2$ memories, excluding the event just passed. If the next event is a target assignment event, the cursor is placed at the corresponding mark in the memory, the slide is advanced until the index of the scale S is at the cursor line, and a target is assigned according to the doctrine being used. Holding the slide fixed, the cursor is moved to the target selected. If the cursor in this position falls to the left of the end-point of the $F_1$ scale, the SAM unit must delay its launch. A delayed launch event for this SAM unit is then entered on the memory $M_1$ by reading off the time spacing between the lead target and the selected target, moving the cursor to that time on the scale S while holding the slide fixed, making a mark on the memory $M_1$, and labeling the mark with the SAM unit number. If, on the other hand, the cursor falls at or to the right of the end-point of the $F_1$ scale, the engagement of the selected target need not be delayed. The operations described for the first engagement above, subsequent to target-SAM unit pairing, are then followed.

If the next event is an engagement completion event, the number of the target involved is read off the memory $M_2$. If the target was killed, an appropriate mark is placed on the scale T over the target. If the target was not killed, the check mark previously placed on the scale T is removed or crossed out to indicate a reduction by one in the target's current engagement status.

The battle thus proceeds, event by event, until all targets have been killed or all missiles exhausted, or until the weapon release passage event for the lead target comes up. No explicit operation is called for at this time; the significance of this event lies in the additional operation that precedes each subsequent target assignment event. Before selecting a target for assignment, the live targets lying to the right of the WR mark on the memory $M_1$ must now be counted; when this number equals the desired number of penetrators, the battle is terminated.

# CHAPTER VII

## Measures of Effectiveness

### A. Choice of a Measure

A measure of effectiveness of a weapon system is commonly thought of as a quantitative index of the tactical worth of the system. Before a proper measure of effectiveness can be selected for use in an analysis, it is essential to understand clearly the aims or objectives of the weapon system. Ideally one would like an air defense system to be capable of destroying all hostile air targets presented to it. This lofty aim is of course not achievable, which fact has erroneously suggested to many that measure of effectiveness should mean the degree to which the ideal goal is achieved. But such a view is meaningless without considerable elaboration of the purpose or objective of the air defense being studied. In one instance, to destroy all but one attacking weapon may mean utter defeat. Yet in different circumstances to destroy but a few targets may suffice for victory. A few illustrations will help to throw light on the importance of air defense objectives in selecting a measure of effectiveness.

Consider a small military force whose immediate mission is vital to the success of a major campaign in a war. The elements of the force may possess some inherent value independent of the mission, but the urgency of the mission temporarily places greatly increased value on the force, lasting until the mission has been completed. Suppose the force expects to be subjected to air attack and so possesses elements to provide it with air defense. The prime objective of the air defense is clearly to prevent damage from air attack that would preclude success of the mission. This objective holds sway only during prosecution of the mission. Activities of the air defense may include destruction of enemy air targets, but destruction of targets is not essential to the air defense objective. To cite an example, successful jamming of bombing radars carried by bomber aircraft could constitute completely successful air defense without a shot being fired. The one paramount requirement is to prevent, by any means, successful delivery of enemy warheads. Whatever form the air defense takes, it fails if enemy air attack prevents the force from carrying out its mission. Otherwise the air defense is successful.

The strategic bombing operations of World War II serve to illustrate a different air defense objective. In these bombing raids the damage inflicted by any one bomber on a single mission was usually rather meager. Indeed, destruction caused by an entire raid was seldom really crippling to the nation receiving it. Thus it was necessary to attack and re-attack each of many targets time and time again, with large numbers of bombers, in order that the bombing operations be effective. The most telling means open to the defense for discouraging these bombing operations was to destroy bombers. Moreover, experience showed that a modest level of bomber attrition (5 to 10%) was usually sufficient to dissuade the enemy from continuing his bombing efforts without first taking steps to negate the air defense fo[illegible] opposing him. Thus, in these operations the primary air defense obje[illegible] was to exact enemy bomber attrition, the higher the level of attriti[illegible] the more pleasing to the defense.

