AFOSR 684

CONTRACT NO AF 49(638)-445
PROJECT NO 9781

APPROXIMATE SOLUTIONS FOR REENTRY TRAJECTORIES WITH AERODYNAMIC FORCES

by

Kenneth Wang and Lu Ting

MAY 1961

POLYTECHNIC INSTITUTE OF BROOKLYN

DEPARTMENT
of
AEROSPACE ENGINEERING
and
APPLIED MECHANICS

PIBAL REPORT NO. 647

AFOSR 684

APPROXIMATE SOLUTIONS FOR REENTRY TRAJECTORIES WITH AERODYNAMIC FORCES

by

Kenneth Wang and Lu Ting

This research was supported by the
United States Air Force through the
Air Force Office of Scientific Research,
Air Research and Development Command,
under Contract No. AF 49(638)-445,
Project No. 9781.

Polytechnic Institute of Brooklyn

Department of

Aerospace Engineering and Applied Mechanics

May 1961

PIBAL Report No. 647

APPRCXIMATE SOLUTIONS FOR REENTRY TRAJECTORIES WITH AERODYNAMIC FORCES[+]

by

Kenneth Wang[*] and Lu Ting[**]

Polytechnic Institute of Brooklyn

SUMMARY

When the motion of the reentry vehicle is expressed in two equations for the components in the instantaneous trajectory plane, and one for the component normal to the plane, the former equations are uncoupled from the latter. Thus, the problem of a nonplanar trajectory is reduced to an equivalent planar trajectory.

In the present paper approximate analytic solutions for planar trajectories with constant lift and drag coefficients are obtained by improving and extending the analytic solution of Allen and Eggers, and that of Lees, Hartwig, and Cohen The present solutions are not subjected to the restrictions of the Allen and Eggers solution, which is valid for drag-only vehicles at large entry angle, nor are they subjected to that of Lees, Hartwig, and Cohen's solution, which

---

[+] This research was supported by the United States Air Force through the Air Force Office of Scientific Research, Air Research and Development Command, under Contract No. AF 49(638)-445, Project No. 9781. The authors are indebted to Dr. Ferri for his invaluable discussions.

[*] Research Associate

[**] Research Professor of Aeronautical Engineering

is valid for lifting vehicle entering the atmosphere at near circular orbit velocity and shallow angles. They are derived in a closed form of simple functions expressing the relations between the velocity, the angle of inclination and the density, or elevation. Using these relations, the acceleration experienced by the pilot can be calculated and the peak value determined. The numerical results calculated for these cases, where the above restrictions are violated, show good agreement with the machine results by direct integration of the equations of motion.

Before reaching the peak acceleration the trajectory of constant lift and drag coefficients can be changed to a trajectory with constant designed acceleration by lift and drag modulation. When the lift-drag ratio remains nearly constant analytic solutions are derived. For a given range of adjustment in the lift, the maximum possible ratio of the peak acceleration to the designed acceleration is determined.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

LIST OF SYMBOLS

A	reference area for drag and lift, ft^2
C_D	drag coefficient
C_L	lift coefficient
D	drag, $\frac{1}{2}\rho V^2 A C_D$, lb
g	gravitation of the earth, ft/sec/sec
h	altitude, ft
L	lift, $\frac{1}{2}\rho V^2 A C_L$, lb
m	vehicle mass, slug
q	heat flux, b.t.u./ft^2
r	radial distance from the center of the earth, ft
t	time, sec
V	vehicle velocity, ft/sec
X, Y, Z	inertial coordinates, rectangular
α, γ, φ	Eulerian Angles
β	1/23,000, ft^{-1}
Δ	$W/C_D A$, lb/ft^2
θ	angle of inclination
ρ	atmospheric density, slug/ft^3 (atmospheric density at sea level, 0.0034 slug/ft^3)
ω	angular velocity, radian/sec

<u>Subscripts</u>

e	value at the entry
ex	value at the exit
m	maximum value
o	value at $\theta = 0$

SECTION I

INTRODUCTION

The problem of reentry trajectory for space vehicle subjected to the aerodynamic forces has been investigated recently by Allen and Eggers[1], Chapman[2,3], Nonweiler[4], Phillips and Cohen[5], Lees, Hartwig and Cohen[6], Grant[7], Moe[8] and Broglio[9]. In most cases a planar reentry trajectory was considered while neglecting the velocity of the atmosphere due to the rotation of the earth.

Allen and Eggers treated the ballistic trajectory with large entry angle. By neglecting the effect of the gravitational and the centrifugal forces the angle of inclination can be taken as constant and is approximated by the entry angle. A closed form analytic solution is then obtained by integration and is good for the drag-only vehicle entering the atmosphere at steep entry angles. Chapman reduced the two components of the equation of motion into one non-linear ordinary differential equation with two justifiable approximations. The numerical solution of the non-linear differential equation obtained through numerical integration depends only on the ratio of lift and drag coefficients. Therefore it is superior to the solution by straight forward integration of the two equations of motion which depend on both the lift and drag coefficients. Nonweiler used a similar method without including the lift forces. Phillips and Cohen obtained an analytic solution for a ballistic vehicle with near circular orbit velocity at the entry by neglecting the term representing the gravitational and the centrifugal forces. Their results are not restricted to steep entry angles as that of reference 1. The effect of drag modulation was also discussed. Lees, Hartwig and Cohen treated the trajectory of vehicle with constant lift and drag coefficients and obtained an approximate analytic solution for entry velocity close to the circular orbit velocity also by neglecting the gravitational and centrifugal forces term. Their solution is valid for shallow entry with the entry velocity near the circular orbit velocity. Grant demonstrated the possibility of reducing the maximum acceleration by employing the high drag side of the drag polar, instead of the low drag side as in usual flight. Moe obtained an analytic solution for drag only vehicle by solving an integral equation in which the angle of inclination is replaced by the entry angle. The numerical results indicated improvements over that of reference 1. Broglio presented the equations of motion in a non-dimensional form and similar solutions were obtained by numerical integration. They depended on the peak acceleration, the value of the density at the peak acceleration and the lift-drag ratio. An approximate solution for circular orbit velocity was also presented.

In general, a trajectory can be divided into the following three regions: (1) the region outside the limit of the sensible atmosphere, (2) the region next to the region 1, and (3) the region well inside the atmosphere. In region 1, the aerodynamic forces are negligible and Kepler's motion prevails as the gravitational force dominates. In region 3 the motion in turn is governed by the aerodynamic forces as they dominate the gravitational and the centrifugal forces. Region 2 which overlaps both regions 1 and 3 is the transition region in the sense that the aerodynamic forces are significant at the beginning and become rapidly dominant as compared to the gravitational and the centrifugal forces. In this investigation, the reentry trajectory is considered to be in region 2.

