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MECHANICS OF GRANULAR MEDIA

by

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MECHANICS OF GRANULAR MEDIA *

For a number of years, there has been under development a mathematical theory of the mechanical behavior of materials composed of discrete elastic grains in direct contact. Eventually, the theory is intended to predict stress-strain relations, stress distributions, vibrations, wave propagation phenomena and criteria of failure for such materials as are found in a bed of dry sand or the pile of grains in the carbon microphone. The line of attack, which has been the most fruitful, begins with a consideration of the local forces and deformations at the contact surfaces between adjacent grains. Because of the extraordinarily complex nature of the problem the grains have been idealized as like spheres in regular arrays. Even with this simplification, at least until recently, only the component of force normal to each contact surface has been taken into account [1,2,3,4]. The relations between normal force, N, contact radius, a, and displacement, α, are obtained from the Hertz theory of contact of elastic bodies [5]:

$$a = \left[\frac{3(1-\nu)RN}{8\mu}\right]^{1/3} \tag{1}$$

$$\alpha = 2\left[\frac{3(1-\nu)N}{8\mu R^{1/2}}\right]^{2/3} \tag{2}$$

where R is the radius of the spheres, ν is Poisson's ratio and μ the shear modulus of the material of the spheres. Of special interest is the normal compliance

$$C = \frac{d\alpha}{dN} = \frac{1-\nu}{2\mu a} \tag{3}$$

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The non-linearity of these relations gives the first inkling of dynamical difficulties in addition to the purely geometrical ones. The behavior of a granular material may be expected to depend strongly on the initial stress which, in turn, affects the role of the elastic constants of the individual grains. The early forms of the theory predict wave velocities proportional to the sixth root of an initial isotropic pressure and the cube root (rather than the usual square root) of the shear modulus of the grain [2,3,4]. These relations have been confirmed experimentally [2,13] but absolute velocities are in poor agreement when the theory does not include the effect of tangential components of force between grains. It is the purpose of this lecture to discuss some of the problems and consequences of including consideration of tangential forces at the contact surfaces.

Corresponding to the Hertz theory, there is a solution of the equations of elasticity [6,7] which takes into account a monotonically increasing tangential force (T) subsequent to the application of a normal force. It is found that a tangential force, no matter how small, produces infinite tangential traction (τ) at the edge of the contact surface (see Fig. 1) if it is assumed that there is no relative displacement of opposing points on the contact surface. Accordingly, it is assumed, in the theory, that such a relative displacement does take place and, because of symmetry, it occurs on an annulus. Further, the outer edge of the annulus is assumed to coincide with the edge of the contact surface, because it is there that the infinite traction would otherwise occur. The boundary conditions of the theory of elasticity require that there be specified, on the annulus, the tangential traction or displacement or a relation between the two. In this case it has been assumed that the tangential component, τ , of traction at each point of the annulus is proportional to the normal component, σ , at that point. Physically, this is to say that slip takes place on the annulus in such a way that Coulomb's law of friction holds at each point, i.e., $\tau = f\sigma$ where σ is the Hertz normal pressure and f is a constant coefficient of friction. The resulting distribution of tangential traction over the entire contact surface is illustrated in Fig. 1.

As the tangential force is increased, the theory predicts that the inner radius (c) of the annulus of slip diminishes according to the law

$$\left(\frac{c}{a}\right)^3 = 1 - \frac{T}{fN} \tag{4}$$

At the same time, the relative tangential displacement, δ , of distant points in the two spheres depends on the tangential force according to

$$\delta = \frac{3(2-\nu)fN}{8\mu a}\left[1 - \left(1 - \frac{T}{fN}\right)^{2/3}\right] \tag{5}$$

This relation is shown in Fig. 2 along with experimental data, obtained by Johnson [8] with steel spheres, which confirm it. The tangential compliance, to be compared with Equation (3), is

$$S = \frac{d\delta}{dT} = \frac{2-\nu}{4\mu a}\left(1 - \frac{T}{fN}\right)^{-1/3} \tag{6}$$

