

THIS REPORT HAS BEEN DELIMITED
AND CLEARED FOR PUBLIC RELEASE
UNDER DOD DIRECTIVE 5200.20 AND
NO RESTRICTIONS ARE IMPOSED UPON
ITS USE AND DISCLOSURE.

DISTRIBUTION STATEMENT A

APPROVED FOR PUBLIC RELEASE;
DISTRIBUTION UNLIMITED.

Armed Services Technical Information Agency

Because of our limited supply, you are requested to return this copy WHEN IT HAS SERVED YOUR PURPOSE so that it may be made available to other requesters. Your cooperation will be appreciated.

AD

42257

NOTICE: WHEN GOVERNMENT OR OTHER DRAWINGS, SPECIFICATIONS OR OTHER DATA ARE USED FOR ANY PURPOSE OTHER THAN IN CONNECTION WITH A DEFINITELY RELATED GOVERNMENT PROCUREMENT OPERATION, THE U. S. GOVERNMENT THEREBY INCURS NO RESPONSIBILITY, NOR ANY OBLIGATION WHATSOEVER; AND THE FACT THAT THE GOVERNMENT MAY HAVE FORMULATED, FURNISHED, OR IN ANY WAY SUPPLIED THE SAID DRAWINGS, SPECIFICATIONS, OR OTHER DATA IS NOT TO BE REGARDED BY IMPLICATION OR OTHERWISE AS IN ANY MANNER LICENSING THE HOLDER OR ANY OTHER PERSON OR CORPORATION, OR CONVEYING ANY RIGHTS OR PERMISSION TO MANUFACTURE, USE OR SELL ANY PATENTED INVENTION THAT MAY IN ANY WAY BE RELATED THERETO.

Reproduced by
DOCUMENT SERVICE CENTER
KNOTT BUILDING, DAYTON, 2, OHIO

UNCLASSIFIED

ON MINIMIZING AN EXPECTATION SUBJECT TO
CERTAIN SIDE-CONDITIONS

BY

STANLEY ISAACSON AND HERMAN RUBIN

TECHNICAL REPORT NO. 25

JULY 30, 1954

This work was sponsored by the Army, Navy, and Air Force through the Joint Services Advisory Committee for Research Groups in Applied Mathematics and Statistics by Contract No. N6onr 25140 (NR-342-022)

APPLIED MATHEMATICS AND STATISTICS LABORATORY
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

ON MINIMIZING AN EXPECTATION SUBJECT TO
CERTAIN SIDE-CONDITIONS

BY

STANLEY ISAACSON AND HERMAN RUBIN

1. Introduction.

Chernoff and Reiter [1] have recently considered the problem of choosing a cumulative distribution function F so as to minimize

$$(1.1) \quad E = \int g(x) d F(x)$$

subject to the conditions

$$(1.2) \quad \int x d F(x) = c_1 ,$$

$$(1.3) \quad \int x^2 d F(x) = c_2 ,$$

where c_1 and c_2 are given positive constants $c_2 \geq c_1^2$ and

$$(1.4) \quad g(x) = 1 - e^{-\beta x} , \quad (\beta > 0) .$$

This problem arose as a result of a bioassay investigation. In this paper we consider a related problem resulting from further consideration of the same bioassay problem. We too are interested in minimizing an expectation involving the function g of (1.4) subject to side-conditions; but in our problem we make certain additional assumptions and hence the lower bound on the expectation which we obtain is greater than that obtained by Chernoff and Reiter.

The problem we consider is the following: We have a countably infinite number of points $u_1, u_2, \dots$ which are uniformly and independently distributed over a two-dimensional region R of infinite area; hence the distribution of the number of points occurring in a subregion of R with unit area is a Poisson random variable with a parameter we shall call λ . The distribution of the number of points occurring in a subregion R_j of finite area A_j is therefore also Poisson with parameter λA_j . A certain type of material emanates from each of the points u_i and spreads over the region R . We consider two different ways by which the material may spread over R :

Case (i): The concentration at an arbitrary point x in R of material emanating from a particular point u_i is given by $f(u_i - x)$ where f is a non-negative fixed function and $u_i - x$ is the ordinary vector difference between u_i and x . Thus the total concentration of material at an arbitrary point x in R is

$$(1.5) \quad T(x) = \sum_{i=1}^{\infty} f(u_i - x) .$$

Case (ii): The concentration at an arbitrary point x in R of material emanating from a particular point u_i is given by $f_i(u_i - x)$ where f_i is now a non-negative random function. We assume that the f_i are identically and independently distributed. Thus the total concentration of material at an arbitrary point x in R is

$$(1.6) \quad T_r(x) = \sum_{i=1}^{\infty} f_i(u_i - x) .$$

We note that T in (1.5) is a random function since the u_i are random, while T_r in (1.6) is a random function due to the fact that the u_i and f_i are both random. Thus Case (ii) deals with a more general situation than Case (i).

In Case (i) we are interested in choosing f so as to minimize

$$(1.7) \quad E g(T(x_0))$$

for some arbitrary fixed point x_0 in R subject to the conditions that

$$(1.8) \quad E T(x_0) = \mu (> 0) ,$$

$$(1.9) \quad \text{Var}(T(x_0)) = \sigma^2 ,$$

where μ and σ^2 are given numbers and g is given by (1.4). We solve this problem in section 2 of this paper. It is shown that the minimum value of (1.7) subject to (1.8) and (1.9) is given by

$$(1.10) \quad E_{\min} = 1 - e^{-\frac{1}{v^2} (1 - e^{-\gamma v^2})}$$

where $v^2 = \frac{\sigma^2}{\mu^2}$ is the squared coefficient of variation and $\gamma = \beta\mu$.

In Case (ii) we are interested in choosing the distribution of the random variables f_i in such a manner as to minimize

$$(1.11) \quad E g(T_r(x_0))$$

subject to the conditions that

$$(1.12) \quad E T_r(x_0) = \mu (> 0) ,$$

$$(1.13) \quad \text{Var}(T_r(x_0)) = \sigma^2,$$

where μ and σ^2 are given numbers and g is again given by (1.4). This problem is solved in section 3. The answer to Case (ii) is the same as to Case (i).

Chernoff and Reiter [1] obtained

$$(1.14) \quad E_{\min} = \frac{1 - e^{-\gamma(1 + v^2)}}{1 + v^2}$$

as the lower bound of (1.1) subject to (1.2) and (1.3). We show in section 4 that (1.10) for any value of v^2 is never less than (1.14) for the same value of v^2 . To compare these lower bounds, the values of γ for which

$$(1.15) \quad 1 - e^{-\gamma} = .90, .85, .80, \dots, .05, .01$$

have been selected. For these values of γ at both (1.10) and (1.14) have been plotted as a function of v^2 , for $0 \leq v^2 \leq 6$. The graphs are given at the end of this paper.

2. Consideration of Case (i).

In this section we consider the problem of choosing f so as to minimize (1.7) subject to side-conditions (1.8) and (1.9). Since x_0 is an arbitrary point in R , we may take it to be the origin without any loss of generality. Our problem thus is to choose a non-negative function f so as to minimize

$$(2.1) \quad E g(T(0)) = E \left(1 - e^{-\beta \left(\sum_{i=1}^{\infty} f(u_i) \right)} \right)$$

subject to

$$(2.2) \quad E(T(0)) = E\left(\sum_{i=1}^{\infty} f(u_i)\right) = \mu (> 0),$$

and

$$(2.3) \quad \text{Var}(T(0)) = \text{Var}\left(\sum_{i=1}^{\infty} f(u_i)\right) = \sigma^2,$$

where μ and σ^2 are given constants.

We will first find the characteristic function of $T(0)$ and use this to get explicit expressions for (2.1), (2.2), and (2.3). Let $\varphi(t)$ be the characteristic function of $T(0)$ and let $\{R_j\}$ be a partition of R into a denumerable number of mutually exclusive and exhaustive subregions R_j such that each R_j has a finite area A_j . Then, using the fact that the u_i are independently distributed and that the number of points in a subregion R_j is a Poisson variate with parameter λA_j , we obtain

$$\begin{aligned} \varphi(t) &= E e^{it T(0)} = E e^{it \sum_{i=1}^{\infty} f(u_i)} = E e^{it \sum_{j=1}^{\infty} \sum_{u_i \in R_j} f(u_i)} \\ &= \prod_{j=1}^{\infty} E e^{it \sum_{u_i \in R_j} f(u_i)} = \prod_{j=1}^{\infty} \sum_{n=0}^{\infty} P_j(n) E_{R_j} e^{it \sum_{i=1}^n f(u_i)} \\ &= \prod_{j=1}^{\infty} \sum_{n=0}^{\infty} e^{-\lambda A_j} \frac{(\lambda A_j)^n}{n!} [E_{R_j} e^{it f(u)}]^n = \prod_{j=1}^{\infty} e^{\lambda A_j (E_j(t) - 1)} \\ &= e^{\lambda \sum_{j=1}^{\infty} (A_j E_j(t) - A_j)} = e^{\lambda \sum_{j=1}^{\infty} \int_{R_j} (e^{it f(u)} - 1) du} = e^{\lambda \int_R (e^{it f(u)} - 1) du}, \end{aligned}$$

where $P_j(n)$ = the probability of n of the points u_i being in the region R_j ,

and

$$E_{R_j} e^{it f(u)} = E_j(t) = \frac{1}{A_j} \int_{R_j} e^{it f(u)} du .$$

Thus we see that

$$(2.4) \quad \varphi(t) = E e^{it T(0)} = e^{\lambda \int_R (e^{it f(u)} - 1) du} .$$

In view of this result, it follows from (2.1) that

$$(2.5) \quad E g(T(0)) = 1 - \varphi(i/\beta) = 1 - e^{-\lambda \int_R (1 - e^{-\beta f(u)}) du} .$$

In order to get the mean and variance of $T(0)$, we take $\Psi(t) = \log \varphi(t)$, and consider the first two derivatives of $\Psi(t)$ at $t = 0$. Differentiating under the integral sign, we get

$$(2.6) \quad E T(0) = \frac{\Psi'(t)|_{t=0}}{i} = \frac{(\log \varphi(t))'|_{t=0}}{i} \\ = \frac{\lambda \int_R i f(u) e^{it f(u)} du |_{t=0}}{i} = \lambda \int_R f(u) du ;$$

and

$$(2.7) \quad \text{Var}(T(0)) = \frac{\Psi''(t)|_{t=0}}{i^2} = \frac{\lambda \int_R i^2 f^2(u) e^{it f(u)} du |_{t=0}}{i^2} = \lambda \int_R f^2(u) du .$$

Substituting (2.6) and (2.7) into (2.2) and (2.3) respectively, we obtain

$$(2.8) \quad \int_{\mathbb{R}} f(u) \, du = \frac{\mu}{\lambda},$$

and

$$(2.9) \quad \int_{\mathbb{R}} f^2(u) \, du = \frac{\sigma^2}{\lambda};$$

here μ and σ^2 are given constants and λ is an unknown constant. Our problem thus is to choose f so as to minimize (2.5) subject to side-conditions (2.8) and (2.9). We shall see that we need not know λ in order to solve this problem. Now it is obvious that (2.5) is a minimum when

$$(2.10) \quad \int_{\mathbb{R}} (1 - e^{-\beta f(u)}) \, du$$

is a minimum. Thus our problem is equivalent to minimizing (2.10) as a function of f subject to (2.8) and (2.9). Since f is non-negative, this problem is equivalent to finding a measure H defined for $a > 0$ so as to minimize

$$(2.11) \quad \int (1 - e^{-\beta a}) \, dH(a)$$

subject to

$$(2.12) \quad \int a \, dH(a) = \frac{\mu}{\lambda}$$

$$(2.13) \quad \int a^2 \, dH(a) = \frac{\sigma^2}{\lambda}.$$

In order to solve this problem, we make use of the Hahn-Banach Theorem (see Munroe [2], pp. 56-57) together with certain unpublished results due to Rubin.

