A SEQUENTIAL INSPECTION PLAN FOR QUALITY CONTROL

BY

M. A. GIRSHICK

TECHNICAL REPORT NO. 16

JULY 23, 1954

PREPARED UNDER CONTRACT N6onr-25126
(NR-042-002)

FOR

OFFICE OF NAVAL RESEARCH

GERALD J. LIEBERMAN, DIRECTOR

APPLIED MATHEMATICS AND STATISTICS LABORATORY
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

A SEQUENTIAL INSPECTION PLAN FOR QUALITY CONTROL

BY

M. A. GIRSHICK

1. General Discussion.

A continuous production process results in a sequence of units of product each of which is classified as either defective or non-defective. The production process may or may not be in a state of statistical control, but in either case it is desired to inspect the finished product in order to (a) improve, when necessary, the quality of the outgoing product by removing and replacing the defective units found and (b) get an estimate of the quality of the product for control purposes.

When the cost of inspection compared with the cost of production is low, and the tests made during inspection are not destructive, 100 percent inspection of the product may be feasible, though not always desirable. But when inspection costs are high or the tests destructive, recourse must be had to sampling inspection. If this is done, the goal can no longer be perfect product.[1] Instead the goal becomes tolerable product. What constitutes tolerable product depends on the problem. For example, the consumer of the product might be employing a lot-by-lot acceptance inspection procedure for judging the quality of the product submitted to him by the producer. In that case the producer might employ the same definition of acceptable quality as that inherent in the consumer's

---

[1] This is not to imply that 100 percent inspection always yields perfect material. Experience shows that the magnitude of such a task usually results in inefficient and careless inspection performance.

inspection scheme. Alternatively, the producer himself might also be the consumer, in which case quality will be defined by the use to which the product is put.

Sampling inspection for quality control implies therefore that the user of the inspection scheme has in mind some number which he considers a desirable upper limit to the fraction defective remaining in the product after inspection. This upper limit is known as the Average Outgoing Quality Limit, usually abbreviated by the letters AOQL. The concept of an AOQL is basic in all inspection schemes for quality control. There is however another concept which plays an important role in determining a particular inspection plan and that is the standard deviation of the outgoing quality (OQ). This standard deviation measures the variability of the fraction defective remaining in the product after the inspection procedure has been applied. The importance of this concept arises from the fact that the decision to inspect or not to inspect a certain portion of the product 100 percent is made on the basis of a random sample. Since the result of a sample is subject to fluctuation, the decision is not always correct. Hence the actual fraction defective remaining in a lot after a single inspection may be larger or smaller than the AOQL even though the average fraction defective (taken over many inspected segments of product) is guaranteed not to exceed the AOQL. The standard deviation of the sample OQ measures the extent of the deviations from the average that may be expected.

An inspection plan of the type discussed here is often applied to a product which is considered to be of acceptable quality before inspection. In such a case, it is desirable to permit the inspection to go on for an indefinitely long time without any decision being reached. This is particularly true if the plan can detect quickly deterioration in the product during an inspection operation.

The proposed plan was developed to meet the need for an improved sample verification plan for controlling the quality of card-punch operations of the Foreign Trade Division of the Bureau of the Census while the author was in the employ at that Bureau. The plan, which has been put into operation by the Foreign Trade Division, has the advantage that its statistical basis and consequences are known, and the procedure insures a reasonable efficiency at low cost. Although this plan was developed for the above purpose, it is equally applicable to industrial problems having similar properties.

2. <u>How the Inspection Plan Operates</u>.

The inspection plan under consideration is defined by the three integers m, N, and k and the positive fraction — the AOQL. The integers m and N jointly define the criterion for terminating inspection and making the appropriate decision. The reciprocal of the integer k defines the partial sampling ratio employed. The AOQL represents the upper limit to the outgoing quality desired. The plan operates as follows:

The units of product in the production sequence (or lots) are divided into segments of size k. Inspection begins by selecting at random one item from each consecutive segment of k items. These items are inspected in sequence and the number of defectives found as well as the number of items examined is cumulated. Inspection terminates when, and only when, the cumulative number of defectives reaches m. At this point, the size of the sample n is compared with the integer N. If $n \geq N$, the product which has passed through inspection is considered acceptable and the inspection procedure is repeated on new product beginning with the segment next to the last segment inspected. If, on the other hand, $n < N$, the product which passed through inspection is considered unacceptable and the following actions are taken: (a) $N-n$ segments (corresponding to $k(N-n)$ items) are inspected 100 percent. These segments are selected from the product beginning with the segment adjacent to the last segment partially inspected; (b) the inspection procedure is repeated on new material, beginning the inspection with the segment following the last segment which was 100 percent inspected.

In mathematical symbols the criterion for making a decision under this plan is as follows: Let $x_1, x_2, \ldots$, represent the quality of the items selected from the first, second, etc. segment. Thus $x_i = 1$ if the item selected from the i^{th} segment is found defective and $x_i = 0$ otherwise. The cumulative number of defectives at the n^{th} stage of inspection is given by $\sum_{i=1}^{n} x_i$. Inspection terminates as soon as $\sum_{i=1}^{n} x_i = m$. If $n \geq N$, the inspection procedure is repeated on new material. If, however, $n < N$, then $N-n$ segments of size k are inspected 100 percent, and the inspection procedure is repeated on new material.

It can be proved by methods similar to those employed in [1] and [2] that whether or not the production is in a state of statistical control, the outgoing quality of the product after this inspection procedure has been applied cannot exceed $\frac{k-1}{k} \frac{m}{N}$ on the average. (This assumes that the defectives found during partial inspection are also removed and replaced by non-defectives. If this is not done the AOQL will equal m/N.)

2.1 Procedure for Employing Reduced and Strict Partial Inspection.

With any plan there is a minimum amount of inspection which cannot be avoided regardless of the quality of the product. This minimum is defined by the partial sampling rate employed. If the quality of the product is subject to variation, a method for automatically adjusting the partial sampling rate to the defective rate in the product is given by reduced and strict partial sampling. This procedure can be briefly described as follows:

As before, the plan is defined by the integers m and N and by the AOQL, but instead of one sampling rate, two are employed. That is, two integers k_1 and k_2 are chosen with $k_1 > k_2$ and the sequence of product is divided into either segments of size k_1 or k_2 depending on whether reduced or strict inspection is called for, respectively.

Inspection begins with strict inspection. That is, one item is chosen at random from each consecutive segment of k_2 items and the number of defective items found, as well as the number of items sampled, is cumulated. Inspection terminates when and only when m defectives have been found. If m defectives have been found with a sample size $n \geq N$, the inspection procedure is repeated on new material, but with the reduced sampling rate, that is, the sequence of items is now divided into segments of k_1 items each and partial inspection consists of selecting and examining one item out of each segment. If, however, m defectives have been found with a sample size $n < N$, the following actions are taken: (a) a batch of $k_2(N-n)$ consecutive items is completely inspected. This batch is selected from the product beginning with the segment adjacent to the last segment partially inspected; (b) the inspection procedure is repeated on new material, beginning with the segment following the last segment that was inspected 100 percent.

In the second and subsequent inspection operations, a decision is always made when (but not before) m defectives have been found. If reduced inspection was in effect and the sample size was $n < N$, a batch of $k_1(N-n)$ items is completely inspected and the inspection procedure is repeated on new material but the sampling rate is changed from reduced to strict. If reduced inspection was in effect and the sample size was $n \geq N$, the inspection procedure is repeated on new material at the reduced sampling rate. If, however, strict inspection was in effect and the sample size was $n < N$, a batch of $k_2(N-n)$ items is completely inspected and the inspection procedure repeated on new material at the strict sampling rate. If strict inspection was in effect and the same size was $n \geq N$, the inspection procedure is repeated but at the reduced sampling rate. Whenever m defectives are found and the sample size is $n \geq N$, the inspection procedure is repeated on new material employing a reduced sampling rate regardless of the sampling rate in effect in the operation just preceding.

The criterion for employing either reduced or strict inspection can be summarized briefly as follows: Whenever an inspection operation terminates and 100 percent inspection of some material is called for, it also calls for strict inspection to follow. Conversely, if no 100 percent inspection is indicated, then reduced inspection is to follow.

It is easily seen that when the material submitted for inspection is of good quality, the partial sampling rate will most often be reduced, while if the material is of poor quality, the partial sampling rate will most often be strict. This will have the effect of reducing the cost of inspection when inspection is least needed and reducing the variability of the sample OQ when such a reduction is of significance.

3. How the Constants Defining the Plan are Determined.

It was mentioned above that the inspection plan is based on the integers m, N, and k, and the AOQL. Not all of these constants can be chosen at will. For example, the AOQL is a direct function of m, N, and k and is given by

$$AOQL = \frac{k-1}{k} \cdot \frac{m}{N} .$$

Thus, if the AOQL is fixed, the constants m, N, and k are, at least in part, also determined by it. Unless the constant k is very small, the AOQL is hardly affected by variations in it. Thus, in many cases it will be found that for all practical purposes fixing the AOQL fixes the ratio of m to N, and conversely.

If the production process is in a state of statistical control, or if the defectives within the lots submitted for inspection are distributed in a random manner, the frequency with which decisions are reached as well as the type of decision made will depend upon m and N but not on the partial sampling rate. In addition, as was seen above, the constants m and N essentially determine the AOQL. Thus, regardless of how the defectives are distributed in the sequence of the product inspected, if the average fraction defective contained in the product exceeds the AOQL, the resulting outgoing quality will depend largely on the values of m and N. However, this is not to imply that the constants m and N are the only determining factors in the operation of the plan. The value assigned to k will often play just as significant a role in determining the consequences of the plan and for the following reasons:

One objective of an intelligently designed inspection plan is to supply current and adequate information on the quality of the product for process control purposes. This sets a limit on the value of k. For if k is very large, inspecting one unit out of each segment of k units may not supply the information necessary to quickly detect assignable variations in the production process. Another consideration is related to the sample OQ among the lots inspected. Even though the plan guarantees that the fraction defective in the product after inspection will not exceed the AOQL in the long run, the long run may be very long indeed if k is very large. A third factor which delimits the value of k is connected with the cost of inspection. From the point of view of the operation of the plan, inspection has a two-fold purpose. One is to obtain information for making decisions. This is accomplished through the process of partial inspection. The other purpose is to reduce the number of defectives if the sample judges it to be more then tolerable. This is accomplished through the 100 percent inspection procedure. Generally speaking, if the defective rate in the material is substantially higher than the AOQL, the total amount of inspection will on the average be no greater than that required to accomplish this reduction. But if the quality of product submitted for inspection is of acceptable quality to begin with, no reduction in the fraction defective is required and any inspection is in a sense wasteful. Thus, even though the plan will in such cases most often require only partial inspection, the cost of inspection may be unnecessarily high if k is small. The above considerations are the most important in determining k.