A third air defense objective can be illustrated [illegible] the defense of cities or other civil targets against nuclear air attac[illegible] A single offensive nuclear warhead [illegible] delivered may cause [illegible]emendous devastation. Thus if an [illegible] should attack a city with enough weapons to assure with high confidence the delivery of two or three warheads, he could fully [illegible] attack to succeed, and would rarely have to re-attack. In [illegible] attack a large part of the attacking force may be lost to air defense action. Indeed if an enemy expects such losses to be high - or at least not negligible - he should allocate to the one attack not only enough weapons to surely absorb the expected losses, but also enough more to insure destruction of the city, rather than allocate fewer weapons and so run a serious risk of having to re-attack another day, with consequent additional losses to air defense action. Having paid his admission price once, he had best see the show rather than have to pay again. The objective of the air defense in this case is to exact a high toll of air targets destroyed.

A further difficulty in selecting a measure of effectiveness arises from the fact that an air defense weapon system may at different times be employed in different ways with different objectives. A naval missile ship in a major nuclear war may have the prime task of protecting a carrier during the short time required for the latter to launch an air strike. In a high explosive war the first objective of the same missile ship may be to exact attrition of attacking aircraft during an extended period when the fleet may be operating within range of enemy air attacks. In carrying out a tactical analysis of the performance of such a missile ship some compromise may be called for in choosing a measure of effectiveness, or resort to two or more measures appropriate to the different potential objectives may be necessary.

In choosing a measure of effectiveness, the tactical analyst must pay heed to certain practical considerations. He is often limited in the time available for his analysis, or in the computational facilities at his disposal, or both. If so, he must limit his choice of measure to those amenable to fairly rapid and simple calculation, striking the best compromise he can between appropriateness to the problem and practicality in computation. Some of the data available to analysts as inputs are usually of such poor quality as to be little better than random guesses. Given great uncertainty in the proper value of a parameter, it becomes necessary to determine the sensitivity of the SAM system effectiveness to the parameters. Is effectiveness critically dependent on the parameter value used? If so, where in the range of possible values does effectiveness vary markedly? The desirability for computa[illegible]ional ease gains added emphasis from this need for ascertaining sensitivity.

Consideration of the purpose to which a tactical analysis is to be put can at times simplify the choice of an effectiveness measure. Frequently the purpose is to make a comparison between weapon systems. If two systems being compared are sufficiently similar, it may be admissible to employ a measure that reflects only those features of the two systems that differ. By way of illustration, suppose two SAM systems have about the same missile kill probability and are capable of about the same level of coordination of fire. The consideration of firepower achievable by each may suffice in making a comparison. However, resort to the use of such a partial measure should be made with caution. The analyst should himself be aware and in the presentation of his results should include some indication of how overall effectiveness of either system will vary with firepower. In short, if one system can achieve twice the firepower of the other, is it then twice as effective, four times as effective, half again as effective? If this question is left open, the inclination of the reader will usually be to interpret firepower as proportional to overall effectiveness and so, in our example, to assume the one system to be twice as effective as the other. The analyst should also satisfy himself that the systems are indeed similar. Two SAM systems differing only in maximum range will obviously have different firepower capabilities. They may also differ in the coordination they can achieve because of their different depths, even though they employ identical firing doctrines.

Regrettably, parameters describing the performance of a weapon system such as range, speed and altitude ceiling, are all too often used as measures of effectiveness. Such use fails to recognize the need for matching performance with opposing performance in ascertaining effectiveness. The ability of a SAM to fly at 100,000 ft. is of little consequence if no target for it will be found there. Nor does the use of performance parameters to measure effectiveness recognize the importance of how the weapon system will be employed. A sparse, distant deployment of SAM units about a surface target will not provide a good defense of that target against low altitude attacks.

## B. Useful Measures of SAM Effectiveness

We are concerned here with measures of effectiveness for SAM systems. The preceding discussion demonstrates that no one general or universal measure can be found. The best that we can do is to present a few measures that can themselves be employed in a fairly wide variety of problems, or from which other suitable measures can be derived.

A SAM system is designed to destroy air targets. Whatever the nature and mission of the military force of which the SAM system is a part, the contribution to air defense of the SAM system is through its ability to destroy targets. In the first illustration of the preceding section (a military force on a critical mission), the SAM role is to kill all targets not otherwise countered. A reasonable measure of SAM success is the probability of annihilation of all of these targets in any one raid. In the second illustration (World War II bombing) the SAM role is to exact bomber attrition day after day. A good measure of success here is the expected number of kills achieved against a raid. In the third illustration (nuclear strike) the SAM role is to take a high toll of bombers during the attack. Success in this instance can be measured by the attack force needed in a raid in order to achieve a high probability of penetration.