The Hahn-Banach theorem may be stated as follows: If $\mathcal{N}$ is a linear subspace of the linear space $\mathcal{M}$, if λ is subadditive functional on $\mathcal{M}$ such that $\lambda(a\varphi) = a\lambda(\varphi)$ for $a \geq 0$, and if L is an additive, homogeneous functional on $\mathcal{N}$ such that $L(\varphi) \leq \lambda(\varphi)$ for every $\varphi \in \mathcal{N}$, then there is an additive, homogeneous extension $\bar{L}$ of L to the whole of $\mathcal{M}$, such that

$$\bar{L}(\varphi) \leq \lambda(\varphi)$$

for every $\varphi \in \mathcal{M}$; furthermore, for some given $\varphi_0 \in \mathcal{N} \subset \mathcal{M}$, the extension $\bar{L}$ of L may be chosen so to minimize $\bar{L}(\varphi_0)$; this may be done by defining

$$(2.14) \quad \bar{L}(\varphi_0) = \sup_{\Psi \in \mathcal{N}} [\lambda(\Psi - \varphi_0) - L(\Psi)] .$$

To apply the Hahn-Banach theorem to our problem, let $\mathcal{N}$ be the totality of functions of the form $\alpha a + \delta a^2$ where α and δ are given constants and $a \geq 0$; then let $\mathcal{M}$ consists of all continuous functions which are defined for $a \geq 0$ and which are bounded by elements of $\mathcal{N}$. Thus $\mathcal{M}$ is a linear space and $\mathcal{N}$ is a linear subspace of $\mathcal{M}$. Let $\varphi_0 = 1 - e^{-\beta a}$; we note that $\varphi_0 \in \mathcal{M}$. Now we define

$$(2.15) \quad L(\alpha a + \delta a^2) = \frac{\alpha \mu}{\lambda} + \frac{\delta \sigma^2}{\lambda} .$$

Thus L is defined for all elements of $\mathcal{N}$ and is obviously an additive, homogeneous functional on $\mathcal{N}$; furthermore if $\alpha a + \delta a^2$ is a positive function, then L is positive. Now if we define for all $\varphi \in \mathcal{M}$,

$$(2.16) \quad \lambda(\varphi) = \inf_{\substack{\Psi \in \mathcal{N} \\ \Psi \geq \varphi}} L(\Psi) ,$$

then λ as can be readily verified satisfies the hypothesis of the Hahn-Banach theorem. Thus an extension $\bar{L}$ of L to the whole of $\mathcal{M}$ such that $\bar{L}(\varphi_0) = \bar{L}(1 - e^{-\beta a})$ is a minimum exists, and by (2.14)

$$(2.17) \quad \bar{L}(\varphi_0) = \sup_{\psi \in \mathcal{N}} [-\lambda(-\psi - \varphi_0) - L(\psi)] .$$

Now, we can simplify (2.17) as follows:

$$\begin{aligned} -\lambda(-\psi - \varphi_0) &= - \inf_{\substack{\xi \in \mathcal{N} \\ \xi \geq -\psi - \varphi_0}} L(\xi) = \sup_{\substack{\xi \in \mathcal{N} \\ \xi \leq \psi + \varphi_0}} L(\xi) = \sup_{\substack{\xi \in \mathcal{N} \\ \xi - \psi \leq \varphi_0}} L(\xi) \\ &= \sup_{\substack{\mathcal{J} \in \mathcal{N} \\ \mathcal{J} \leq \varphi_0}} L(\psi + \mathcal{J}) = \sup_{\substack{\mathcal{J} \in \mathcal{N} \\ \mathcal{J} \leq \varphi_0}} \{L(\psi) + L(\mathcal{J})\} \\ &= L(\psi) + \sup_{\substack{\mathcal{J} \in \mathcal{N} \\ \mathcal{J} \leq \varphi_0}} L(\mathcal{J}) = L(\psi) - \lambda(-\varphi_0) . \end{aligned}$$

Hence in (2.17) we have

$$\bar{L}(\varphi_0) = \sup_{\psi \in \mathcal{N}} \{-\lambda(-\varphi_0)\} = -\lambda(-\varphi_0) ,$$

so that

$$(2.18) \quad \bar{L}(\varphi_0) = -\lambda(-\varphi_0) = \sup_{\substack{\mathcal{J} \in \mathcal{N} \\ \mathcal{J} \leq \varphi_0}} L(\mathcal{J}) .$$

Now we have seen that L is positive on $\mathcal{N}$ (i.e., L is positive for any positive function on $\mathcal{N}$); hence in order to calculate $\bar{L}(\varphi_0)$, we must

take $f \in \mathcal{M}$ so that $L(f)$ is as large as possible subject to the constraint that $f \leq \varphi_0$. But the condition $f \leq \varphi_0$ is in our case the conditions

$$(2.19) \quad \alpha a + \delta a^2 \leq 1 - e^{-\beta a}.$$

We must therefore choose α and δ so that $L(\alpha a + \delta a^2)$ is as large as possible subject to the constraint that $\alpha a + \delta a^2$ is always less than $1 - e^{-\beta a}$, since $a > 0$, (2.19) is equivalent to

$$(2.20) \quad \alpha + \delta a \leq \frac{1 - e^{-\beta a}}{a}.$$

But now we are in a position to choose α and δ so that $L(f)$ will be a maximum subject to the constraint (2.20). On the left-hand side we have a linear function of a ; on the right-hand side, a monotone-decreasing concave function of a . Hence we choose α and δ so that the straight line will be tangent to the concave function at some point a_0 ; and then choose a_0 so that $L(\varphi_0)$ will be a maximum. Now if $\alpha + \delta a$ is to be tangent to $\frac{1 - e^{-\beta a}}{a}$ at a_0 , these functions and their first derivations must be equal at a_0 . Hence we have

$$(2.21) \quad \alpha + \delta a_0 = \frac{1 - e^{-\beta a_0}}{a_0}$$

and

$$(2.22) \quad \delta = - \frac{1 - e^{-\beta a_0} (1 + a_0 \beta)}{a_0^2}$$

consequently

$$(2.23) \quad \alpha = \frac{1 - e^{-\beta a_0}}{a_0} - \delta a_0 = \frac{2 - e^{-\beta a_0} (2 + \beta a_0)}{a_0}.$$

Using these values of α and δ and (2.23) and being the definition of L in (2.15), we obtain

$$(2.24) \quad \bar{L}(\varphi_0) = \bar{L}(1 - e^{-\beta a_0})$$

$$= \left(\frac{2 - e^{-\beta a_0}(2 + \beta a_0)}{a_0} \right) \frac{\mu}{\lambda} - \frac{\sigma^2}{\lambda} \left(\frac{1 - e^{-\beta a_0}(1 + a_0\beta)}{a_0^2} \right).$$

Now we must choose a_0 so to make (2.24) as large as possible. It is easy to verify that (2.24) is maximized by taking

$$(2.25) \quad a_0 = \frac{\sigma^2}{\mu}.$$

For this choice of a_0 , we obtain from (2.24),

$$(2.26) \quad \bar{L}(\varphi_0) = \frac{\mu^2}{\lambda \sigma^2} (1 - e^{-\beta \frac{\sigma^2}{\mu}}).$$

Thus we have used the Hahn-Banach to extend the linear functional given in (2.15) from $\mathcal{N}$ to $\mathcal{M}$ in such a manner as to obtain the minimum value for $\bar{L}(\varphi_0)$ which is given in (2.26).

The question remains whether the extended linear functional constitutes a measure; that it does follows from results of an unpublished paper by Rubin which is applicable to this situation. Furthermore, it follows from Rubin's results that this measure concentrates all its weight on the single point a_0 .

Thus we have seen that there exists a measure which subject to the side-conditions (2.12) and (2.13) assigns the minimum value

$$(2.27) \quad \frac{\mu^2}{\lambda \sigma^2} (1 - e^{-\beta \frac{\sigma^2}{\mu}})$$

to (2.11). We are really interested in the minimum of (2.5) subject to (2.8) and (2.9); we now see immediately that this minimum is given by

$$(2.28) \quad 1 - e^{-\frac{\mu^2}{\sigma^2} (1 - e^{-\beta \frac{\sigma^2}{\mu}})} = 1 - e^{-\frac{1}{v^2} (1 - e^{-\gamma v^2})},$$

where v^2 is the squared coefficient of variation defined by

$$(2.29) \quad v^2 = \frac{\sigma^2}{\mu^2},$$

and

$$(2.30) \quad \gamma = \beta \mu.$$

In passing, we note that an argument similar to the one above will show that no measure exists which maximizes (2.5) subject to (2.8) and (2.9); but that a least upper bound for (2.5) subject to (2.8) and (2.9) is given by $(1 - e^{-\gamma})$.

3. Consideration of Case (ii).

In this section we consider the problem of choosing a distribution for the independently and identically distributed random functions f_i in such a manner as to minimize (1.11) subject to (1.12) and (1.13). Much of the argument is the same as given in section 2 for Case (i); we shall only sketch these portions of the argument in this section.

We again take x_0 as the origin. We then have

$$(3.1) \quad T_r(0) = \sum_{i=1}^{\infty} f_i(u_i).$$

By an argument exactly like the one employed to obtain (2.4), we can show that the characteristic function $\varphi_r(t)$ of $T_r(0)$ is given by

$$(3.2) \quad \varphi_r(t) = E e^{it T_r(0)} = e^{\lambda \int_R (E_f e^{it f(u)} - 1) du}$$

where by E_f we mean the expectation taken with respect to the distribution of the random function f . Using (3.2) we obtain,

$$(3.3) \quad E(g(T_r(0))) = 1 - \varphi(i/\beta) = 1 - e^{-\lambda \int_R (1 - E_f e^{-\beta f(u)}) du};$$

$$(3.4) \quad \int_R (E_f f(u)) du = \frac{\mu}{\lambda};$$

$$(3.5) \quad \int_R (E_f f^2(u)) du = \frac{\sigma^2}{\lambda},$$

where μ and σ^2 are given numbers.

Our problem is to choose the distribution of f so as to minimize (3.3) subject to the side-conditions (3.4) and (3.5). This is equivalent to minimizing

$$(3.6) \quad \int_R E_f (1 - e^{-\beta f(u)}) du$$

subject to (3.4) and (3.5) since f is non-negative, we can solve this problem in two stages. First holding u fixed, we will choose a distribution function for $f(u)$ minimizing

$$(3.7) \quad E_f (1 - e^{-\beta f(u)})$$

subject to

$$(3.8) \quad E_f f(u) = m_1(u)$$

and

$$(3.9) \quad E_f f^2(u) = m_2(u) ,$$

where m_1 and m_2 are for the present considered as given functions ($m_2 \geq m_1^2$). Then, having solved this problem, we will choose $m_1(u)$ and $m_2(u)$ so as to minimize the expression obtained by substituting the minimum value of (3.7) into (3.6) subject to the conditions

$$(3.10) \quad \int_R m_1(u) du = \frac{\mu}{\lambda} ,$$

and

$$(3.11) \quad \int_R m_2(u) du = \frac{\sigma^2}{\lambda} .$$

The problem of minimizing (3.7) subject to (3.8) and (3.9) is the one solved by Chernoff and Reiter [1]. Their solution is to take the c.d.f. $\tilde{G}(u)$ of $f(u)$ to be

$$(3.12) \quad \tilde{G}(u) = \begin{cases} 0 & \text{for } f(u) < 0 \\ 1 - \frac{m_1^2(u)}{m_2(u)} & \text{for } 0 \leq f(u) < \frac{m_2(u)}{m_1(u)} \\ 1 & \text{for } f(u) \geq \frac{m_2(u)}{m_1(u)} \end{cases} .$$

Using the solution given by (3.12), the minimum value of (3.7) is given by

$$(3.13) \quad (1 - e^{-\beta \frac{m_2(u)}{m_1(u)}}) \frac{m_1^2(u)}{m_2(u)}.$$

Thus our problem now becomes that of choosing the functions m_1 and m_2 ($m_2 \geq m_1^2$) so as to minimize

$$(3.14) \quad \int_R (1 - e^{-\beta \frac{m_2(u)}{m_1(u)}}) \frac{m_1^2(u)}{m_2(u)} du$$

subject to (3.10) and (3.11).

Since $m_2 \geq m_1^2 \geq 0$, if we define

$$(3.15) \quad \begin{aligned} x &= m_1 \\ y &= m_2 - m_1^2, \end{aligned}$$

then we see that our problem is that of finding a measure $F(x,y)$ defined for $x > 0, y > 0$ so as to minimize

$$(3.16) \quad \int (1 - e^{-\beta(x + \frac{y}{x})}) \frac{x^2}{x^2 + y} d F(x,y)$$

subject to

$$(3.17) \quad \int x d F(x,y) = \frac{\mu}{\lambda},$$

and

$$(3.18) \quad \int (x^2 + y) d F(x,y) = \frac{\sigma^2}{\lambda}.$$

We can again apply the Hahn-Banach theorem and Rubin's results. $\mathcal{N}$ will be the totality of functions of the form $\alpha x + \delta(x^2 + y)$ where

α and δ are constants and $x > 0, y > 0$; $\mathcal{M}$ will consist of all continuous functions defined for $x > 0, y > 0$ which are bounded by elements of $\mathcal{N}$.

We let

$$(3.19) \quad \varphi_0 = \frac{x^2}{x^2 + y} (1 - e^{-\beta(x + \frac{y}{x})}).$$

We then define

$$(3.20) \quad L(\alpha x + \delta(x^2 + y)) = \frac{\alpha}{\lambda} + \frac{\delta \sigma^2}{\lambda},$$

and

$$(3.21) \quad \lambda(\varphi) = \inf_{\substack{\psi \in \mathcal{N} \\ \psi \geq \varphi}} L(\psi).$$

Then with these definitions the minimum value of $\bar{L}(\varphi_0)$ subject to (3.17) and (3.16) is again given by

$$(3.22) \quad \bar{L}(\varphi_0) = \sup_{\substack{\mathcal{J} \in \mathcal{N} \\ \mathcal{J} \leq \varphi_0}} L(\mathcal{J}).$$

The condition $\mathcal{J} \leq \varphi_0$ is in our case

$$(3.23) \quad \alpha x + \delta(x^2 + y) \leq \frac{x^2}{x^2 + y} (1 - e^{-\beta(x + \frac{y}{x})}),$$

or, since $x > 0$,

$$(3.24) \quad \alpha + \delta(x + \frac{y}{x}) \leq \frac{x}{x^2 + y} (1 - e^{-\beta(x + \frac{y}{x})}).$$

Now if we let

$$(3.25) \quad a = x + \frac{y}{x},$$

then $a > 0$, and (3.24) becomes

$$(3.26) \quad \alpha + \delta a \leq \frac{1 - e^{-\beta a}}{a}.$$

But (3.26) is exactly the same condition as (2.20). Hence α and δ are again given by (2.22) and (2.23), a_0 is given by (2.25), and $L(\varphi_0)$ by (2.26). Once again Rubin's results are applicable and tell us that the extension of L which we have obtained actually defines a measure. Thus we see that the minimum of (3.3) subject to (3.4) and (3.5) is given by

$$(3.27) \quad 1 - e^{-\frac{\mu^2}{\sigma^2} (1 - e^{-\beta \frac{\sigma^2}{\mu}})} = 1 - e^{-\frac{1}{v^2} (1 - e^{-\gamma v^2})},$$

which, of course, is the same expression obtained for the lower bound in Case (i). Thus allowing f to be random does not make matters worse and the lower bound for the expectation remains the same.