The next step, in the study of local effects at the contact surfaces, was to determine the consequences of reversal of the sense of the tangential force [9]. If the tangential force, after reaching a magnitude $T = T^* < fN$, is diminished, the force-displacement relation is

$$\delta = \frac{3(2-\nu)fN}{8\mu a}\left[2\left(1 - \frac{T^*-T}{2fN}\right)^{2/3} - \left(1 - \frac{T^*}{fN}\right)^{2/3} - 1\right] \tag{7}$$

This relation is shown as the curve PRS in Fig. 3. Here a new complication is seen to enter, namely, the inelastic (as distinguished from non-linear elastic) character of the tangential load-displacement relation. In the case where the tangential force oscillates between $\pm T^*$ (where $|T^*| < fN$), three important conclusions were reached: (1) slip is confined to an annulus whose inner radius is given by Equation (4) with c and T replaced by c^* and T^*; (2) the amplitude of the relative displacement of the spheres is given by Equation (5) with T replaced by T^*; (3) the force-displacement curve is a loop (Fig. 3) enclosing an area which represents the energy dissipation per cycle:

$$F = \frac{9(2-\nu)f^2N^2}{5\mu a}\left\{1 - \left(1 - \frac{T^*}{fN}\right)^{5/3} - \frac{5T^*}{6fN}\left[1 + \left(1 - \frac{T^*}{fN}\right)^{2/3}\right]\right\} \approx \frac{(2-\nu)(T^*)^3}{18\mu afN}, \quad T^* \ll fN \qquad (8)$$

All of these conclusions have been subjected to experimental test.

Tests by Mindlin, Mason, Osmer and Deresiewicz [10] were made with a pile of three polished glass lenses, pressed together with a normal force following which an oscillating transverse force was applied to the central lens at 60 c.p.s. (Fig. 4). According to the theory, relative displacement at the contact surface occurs only on an annulus, so that wear patterns should be observed only there and with inner radius given by Equation (4). Such patterns were observed (Fig. 5) and the comparison of their dimensions with those predicted by the theory is shown in Fig. 6. Measurements were also made of energy dissipation. At large amplitudes these conformed with Equation (8) but, at small amplitudes, the energy dissipation varied as the square of the tangential force rather than the cube as the theory requires. This was evidence that a velocity dependent factor might contribute to energy dissipation in addition to the static considerations on which Equation (8) is based.

An extensive series of both static and dynamic tests by Johnson [8] bear on many aspects of the theory. His static experiments included loading, unloading, overloading and cyclic loading: all confirming the behavior predicted by the theory. In his dynamic tests, conducted at 46.5 c.p.s. with a variety of sphere diameters and normal loads, Johnson obtained the relations between tangential force and displacement amplitudes shown in Fig. 7. As may be seen, the theory is very good in this respect. The same series of tests yielded data on energy dissipation (Fig. 8) and in this case the theory is not satisfactory. As may be seen, in Fig. 8, the energy dissipation per cycle at small amplitudes is again found to vary as the square of the amplitude, indicating the presence of a velocity dependent mechanism which completely overshadows the static mechanism at very small amplitudes. In addition, there appears to be a geometrical factor, missing in the theory, which is important at intermediate amplitudes, since, in that region, Johnson's experiments reveal a dependence of energy dissipation on both sphere diameter and normal load, which is not accounted for in the theory. It is only at large amplitudes (near gross sliding) that the theory appears to give good results for energy dissipation per cycle.

In addition to normal and tangential forces on the contact surfaces, twisting couples can also be present in a significant amount in certain types of deformation of granular materials. The problems analogous to those described above for tangential forces have also been solved for twisting couples [7,11,12].