The equations of motion in space will have three components in describing the motion of the vehicle during the reentry. In general, they will be coupled and difficult to analyze. However, by describing the vehicle motion in the instantaneous plane and normal to the plane as in reference 10, it is shown that the equations of motion are uncoupled; that is, the motion in the instantaneous plane can be dealt with independently of the motion normal to the plane. The problem is then greatly simplified and the analysis can be reduced to that applied to the planar trajectory.

In the result of Allen and Eggers the effect of the change in the angle of inclination is neglected, and the approximation is good only for large entry angles. In the result of Lees, Hartwig, and Cohen the term representing the gravitation and the centrifugal forces is dropped, and the analysis is limited to entry velocity close to the circular orbit velocity. In the previous report[11,12], the velocity in the centrifugal force term was approximated by the velocity at the entry, and the simple solutions obtained are restricted to the lifting vehicle entering the atmosphere at a shallow angle. In the present report the effect of the change in the angle of inclination on the velocity, and the effects of the variations of the velocity in the centrifugal forces on the variation of the angle, have been taken into consideration. The approximate analytic solutions obtained are then valid for the general entry case and are derived in two simple functions relating the velocity, the angle of inclination and the density of the atmosphere or the elevation. For the particular case of the lifting vehicle entering the atmosphere at shallow angle, the approximate solutions can be reduced to simpler forms which have been presented in reference 12 in detail and used in the error analysis and the aerodynamic heating calculations in reference 13 and 14. Application of the present general solutions to the error analysis and the aerodynamic heating calculation will be reported later.

SECTION II

EQUATIONS OF MOTION AND APPROXIMATIONS

For the description of the motion of a reentry space vehicle, a system of moving coordinates r, α, φ, γ, based on the "instantaneous plane" technique was used as shown in Fig. 1.

The position of the reentry vehicle in the instantaneous orbital plane is completely defined by the radial distance r and the polar angle α, whereas the instantaneous orbital plane itself is specified by the angles γ and φ. Since there are four instead of three quantities in specifying the vehicle position, a kinematic relation involving α, γ, and φ must exist among them. This relation represents the vanishing of the velocity component normal to the plane r-α, or

$$\dot{\gamma} \sin \varphi \cos \alpha - \dot{\varphi} \sin \alpha = 0 \ . \tag{1}$$

The external forces acting on the reentry vehicle are the aerodynamic and the gravitational forces. Resolving their components along the unit vectors $\bar{r}$, $\bar{\theta}$, and $\bar{k}$ directions, and equating with the inertia forces, yields the following set of equations of motion:

$$\frac{dV}{dt} = -\frac{D}{m} + g \sin \theta , \tag{2a}$$

$$V \frac{d\theta}{dt} = -\frac{L}{m} - \left(\frac{V^2}{r} - g\right) \cos \theta , \tag{2b}$$

$$V \dot{\varphi} \frac{\cos \theta}{\cos \alpha} = \frac{B}{m} \ . \tag{2c}$$

From the exponential approximate atmosphere $\rho = \rho_s \exp(-\beta h)$ and the relation between the velocity and the height $\frac{dh}{dt} = - V \sin \theta$, the following relation is obtained by differentiation with respect to time:

$$\frac{d\rho}{dt} = \beta \rho V \sin \theta \qquad (3)$$

Together with the lift and drag forces in the following forms:

$$L = \frac{1}{2} \rho V^2 C_L A \quad , \quad D = \frac{1}{2} \rho V^2 C_D A$$

Eqs. (2a), (2b), and (3) consist of the governing equations for the reentry trajectory of the vehicle in the present analysis.

Eqs. (2a) and (2b) are identical with the equations describing a planar reentry trajectory, which indicates the independence of the motion in the instantaneous orbital plane from the motion normal to the plane. This result considerably simplifies the problem of reentry trajectory, and, as a consequence, the analysis will be restricted to the investigation of Eqs. (2a) and (2b). Since all the interested quantities, such as the velocity, the acceleration and the height, etc., can be obtained without solving Eq. (2c), it will not be considered further in the present analysis.

In deriving the equations of motion, i.e., Eqs. (2a), (2b), and (2c), the simplifying assumptions of a nonrotating spherical earth and constant gravity are adopted.

Before going into detailed mathematical manipulations of these equations of motion some physical justification of the approximations will be discussed. In Eq. (2a) the last term represents the gravitational forces which is small compared with the aerodynamic forces. In Eq. (2b) the last term represents the gravitational and the centrifugal forces which have small effect on the angle of inclination for the trajectory of the drag-only vehicle entering the atmosphere at finite entry angle. For entry of lifting vehicles at shallow angle their effects are small compared with that of the lift. In the previous report, (references 11 and 12), the velocity in the centrifugal force term was approximated by the entry velocity, and the results are good for lifting vehicle entering at shallow angles. In the analysis of Allen and Eggers, gravitational and centrifugal terms in both equations have been neglected in their approximation, and the resulting expression for the velocity is good for vehicle with drag-only entering at finite entry angle, since in this case the change in the angle is small compared with the entry angle itself. In the present analysis all terms in both equations have been retained, while approximations have been made for the gravitational and centrifugal terms. The velocity in the centrifugal force in Eq. (2b) is replaced by the velocity and density relation of Allen and Eggers. This approximation is good for the following reasons: (1) For drag-only vehicles, entering at finite angle, the approximation is obviously good; (2) for shallow entry, the velocity changes very little from the entry velocity and so does the result of this approximation, as the density variation is small; (3) for steep entry of lifting vehicles, the velocity also changes very little in the earlier part of the trajectory and can be approximated similarly. In the latter part of trajectory where the velocity changes considerably, the lift force becomes predominant and the effect of the centrifugal and the gravitational forces on the angle of inclination are small by comparison.

Using the above approximation Eq. (2b) can be evaluated to yield an algebraic relation between the angle of inclination and the density. With this result, Eq. (2a) is integrated to yield the expression for the velocity.