4. Comparisons of the Lower Bound.

Chernoff and Reiter [1] obtained the lower bound

$$(4.1) \quad G(v^2, \gamma) = \frac{1 - e^{-\gamma(1 + v^2)}}{1 + v^2}.$$

We obtain the lower bound,

$$(4.2) \quad H(v^2, \gamma) = 1 - e^{-\frac{1}{v^2} (1 - e^{-\gamma v^2})}.$$

In both (4.1) and (4.2), $\gamma > 0$.

In this section we will show that for a fixed value of v^2 , (4.2) is never less than (4.1).

First we note that

$$(4.3) \quad G(v^2, 0) = H(v^2, 0) = 0.$$

Now let G_γ and H_γ denote the partial derivatives of G and H with respect to γ . Then

$$(4.4) \quad G_\gamma = e^{-\gamma(1+v^2)}$$

and

$$(4.5) \quad H_\gamma = e^{-\gamma v^2 - \frac{1}{v^2}(1 - e^{-\gamma v^2})}.$$

If for fixed v^2 , we can show that $H_\gamma \geq G_\gamma$ for all values of γ , then we have, of course, completed our proof. But $H_\gamma \geq G_\gamma$ is equivalent to

$$(4.6) \quad \gamma v^2 + \frac{1}{v^2}(1 - e^{-\gamma v^2}) \leq \gamma(1 + v^2)$$

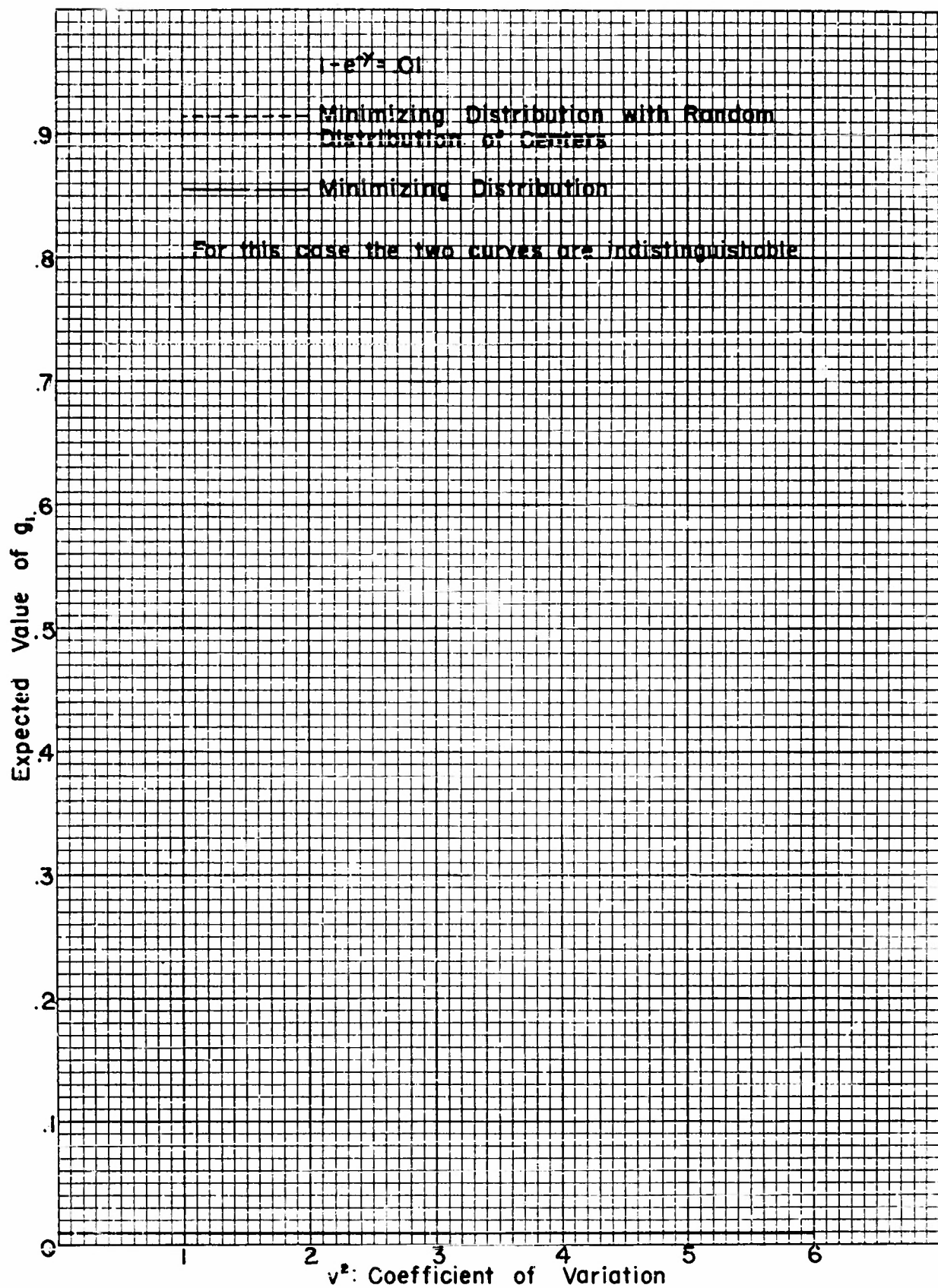
or

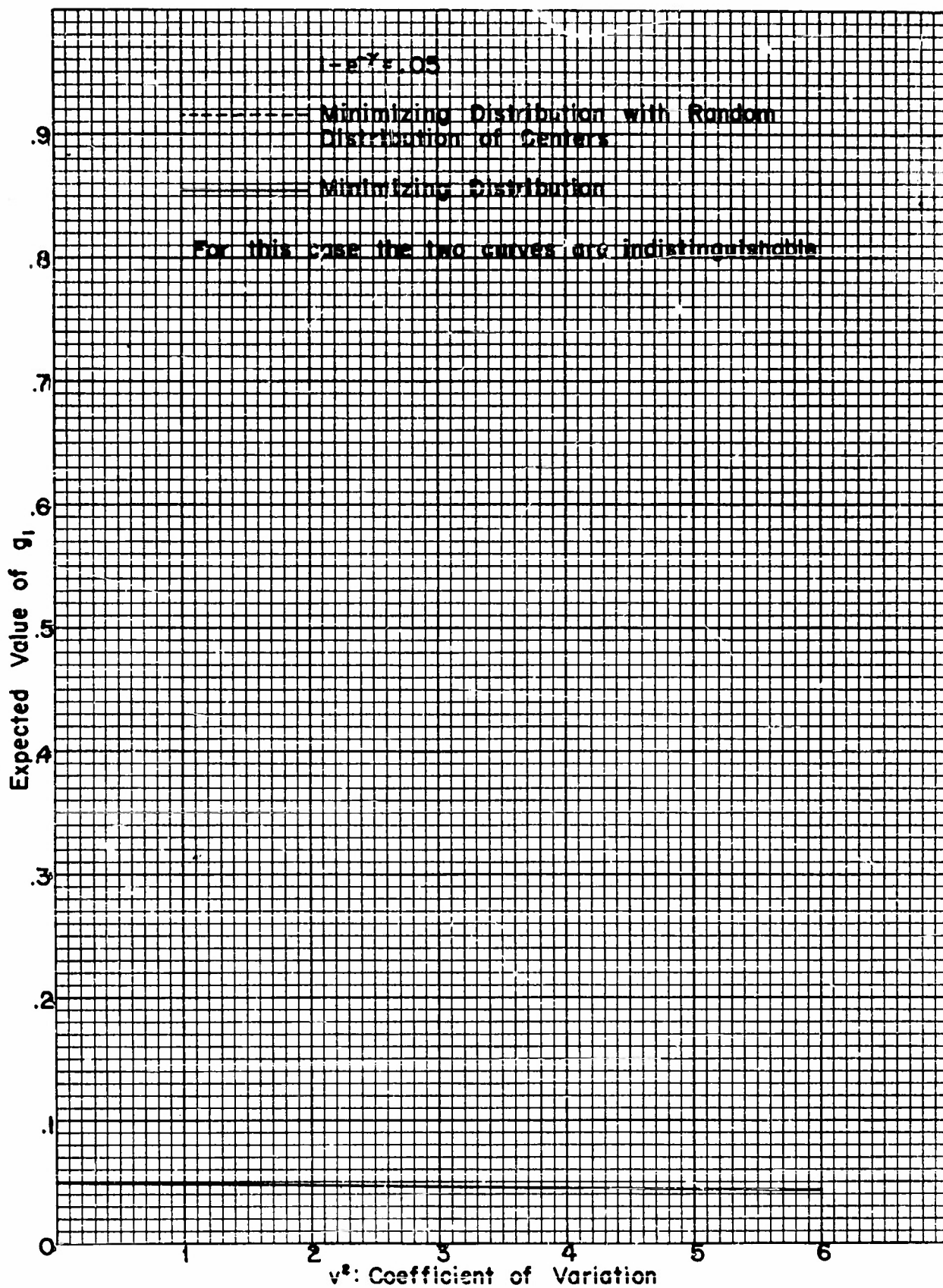
$$(4.7) \quad \frac{1}{v^2}(1 - e^{-\gamma v^2}) \leq \gamma.$$

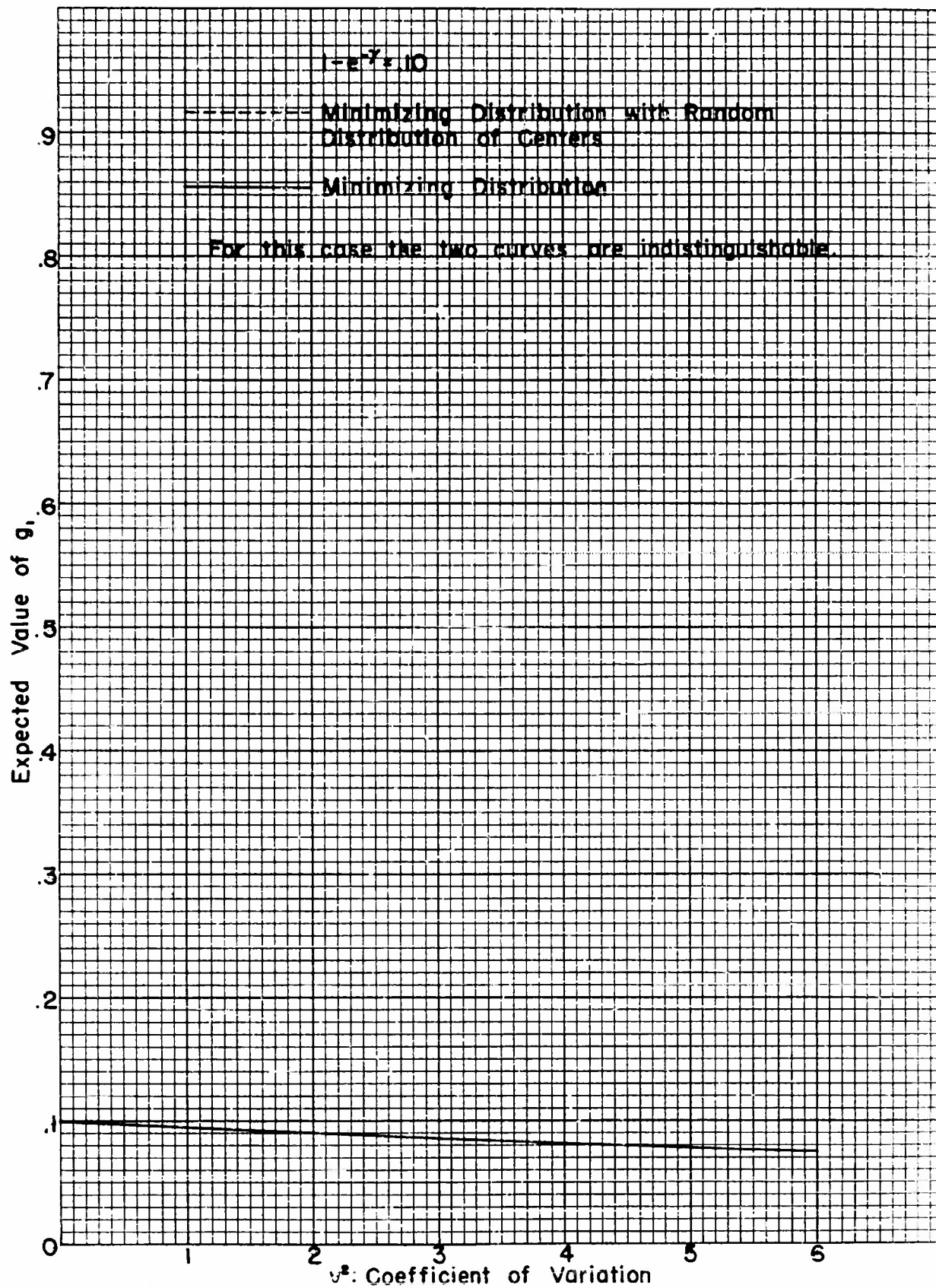
But for $\gamma > 0$, (4.7) is a well-known inequality. Hence our result is established.