Before proceeding to assemblages of spheres it was necessary to carry the theory of pairs of spheres one step farther. Thus far, in both theory and experiment, the normal force was held constant during variation of the tangential force. However, in an assemblage of spheres under varying external load or internal vibration, the normal and tangential forces on a single contact surface vary simultaneously. In this case the inelastic character of the relation between tangential load and displacement introduces a very great complication in that it causes the instantaneous tangential force-displacement relation to depend on the entire past history of normal and tangential loading. Different phenomena are involved and different results obtained depending upon whether the normal or the tangential force is held constant, while the other varies; whether they both vary, and whether the sense of the variation is such that one increases while the other decreases, both increase, or both decrease; whether their relative rate of change is greater or less than the coefficient of friction; whether the immediate past history of loading was in the same or opposite sense as the current loading. For example, suppose that, after applying a normal force N_o, both N and T are increased at an arbitrary relative rate. Then, in place of Equation (6), the tangential compliance is [9]

$$S = \frac{2-\nu}{4\mu a}\left[f\frac{dN}{dT} + \left(1 - f\frac{dN}{dT}\right)\left(1-\frac{T}{fN}\right)^{-1/3}\right], \quad 0 < \frac{dN}{dT} < \frac{1}{f}$$

$$S = \frac{2-\nu}{4\mu a}, \quad \frac{dN}{dT} > \frac{1}{f} \tag{9}$$

where a is the instantaneous radius of the contact surface. Compliances of this type enter into the prediction of failure loads of granular materials. The implications of the form of Equation (9) are discussed below.

Another case, of interest in connection with vibrations of granular materials, is that in which, after an initial normal force N_o is applied, the tangential force oscillates between $\pm T^*$ while the normal force varies in such a way that dN/dT is constant. The tangential compliance during the loading part of the cycle is

$$S = \frac{2-\nu}{4\mu a}\left\{\theta + (1-\theta)\left[1-(1+\theta)\frac{L^*+L}{2(1+\theta L)}\right]^{-1/3}\right\} \tag{10}$$

where

$$L = T/fN_0$$
$$L^* = T^*/fN_0$$
$$\theta = f/\beta$$
$$\beta = dT/dN > f$$

For the unloading part of the cycle the signs of θ and L are reversed in Equation (10). The associated "static" energy dissipation per cycle is

$$F = \frac{9(2-\nu)(fN_0)^2}{10\mu a_0}\left\{\frac{1}{4\theta}\left[\frac{1+\theta}{1-\theta}(1-\theta L^*)^{5/3} - \frac{1-\theta}{1+\theta}(1+\theta L^*)^{5/3}\right.\right.$$
$$\left.\left. - \frac{1}{1-\theta^2}\left(1-\frac{1+5\theta^2}{6}L^*\right)(1-L^*)^{2/3}\right\} \qquad (11)$$

Consider, now, a granular body composed of like spheres. If the body is fully consolidated the arrangement of the spheres is a face-centered cubic or hexagonal array, both of these being arrangements of densest packing. An incompletely consolidated body contains clusters of spheres having such packing. We begin by considering an element of a face-centered cubic array of spheres in equilibrium under an arbitrary state of initial stress and ask what deformation will result from an arbitrary additional increment of stress. This question has been explored in detail recently [13].

The elementary block of the face-centered cubic array is shown in Fig. 9, and the components of incremental force, dP_{ij}, acting on it are shown in Fig. 10. The incremental stress $d\sigma_{ij}$ is defined as the ratio of the incremental force to the area of a face of the block, i.e., $d\sigma_{ij} = dP_{ij}/8R^2$ where R is the radius of the spheres. The deformation of the block, resulting from the application of $d\sigma_{ij}$, can be obtained if the increments of contact force between spheres are known; for then the relative incremental displacements of the spheres can be found by multiplying by the contact compliances.