There are several considerations involved in the choice of the AOQL-- the desired upper limit to the outgoing quality. Generally speaking, the stricter the standard set, the greater will be the cost of inspection required to attain it. If nothing is known about the production process and protection is sought for every possible eventuality, only cost consideration can enter into the choice of the AOQL. However, often information which delimits the fraction defective inherent in the process will be available. In that case, it will be uneconomical to set the AOQL at a desired low level and pay for protection against an event which is not expected to occur very often.

It must be emphasized again that one major purpose of inspection is to obtain information on the production process. Thus, even if such information is not available at the start, the inspection plan should be designed to supply it and should be modified accordingly.

4. Operating Characteristics of the Plan.

In Section 3 an attempt was made to describe briefly the various considerations involved in the choice of a particular plan of the type under discussion. It was found that the construction of a plan necessarily involves a compromise between the goal aimed at and the cost of attaining that goal. In order to bring out more clearly the issues involved, a more detailed study of the statistical consequences of the plan is required. Such a study will be undertaken in this section.

4.1 The Operating Characteristic (OC) Curve for the Plan.

In the plan under consideration the criterion for making a decision to inspect or not to inspect product 100 percent is given by the two integers m and N. The type of decision reached will depend of course on the results of the sample randomly selected from each of the segments into which the production has been divided. Since a decision is always reached when the number of defectives found equals m, knowing the probability of reaching one of the two decisions determines the probability of reaching the other. Purely as a convention, therefore, the probability considered here will refer to the decision not to inspect any product 100 percent. This is equivalent to the probability that when m defectives have been found, the sample size n is greater than or equal to N. The magnitude of this probability will depend on the values of m and N and on the quality of the product submitted for inspection.

The fact that the sample is selected in a stratified manner will in some situations impose serious difficulties in determining the probability of reaching the designated decision. Such difficulties will arise, for example, in cases where the production process is not stable and the inspection procedure is applied to the production sequence in the order in which the items are produced. The probability under consideration will in that case generally be a function of n parameters, namely, the fractions defective in each of the n segments accumulated up to the point where m defectives are found. The large number of parameters involved will make the computation of this probability impractical and of dubious value.

If the production process is in a state of statistical control, however, the above mentioned difficulties do not arise. For in that case the probability depends only on the fraction defective yielded by the process. This dependence, moreover, can be easily exhibited and computed without much difficulty.

The relationship between the fraction defective p and the probability that a given inspection will terminate and no complete inspection will be required will be designated by the symbol L(p). When L(p) is plotted against p, the resulting curve is known as the Operating Characteristic (OC) Curve.

The functional relationship between L(p) and p is given by the equation

$$(4.1.1) \qquad L(p) = \sum_{j=0}^{m-1} \binom{N}{j} p^j q^{N-j} + \binom{N-1}{m-1} p^m q^{N-m}$$

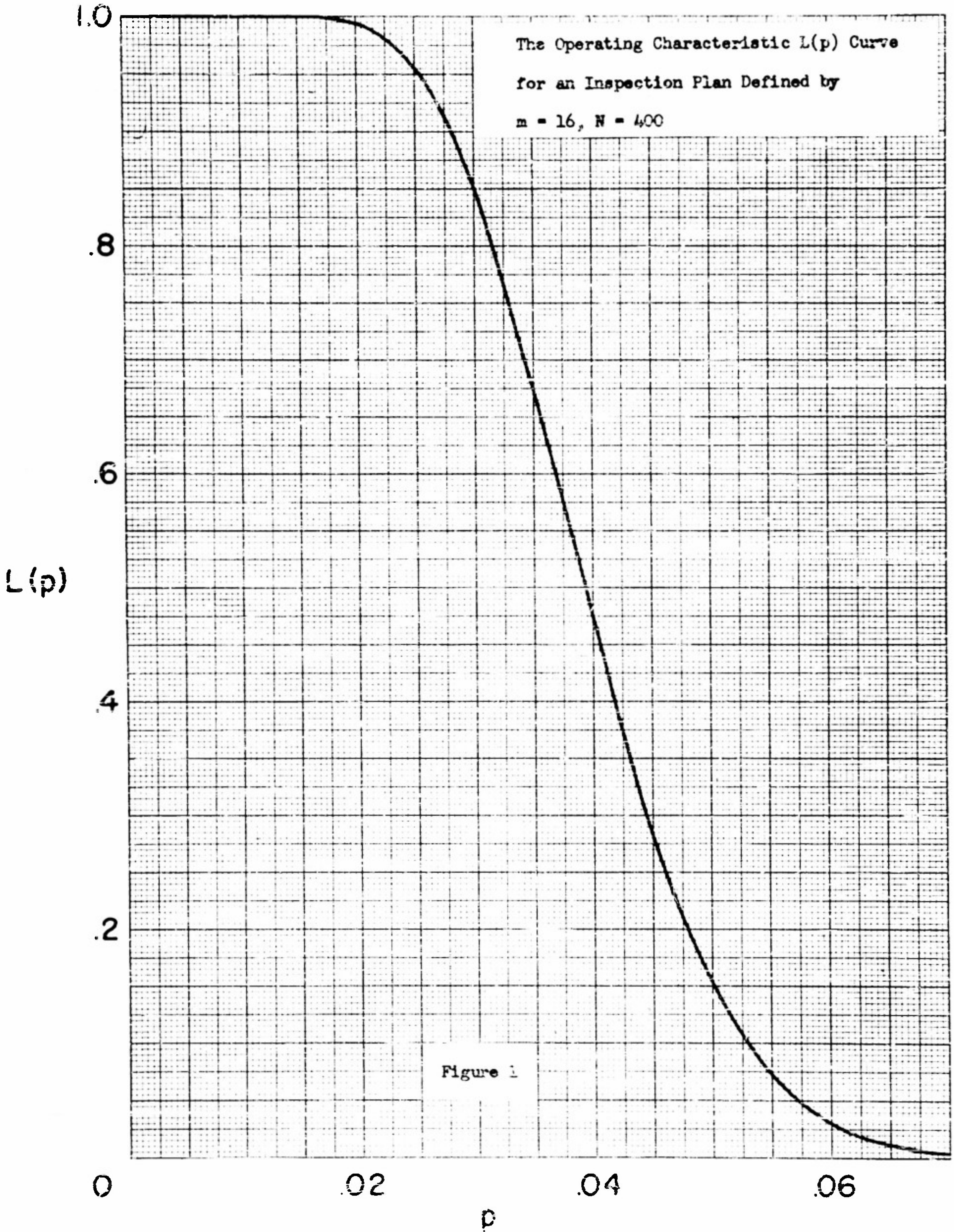
$$= \sum_{j=0}^{m-1} \binom{N-1}{j} p^j q^{N-1-j} .$$

The first expression on the right of (4.1.1) represents the probability that $n > N$ and the second expression represents the probability that $n = N$.

As was stated above this plan was developed to institute a more efficient sampling inspection plan for controlling errors in card-punch operations. Knowledge of the operations was such that it was known that the average error rate would almost never exceed 2 percent, and that the production process was stable. For these reasons the AOQL was set high enough that an amount of inspection disproportionate to the number of errors expected to be found would not be required by the plan. The upper limit of the AOQ was set at .0380 even though a percentage error of this size was not expected. It was also decided that the product would be subject to strict and reduced partial inspection according to the quality of output of the individual operators. Integers chosen to determine the plan were $m = 16$, $N = 400$, $k_1 = 50$, and $k_2 = 20$. Figure 1 illustrates the OC curve of this plan.

4.2 The Average Outgoing Quality (AOQ) Curve.

The average proportion of defective items remaining in the product after the inspection plan has been applied to products of quality p is a function of p, L(p), the expected value of n, and the conditional



1.0
.8
.6
.4
.2
L(p)
0
.02
.04
.06
p
The Operating Characteristic L(p) Curve
for an Inspection Plan Defined by
m = 16, N = 400

Figure 1

expected value of n given that $n \leq N$. As a function of p, the AOQ will be written as Q(p). We shall designate by $E(n|p)$, $E_1(n|p)$, and $E_2(n|p)$, respectively, the expected value of n for a given p, the conditional expected value of n given that $n \geq N$, and the conditional expected value of n given that $n < N$. The relationship between $E(n|p)$, $E_1(n|p)$, and $E_2(n|p)$ is given by

$$(4.2.1) \qquad E(n|p) = L(p)E_1(n|p) + (1-L(p))E_2(n|p)$$

where L(p) is given in (4.1.1),

$$(4.2.2) \qquad E(n|p) = \frac{m}{p}$$

$$(4.2.3) \qquad E_1(n|p) = \frac{m}{p} \frac{\sum_{j=0}^{m} \binom{N}{j} p^j q^{N-j}}{\sum_{j=0}^{m-1} \binom{N-1}{j} p^j q^{N-1-j}}$$