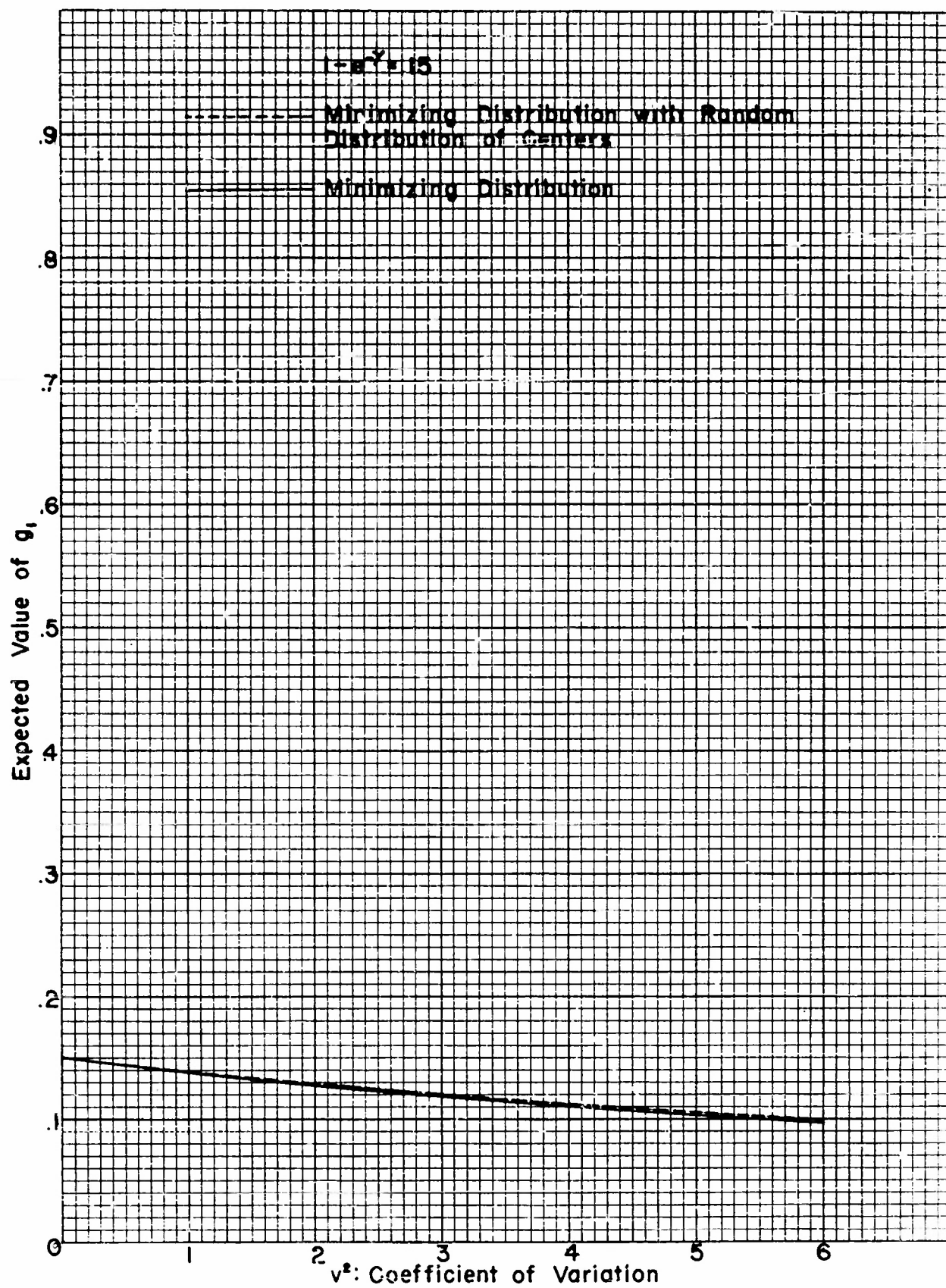
References

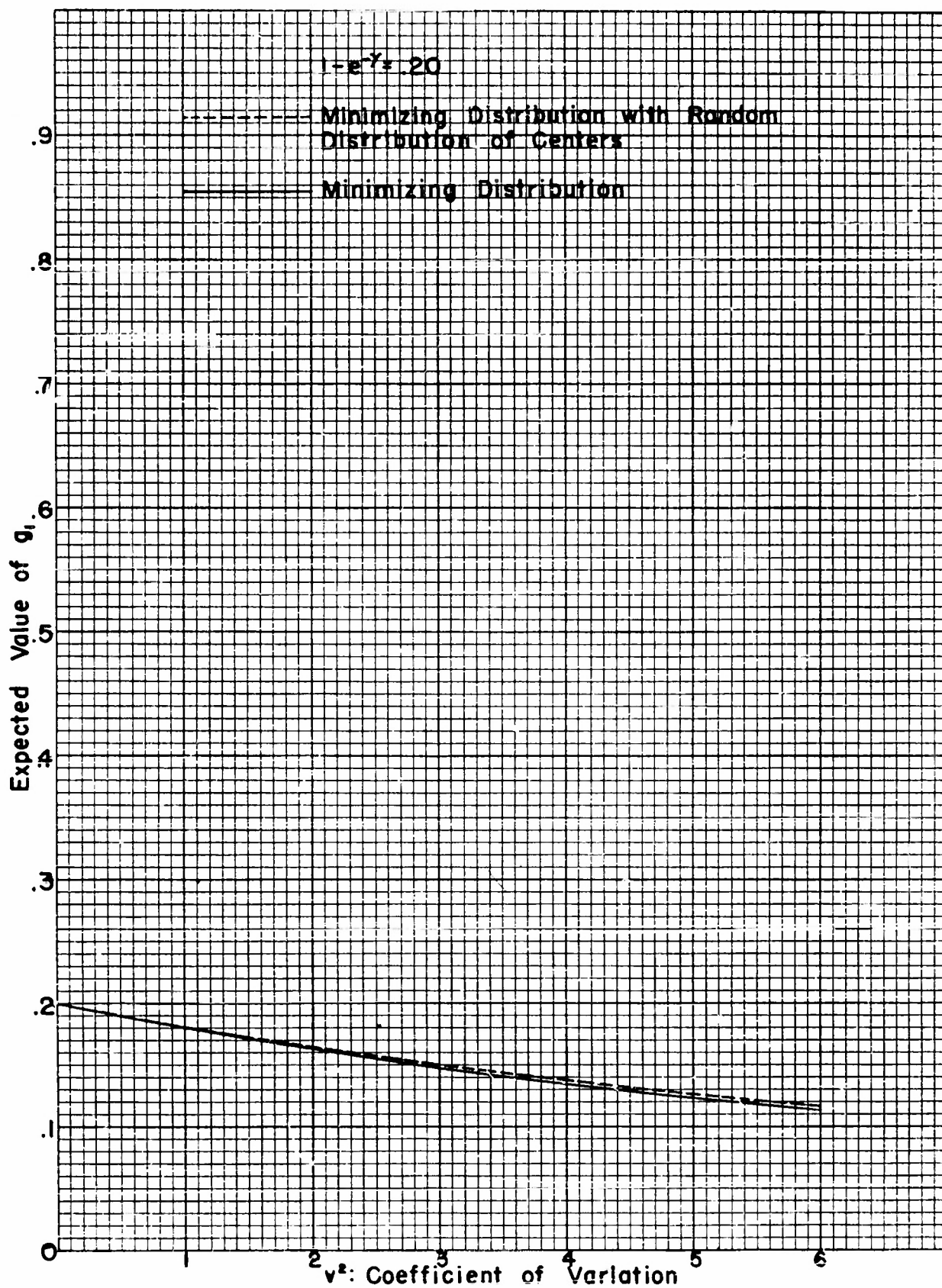
- [1] Chernoff, H., and Reiter, S., "Selection of a Distribution Function to Minimize an Expectation Subject to Side Conditions," Technical Report No. 23, March 12, 1954, Applied Mathematics and Statistics Laboratory, Stanford University.
- [2] Munroe, M. E., Introduction to Measure and Integration, Addison-Werley Publishing Co., Cambridge, Mass., 1953.