In each of the above illustrations a battle is presumed to take place. Of importance, too, is the deterrent role of the air defense, and the SAM contribution to it. In each of the above illustrations a credible threat that the SAM system will perform well if called upon may deter an enemy from attacking. Thus the same measures of effectiveness will serve, but the emphasis now will rest on what the enemy may believe the SAM system could do, rather than on the system's true capability. (The two may, of course, be the same.)

Although the three measures: annihilation probability, expected number of kills and penetration probability are by no means the only possibilities, one or another of them will serve adequately in many analyses.

For any postulated battle situation, there is a probability $P_k$, however poorly known, that the SAM system will kill precisely k out of b targets. Each of the above measures can be expressed in terms one or more of the $P_k$'s:

Annihilation probability is $P_b$; i.e., $P_k$ with k set equal to b.

Expected number of kills is

$$E = \sum_{k=0}^{b} k\, P_k \quad .$$

Penetration probability (probability that the number of survivors will be at least as great as a specified number a) is

$$Q_{s \geq a} = \sum_{k=0}^{b-a} P_k \quad .$$

A number of other useful measures are also functions of the $P_k$'s; the probability $P_D$ that an air attack will succeed in destroying a defended surface target is given by

$$P_D = \sum_{k=0}^{b} D(b-k)P_k$$

where $D(j)$ is the conditional probability that the surface target will be destroyed if $j$ attackers penetrate the defense. (Obviously $D(0)=0$.) As was shown in some of the derivations of Chapter V, and in the discussion of Monte Carlo simulation of Chapter VI, a measure can sometimes be calculated without explicit calculation of the $P_k$'s.

A brief discussion of the behavior of the various measures, and of certain approximate relations between them may be helpful. Figure 50a shows plots of penetration probability $Q_{s \geq a}$ against number of attackers b, for several values of least number of survivors a. The curves were computed for fixed kill probability $p = .5$, fixed firepower $N = 20$, and perfect coordination. Notice that the curves have very nearly the same slopes. The effect of increasing a is to translate the entire curve to the right; moreover, the extent of the translation is just the difference in the values of a. Much the same effect obtains quite generally, although for poorer coordination the extent of the translation is somewhat greater than shown in the figure. Because of this effect, it will sometimes suffice to calculate $Q_{s \geq 1}$ only, and then perform the appropriate translations to obtain $Q_{s \geq a}$ for other values of a. It should be noted that $Q_{s \geq 1}$ bears a simple and useful relation to annihilation probability:

$$Q_{s \geq 1} = 1 - P_b \quad .$$

If the conditional probabilities of destruction $D(b-k)$ are nearly equal to 1, at least where $P_k$'s are not negligible, the probability of defended surface target destruction $P_D$ is approximately the same as $Q_{s \geq 1}$. ($P_D$ is identical to $Q_{s \geq 1}$ if the $D(b-k)$'s are all equal to 1.) Figure 50b illustrates the effect of lower conditional probabilities $D(b-k)$ on $P_D$. These curves were computed for the same kill probability, firepower, and coordination as above, and assuming

$$D(b-k) = 1 - c^{b-k}$$

where c is an arbitrary constant. Notice that as c increases and so $D(b-k)$ decreases (i.e., the enemy weapons are less effective), the slope of the $P_D$ versus b curve decreases. Comparison of Figures 50a and b suggests that only when the $D(b-k)$'s are high will $Q_{s \geq 1}$ serve as a reasonable approximation to $P_D$.

The probabilities $P_k$, and so any measure that depends upon them, are functions of the kill probability p, the firepower N, and the coordination of fire associated with the SAM system in the air battle in question. It is of some interest to note how variations in p, N, and the level of coordination affect certain measures.

Figure 51a shows three plots of expected number of kills E versus number of attackers b, assuming kill probability $p = .5$, firepower $N = 20$, and three different levels of coordination. For sufficiently small b, the defense saturates the attack, and $E \approx b$ independent of the coordination level. Changes in p and N simply alter the definition of "sufficiently small b[illegible]. For suffi[illegible] [illegible] the de[illegible], and E ≈ Np, again independent of [illegible]. In between these limits the consequences of the [illegible] become significant; the higher the level of coordination [illegible] knee of the curve.