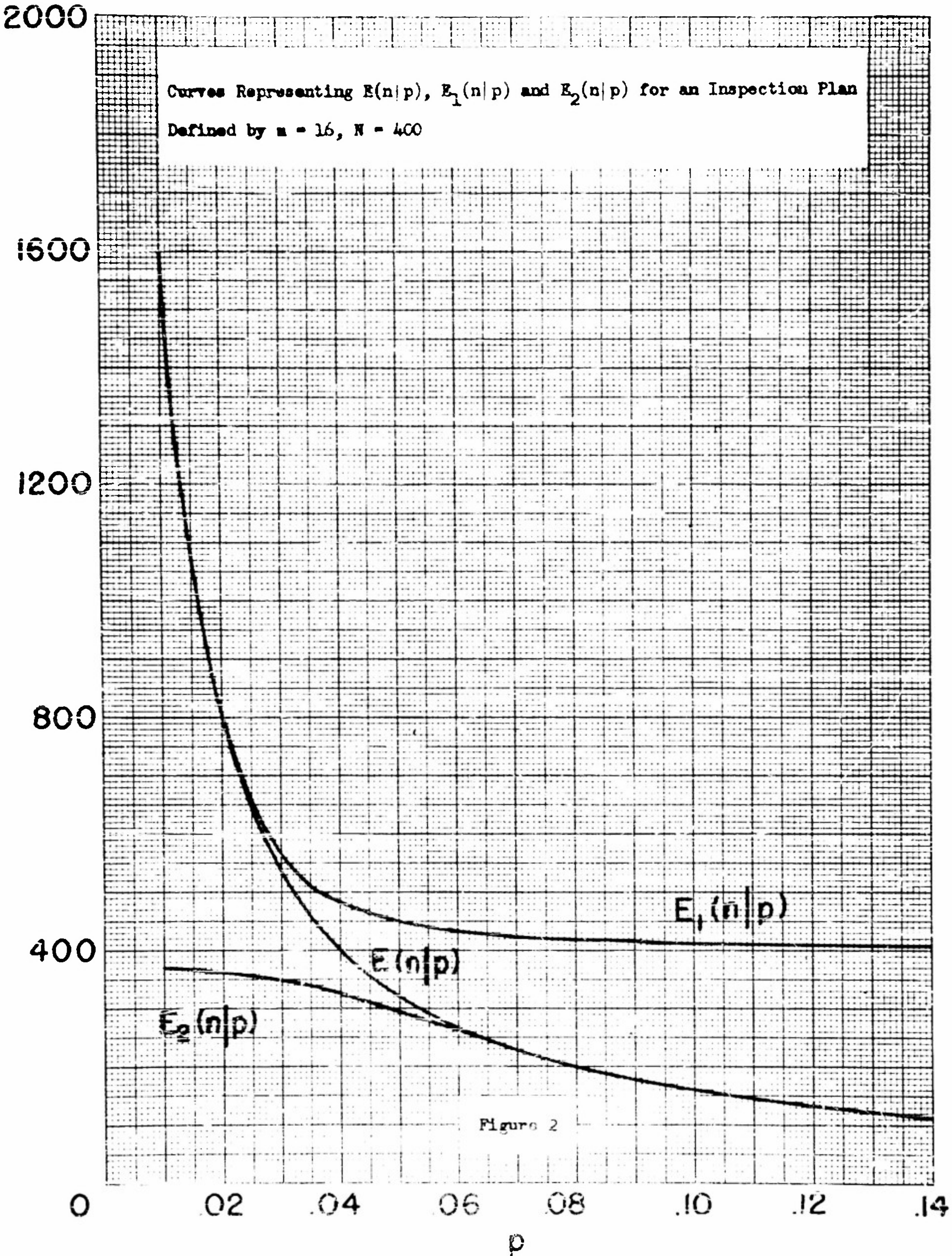
and

$$(4.2.4) \qquad E_2(n|p) = \frac{m}{p} \frac{\left(1 - \sum_{j=0}^{m} \binom{N}{j} p^j q^{N-j}\right)}{\left(1 - \sum_{j=0}^{m-1} \binom{N-1}{j} p^j q^{N-1-j}\right)} .$$

Graphs of the quantities $E(n|p)$, $E_1(n|p)$, and $E_2(n|p)$ for the plan with $m = 16$, $N = 400$, and AOQL = .0380 are given in Figure 2.

In terms of the above quantities, the AOQ function Q(p) is given by

$$(4.2.5) \qquad Q(p) = \frac{p(k-1)E(n|p)}{k[E(n|p) + (1-L(p))(N-E_2(n|p))]}$$



Curves Representing E(n|p), E1(n|p) and E2(n|p) for an Inspection Plan
Defined by m = 16, N = 400
2000
1600
1200
800
400
0
.02
.04
.06
.08
.10
.12
.14
p
E1(n|p)
E(n|p)
E2(n|p)

Figure 2

where k_1 or k_2 may be substituted for k depending upon the sampling rate employed. The AOQ curve for the design m = 16, N = 400, AOQL = .0380, and k = 20 is illustrated in Figure 3. It will be noted from the graph that the AOQ curve is monotonically increasing and approaches rapidly the AOQL. This is a general property of this function, as can be seen from a careful examination of (4.2.5).

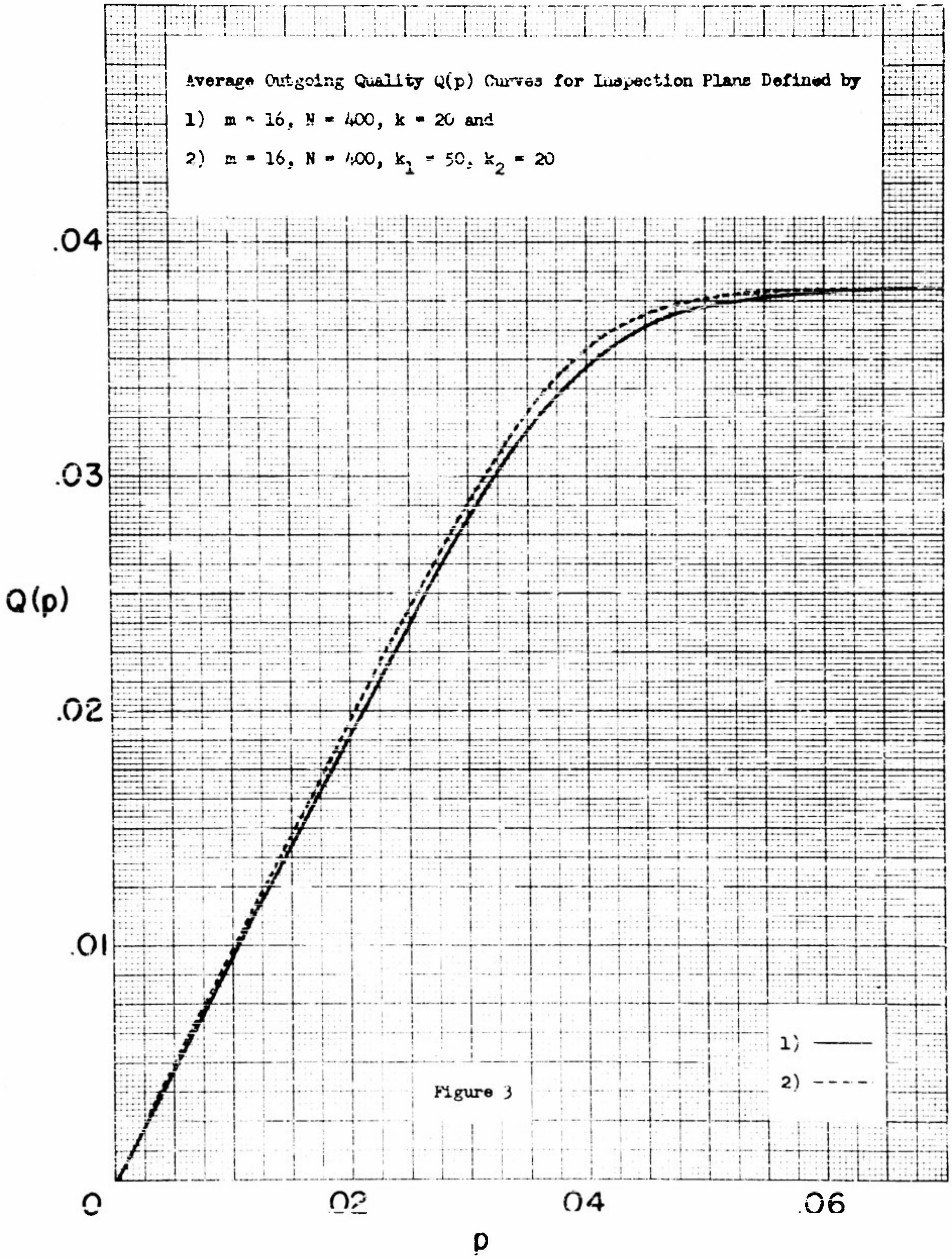
It was pointed out above that the product inspected might be subjected to strict and reduced partial inspection dependent upon the quality of the product. If the two sampling rates for reduced and strict inspection are k_1 and k_2, the Q(p) function is given by (4.2.5) with k replaced by $E(k|p)$ where

$$(4.2.6) \qquad E(k|p) = L(p)k_1 + (1-L(p))k_2 \ .$$

It should be pointed out that when strict and reduced partial sampling rates are employed, the resulting AOQL is that of the strict partial inspection rate. Thus, for example, if m = 16, N = 400, and k = 50, the AOQL = .0392; if k = 20, the AOQL = .0380; but if reduced and strict partial inspection are employed with k_1 = 50, k_2 = 20, we have AOQL = .0380. The reason that this smaller value is attained is clear since as the product deteriorates, we tend to be operating with strict partial inspection. A graph of the Q(p) function for m = 16, N = 400, k_1 = 50, and k_2 = 20 is also given in Figure 3.