SECTION III

ANALYTIC SOLUTION

For shallow entry, the angle of inclination remains small and cos θ is approximately equal to unity and can be approximated by cos θ_e in Eq. (2b). For steep entry, the angle of inclination changes little from the entry angle, and it can also be approximated by the entry angle. With cos θ replaced by cos θ_e, Eqs. (2a) and (2b) are reduced to the following form, using the given expressions for the lift and drag:

$$\frac{dV}{dt} = -\frac{C_D A \rho V^2}{2m} + g \sin\theta, \tag{4a}$$

$$V\frac{d\theta}{dt} = -\frac{C_L A \rho V^2}{2m} - \left(\frac{V^2}{r} - g\right)\cos\theta_e. \tag{4b}$$

With Eq. (3), Eqs. (4a) and (4b) become

$$\frac{dV}{V} = -\frac{C_D A}{2m\beta}\frac{d\rho}{\sin\theta} + \frac{g\,d\rho}{\beta\rho V^2} \tag{5}$$

$$\sin\theta\, d\theta = -\frac{C_L A}{2m\beta}\, d\rho - \left(\frac{1}{r} - \frac{g}{V^2}\right)\frac{\cos\theta_e}{\beta}\frac{d\rho}{\rho} \tag{6}$$

The quantities in the parenthesis represent the terms due to the centrifugal and the gravitational forces. They were neglected in reference 6 on the assumption that the entry velocity is close to the circular orbit velocity and must be taken into consideration for entry velocity that differs considerably from the circular orbit velocity.

As discussed in the preceding section, the velocity in the centrifugal force term can be approximated with good accuracy by the velocity-density relationships of Allen and Eggers' solution

$$\ln \frac{V}{V_e} = - \frac{C_D A}{2m\beta \sin \theta_e} (\rho - \rho_e) \tag{7}$$

Direct integration of Eq. (6) with the aid of Eq. (7) proves to be difficult. The term g/V^2 in Eq. (6) is next expanded in terms of $\ln \frac{V}{V_e}$, keeping only the first three terms

$$\frac{g}{V^2} = \frac{g}{V_e^{\,2}} [1 + C_1 \ln \frac{V}{V_e} + C_2 (\ln \frac{V}{V_e})^2] . \tag{8}$$

In the above expression the constant coefficients can be determined by the collocation method in accordance with range of the velocity change required. For most cases the maximum acceleration is reached before the velocity has been reduced to one-half of its value at the entry. In the numerical computations presented later, the constant coefficients are determined for this velocity range. They will be shown to be satisfactory for most trajectories of interest by comparison with the exact results from machine calculations.

With Eq. (7), Eq. (8) becomes

$$\frac{g}{V^2} = \frac{g}{V_e^{\,2}} [1 - \frac{C_D A C_1}{2m\beta \sin \theta_e} (\rho - \rho_e) + C_2 (\frac{C_D A}{2m\beta \sin \theta_e})^2 (\rho - \rho_e)^2], \tag{9}$$

and integration of Eq. (6) with $\theta = \theta_e$ at $\rho = \rho_e$, yields for constant C_L*

*For the case where C_L or C_D is a given function of ρ and hence of the altitude, Eq. (18) can still be evaluated.

$$\cos\theta = \cos\theta_e + B_1(\rho-\rho_e) + B_2 \ln\frac{\rho}{\rho_e} + B_3 f_1(\rho) , \tag{10}$$

where

$$B_1 = \frac{C_L A}{2m\beta} , \quad B_2 = \frac{\cos\theta_e}{\beta r} , \quad B_3 = -\frac{g\cos\theta_e}{\beta V_e^2} , \quad B_4 = \frac{C_D A\rho_e}{2m\beta\sin\theta_e} ,$$

$$f_1(\rho) = \left[(1+B_4C_1+B_4^2C_2)\ln\frac{\rho}{\rho_e} - (B_3C_1+2B_4^2C_2)\frac{(\rho-\rho_e)}{\rho_e} + \frac{1}{2}B_4^2C_2\left(\frac{\rho-\rho_e}{\rho_e}\right)^2\right]$$

The relationship between V and ρ can now be obtained by integrating Eq. (5). With C_D constant,* it becomes

$$\ln\frac{V_e}{V} = \frac{C_D A}{2m\beta}\int_{\rho_e}^{\rho}\frac{d\bar{\rho}}{\sin\theta(\bar{\rho})} - \frac{g}{\beta}\int_{\rho_e}^{\rho}\frac{d\bar{\rho}}{\bar{\rho}V^2} . \tag{11}$$

The denominator $\sin\theta$ will be expanded in terms of θ in two different forms according to the value of the entry angle. For entry angle less than 60°, the term $\frac{1}{\sin\theta}$ will be written as follows:

$$\frac{1}{\sin\theta} \doteq \frac{1}{\theta-\frac{1}{6}\theta^3} \doteq \frac{1}{\theta} + \frac{1}{6}\theta. \tag{12}$$

The maximum error will be less than 3% in using Eq. (12).

For entry angle greater than 45° the same term can be expanded in terms of $(\theta_e-\theta)$, which, before reaching the peak acceleration, is small in most cases compared with θ_e.

---

*See footnote on page 9.

$$\frac{1}{\sin\theta} = \frac{1}{\sin[\theta_e-(\theta_e-\theta)]} \doteq \frac{1}{\sin\theta_e-(\theta_e-\theta)\cos\theta_e}$$

$$\doteq \frac{1}{\sin\theta_e} + \frac{\cos\theta_e}{\sin^2\theta_e}(\theta_e-\theta) . \tag{13}$$

The maximum error in this case for θ_e=45° with $\theta_e-\theta$=2°, is less than 5%.

Direct integration of Eq. (11) with the aid of Eq. (12) or Eq. (13) is still difficult. However, inspection of Eq. (11) indicates that the first term of the integral depends predominately on the minimum value of $\theta(\bar{\rho})$ in the range of integration. For most cases of interest the entry velocity will be greater than the circular orbit velocity. The angle of inclination θ will decrease monotonously as ρ increases, until its minimum value is reached. Thus, for the evaluation of the velocity at any point $\bar{\rho} = \rho$ before θ reaches its minimum value, the term ln $\bar{\rho}/\rho$ in Eq. (10) can be approximated by a series expansion in terms of $\frac{\rho-\bar{\rho}}{\rho}$, keeping only the first two terms. With $\cos\theta$ approximated by $(1-\frac{\theta^2}{2})$ Eq. (10) becomes

$$\theta = [\bar{\theta}^2 + K_1\xi + K_2\xi^2]^{\frac{1}{2}} \tag{14}$$

where $\xi = \frac{\rho-\bar{\rho}}{\rho}$,

$$\bar{\theta}^2 = \theta_e^2 - [2B_1 - \frac{2C_1B_3B_4}{\rho_e} + C_2B_3B_4^2\frac{(\rho-3\rho_e)}{\rho_e^2}](\rho-\rho_e)$$

$$- 2[B_3+B_2+C_1B_3B_4+C_2B_3B_4^2]\ln\frac{\rho}{\rho_e} ,$$

$$K_1 = 2[B_1\rho + B_3+B_2-C_1B_3B_4\frac{(\rho-\rho_e)}{\rho_e} + C_2B_3B_4^2\frac{(\rho-\rho_e)^2}{\rho_e^2}],$$

$$K_2 = B_3 + B_2 + C_1 B_3 B_4 - C_2 B_3 B_4^2 \frac{(\rho^2 - \rho_e^2)}{\rho_e^2}$$