Figure 5ib [illegible] plot of [illegible] probability [illegible] versus b for the same situation [illegible] preceding paragraph. As above, for sufficiently small b the [illegible], independent of the coordination level. [illegible] if b is sufficiently large [illegible] ≈ 1, again independent [illegible] the coordination level. In between [illegible] level of coordination achieved assumes importance, [illegible] predominant [illegible]. In the position of the curve; the [illegible] level of coordination, [illegible] further to the right the curve [illegible].

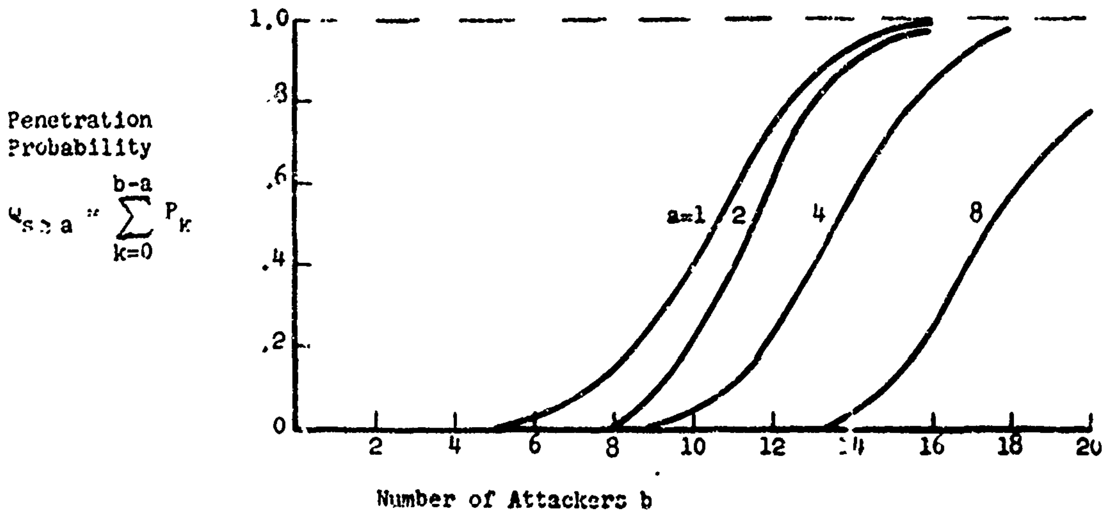



Figure 50a

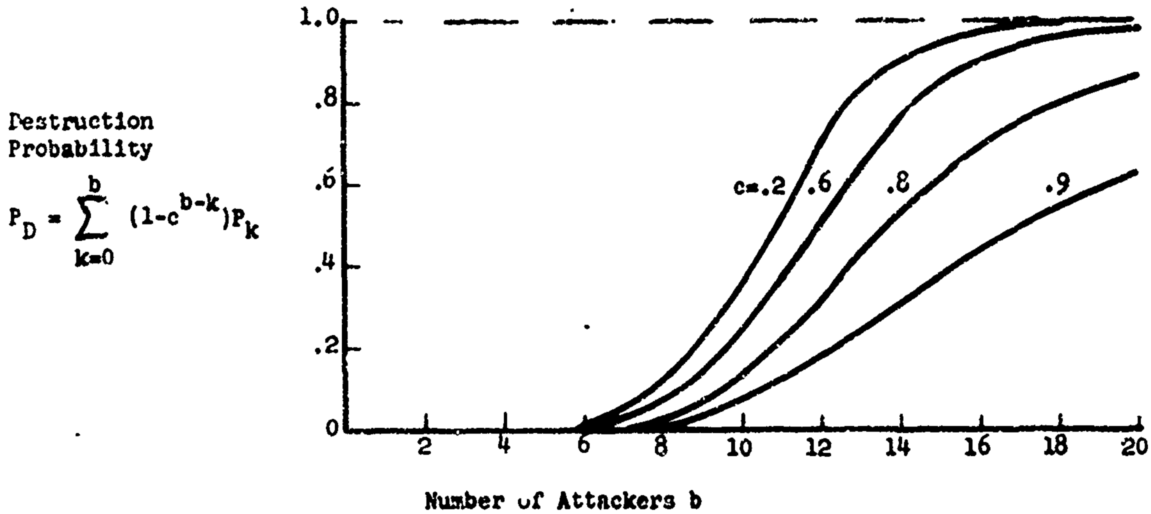



Figure 50b

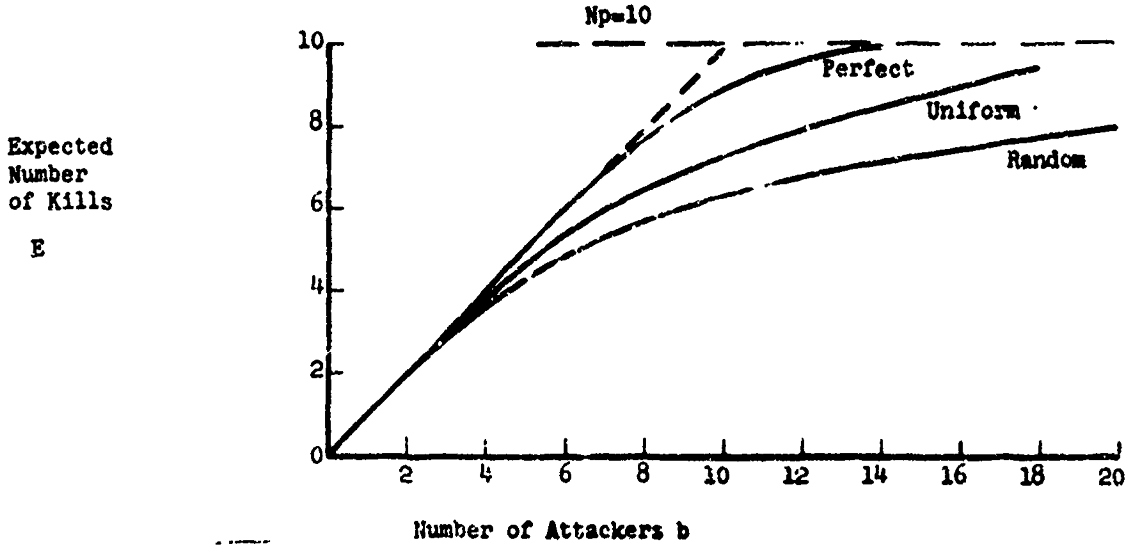



Figure 51a

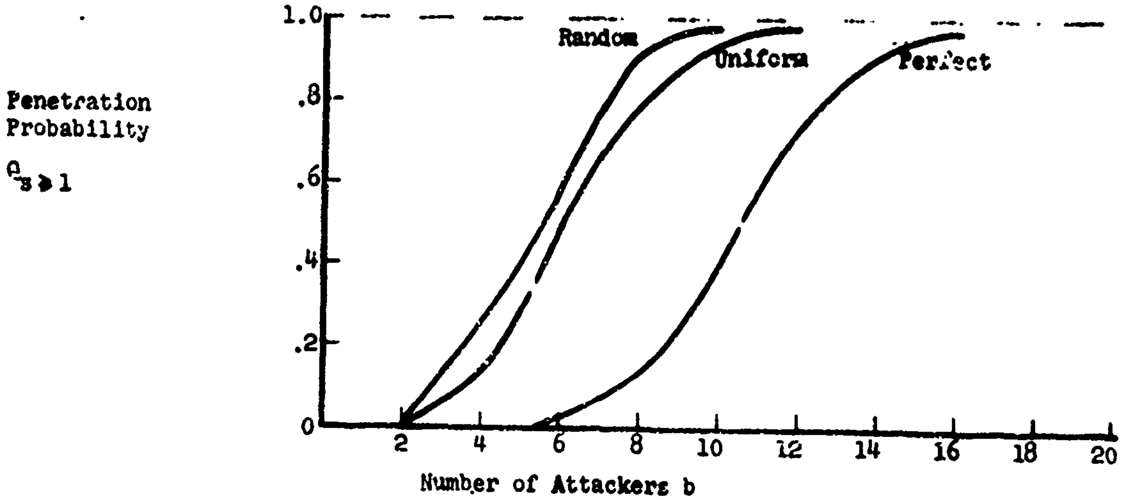



Figure 51b

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