4.3 The Average Fraction Inspected Curve.

The inspection plan under consideration improves the outgoing product by a screening process whereby the defective items found are replaced by non-defective items. Since the aim is to attain a given AOQL,


Average Outgoing Quality Q(p) Curves for Inspection Plans Defined by
1) m = 16, N = 400, k = 20 and
2) m = 16, N = 400, k1 = 50, k2 = 20
.04
.03
.02
.01
Q(p)
0
.02
.04
.06
p
1)
2)
Figure 3

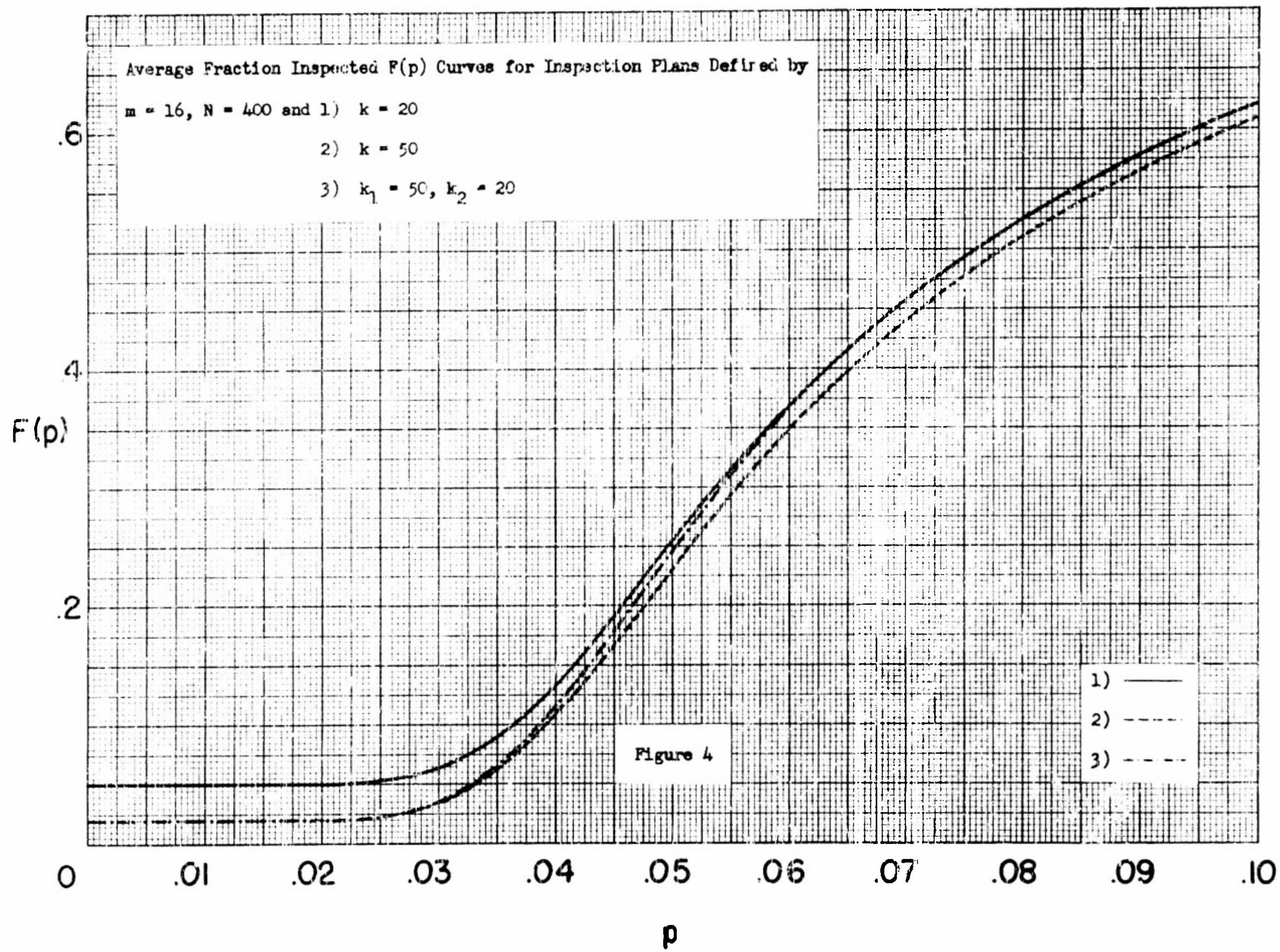
it follows that the poorer the quality of the submitted product, the greater is the fraction of the material inspected, on the average. The average fraction inspected under a given plan as a function of p will be designated by F(p) and is related to the function Q(p) by the following formula:

$$F(p) = 1 - \frac{Q(p)}{p} . \tag{4.3.1}$$

From the point of view of cost of inspection, an efficient inspection plan has the property that for $p < AOQL$, $F(p)$ is equal to the partial sampling rate, i.e., $F(p) = 1/k$, and for $p > AOQL$, $Q(p) = AOQL$ so that the fraction inspected is exactly equal to what is required to bring the product to the desired limit. The plan under consideration does not satisfy this condition of efficiency exactly, but comes close to it for reasonable values of m, N, and k. The average fraction inspected curve for the design $m = 16$, $N = 400$, $AOQL = .0380$, and $k = 20$ is given in Figure 4. The curve for the same values of m, N, with $k = 50$ and $AOQL = .0392$ as well as for the design $m = 16$, $N = 400$, $AOQL = .0380$, $k_1 = 50$, and $k_2 = 20$ are also given in Figure 4.

4.4 Variance of the Sample AOQ.

The plan under consideration is defined by these constants k, m, and N. However, the only quantity that is fixed by design is the AOQL which is essentially the ratio of m to N. Thus, in addition to requiring a given AOQL, we might impose other conditions. For example, suppose we consider the variance of the outgoing quality Q from lot to lot where a lot is defined in the following manner: If $n < N$, the lot consists of the $M = kN$ items subject to partial and complete inspection. If $n \geq N$, the lot consists only of the first N segments (of $M = kN$ items) subject



Average Fraction Inspected F(p) Curves for Inspection Plans Defined by
m = 16, N = 400 and 1) k = 20
2) k = 50
3) k1 = 50, k2 = 20
F(p)
.6
.4
.2
0
.01
.02
.03
.04
.05
.06
.07
.08
.09
.10
p
1)
2)
3)

Figure 4

to partial inspection. (It will be observed that by this definition of the lot, whenever $n > N$, $k(n-N)$ items are not included in the computation of the variance of the outgoing quality. However, the event $n > N$ will generally occur when the quality is good, in which case omission of part of the product from the consideration of the variance may not be very serious.) Then, if the production process is in control, we might require that the variance of the outgoing quality of the lots shall not exceed a given number for some specified value of p, say the AOQL. Alternatively, the constants may be chosen so that the maximum value of the variance shall not exceed a specified number. When the process is in control, the variance of $\hat{Q}$ in a lot of size M of the type defined above is given by

$$(4.4.1) \qquad \sigma^2_{\hat{Q}} = \frac{pq(k-1)E^*(n|p) + (k-1)^2p^2\sigma^2_n}{M^2}$$

where

$$(4.4.2) \qquad E^*(n|p) = \frac{1}{p}\left(m - \sum_{j=0}^{m-1}(m-j)\binom{N}{j}p^jq^{N-j}\right)$$

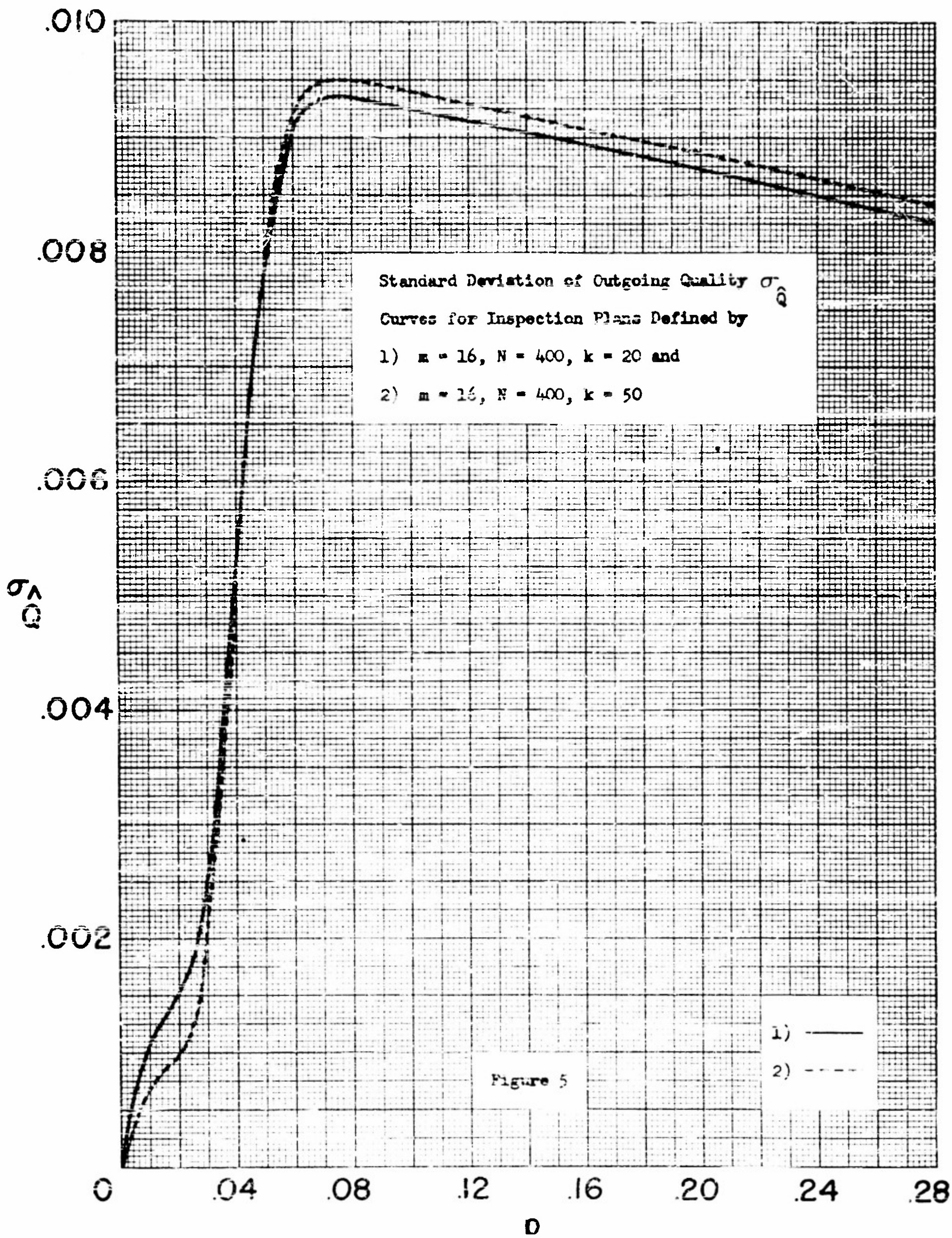
and

$$(4.4.3) \qquad \sigma^2_n = E^*[n(n+1)|p] - E^*[n|p) + 1]$$

with

$$(4.4.4) \qquad E^*[n(n+1)|p] = \frac{1}{p^2}\left(m(m+1) - \sum_{j=0}^{m}[m(m+1)-j(j-1)]\binom{N+1}{j}p^jq^{N+1-j}\right) .$$