With Eq. (14) the integration of Eq. (11), using Eq. (12), yields, for K_2 positive,

$$\ln \frac{V_e}{V} = \frac{B_3}{\cos \theta_e} f_1(\rho) + \frac{B_5}{\sqrt{K_2}} f_2(\rho) + f_3(\rho) , \qquad (15)$$

where

$$B_5 = \frac{C_D A}{2m\beta} \left(1 + \frac{4K_2\overline{\theta}^2 - K_1^2}{48K_2}\right) ,$$

$$f_2(\rho) = \rho \ln\left[\frac{K_1 + 2(1-\sigma)K_2 + 2\sqrt{K_2\overline{\theta}^2 + (K_1 + K_2 - \sigma K_2)(1-\sigma)K_2}}{2\sqrt{K_2}\,\overline{\theta} + K_1}\right]$$

$$f_3(\rho) = \frac{C_D A \rho}{48 m \beta K_2} \left\{ [2K_2(1-\sigma) + K_1] \sqrt{\overline{\theta}^2 + K_1(1-\sigma) + K_2(1-\sigma)^2} - K_1\overline{\theta} \right\} ,$$

$$\sigma = \frac{\rho_e}{\rho} .$$

For negative K_2, integration yields

$$\ln \frac{V_e}{V} = \frac{B_3}{\cos \theta_e} f_1(\rho) + \frac{B_5}{\sqrt{-K_2}} f_4(\rho) + f_3(\rho) , \qquad (16)$$

where

$$f_4(\rho) = \rho\left[\sin^{-1}\frac{K_1}{\sqrt{K^2-4K_2\overline{\theta}^2}} - \sin^{-1}\frac{K_1+2(1-\sigma)K_2}{\sqrt{K_1^2-4K_2\overline{\theta}^2}}\right].$$

Eq. (15) and Eq. (16) can be used to calculate the velocity at any point between the entry and the point where θ becomes minimum for the entry angle less than 60°.

For entry angle greater than 45°, Eq. (13) will be used in the integration of Eq. (11). For K_2 positive the integration of Eq. (11) yields

$$\ln\frac{V_e}{V} = \left(\frac{1}{\sin\theta_e} + \frac{\theta_e\cos\theta_e}{\sin^2\theta_e}\right)\frac{C_DA}{2m\beta}(\rho-\rho_e) - \frac{B_6}{\sqrt{K_2}}f_3(\rho) - 6f_2(\rho), \tag{17}$$

where

$$B_6 = \frac{C_DA\cos\theta_e(4K\overline{\theta}^2-K^2)}{16m\beta\sin^2\theta_eK_2}.$$

For K_2 negative the integration of Eq. (11) yields

$$\ln\frac{V_e}{V} = \left(\frac{1}{\sin\theta_e} + \frac{\theta_e\cos\theta_e}{\sin^2\theta_e}\right)\frac{C_DA}{2m\beta}(\rho-\rho_e) - \frac{B_6}{\sqrt{-K_2}}f_4(\rho) - 6f_2(\rho) \tag{18}$$

<u>Maximum Acceleration:</u>

With the velocity and the density of the air known at any point of the trajectory, the acceleration exerted on the pilot in addition to the gravity can be computed.

In unit of g's the acceleration is

$$G = \frac{1}{2} \frac{C_D A}{W} \rho V^2 \sqrt{1 + \left(\frac{C_L}{C_D}\right)^2} \qquad (19)$$

for the planar trajectory.

The condition for the maximum acceleration is $\frac{dG}{dt} = 0$, or by Eq. (19)

$$V^2 \frac{d\rho}{dt} + 2\rho V \frac{dV}{dt} = 0 \, .$$

It follows then from Eqs. (3) and (4) that at the maximum acceleration

$$\left(1 + \frac{2g}{\beta V_m^2}\right) \sin \theta_m = \frac{C_D A}{m\beta} \rho_m \, . \qquad (20)$$

It has been shown in reference 15, on the approximate technique for variational problems, that the error in the extreme of a function is of the order ϵ^2 if the terms of the order ϵ are omitted in the conditions of extreme. Applying to this particular case, Eq. (20) becomes

$$\sin \theta_m = \frac{C_D A}{m\beta} \rho_m \qquad (21)$$

for the determination of the values of ρ and θ at the maximum acceleration which will be found to differ from the exact value only by the order of ϵ^2, where $\frac{2g}{\beta V_m^2}$ is of the order ϵ.

As discussed in referencd 12, for the trajectory whose angle of inclination θ decreases from entry angle to zero, the first peak acceleration can be approximated by the value of the acceleration at $\theta=0$ with an error of the order of θ_m^2, where θ_m is of an order smaller than the entry angle.

Numerical Results:

The machine results in reference 6 for the case of lifting vehicle were used for the purpose of comparison. The numerical computations using the analytic solutions for the velocity and the acceleration at the peak acceleration compare very favorably. As shown in Figs. 2 and 3 the maximum deviation in the velocity and the peak acceleration come to less than 1% and 5%, respectively. The same entry height at 400,000 ft is used here as in reference 6.

Numerical computations were also made for the case of drag-only vehicle entering the atmosphere at small entry angle. The example in reference 8 for vehicle with $C_D A/W = .01$ ft^2/lb, entry velocity at 30,000 fps and entry angle at 5^o, is chosen for comparison. The numerical results from Eqs.(10) and (15) are listed below together with those of reference 8.

	Reference 8				Results from	
h	Approx.		Exact		Eq. (10) & Eq. (15)	
10^3 ft	θ	$V \times 10^{-3}$ fps	θ	$V \times 10^{-3}$ fps	θ	$V \times 10^{-3}$ fps
250	5°	30.00	5°	30.00	5°	30.00
200	4.60°	29.40	4.60°	29.36	4.59°	29.28
150	4.37°	24.66	4.40°	24.06	4.34°	24.04
100	8.19°	5.98	8.06°	7.00	7.30°	6.61
50	89.99°	0.247	51.63°	.987	43.17°	.515
0	90.00°	0.000	89.97°	.299	—	—

Note that the numerical results obtained from the approximate solution show an improvement over that of reference 8, particularly at lower altitude, where the changes in the velocity and angle becomes rather large.