Each sphere in a face-centered cubic array is in contact with twelve other spheres. Hence there are thirty-six components of contact force on each sphere. However, since we consider, temporarily, a homogeneous state of incremental stress, eighteen of the components of contact force are equal in pairs, leaving only eighteen to be found, of which six are normal components and twelve tangential. The latter are, in turn, related through three equations of moment equilibrium. The eighteen contact forces are related to the stresses $d\sigma_{ij}$ through six independent equilibrium equations so that, in all, there are only nine equations of equilibrium from which to determine eighteen contact forces; that is, the problem is statically indeterminate. It may be solved either by introducing equations of compatibility of relative displacements of spheres (there are nine such equations) or by starting with a set of compatible incremental strains $d\epsilon_{ij}$ and calculating the corresponding contact forces. The latter procedure is simpler since it does not involve the solution of eighteen simultaneous equations. In either case the incremental stress-strain relation is found in the form

$$d\sigma_{ij} = c_{ijkl}\, d\epsilon_{kl} \tag{12}$$

where, for the most general state of initial stress, c_{ijkl} is a non-symmetric tensor having thirty non-zero components when referred to the principal axes of the cubic array. These components are linear functions of the reciprocals of the eighteen initial compliances associated with the twelve contact surfaces. Each of the initial compliances depends, in turn, on the history of the initial stress according to relations such as Equations (9) in which N and T are themselves functions of the stress. Thus the problem of solving Equation (12) to obtain a finite stress-strain relation is a formidable one involving, as it does, the solution of simultaneous, non-linear, integro-differential equations. However, in certain special cases, which can be realized in the laboratory, the integration of the incremental stress-strain relation either can be accomplished or is not necessary.

An example of a test in which the incremental stress-strain relations may be used without integration is that of small vibrations in the presence of high initial stress. In this case the change in stress during vibration can be made so small in comparison with the initial stress that the contact compliances remain essentially constant. Furthermore, if the initial stress is isotropic the incremental stress-strain relation reduces to one of simple cubic symmetry with only three coefficients:

$$\begin{aligned}
d\sigma_{xx} &= c_{11}\, d\epsilon_{xx} + c_{12}(d\epsilon_{yy} + d\epsilon_{zz}) \\
d\sigma_{yy} &= c_{11}\, d\epsilon_{yy} + c_{12}(d\epsilon_{zz} + d\epsilon_{xx}) \\
d\sigma_{zz} &= c_{11}\, d\epsilon_{zz} + c_{12}(d\epsilon_{xx} + d\epsilon_{yy}) \\
d\sigma_{yz} &= 2c_{44}\, d\epsilon_{yz} \\
d\sigma_{zx} &= 2c_{44}\, d\epsilon_{zx} \\
d\sigma_{xy} &= 2c_{44}\, d\epsilon_{xy}
\end{aligned} \qquad (13)$$

where

$$c_{11} = 2c_{44} = \frac{4-3\nu}{\nu}\, c_{12} = \frac{4-3\nu}{2-\nu}\left[\frac{3\mu^2\sigma_0}{2(1-\nu)^2}\right]^{1/3} \qquad (14)$$

in which σ_0 is the initial isotropic stress. In the case of a high frequency vibration, c_{11}, c_{12} and c_{44} must also have imaginary parts; but the theory is not sufficiently developed to write them explicitly, although Johnson's experiments give a good indication of what their form should be. At present, the imaginary parts are omitted. It is then a simple matter to calculate wave velocities or frequencies of vibration of a bar. Such bars were constructed in the following manner [13]. A long rectangular box, lined with a loose rubber sheet, was carefully filled with 1/8" steel balls arranged in face-centered cubic array. The sheet was then folded over, sealed and evacuated. The external pressure locked the balls in place so that the solid "granular bar" could be removed from the box (see Fig. 11). The balls were arranged, in various bars, so that either the [100] or the [110] direction was parallel to the length of the bar so as to eliminate coupling between longitudinal and flexural modes. Thus the bars could be excited in simple axial vibration and their natural frequencies measured as a function of the external pressure. Results of such experiments are shown in Fig. 12. Two sets of data are given: one with balls having a dimensional tolerance of $\pm 50 \times 10^{-6}$ in. and the other $\pm 10 \times 10^{-6}$ in. As may be seen, the frequencies of the bar made with the better balls are closer to the theoretical frequencies and the agreement improves in both cases with increasing pressure. The reason for this becomes apparent when the dimensional tolerances are compared with the relative approach of the balls under the initial pressure. When $\sigma_o = 2$ psi, $\alpha = 1.955 \times 10^{-6}$ in. and when $\sigma_o = 14.7$ psi, $\alpha = 7.39 \times 10^{-6}$ in. Thus many spheres may be expected to be under larger and smaller initial contact forces than if all spheres were identical in size and, also, some spheres may be loose. It may be shown that the presence of off-size or loose spheres diminishes the stiffness (and hence the frequency of vibration) of the array and the diminution becomes greater with increased spread of the dimensional tolerance and reduction of pressure. These effects are reflected in the data shown in Fig. 12.