The graph of $\sigma_{\hat{Q}}$ for the plan $m = 16$, $N = 400$, AOQL = .0380, and $k = 20$ is given in Figure 5. The graph for the same values of m and N, but with AOQL = .0392, $k = 50$ is also given in Figure 5. We observe that the maximum


Standard Deviation of Outgoing Quality $\sigma_{\hat{Q}}$
Curves for Inspection Plans Defined by
1) m = 16, N = 400, k = 20 and
2) m = 16, N = 400, k = 50
.010
.008
.006
.004
.002
$\sigma_{\hat{Q}}$
0
.04
.08
.12
.16
.20
.24
.28
D
1)
2)
Figure 5

standard deviation for $\hat{Q}$ for each of these two plans is approximately .0093 and .0095, respectively.

Even if the production process is not in control, various reasonable models of the possible distribution of defectives among the segments of size k in the lots may be constructed, and either the variance or an upper limit for the variance of $\hat{Q}$ obtained as a function of the parameters k, m, and N. For example, one model which appears reasonable is to assume that the defective items produced by the machine tend to occur in clusters but that the position of the clusters in the lot is random. The degree of clustering within the segments into which the lot is divided may be measured by the intraclass correlation

$$\rho = \frac{\sum_{i=1}^{N} (p_i - p)^2 - \frac{1}{k-1} \sum_{i=1}^{N} p_i q_i}{N\, pq} \tag{4.4.5}$$

where the p_i's are the fraction defective in the segments of size k, and p is the average fraction defective in the lot. When each p_i is either 0 or 1, $\rho = 1$ and the variance of $\hat{Q}$ in that case is zero. When all the p_i's are equal to the fixed average p, ρ takes on its minimum value $-\frac{1}{k-1}$. But it can be shown that when ρ is a minimum, the variance of $\hat{Q}$ is a maximum. Thus the variance of $\hat{Q}$ in this special case gives a convenient upper bound for the variability of the outgoing quality even if the production process is not in statistical control. Moreover the variance of $\hat{Q}$ in this special case is computable and is given by

$$\sigma_{\hat{Q}}^2 = \frac{k^2 p^2 \sigma_n^2 + (2k-1)p^2 (E^*(n|p))^2 - 2kpE(nd_n|p) + E(d_n^2|p)}{M^2} \tag{4.4.6}$$

where σ_n^2 and $E^*(n|p)$ are given by (4.4.3) and (4.4.2), respectively, d_n is the total number of defectives found in the partial inspection of the lot, i.e., $d_n = m$ for $n < N$ and $d_n = 0,1,\ldots,m$ for $n = N$,

$$(4.4.7) \qquad E(nd_n|p) = mE^*(n|p) - N\sum_{j=0}^{n-1}(m-j)\binom{N}{j}p^jq^{N-j}$$

and

$$(4.4.8) \qquad E(d_n^2|p) = m^2 - \sum_{j=0}^{m-1}(m^2-j^2)\binom{N}{j}p^jq^{N-j} \quad .$$

The graph of the standard deviation for the worst possible distribution of the defectives for the plan $m = 16$, $N = 400$, $k = 20$, and AOQL = .0380 is given in Figure 6. The graph for the same values of m and N, AOQL = .0392 and $k = 50$ is approximately the same as this one.

4.5 Variance of Sample Fraction Inspected.

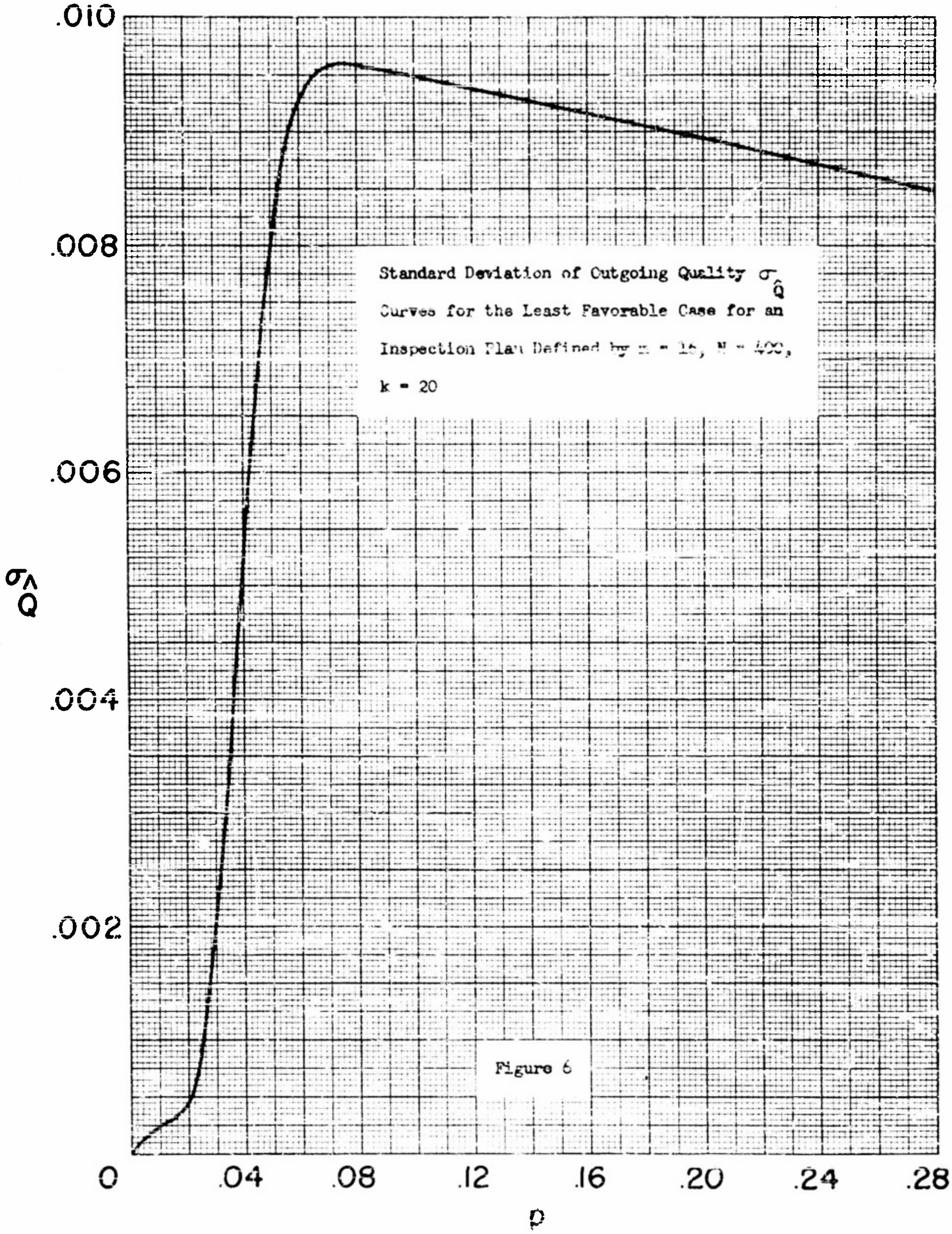
The fraction of material inspected in a lot of size kN during a single inspection operation, both partial and complete, is given by

$$(4.5.1) \qquad \hat{F} = 1 - \frac{(k-1)}{k}\frac{n}{N}$$

where n is the number of segments partially inspected within the lot. It may be of interest, therefore, to see how variable the inspection load is as a function of the production quality p. This is measured by the variance of $\hat{F}$ and is clearly given by

$$(4.5.2) \qquad \sigma_{\hat{F}}^2 = \left(\frac{k-1}{k}\right)^2 \frac{\sigma_n^2}{N^2}$$

where σ_n^2 is given in equation (4.4.3). The graph of the standard deviation



Standard Deviation of Outgoing Quality $\sigma_{\hat{Q}}$ Curves for the Least Favorable Case for an Inspection Plan Defined by n = 16, N = 400, k = 20

Figure 6

for $m = 16$, $N = 400$, AOQL = .0380, $k = 20$ is given in Figure 7. The corresponding graph for $m = 16$, $N = 400$, AOQL = .0392, $k = 50$ is also given in Figure 7.