SECTION IV

CONSTANT ACCELERATION TRAJECTORIES WITH LIFT AND DRAG MODULATIONS

In this section approximate solution is presented for the constant acceleration trajectories with the lift and drag modulation for entry at shallow angles. The constant acceleration is achieved by modulating the lift and drag coefficients, while keeping their ratio, C_L/C_D, constant. Mechanically, this type of modulation requires some auxiliary adjusting surfaces in addition to the main body of the vehicle. By programming the adjustment of these surfaces in concert with the angle of attack it is not difficult to fulfill these requirements. In addition, from Eq. (19) it is clear that a reasonable change in the ratio C_L/C_D has less effect on the acceleration than the change in C_D. Therefore, a constant acceleration trajectory can at least be approximated by modulating only the drag. For shallow entry the last term in the Eq. (5) can be neglected in comparison with the drag forces, and it becomes

$$\frac{dV}{V} = - \frac{C_D A}{2m\beta} \frac{d\rho}{\theta} \qquad (21)$$

During the early part of the entry trajectory will be flown at constant lift and drag coefficients until the limiting value of the acceleration is reached. The modulation will then take place and maintain the acceleration at this value until it can be turned off without the acceleration subsequently exceeding the given value. For the part of the trajectory with constant lift and drag coefficients, the solutions obtained in the previous chapter are applicable. Thus, for any given entry conditions and limiting value of acceleration the data at the start of modulation can be determined.

In Eq. (17) the factor before the square root sign is the drag force $D = \frac{1}{2} \frac{C_D A}{W} \rho V^2$, which may be kept constant by adjusting the drag coefficient as ρ and V varies. Hence, the acceleration may be kept constant, provided the modulation of C_D is such that the lift-drag ratio does not change. Under these conditions the equations of motion as represented by Eq. (6) and Eq. (21) may be integrated.

Let the subscripts 1 and 2 denote the start and the end of the modulation, respectively. The drag force, remaining constant during the modulation period, is

$$D = \frac{1}{2} \frac{C_D A}{W} \rho V^2 = \frac{1}{2} \frac{C_{D1} A}{W} \rho_1 V_1^2 = D_1 , \tag{22}$$

and Eq. (6), with the help of Eq. (21), becomes

$$d\theta = \frac{C_L}{C_D} \frac{dV}{V} - \frac{\cos\theta_e}{D_1} \frac{dV}{V} + \frac{\cos\theta_e}{grD_1} VdV . \tag{23}$$

With the values of θ, ρ, and V at the start of the modulation, i.e., $\theta = \theta_1$, $\rho = \rho_1$, $V = V_1$, the integration of Eq. (23) yields

$$\theta = \theta_1 - \left(\frac{C_L}{C_D} - \frac{\cos\theta_e}{D_1}\right) \ln \frac{V_1}{V} - \frac{\cos\theta_e}{2grD_1} (V_1^2 - V^2) . \tag{24}$$

Rewrite Eq. (21) with the help of Eq. (22)

$$\theta V dV = -\frac{gD_1}{\beta}\frac{d\rho}{\rho}. \qquad (25)$$

With Eq. (24) the integration yields

$$\frac{D_1 g}{\beta}\ln\frac{\rho}{\rho_1} = \frac{1}{2}(V_1^2-V^2)\theta_1 - \left(\frac{C_L}{C_D} - \frac{\cos\theta_e}{D_1}\right)\left(\frac{V_1^2-V^2}{4} - \frac{V^2}{2}\ln\frac{V_1}{V}\right) - \frac{\cos\theta_e}{8grD_1}(V_1^2-V^2)^2. \qquad (26)$$

Eqs. (24) and (26) represent the relations between θ, ρ, and V for the trajectory during the modulation period with the acceleration kept constant.

As mentioned earlier, the modulation can be stopped when the subsequent acceleration will not exceed the limiting value or G_1. In reference 6, it was shown that with a smaller terminal lift the acceleration will not exceed the limiting value, and numerical computations confirmed this conclusion. In addition, from the discussion in reference 13 for the constant lift and drag coefficients trajectory the maximum acceleration will in general take place near the point where $\theta=0$. By choosing the point of the trajectory for the end of the modulation at $\theta=0$ the acceleration certainly will not exceed the limiting value if a small positive lift is maintained.

With the end of the modulation period thus chosen at $\theta=0$, the amount of acceleration that can be reduced from the maximum acceleration of the constant C_L and C_D trajectory will be limited by the range of adjustment available in the vehicle. For a given entry condition, the range of adjustment necessary for any required reduction in acceleration can be computed by using Eqs. (22), (24), and (26).

In Fig. 4 the ratio of the acceleration, i.e., G_1/G_m vs. the ratio of the drag coefficients, or C_{D_1}/C_{D_2}, is plotted for a particular entry condition. For the case of the drag coefficient ratio equals $4\frac{1}{2}$, the reduction in the acceleration comes to 61%. However, by increasing the ratio of the drag coefficient the increase in the reduction of acceleration is rather slight. For example, a ratio of 9 results in only 65% reduction in the acceleration. Therefore, for this particular case, the increase in the range of the drag coefficients will not be very profitable beyond certain limits.

Numerical Results

In the following example the velocity, the density, and the acceleration were computed for the modulated trajectory. Except where otherwise noted, the same constants were used as in the previous numerical examples.

Given Data: Entry velocity 36,000 fps, entry angle 9°,

$$\frac{C_L A}{W} = \frac{C_D A}{W} = 0.01 \text{ ft}^2/\text{lb}$$

Here the maximum acceleration for the constant lift and drag trajectory was computed to be 16.54g. Suppose it is required to limit the acceleration to 7.89g by the modulation of lift and drag coefficients. From Eq. (10), with the use of Eq. (15), the values of ρ and V at the start of modulation are 8.969×10^{-7} slug/ft^3 and 35,262 ft/sec, respectively.

Next, the values of ρ and V at the end of modulation or $\theta=0$ are computed, using Eqs. (24) and (26) and are found to be 3.057×10^{-8} slug/ft^3 and 32,420 ft/sec, respectively.