Measurements of logarithmic decrement of the vibrations were also made, but they cannot be compared with the theory until the imaginary parts of the compliances are introduced into Equations (14).

Regarding integration of incremental stress-strain relations, there is a case which can be handled without difficulty. This is the problem of a simple cubic array of spheres under an initial isotropic stress, subjected subsequently to homothetic loading. The simple cubic array is statically determinate, so that the contact forces can be calculated without reference to the loading history. Furthermore dN/dT , in Equation (9), is a constant for homothetic loading, i.e., if the additional stress quadric is always similar and similarly oriented with respect to its previous form. Accordingly, the general system of simultaneous integro-differential equations reduces to a set of quadratures and these, it turns out, are expressible in closed form [14].

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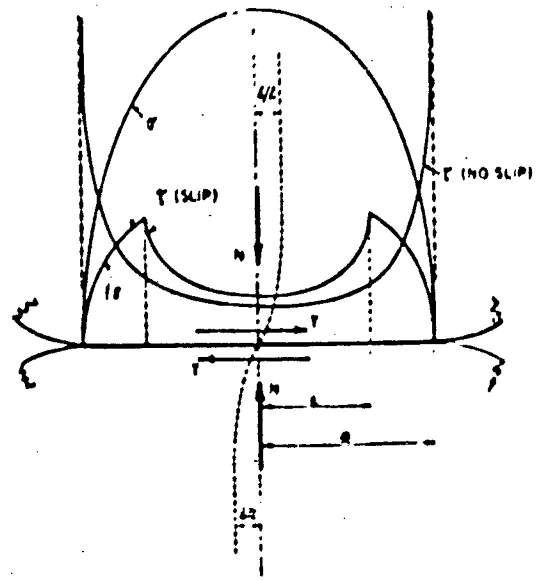

Fig. 1: Distributions of normal (σ) and tangential (τ) tractions on the contact surface of a pair of spherical grains.