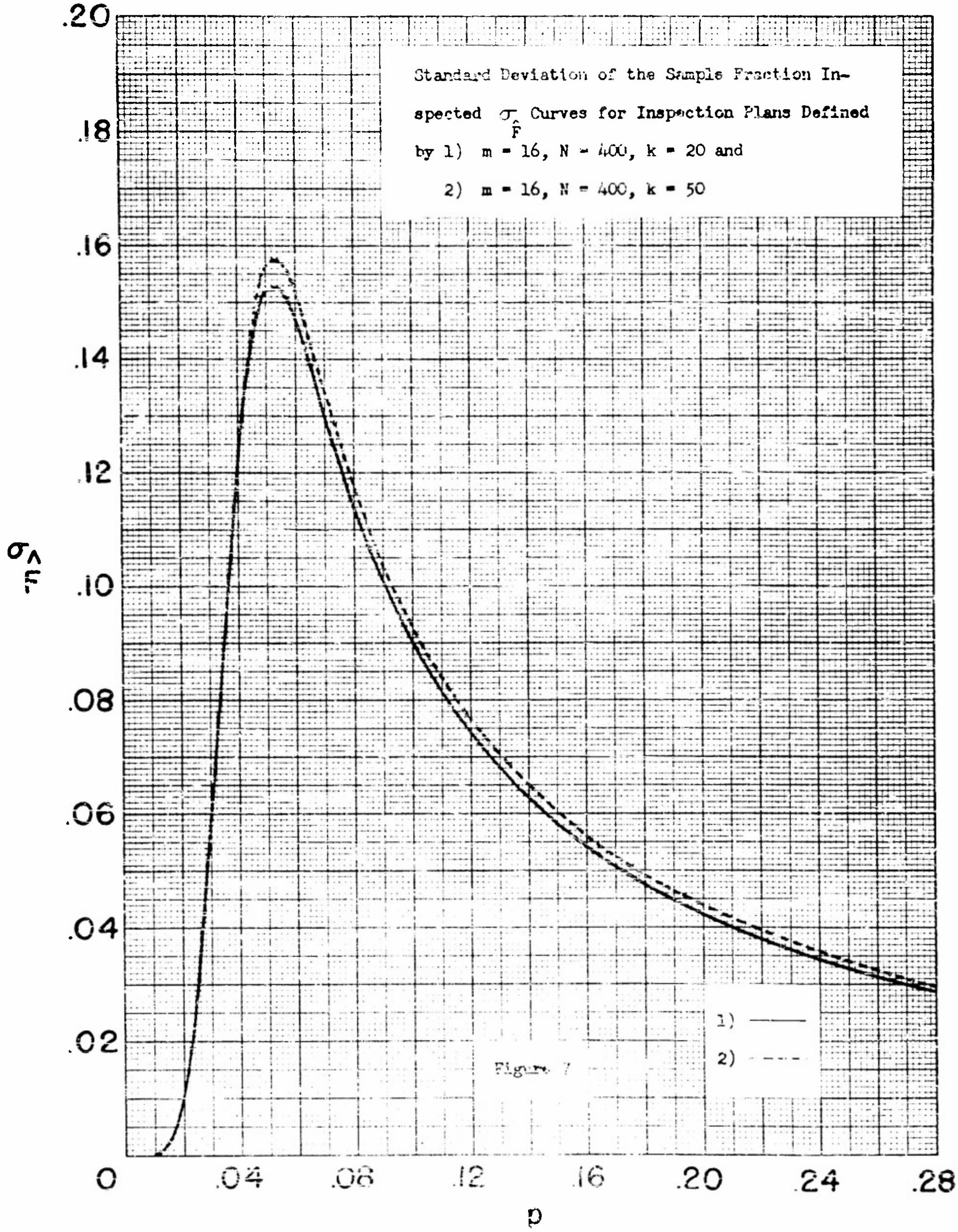
4.6 Estimation of the Process Average.

As was previously indicated, a continuous sampling inspection plan aims to accomplish two objectives, (1) to improve the quality of product through a screening process, and (2) to supply information for quality control purposes. It is probably most often the case that the best way to improve the quality of the product is to do it by quality control before the product is manufactured rather than by screening after the product is manufactured. To supply current information on the quality of the manufactured product requires an estimate of the process average. The sampling plan under consideration gives two types of information, that obtained from partial inspection, and that obtained from 100 percent inspection when this is required. It is clear that the information obtained from partial inspection gives a more representative picture of the quality of the product and we propose to base our estimate of p entirely on this information.

When partial inspection terminates, the number of defectives obtained is necessarily m, a constant. The only random variable involved is n, the number of segments partially inspected. There are two possible estimates of p based on these quantities m and n. One estimate is given by

$$(4.6.1) \qquad \tilde{p}_m = \frac{m}{n} .$$

This estimate is biased with an expectation given by



Standard Deviation of the Sample Fraction Inspected $\sigma_{\hat{F}}$ Curves for Inspection Plans Defined by 1) m = 16, N = 400, k = 20 and
2) m = 16, N = 400, k = 50
.20
.18
.16
.14
.12
.10
.08
.06
.04
.02
0
.04
.08
.12
.16
.20
.24
.28
$\sigma_{\hat{F}}$
p
1)
2)

Figure 7

$$(4.6.2) \qquad E(\tilde{p}_m) = (-1)^m \, m\left(\frac{p}{q}\right)^m \left[\sum_{r=1}^{m-1} (-1)^{r-1} \frac{1}{r} \left(\frac{p}{q}\right)^r + \log p\right] .$$

The bias of this estimate is always positive but approaches zero with large values of m. A graph showing $E(\tilde{p}_m)$ as a function of p for $m = 2,3,4,$ and 5, is given in Figure 8. This estimate, though biased, is probably good. Unfortunately, no one has as yet obtained the variance of it. An unbiased estimate of p for the plan under consideration originally suggested by J. B. S. Haldane [3] is given by

$$(4.6.3) \qquad \hat{p}_m = \frac{m-1}{n-1} .$$

The variance of this estimate is known and is given in closed form by

$$(4.6.4) \qquad \sigma^2_{\hat{p}_m} = pq + (-1)^{m-1} p(m-1)\left(\frac{p}{q}\right)^{m-1}\left[\sum_{r=1}^{m-1} (-1)^{r-1} \frac{1}{r} \left(\frac{q}{p}\right)^r + \log p\right] .$$

The variance of the same quantity was obtained by Haldane as the following infinite series which may be more convenient for computational purposes,

$$(4.6.5) \qquad \sigma^2_{\hat{p}_m} = \frac{p^2 q}{m}\left[1 + \frac{2!q}{m+1} + \frac{3!q^2}{(m+1)(m+2)} + \ldots\right] .$$

The graph of the variance of $\hat{p}_m$ for $m = 2,3,4,$ and 5 is given in Figure 9.

4.7 Continuous Inspection Plan for Fixed Lot Sizes.

Often the product submitted for continuous inspection comes in naturally-formed lots of size M, and it may be desirable to complete an inspection operation with each lot. The plan previously discussed can be easily modified to accomplish this.

Expected Value $E(\tilde{p}_m)$ Curves of the Biased Estimate of the Process Average Fraction Defective for $m = 2, 3, 4, 5$