Finally, the value of $\frac{C_D A}{W}$ at the end of modulation is found to be, from the requirement that $D_1=D_2=$ constant,

$$\frac{C_{D_2} A}{W} = 0.00347 \text{ ft}^2/\text{lb}.$$

This result indicates that the ratio of the drag coefficient C_{D_1}/C_{D_2} must be equal to or greater than 2.88 if the acceleration is to be limited to 7.89g.

SECTION V

RESULTS AND CONCLUSIONS

Approximate solutions are derived for the reentry trajectory of space vehicle with lift and drag. The restrictions of references 1 and 6 on the entry angle and on the entry velocity are removed in the approximation, and the solutions are therefore applicable to trajectories in general. The solutions are written as a set of explicit simple functions expressing the relations between the velocity, the angle of inclination and the atmospheric density.

For a given elevation, Eq. (10) gives the angle of inclination. Eqs. (15)-(18) give the value of velocity at any elevation according to the value of the constant k_a and the entry angle. With the velocity and the density computed, the acceleration is calculated from Eq. (19). Eq. (20) represents the condition for the peak acceleration in terms of the relation between the angle of inclination and the density. Together with Eq. (10) the value of the density at peak acceleration can be determined. From this the peak acceleration can be evaluated using Eq. (19).

For lifting vehicle entering the atmosphere at shallow angles and near circular orbit velocity, the approximate solution reduces to that of Lees. For drag-only vehicle entering the atmosphere at large angles, the approximate solution approaches that of Allen and Eggers. Therefore, in order to show its validity in general cases, numerical examples for the trajectory of drag-only vehicle entering the atmosphere at shallow angle, and lifting vehicle entering the atmosphere at parabolic velocity, are presented. The results are in good agreement with the exact values from the numerical integration.

The trajectory of lifting vehicle entering the atmosphere at shallow angle is of practical importance and interest. In this case the approximate solutions can be simplified to the form presented in references 13 and 14.

For a vehicle with nearly constant lift-drag ratio the acceleration is relatively insensitive to the changes in the lift-drag ratio. An analytic solution is therefore derived for a constant acceleration trajectory by modulating the lift and the drag coefficient. Eqs. (24) and (26) express the relations between the velocity, the angle of inclination and the elevation. With these relations, the ratio of the constant acceleration during modulation to the peak acceleration of the constant lift and drag trajectory is determined for a given range of adjustment in the drag coefficient. For the trajectory of a particular entry presented in Fig. 4, a ratio of $4\frac{1}{2}$ in the drag coefficient, or C_{D_1}/C_{D_2}, gives 61% reduction in the acceleration.

SECTION VI

REFERENCES

1. Allen, H. J. and Eggers, A. J. Jr.: A Study of the Motion and Aerodynamic Heating of Ballistic Missiles Entering the Earth's Atmosphere at High Supersonic Speed. NACA Report 1381, 1958.
2. Chapman, D. R.: An Approximate Analytic Method for Studying Entry into Planetary Atmospheres. NACA TN 4276, May 1958.
3. Chapman, D. R.: An Analysis of the Corridor and Guidance Requirements for Supercircular Entry into Planetary Atmospheres. NASA TR R-55, 1959.
4. Nonweiler, T.: The Motion of an Earth Satellite on Reentry to the Atmosphere. Astronautica Acta, Vol. 5, Fasc. 1, 1959.
5. Phillips, Richard L. and Cohen, Clarence B.: Use of Drag Modulation to Reduce Deceleration Loads During Atmospheric Entry. ARS Journal, Vol. 29, No. 6, pp. 414-422, June 1959.
6. Lees, L., Hartwig, F. W., and Cohen, C. B.: Use of Aerodynamic Lift During Entry into the Earth's Atmosphere. ARS Journal, Vol. 29, No. 9, pp. 633-641, September 1959.
7. Grant, Frederick C.: Importance of the Variation of Drag with Lift in Minimization of Satellite Entry Acceleration. NASA TN D-120, October 1959.
8. Moe, Mildred M.: An Approximation to the Re-entry Trajectory. ARS Journal, Vol. 30, No. 1, pp. 50-53, January 1960.

9. Broglio, L.: Similar Solutions in Re-entry Lifting Trajectories. Space Research, Proceedings of the First International Space Science Symposium, Nice, pp. 564-577, January 11-16, 1960, Interscience Publishers, Inc., New York.

10. Nielsen, J. N., Goodwin, F. K., and Mersman, W. A.: Three-Dimensional Orbit of Earth Satellites, Including Effects of Earth Oblateness and Atmospheric Rotation. NASA MEMO 12-4-58A, December 1958.

11. Wang, K. and Ting, L.: An Approximate Analytic Solution of Re-entry Trajectory with Aerodynamic Forces. ARS Journal, Vol. 30, No. 6, pp. 565-566, June 1960.

12. Wang, K. and Ting L.: Analytic Solutions of Planar Reentry Trajectories with Lift and Drag. Polytechnic Institute of Brooklyn, PIBAL Report No. 601, AFOSR TN 60-508, April 1960.

13. Wang, K. and Ting, L.: Aerodynamic Heating of Re-Entry Vehicle. ARS Journal, Vol. 30, No. 12, pp. 1180-1181, December 1960.

14. Wang, K. and Feitis, P.: Error Analysis of Orbit Transfer and Reentry Trajectory. Polytechnic Institute of Brooklyn, PIBAL Report No. 622, October 1960.

15. Ting, L.: Approximations in Variational Problems. Polytechnic Institute of Brooklyn, PIBAL Report No. 610, AFOSR TN 60-789, July 1960.