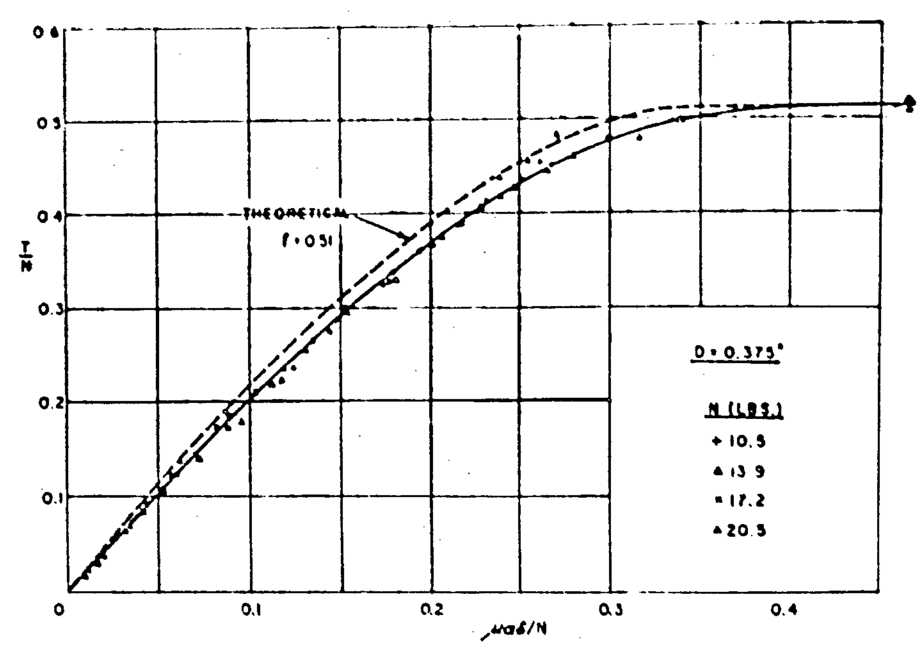

Fig. 2: Static, tangential force-displacement relation. Comparison of Equation (5) with experimental data by Johnson.

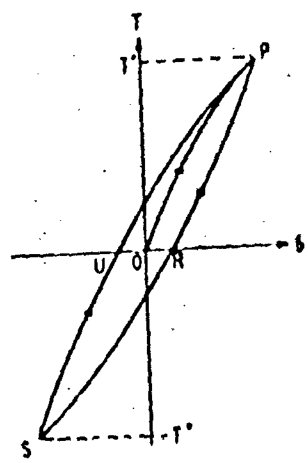

Fig. 3: Inelastic character of static, tangential force-displacement relation.

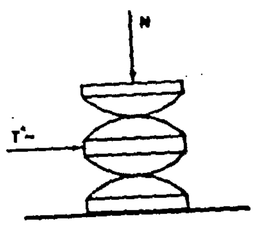

Fig. 4: Arrangement of glass lenses in tests (Ref. [10]).

Fig. 5: Annulus obtained in tests with glass lenses (Ref. [10]).

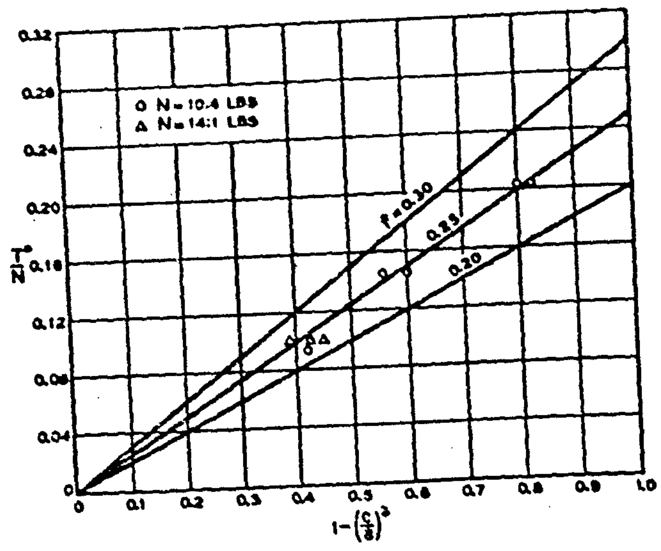

Fig. 6: Dimensions of annuli obtained in tests with glass lenses. Comparison of experimental data with Equation (4).