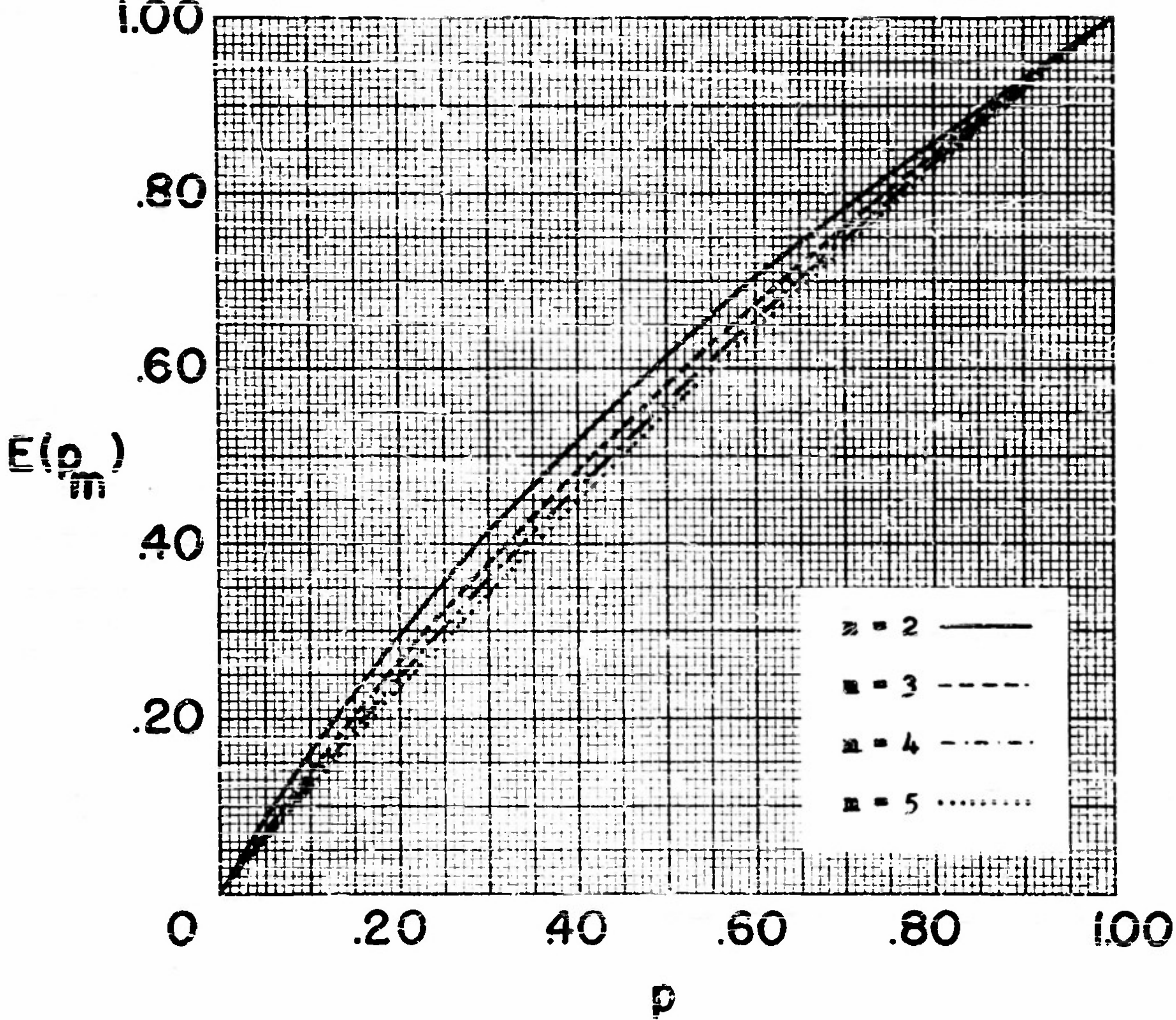


Figure 8

The Variance $\sigma^2_{\hat{p}_m}$ Curves of the Unbiased Estimate $\hat{p}$

of the Process Average Fraction Defective for

$m = 2, 3, 4, 5$

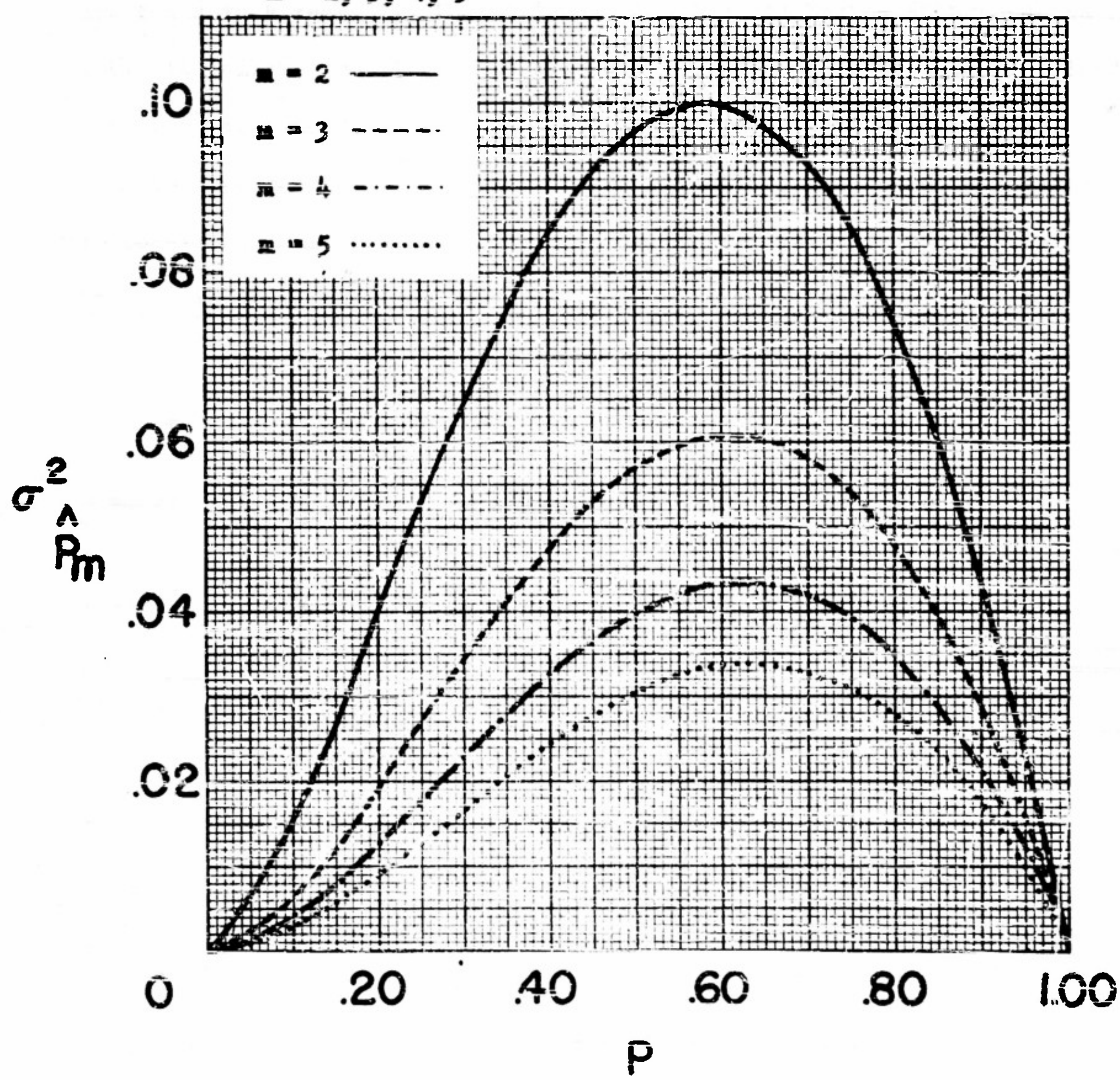


Figure 9

The lot of product of size M is divided into N segments of k items each. The partial and complete inspection procedure is identical with that previously described except that now if fewer than m defective items are found in N segments, inspection terminates and begins anew on a new lot. It follows therefore that whenever 100 percent inspection is called for, the two procedures are identical. The present procedure differs from the preceding only if fewer than m defectives are found in N segments. In the former case partial inspection terminates while in the latter case it goes on until m defectives are found. The AOQL for this plan is still

$$(4.7.1) \qquad AOQL = \frac{k-1}{k} \cdot \frac{m}{N} .$$

However, the AOQ function is somewhat different and is given by

$$(4.7.2) \qquad Q(p) = \frac{k-1}{k} \frac{m}{N} \left(1 - \frac{1}{m} \sum_{j=0}^{m-1} (m-j)\binom{N}{j} p^j q^{N-j}\right) .$$

A careful inspection will show that the variances of $\hat{Q}$ given in equations 4.4.1 and 4.4.6 as well as the variance for $\hat{F}$ in 4.5.2 are precisely the variances for the case under consideration. Thus, these formulas can be used to compute the variance of the sample outgoing quality and fraction inspected for controlled production, and an upper bound for the variance of the sample outgoing quality in the general case for this plan.

For the present case, an unbiased estimate of the lot fraction defective based on the information from partial inspection is given by

$$(4.7.3) \qquad \hat{p} = \frac{m-1}{n-1}$$

in case m defectives are found during partial inspection of the lot and if d defectives are found with $d < m$, it is given by

$$\hat{p} = d/N \qquad d = 0,1,\ldots,m-1 \quad . \tag{4.7.4}$$

In conclusion I wish to express my indebtedness to A. Wald and J. Wolfowitz [2] as well as to H. F. Dodge [4], [5] from whom many of the ideas contained herein have been borrowed. I also wish to thank Jack J. Ingram of the Bureau of the Census and Rosedith Sitgreaves of the Applied Mathematics and Statistics Laboratory of Stanford for the generous help they have given me in the preparation of this report.

References

[1] Gerald J. Lieberman, "A note on Dodge's continuous inspection plan," Ann. Math. Stat., Vol. 24 (1953).

[2] A. Wald and J. Wolfowitz, "Sampling inspection plans for continuous production which insure a prescribed limit on the outgoing quality," Ann. Math. Stat., Vol. 16 (1945).

[3] J. B. S. Haldane, "On a method of estimative frequancies," Biometrica, Vol. 33 (1945).

[4] H. F. Dodge, "A sampling inspection plan for continuous production," Ann. Math. Stat., Vol. 14 (1943).

[5] H. F. Dodge, "Sampling plans for continuous production," Industrial Quality Control, Vol. 4 (1947).

STANFORD UNIVERSITY
Technical Reports Distribution List
Contract N6onr 25126

ASTIA, Western Regional Office
5504 Hollywood Blvd.
Los Angeles 28, California 1

Armed Services Technical Information Agency
Documents Service Center
Knott Building
Dayton 2, Ohio 5

Asst. Chief of Staff, G-4
United States Army
Procurement Division
Standards Branch
Washington 25, D. C. 15

Chief, Bureau of Aeronautics
Department of the Navy
Washington 25, D. C. 1

Chief, Bureau of Ordnance QCC
Department of the Navy
Quality Control Division
Washington 25, D. C. 1

Chief, Bureau of Ordnance QCD
Department of the Navy
Quality Control Division
Washington 25, D. C. 1

Chief, Bureau of Ordnance QC5
Department of the Navy
Quality Control Division
Washington 25, D. C. 2

Chief, Bureau of Ships
Asst. Chief for Research and Development
Department of the Navy
Washington 25, D. C. 2

Chief, Statistical Engineering Laboratory
National Bureau of Standards
Washington 25, D. C. 1

Chief of Naval Research
Office of Naval Research
Washington 25, D. C.
Attn: Code 432
(Mathematics Branch) 1

Planning Research Division
Deputy Chief of Staff
Comptroller, U. S. Air Force
The Pentagon
Washington 25, D. C. 1

Chief of Naval Research
Office of Naval Research
Washington 25, D. C.
Attn: Code 433
(Statistics Branch) 2

Chief of Ordnance
United States Army
Research and Development Division
Washington 25, D. C.
Attn: Brig. General L. E. Simon 1
Mr. Charles Bicking 1

Commander
U. S. Naval Ordnance Test Station
Inyokern, China Lake, Calif. 1

Commanding General
Army Chemical Center
Quality Assurance Branch
Edgewood, Maryland 2

Commanding Officer
Ballistics Research Lab.
U. S. Proving Grounds
Aberdeen, Maryland
Attn: Mr. R. H. Kent 2

Commanding Officer
Office of Naval Research
Branch Office
1000 Geary Street
San Francisco 9, California 1
Attn: Dr. J. D. Wilkes 1

Commanding Officer
Office of Naval Research
Branch Office
Navy No. 100
Fleet Post Office
New York, N. Y. 2

Director of Research
Operations Research Office
U. S. Army
Fort McNair
Washington 25, D. C. 1

Inspection Division
Army Quartermaster Corps
111 East 16th St.
New York, N. Y.
Attn: Mr. David Schwartz 1

Director, Naval Research Lab.
Washington 25, D. C.
Attn: Technical Information Officer 6

Commanding Officer
Signal Corps Supply Agency
225 South 18th St.
Philadelphia 3, Pa.
Attn: Chief, Quality Control Branch, SIGSU-H6d
Inspection Engineering Division 2

Commanding General
New York Quartermaster Procurement Agency
Inspection Division
111 East 16th St.
New York, N. Y. 2

Dr. Clifford J. Maloney
Chief, Statistics Branch
Chemical Corps Biological Labs.
Physical Sciences Division
Camp Detrick, Maryland 1

Commanding Officer
9926 Technical Service Unit
Armed Services Medical Procurement Agency
Inspection Division
84 Sands Street
Brooklyn, N. Y. 1

Chief, Inspection Division
Office of Standardization
Defense Supply Management Agency
The Pentagon
Washington 25, D. C. 2