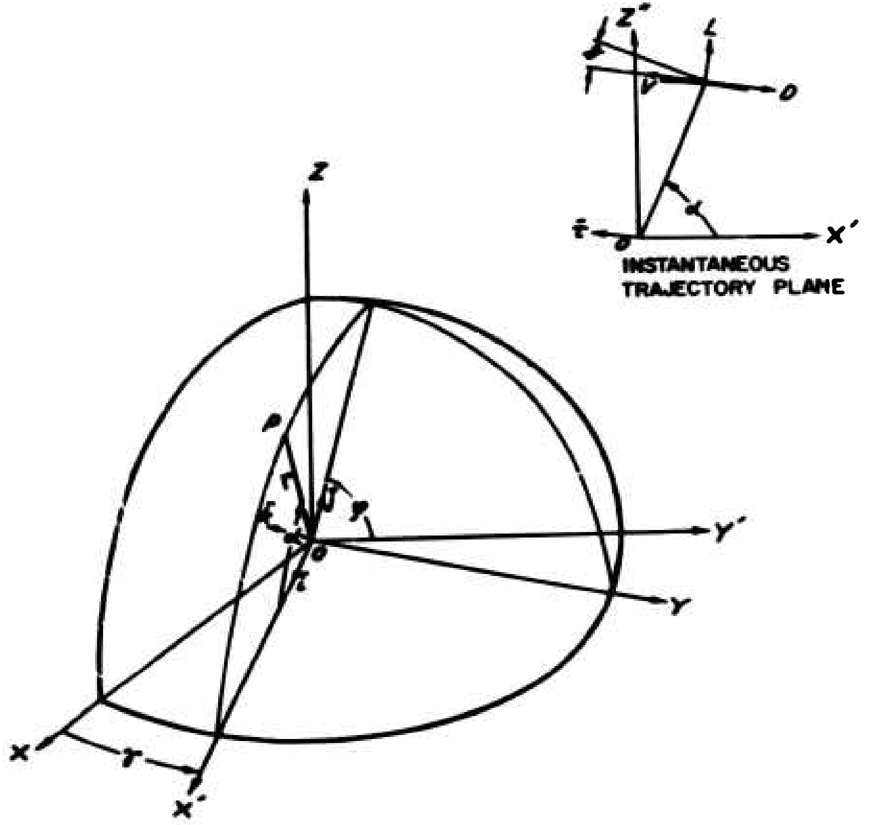

FIG.I COORDINATE SYSTEM

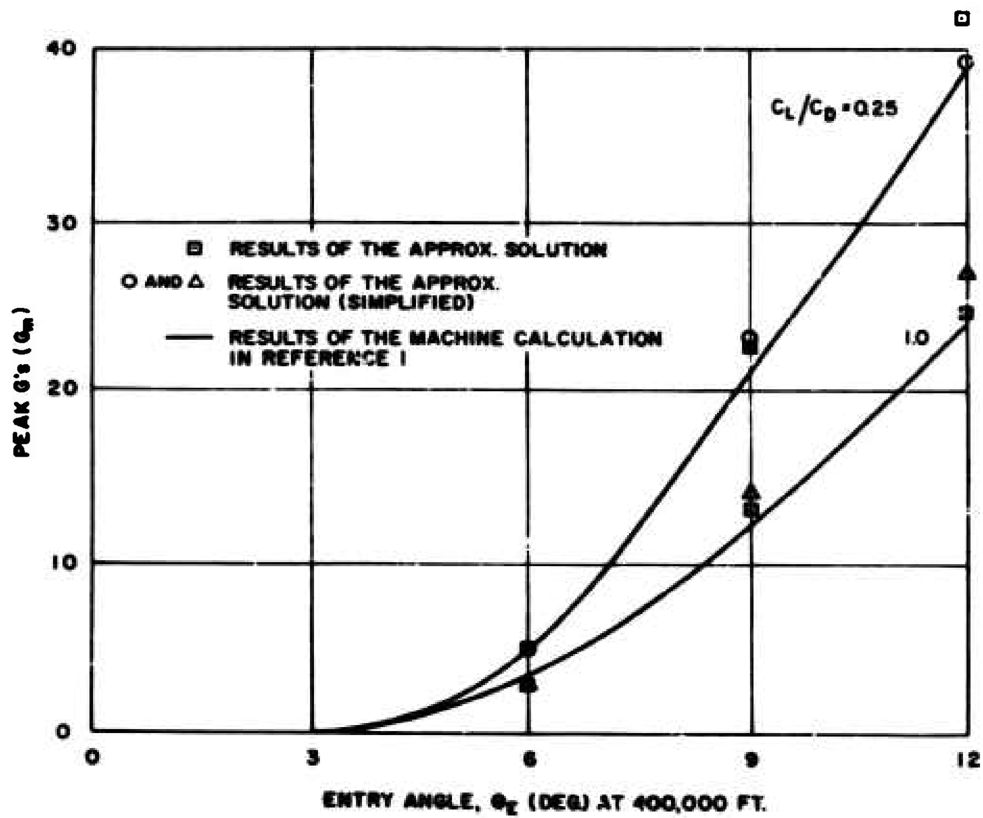

FIG. 2 PEAK ACCELERATION WITH CONSTANT C_L/C_D, V_E = 35,000 FPS.

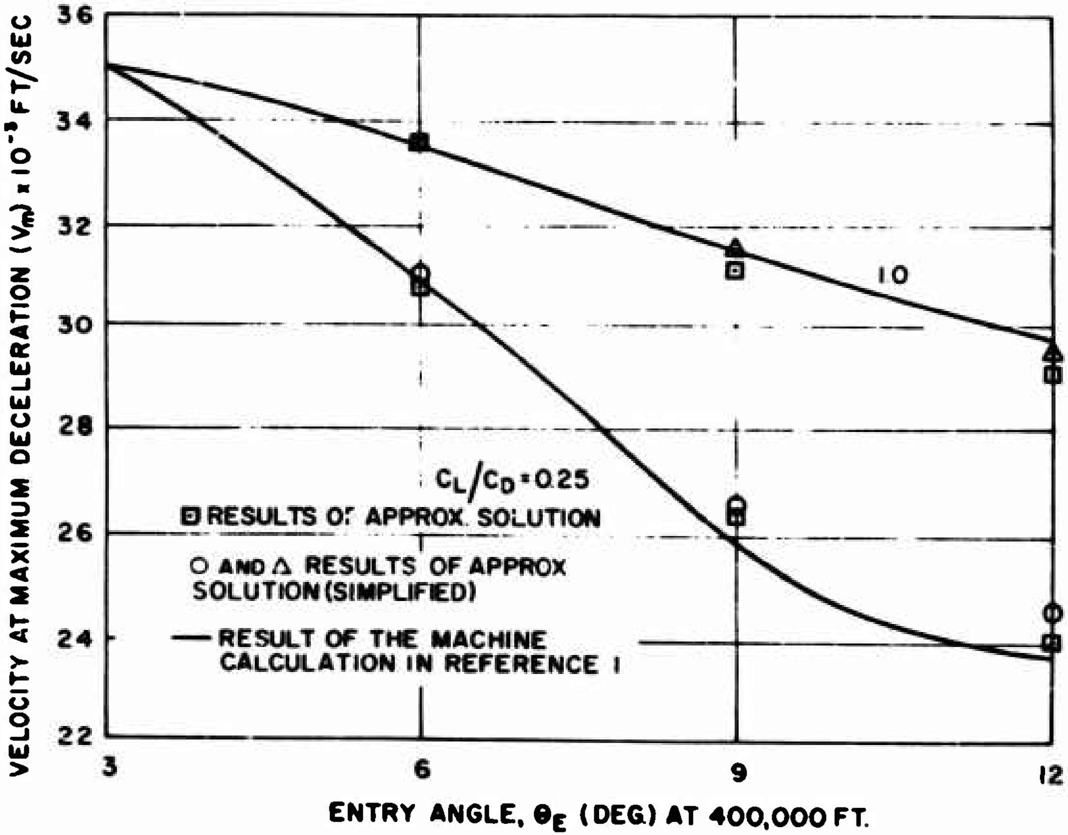

FIG.3 VELOCITY AT PEAK ACCELERATION V_z = 35,000 FPS.