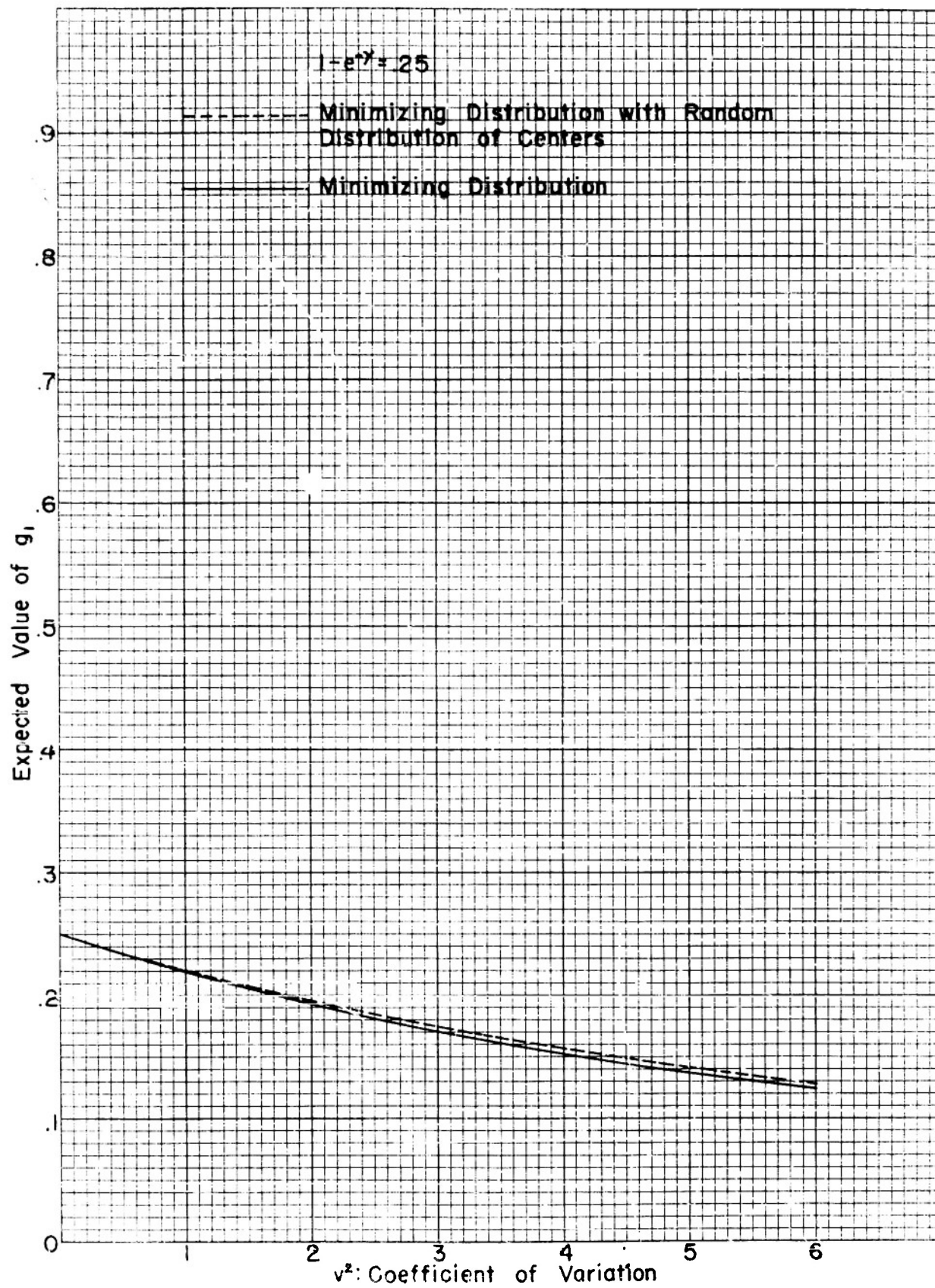


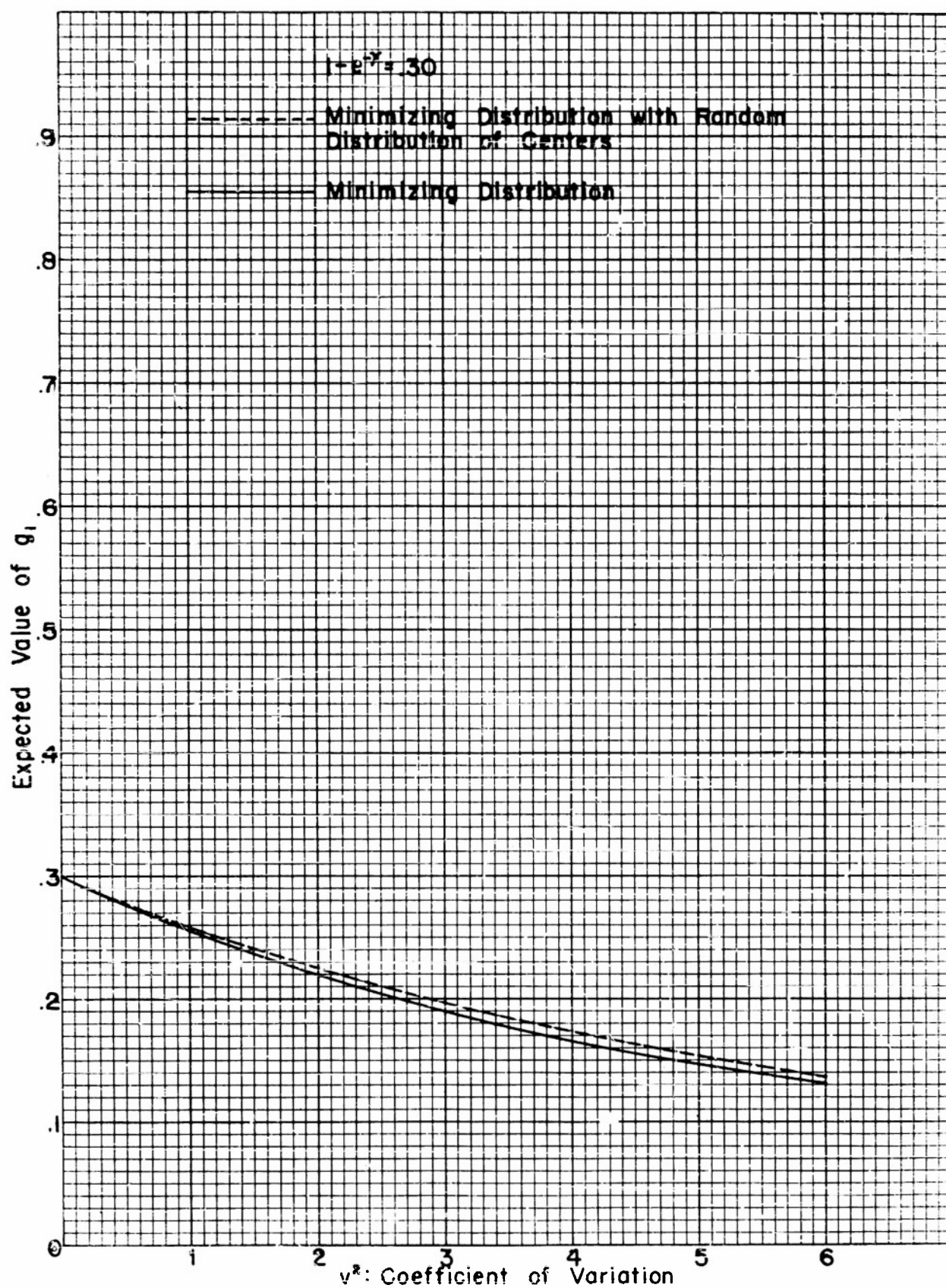


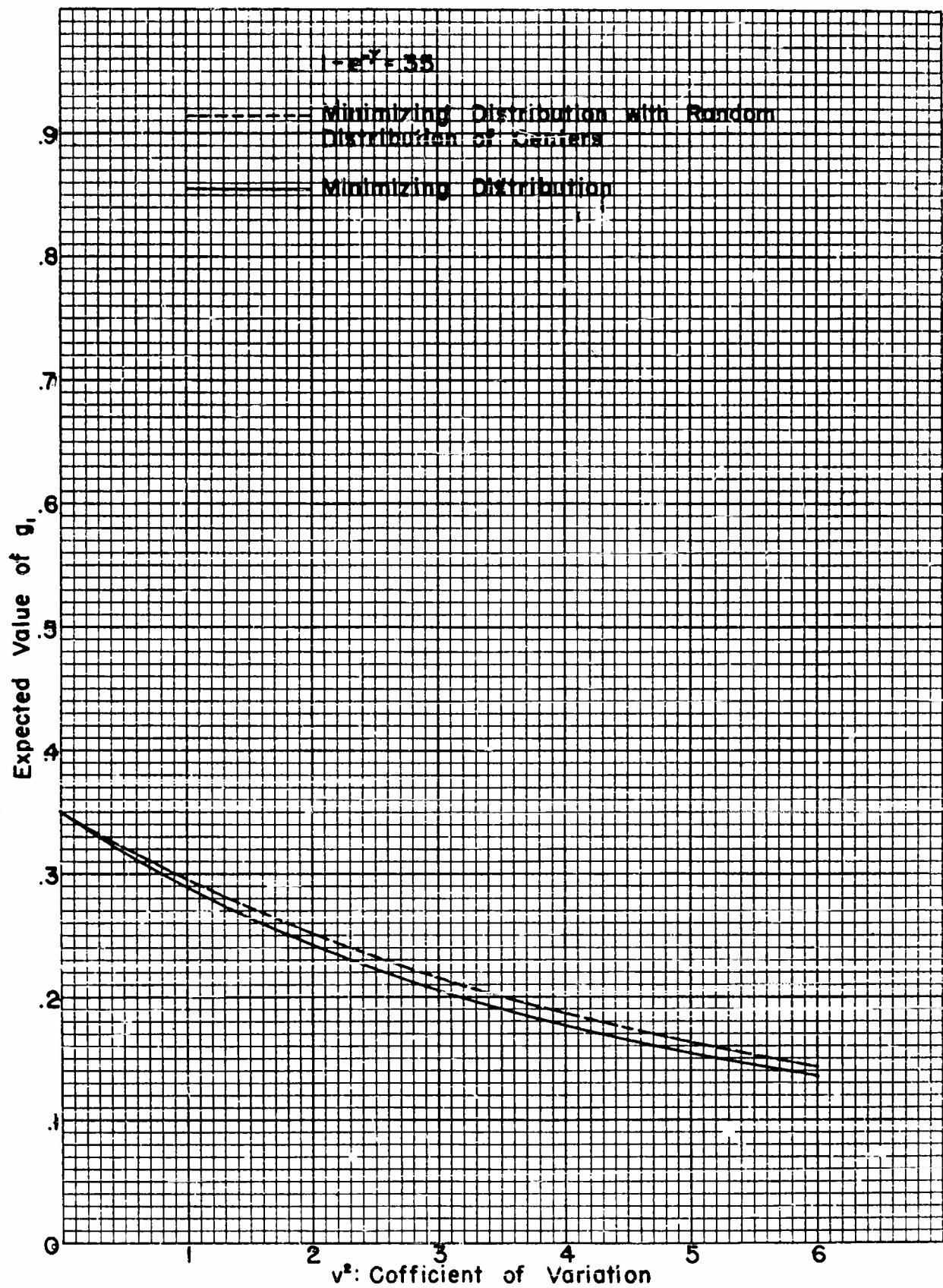


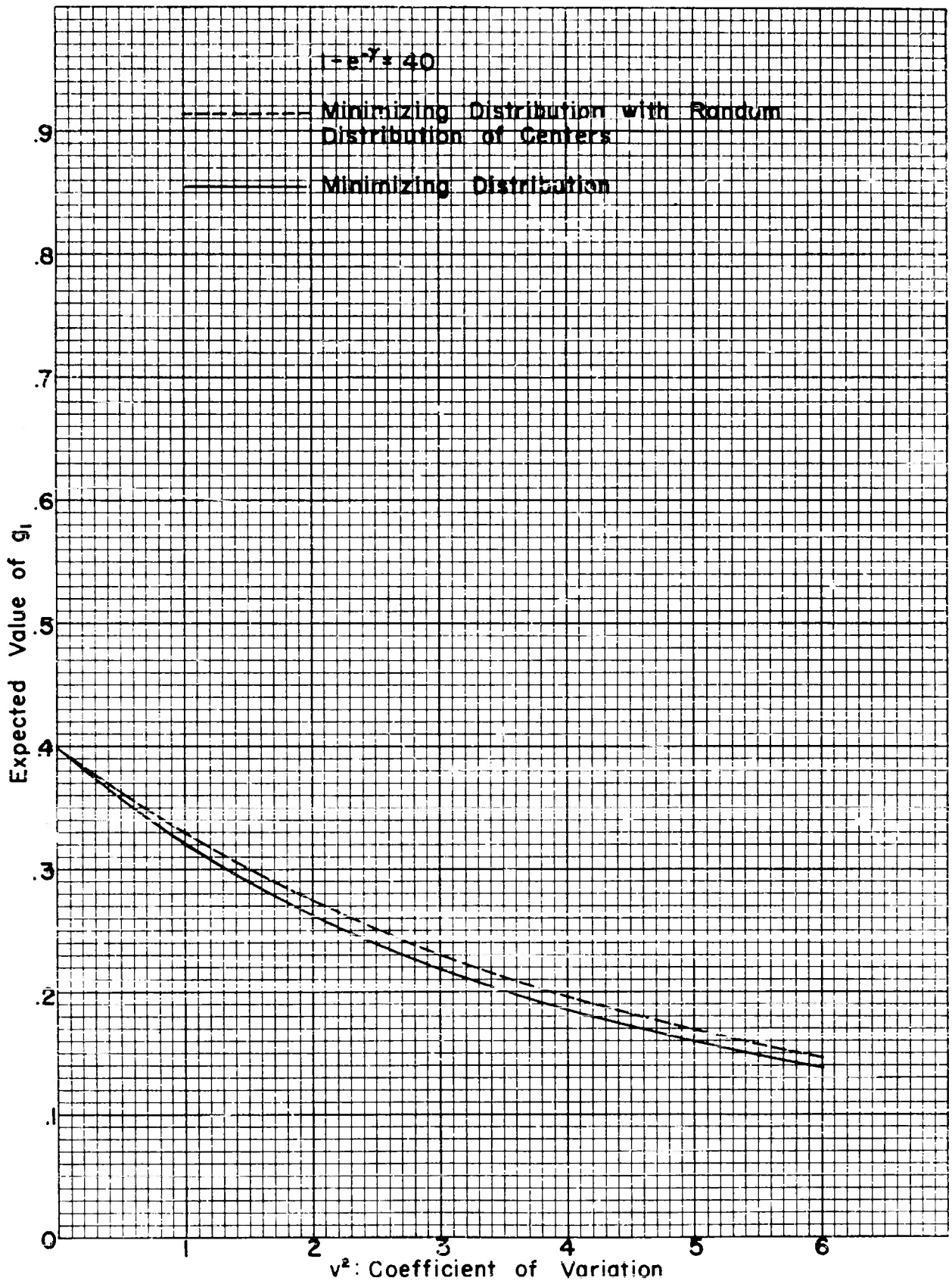


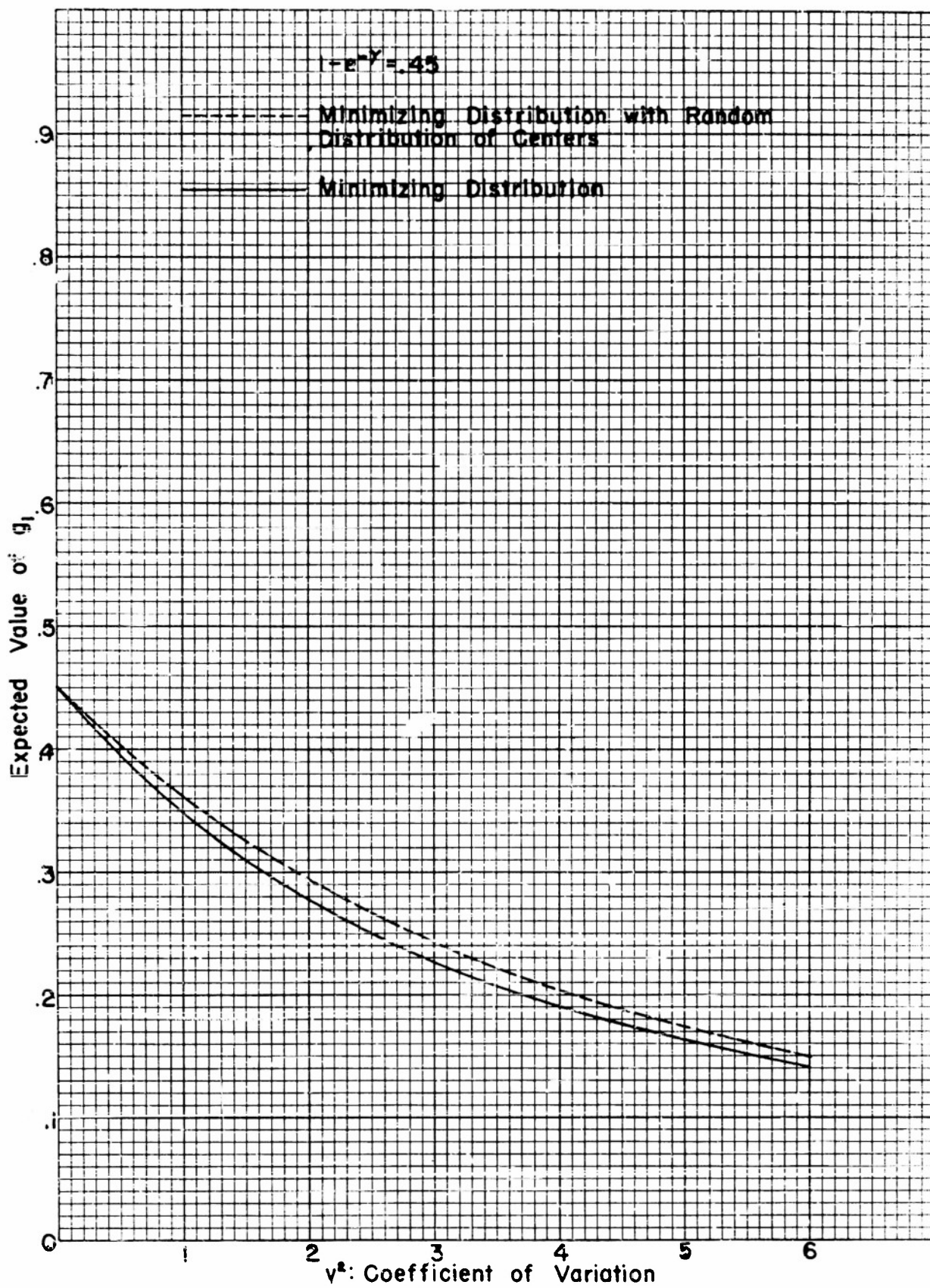


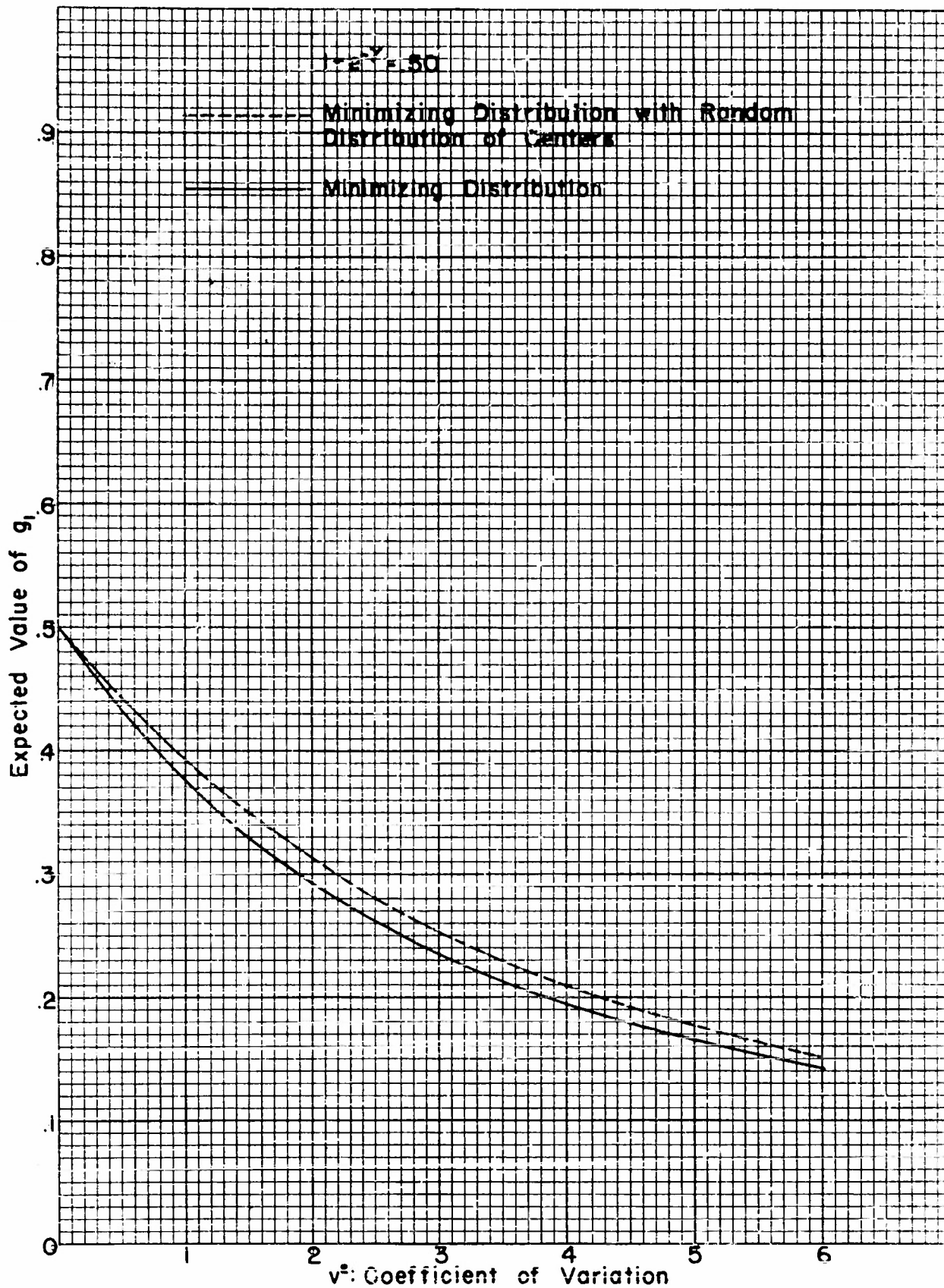


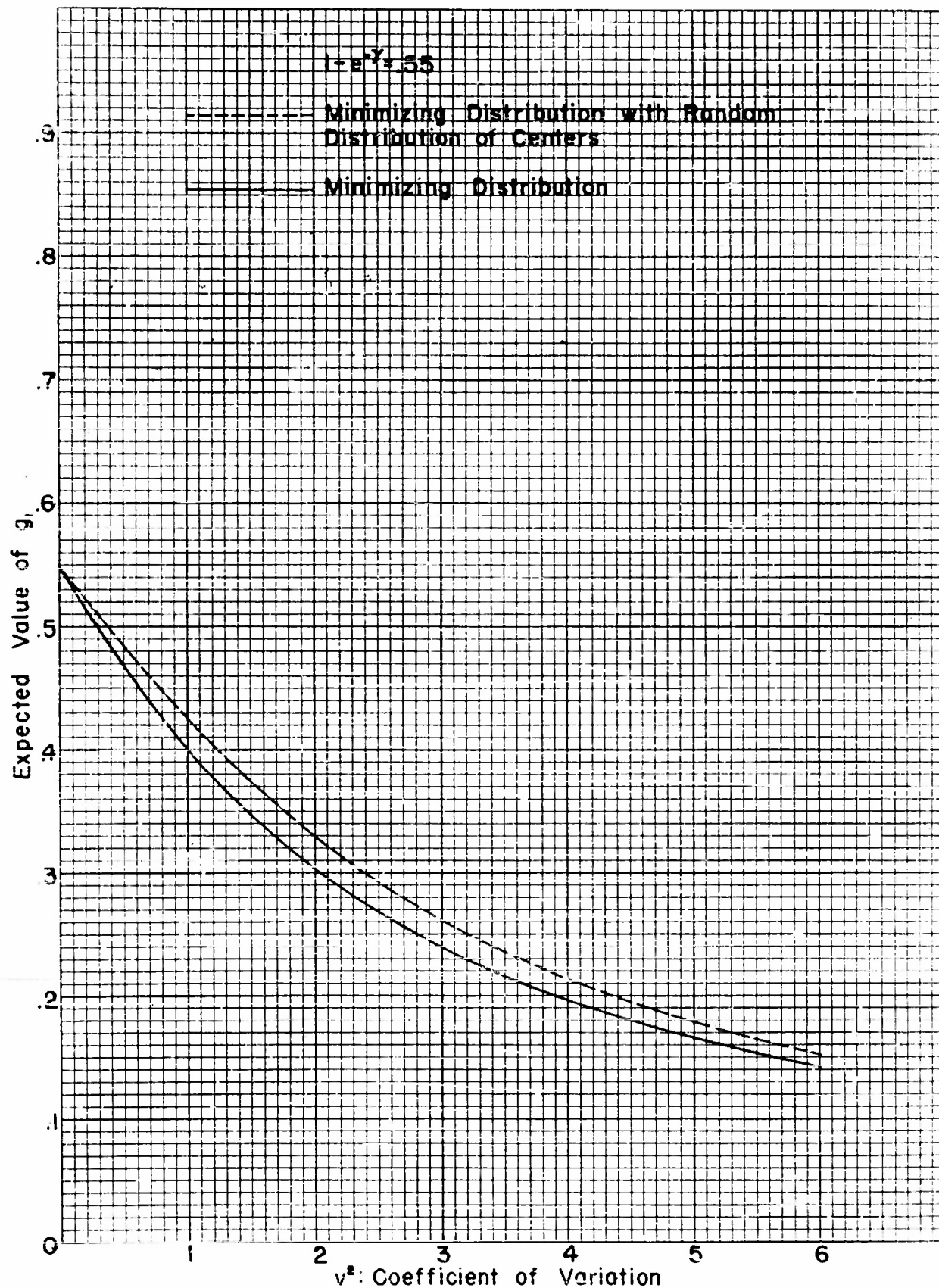


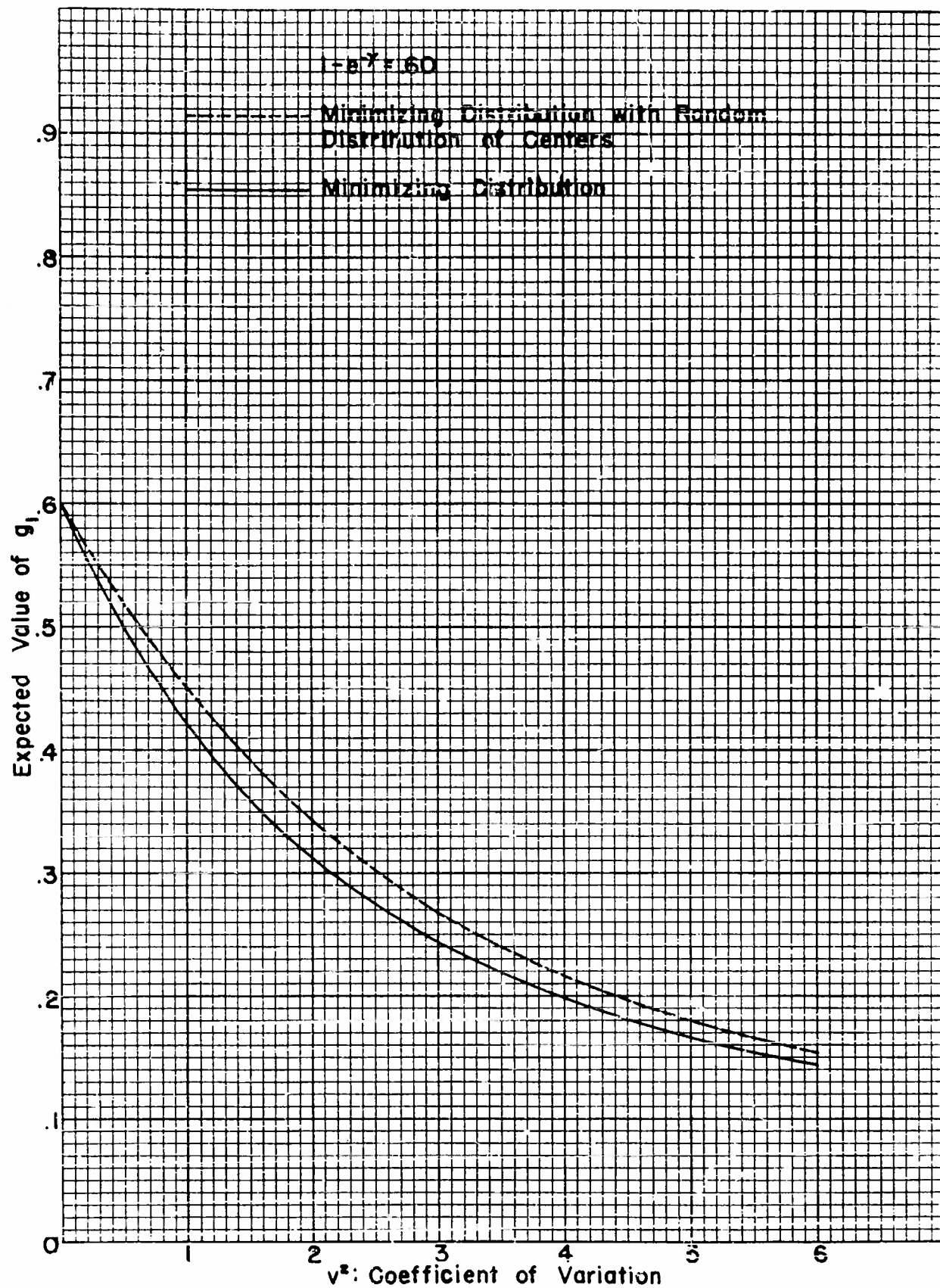


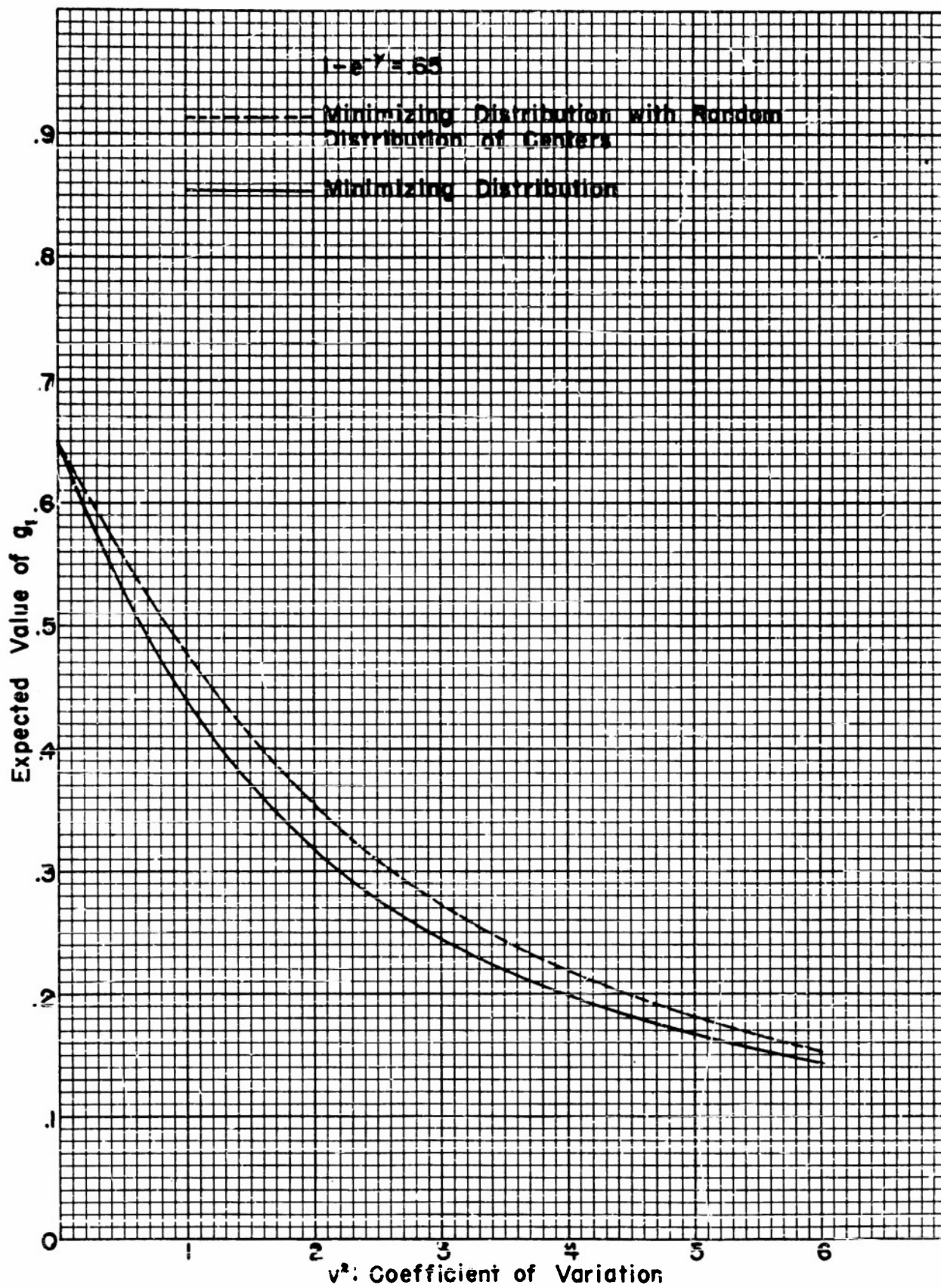


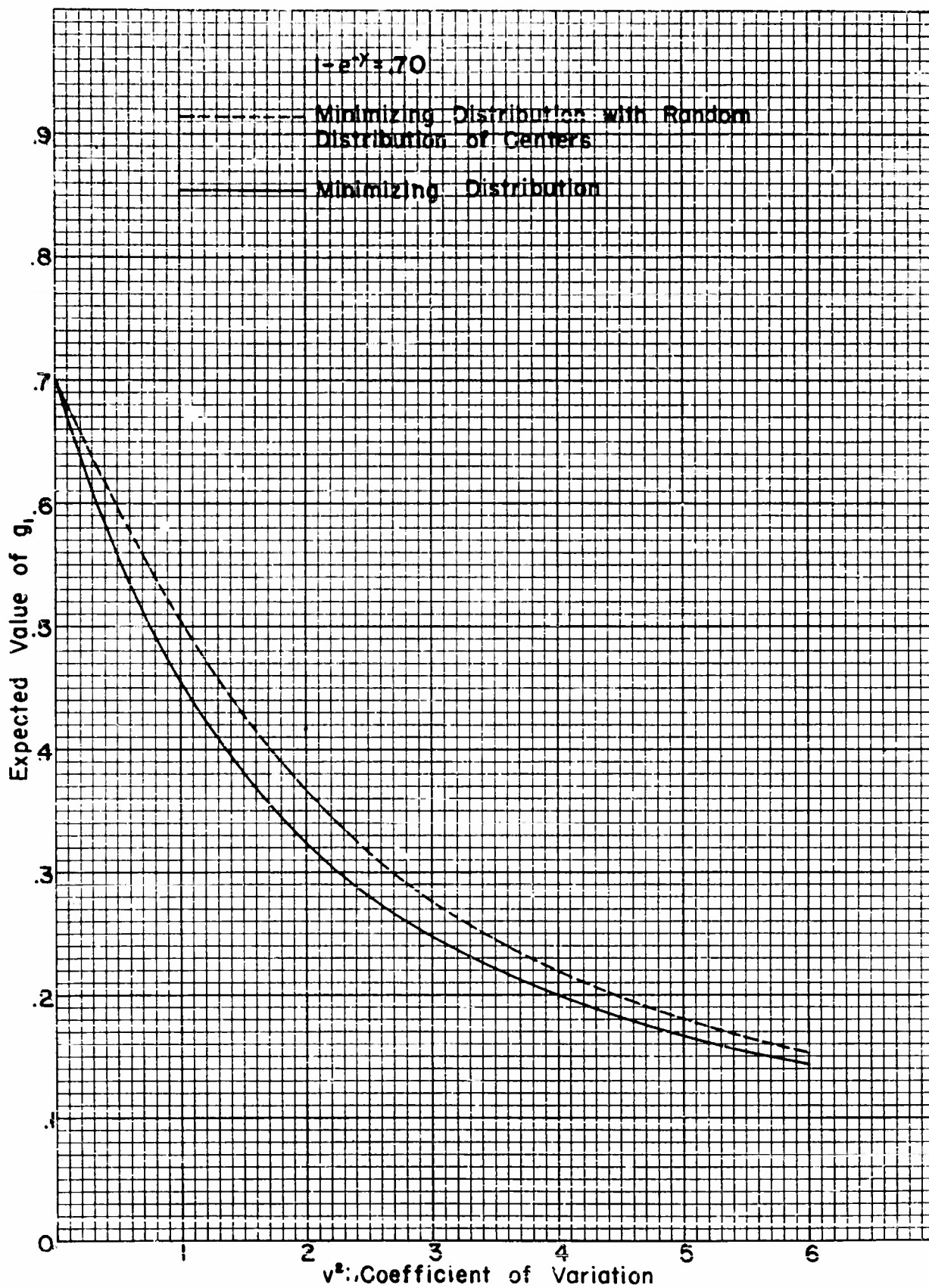


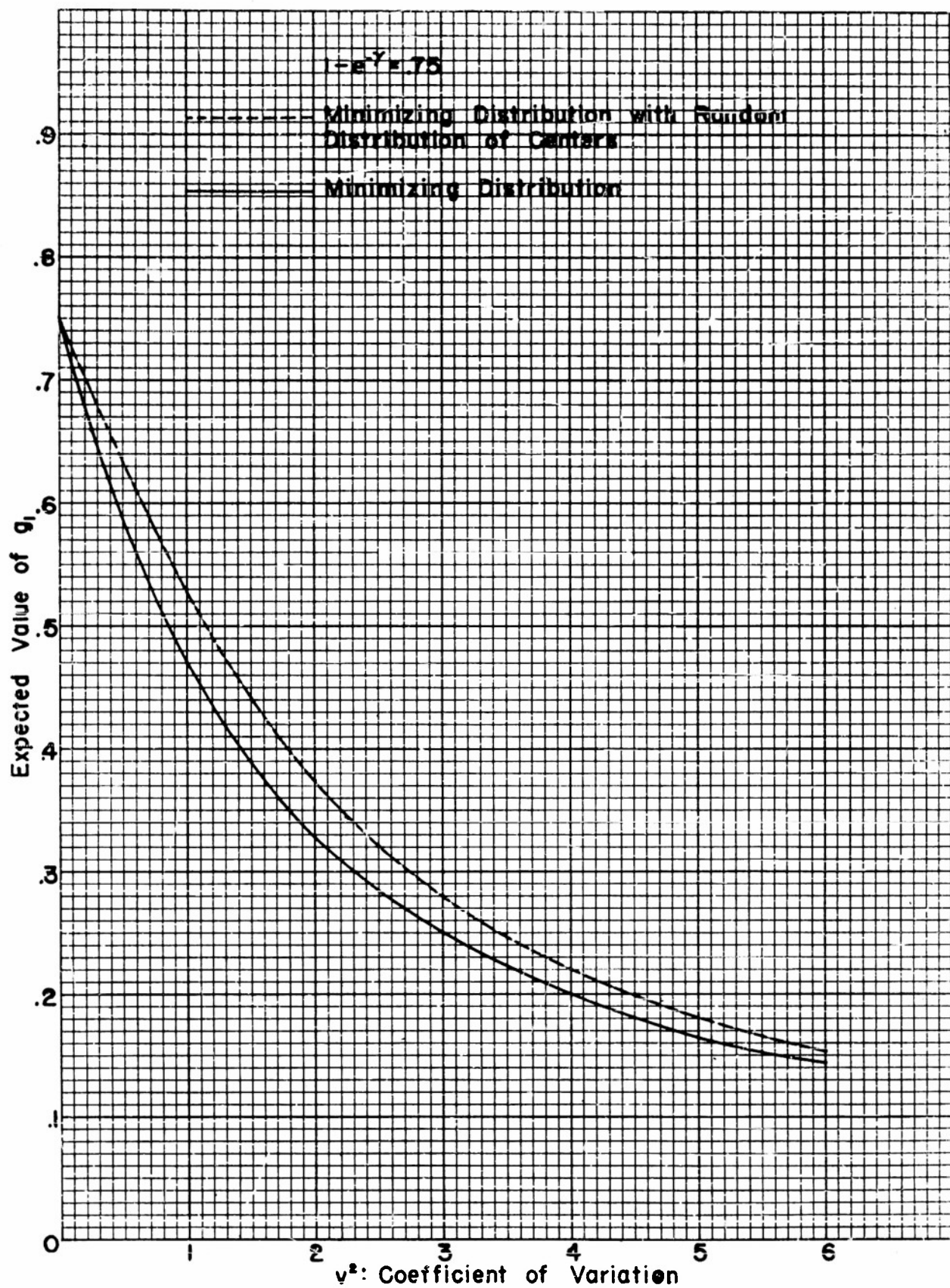


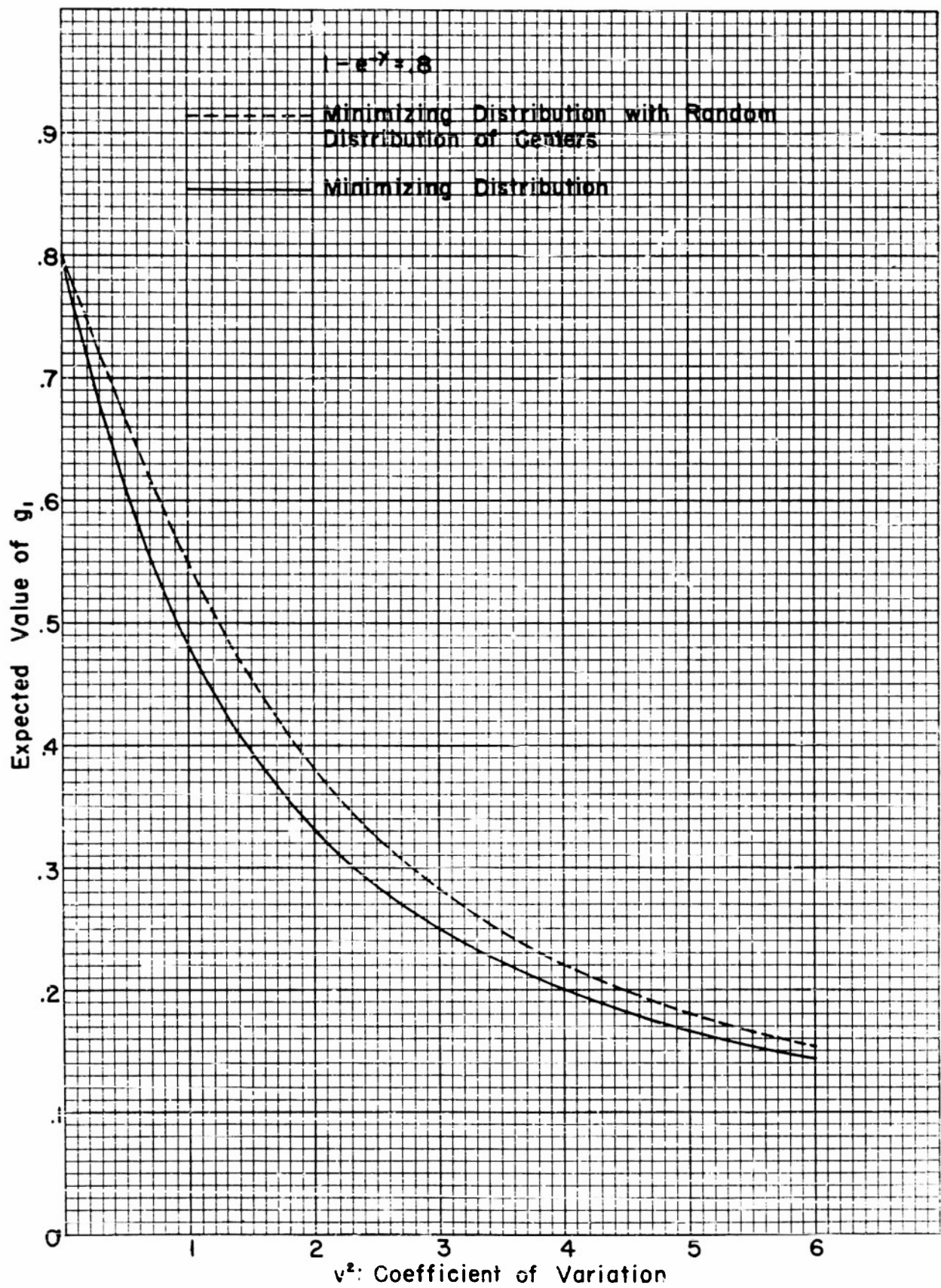


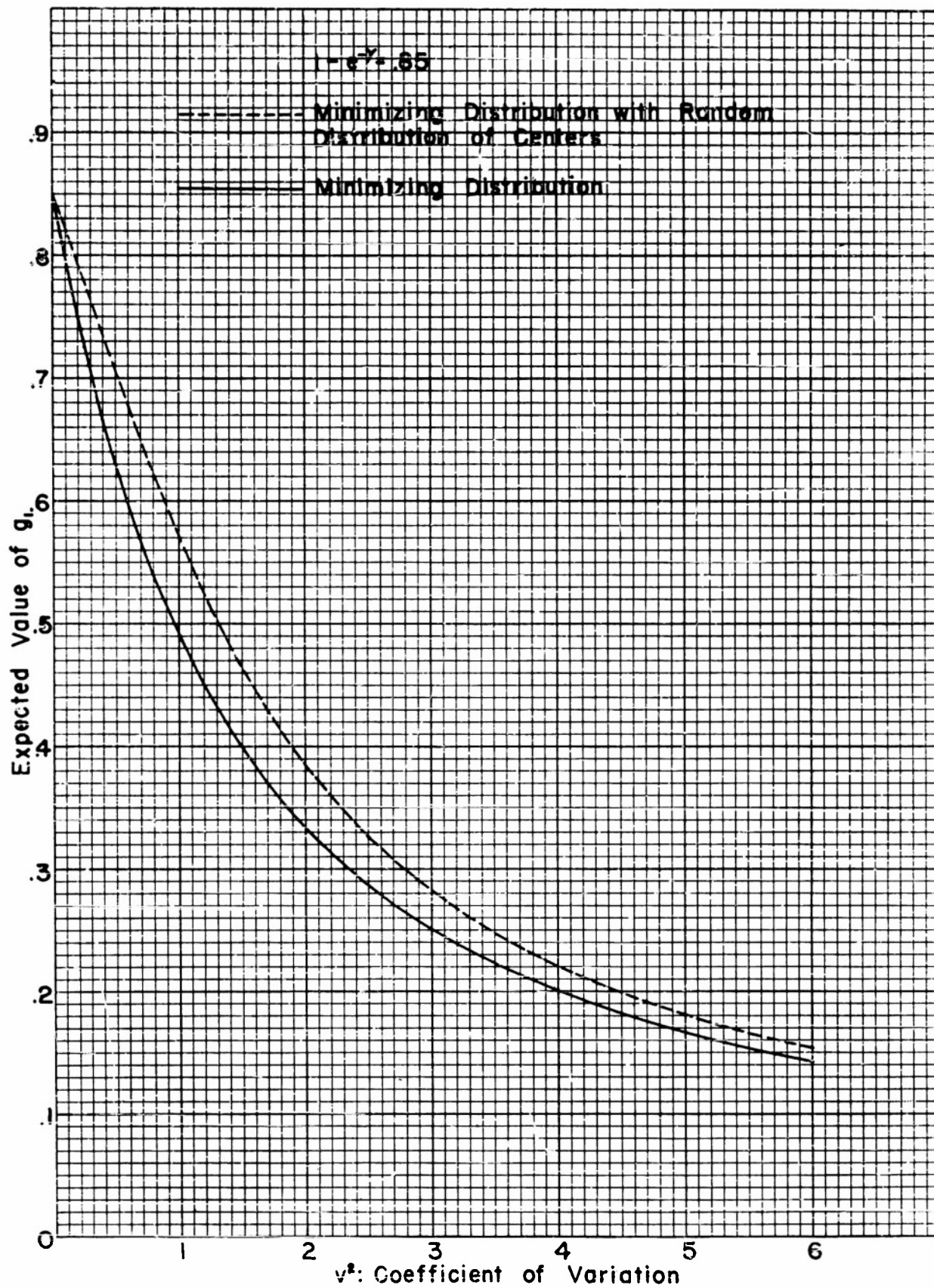


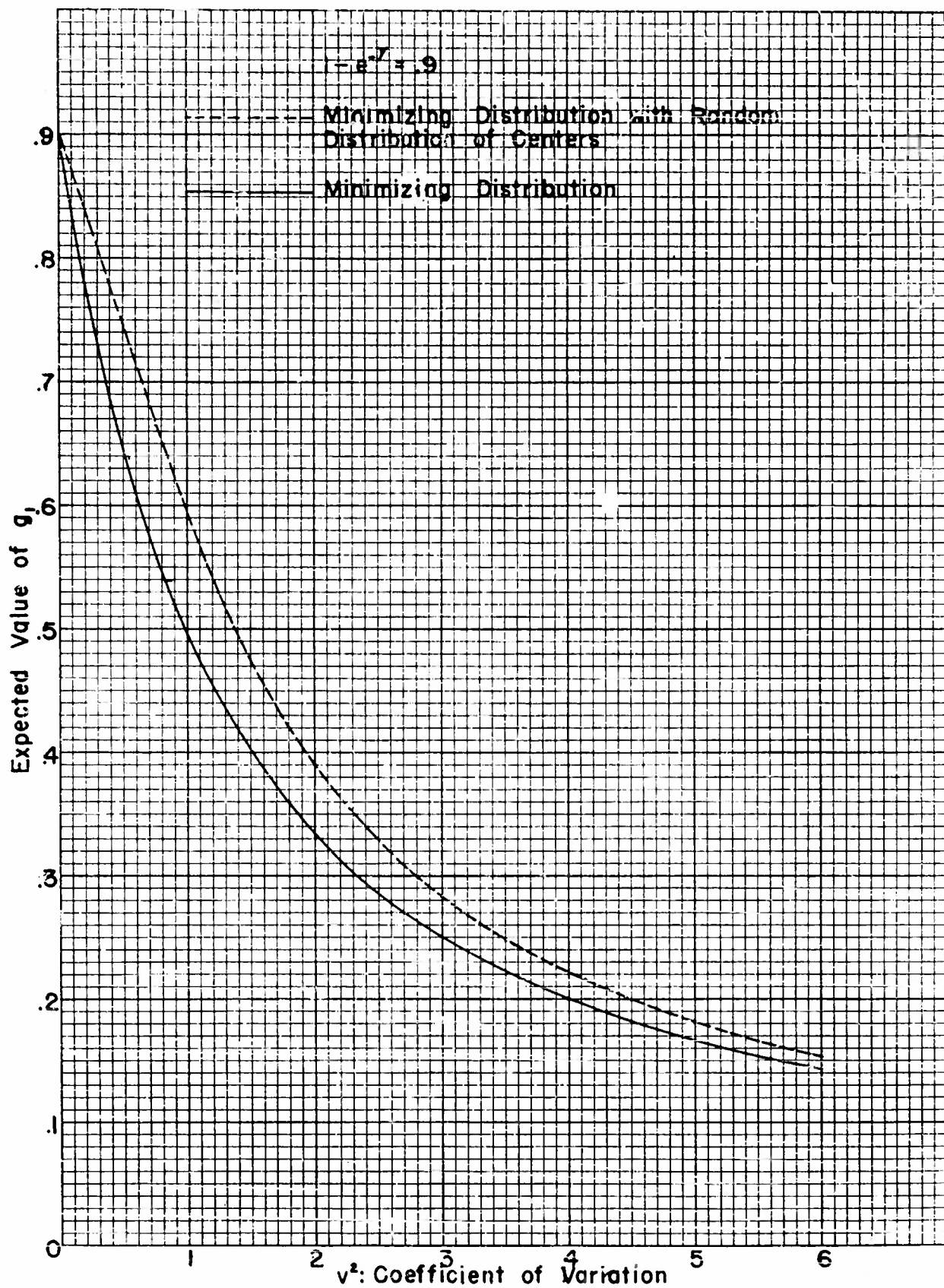












STANFORD UNIVERSITY
 Technical Reports Distribution List
 Contract N6onr 25140
 (NR-342-022)

Armed Services Technical Information Agency Documents Service Center Knott Building Dayton 2, Ohio	5	Dr. Merle M. Andrew Hq. Air Research & Development Command Post Office Box 1395 Baltimore 3, Maryland Attn: RDRRM	1
Commander Wright Air Development Center Wright-Patterson Air Force Base Ohio Attn: ARL Technical Library WCRR	1	Prof. Z. W. Birnbaum Department of Mathematics University of Washington Seattle 5, Washington	1