Commanding Officer
Central Air Procurement District
West Warren Ave. and Lonyo Blvd.
Detroit 32, Michigan
Attn: Director, Quality Control 1

Commanding General
Air Materiel Command
Quality Control Division MCPLXP
Wright-Patterson Air Force Base
Dayton, Ohio 15

Director of Research & Development
AFDRD-RE-3
Washington 25, D. C. 1

Chief, Statistical Engineering Laboratory
National Bureau of Standards
Washington 25, D. C. 1

Chief of Naval Materiel
Code M553
Department of the Navy
Washington 25, D. C. 1

Director
Office of Naval Research
Branch Office
844 North Rush St.
Chicago 11, Illinois 1

Director
Office of Naval Research
Branch Office
346 Broadway
New York 13, N. Y. 1

Director
Office of Naval Research
Branch Office
1030 E. Green St.
Pasadena 1, California 1

Chairman
Research & Development Board
The Pentagon
Washington 25, D. C. 2

Assistant Chief of Staff, G-4 for Research & Development
U. S. Army
Washington 25, D. C. 1

Chief of Naval Operation
Operations Evaluation Group-OP342E
The Pentagon
Washington 25, D. C. 1

Commanding General
Air Proving Ground
Eglin Air Force Base
Eglin, Florida 1

Commander
U. S. Naval Proving Ground
Dahlgren, Virginia 1

Office of Naval Research
Logistics Branch
Code 436
T-3 Building
Washington, D. C. 1

Logistics Research Project
George Washington University
707 22nd Street, N.W.
Washington 7, D. C. 1

Asst. for Operations Analysis
Headquarters, U. S. Air Force
Washington 25, D. C. 1

Ames Aeronautical Laboratory
Moffett Field, California
Attn: Technical Librarian 1

N.A.C.A.
1724 F Street, N.W.
Washington 25, D. C.
Attn: Chief, Office of
Aeronautical Intelligence 1

Los Angeles Engineering Field
Office
Air Research and Development
Command
5504 Hollywood Blvd.
Los Angeles 28, California
Attn: Mr. Chester Pierce 1

U. S. Air Force
Engineering Liaison Office
Ames Aeronautical Laboratory
Moffett Field, California
Attn: Mr. Carl Tusch 1

Frankford Arsenal
VT Fuze Department
Inspection Division
Philadelphia 37, Pa.
Attn: ORDBA-VK 2

Frankford Arsenal
Artillery Ammunition Dept.
Inspection Engineering Division
Philadelphia 37, Pa. 1

U. S. Naval Ordnance Plant
Department of the Navy
Indianapolis 18, Indiana 1

Dr. Paul R. Rider
Chief, Mathematical Statistics Group
Flight Research Laboratory
Wright Air Development Center
Wright-Patterson Air Force Base
Dayton, Ohio 1

Mr. T. M. Vining
Chief, Test Projects Branch
Products Division
Chemical Corps Engineering Agency
Army Chemical Center
Maryland 1

Commanding General
Ordnance Ammunition Center
Joliet, Illinois
Attn: ORDLY-I-V 3

Chief, Procurement Maintenance
Engineering Division
Fort Monmouth, New Jersey
Attn: Procurement Data Branch
SIGEL-PMP-1 1

Northeastern Air Procurement District
Attn: NEQC
14 Court Square
Boston 8, Massachusetts 1

Central Air Procurement District
Attn: CEQC
W. Warren Ave. & Lonyo Blvd.
Detroit 32, Michigan 1

Western Air Procurement District
Attn: WEQC
155 W. Washington Blvd.
P. O. Box 3849, Terminal Annex
Los Angeles 54, California 1

Midcentral Air Procurement District
Attn: MIQC
165 North Canal Street
Chicago 6, Illinois 1

Southern Air Procurement District
Attn: SOQC
3309 Winthrop
P. O. Box 9038
Fort Worth 7, Texas 1

Eastern Air Procurement District
Attn: EAQC
655 Madison Avenue
New York 21, N. Y. 1

U S. Naval Ordnance Test Station
125 South Grand Ave.
Mail Station 54
Pasadena, California 1

Commanding General
Marine Corps Depot of Supplies
1100 South Broad Street
Philadelphia 46, Pa. 1

Naval Inspector of Ordnance
50 W. Main Street
Rochester 4, N. Y.
Attn: Mr. Wollman 1

Chief, Thermionics Branch
Evans Signal Laboratory
Belmar, New Jersey
Attn: Mr. Ross Kilgore 1

Test and Evaluation Laboratory
U. S. N. Underwater Ordnance
Station
Newport, Rhode Island 1

Q. E. Laboratory
U. S. Naval Ammunition Depot
Crane, Indiana 1

Commanding Officer
Rock Island Arsenal
Rock Island, Illinois
Attn: Engineering Officer 1

Scranton Ordnance Plant
156 Cedar Avenue
Scranton, Pa.
Attn: Mr. Carl D. Larson
Chief Inspector 1

Office of Technical Services
Department of Commerce
Washington 25, D. C. 1

Captain Burdette T. Person
Chief, Quality Control Branch
Central Air Procurement District
1279 W. Third St.
Cleveland 13, Ohio 2

Air Force Plant Representative Office
Central Air Procurement District
General Electric Company
Lockland 15, Ohio 1

RAND Corporation
1500 Fourth St.
Santa Monica, California 1

Statistical Laboratory
Department of Mathematics
University of California
Berkeley 4, California 1

Kimberly Corporation
8476 Warner Drive
Culver City, California
Attn: Miss Lucille M. Leis
Office Manager 1

Dr. Adam Abruzzi
Dept. of Economics of Engineering
Stevens Inst. of Technology
Hoboken, New Jersey 1

Prof. Stephen G. Allen
School of Business Administration
University of Minnesota
Minneapolis, Minn. 1

Prof. Robert Bechhafer
Dept. of Industrial Engineering
Columbia University
New York 3, N. Y. 1

Prof. Maurice H. Belz
University of Melbourne
Carlton N. 3
Victoria, Australia 1

Prof. J. N. Berrettoni
Western Reserve University
Cleveland, Ohio 1

Mr. P. M. Blunk
Quality Evaluation Laboratory
U S. Naval Magazine
Port Chicago, California 1

Mr. Milton N. Bradley
632 Van Cortlandt Park Ave.
Yonkers 5, N. Y. 1

Prof. Russell Bradt
Dept. of Mathematics
University of Kansas
Lawrence, Kansas 1

Prof. Irving W. Burr
Dept. of Mathematics
Purdue University
Lafayette, Indiana 1

Prof. Edward P. Coleman
Engineering Dept.
University of California
Los Angeles 24, California 1

Dr. Louis Court
Division 17
National Bureau of Standards
Washington 25, D. C. 1

Mrs. J. Henley Crosland
Director of Libraries
Georgia Inst. of Technology
Atlanta, Ga. 1

Mr. H. F. Dodge
Bell Telephone Labs, Inc.
463 West Street
New York, N. Y. 1

Prof. Acheson J. Duncan
School of Business
The Johns Hopkins University
Baltimore 18, Md. 1

Dr. Benjamin Epstein
Dept. of Mathematics
Wayne University
Detroit, Michigan 1

Dr. George E. Forsythe
National Bureau of Standards
Inst. for Numerical Analysis
University of California
405 Hilgard Ave.
Los Angeles 24, California 1

Mr. S. Gaspar
U. S. N. O. T. S.
Pasadena Annex
125 S. Grand Ave.
Pasadena, California 1

Mr. Bernard P. Goldsmith
Quality Control Engineer
Raytheon Mfg. Co.
55 Chapel St.
Newton 58, Mass. 1

Prof. Eugene L. Grant
Civil Engineering Dept.
Stanford University
Stanford, California 1

Mr. Brent C. Jacob, Jr.
Supervisor of Quality Inspection
Chrysler Corp.
Detroit 31, Michigan 1

Mr. Calvin J. Kirchen
Statistical Quality Control Analyst
AF Plant Representative
Central Air Procurement Division
Kaiser Mfg. Corp.
Willow Run, Michigan 1

Prof. Sebastian Littauer
Dept. of Industrial Engineering
Columbia University
New York, N. Y. 1

Dr. William G. Madow
Dept. of Mathematics
University of Illinois
Urbana, Illinois 1

Dean Paul E. Mohn
School of Engineering
The University of Buffalo
Buffalo, N. Y. 1

Mr. L. E. Morris
Q. E. Laboratory
U. S. M. A. D.
Bangor, Washington 1

Prof. E. G. Olds
Dept. of Mathematics
San Jose State College
San Jose 14, California 1

Mr. John J. Riordan
Chief, Quality Analysis Office
Wright-Patterson Air Force Base
Dayton, Ohio 1

Mr. Harry G. Romig
Quality Manager
Hughes Aircraft Co.
Culver City, California 1

Prof. Norman Rudy
Statistics Dept.
Sacramento State College
Sacramento, California 1

Prof. Henry Scheffe
Statistical Laboratory
University of California
Berkeley, California 1

Mr. R. H. Shaw
U. S. Naval Ordnance Plant
Indianapolis, Indiana 1

Mr. Walter Shewhart
Bell Telephone Labs., Inc.
Murray Hill, New Jersey 1

Prof. Herbert Solomon
Teachers College
Columbia University
New York 27, N. Y. 1

Mr. M. D. Springer
U. S. Naval Ordnance Plant
Indianapolis, Indiana 1

Mr. Arthur Stein
Ordnance Ammunition Center
Joliet, Illinois 1

Miss Elizabeth Vaughan
2325 7th St.
Bremmerton, Washington 1

Prof. W. Allen Wallis
Committee on Statistics
University of Chicago
Chicago 37, Illinois 1

Mr. Joseph Weinstein
Evans Signal Laboratory
Belmar, New Jersey 1

Prof. M. A. Woodbury
Statistics Dept.
Wharton School
University of Pennsylvania
Philadelphia 4, Pa. 1

Additional copies for project
leader and assistants, office
file, and reserve for future
requirements 70