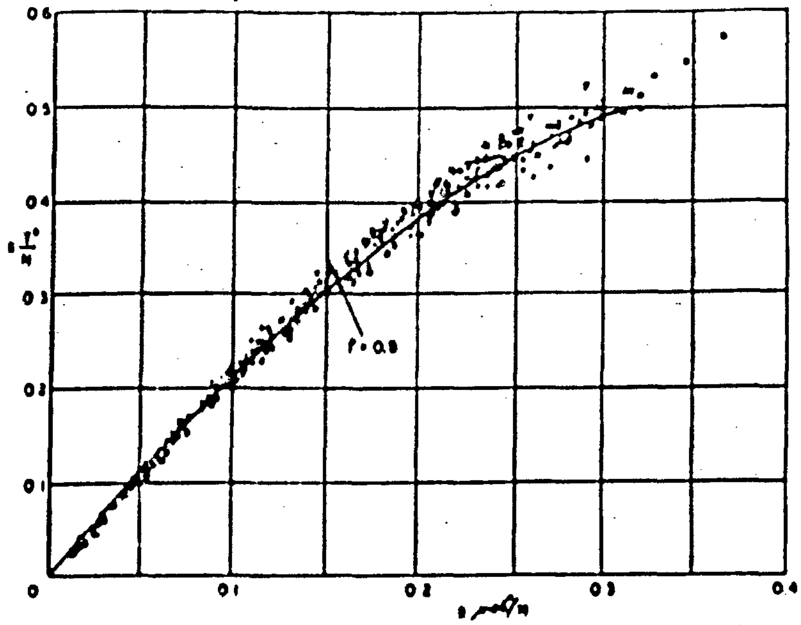

Fig. 7: Dynamic, tangential, force-displacement relation. Comparison of Equation (5) with experimental data by Johnson. Ball diameters and normal loads same as in Fig. 8.

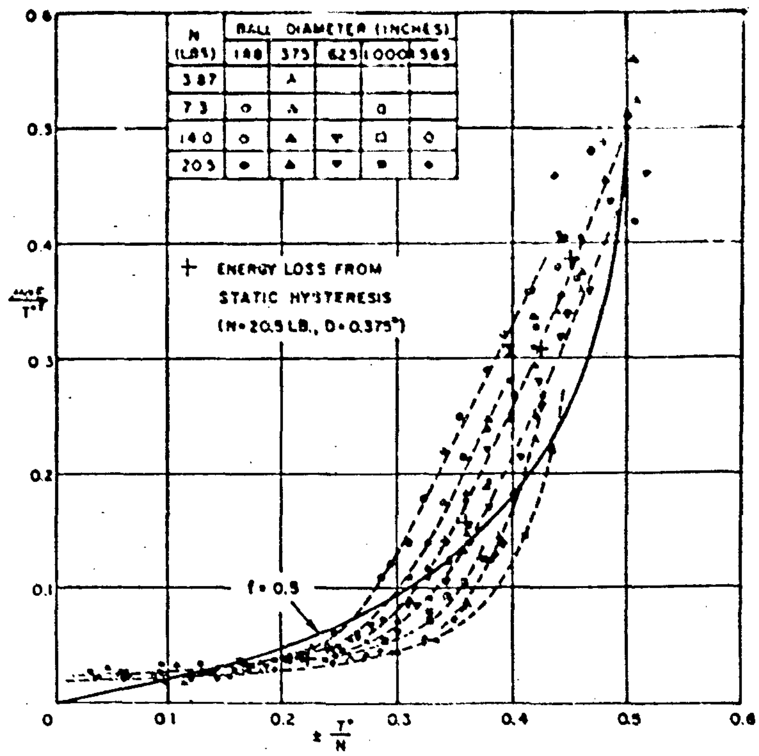

Fig. 8: Energy dissipation per cycle as a function of tangential force amplitude. Comparison of Equation (8) with experimental data by Johnson.

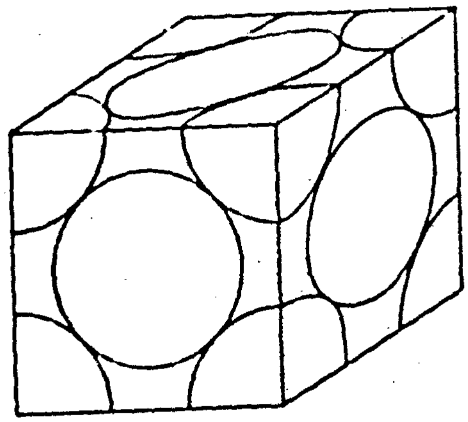

Fig. 9: Element of volume of a face-centered cubic array of spheres.