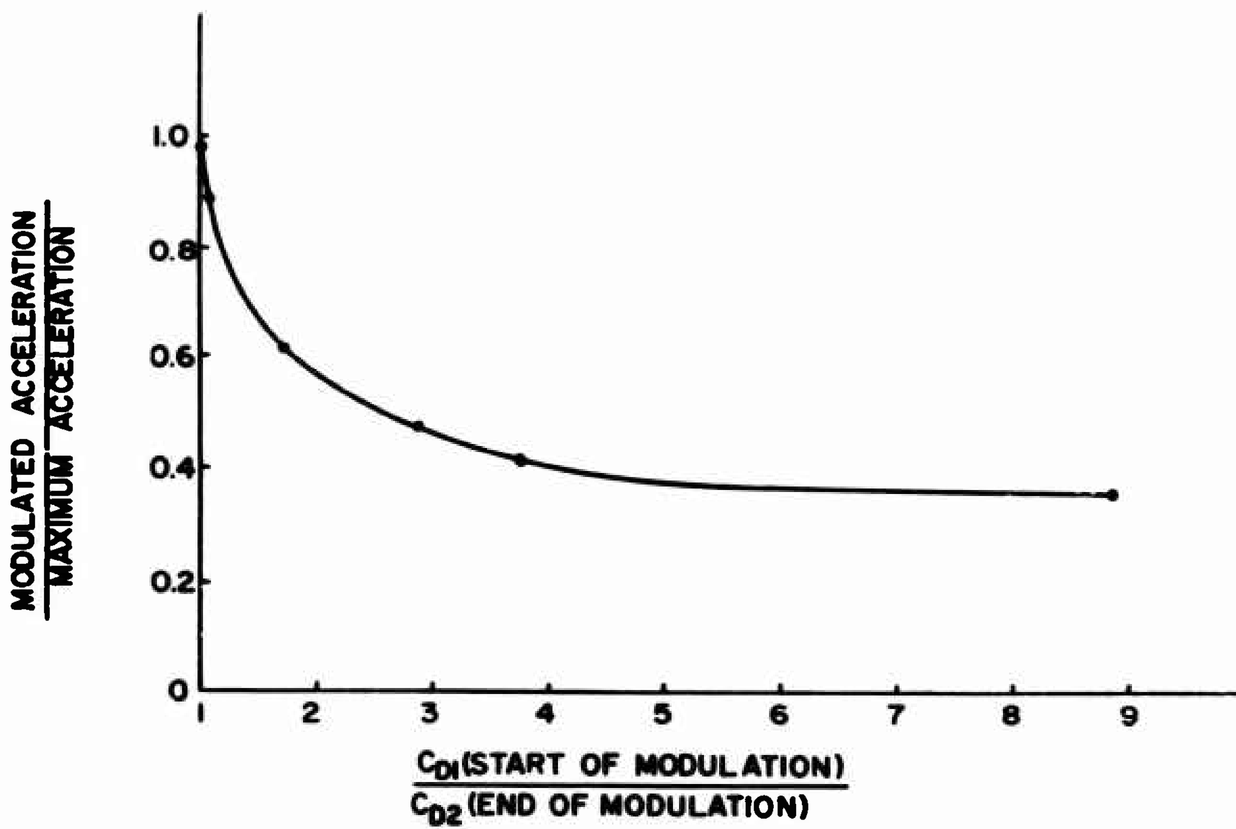

FIG.4 ACCELERATION RATIO VS DRAG COEFFICIENT RATIO

Bibliographical Control Sheet

1. Originating agency and/or monitoring agency:
   O. A.: Polytechnic Institute of Brooklyn, Brooklyn, New York
   M.A.: Mechanics Division, Air Force Office of Scientific Research
2. Originating agency and/or monitoring agency report number:
   O. A.: PIBAL Report No. 647
   M.A.: AFOSR 684
3. Title and classification of title: APPROXIMATE SOLUTIONS FOR REENTRY TRAJECTORIES WITH AERODYNAMIC FORCES (UNCLASSIFIED)
4. Personal authors: Kenneth Wang and Lu Ting
5. Date of report: May 1961
6. Pages: 29
7. Illustrative material: 4 figures
8. Prepared for Contract No.: AF 49(638)-445
9. Prepared for Project Code and/or No.: 9781
10. Security classification: UNCLASSIFIED
11. Distribution limitations: In accordance with the approved distribution list for unclassified reports.
12. Summary: When the motion of the reentry vehicle is expressed in two equations for the components in the instantaneous trajectory plane, and one for the component normal to the plane, the former equations are uncoupled from the latter. Thus, the problem of a nonplanar trajectory is reduced to an equivalent planar trajectory.

In the present paper approximate analytic solutions for planar trajectories with constant lift and drag coefficients are obtained by improving and extending the analytic solution of Allen and Eggers, and that of Lees, Hartwig, and Cohen. The present solutions are not subjected to the restrictions of the Allen and Eggers solution, which is valid for drag only vehicles at large entry angle, nor are they subjected to that of Lees, Hartwig, and Cohen's solution, which is valid for lifting vehicle entering the atmosphere at near circular orbit velocity and shallow angles. They are derived in a closed form of simple functions expressing the relations between the velocity, the angle of inclination and the density, or elevation. Using these relations, the acceleration experienced by the pilot can be calculated and the peak value determined. The numerical results calculated for these cases, where the above restrictions are violated, show good agreement with the machine results by direct integration of the equations of motion.

Before reaching the peak acceleration the trajectory of constant lift and drag coefficients can be changed to a trajectory with constant designed acceleration by lift and drag modulation. When the lift-drag ratio remains nearly constant analytic solutions are derived. For a given range of adjustment in the lift, the maximum possible ratio of the peak acceleration to the designed acceleration is determined.