Headquarters, USAF Director of Operations Washington 25, D. C. Attn: Operations Analysis Division, AFOOP	1	Dr. David Blackwell Department of Mathematics Howard University Washington 1, D. C.	1
Institute of Statistics North Carolina State College of A & E Raleigh, North Carolina	1	Prof. Albert Bowker Applied Math. & Stat. Lab. Stanford University Stanford, California	1
National Applied Mathematics Lab. National Bureau of Standards Washington 25, D. C.	1	Prof. William G. Cochran Department of Biostatistics School of Hygiene and Public Health The Johns Hopkins University 615 North Wolfe Street Baltimore 5, Maryland	1
Office, Chief of Ordnance Department of the Army Washington 25, D. C. Attn: ORDTX-P	1	Prof. W. J. Dixon Department of Mathematics University of Oregon Eugene, Oregon	1
Office of Assistant Chief of Staff, G-4 Research Branch, R & D Division Department of the Army Washington 25, D. C.	1	Prof. M. A. Girshick Applied Math. & Stat. Lab. Stanford University Stanford, California	1
Office of Technical Services Department of Commerce Washington 25, D. C.	1	Dr. I. R. Hershner Department of Mathematics University of Vermont Burlington, Vermont	1
Dr. Forman S. Acton, Director Analytical Research Group Princeton University Box 710 Princeton, N. J.	1	Prof. Harold Hotelling Dept. of Mathematical Statistics University of North Carolina Chapel Hill, North Carolina	1

Prof. Carl F. Kossack Department of Mathematics Purdue University Lafayette, Indiana	1	Mr. Joseph Weinstein Applied Physics Branch Evans Signal Laboratory Fort Monmouth, New Jersey	2
Dr. W. W. Leutert Chief, Computing Laboratory Ballistic Research Laboratory Aberdeen Proving Ground Maryland	1	Prof. Jacob Wolfowitz Department of Mathematics Cornell University Ithaca, New York	1
Dr. Eugene Lukacs Head, Statistics Branch Office of Naval Research Washington 25, D. C.	1	Additional copies for project leader and assistants and re- serve for future requirements	38
Dr. Clifford J. Maloney Chief, Statistics Branch Headquarters, Camp Detrick Frederick, Maryland	1		
Dr. Knox T. Millsaps Wright Air Development Center Wright-Patterson Air Force Base Ohio Attn: WGRRM	1		
Prof. J. Neyman Statistical Laboratory Department of Mathematics University of California Berkeley 4, California	1		
Mr. R. H. Noyes Coles Signal Laboratory Fort Monmouth, New Jersey	1		
Prof. Herbert Robbins Dept. of Mathematical Statistics Columbia University New York 27, N. Y.	1		