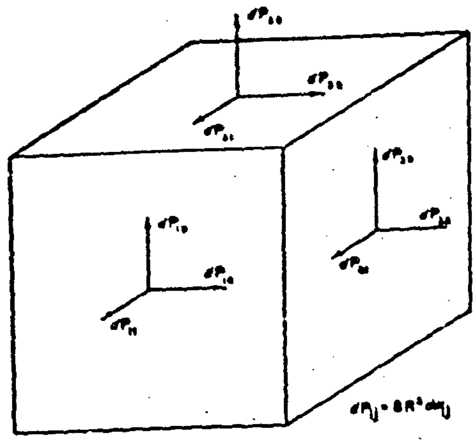

Fig. 10: Incremental forces acting on the faces of an element of volume of a face-centered cubic array of spheres.

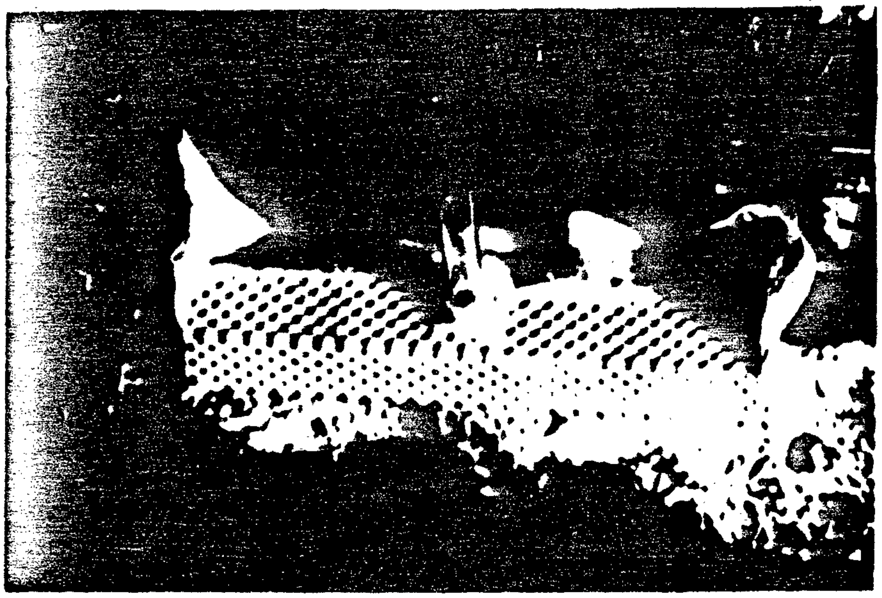

Fig. 11: "Granular bar" made of 1/8" steel balls.

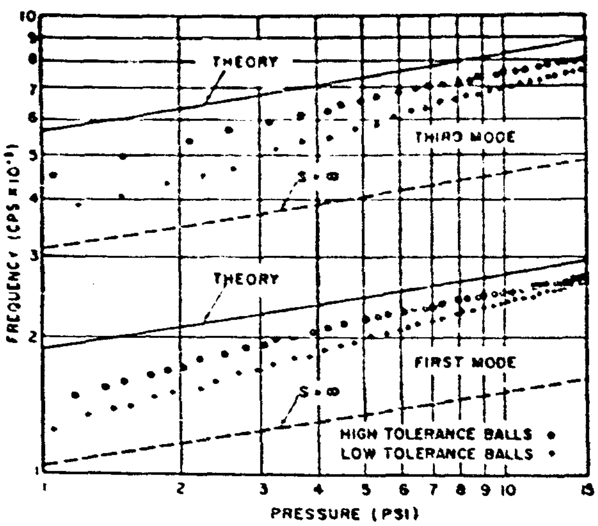

Fig. 12: Frequencies of first and third modes of vibration of granular bar as a function of the initial pressure. Comparison of theory and experiment.

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