Dr. Horace M. Trent Head, Applied Mathematics Branch Naval Research Laboratory Washington 20, D. C.	1		
Prof. W. Allen Wallis Statistical Research Center University of Chicago Chicago 37, Illinois	1		

Armed Services Technical Information Agency

Because of our limited supply, you are requested to return this copy WHEN IT HAS SERVED YOUR PURPOSE so that it may be made available to other requesters. Your cooperation will be appreciated.

AD

42257

NOTICE: WHEN GOVERNMENT OR OTHER DRAWINGS, SPECIFICATIONS OR OTHER DATA ARE USED FOR ANY PURPOSE OTHER THAN IN CONNECTION WITH A DEFINITELY RELATED GOVERNMENT PROCUREMENT OPERATION, THE U. S. GOVERNMENT THEREBY INCURS NO RESPONSIBILITY, NOR ANY OBLIGATION WHATSOEVER; AND THE FACT THAT THE GOVERNMENT MAY HAVE FORMULATED, FURNISHED, OR IN ANY WAY SUPPLIED THE SAID DRAWINGS, SPECIFICATIONS, OR OTHER DATA IS NOT TO BE REGARDED BY IMPLICATION OR OTHERWISE AS IN ANY MANNER LICENSING THE HOLDER OR ANY OTHER PERSON OR CORPORATION, OR CONVEYING ANY RIGHTS OR PERMISSION TO MANUFACTURE, USE OR SELL ANY PATENTED INVENTION THAT MAY IN ANY WAY BE RELATED THERETO.

Reproduced by
DOCUMENT SERVICE CENTER
KNOTT BUILDING, DAYTON, 2, OHIO

UNCLASSIFIED