NAVY DEPARTMENT
THE DAVID W. TAYLOR MODEL BASIN
WASHINGTON 7, D.C.

THE DEAERATION OF WATER BY A SOUND BEAM

by

Leonard Pode, M.S.

May 1953

Report 854

TABLE OF CONTENTS

NOTATION

a_0	$a_0 = \dfrac{-1}{3\gamma\left(1-\frac{\omega^2}{\omega_0^2}\right)}$
a_1	$a_1 = \left(3\gamma + 1 - \frac{\omega^2}{\omega_0^2}\right)\frac{a_0^2}{4}$
a_2	$a_2 = \dfrac{\left(3\gamma + 1 + 5\,\frac{\omega^2}{\omega_0^2}\right)}{\left(4\,\frac{\omega^2}{\omega_0^2} - 1\right)}\,\frac{a_0^2}{4}$
$c\,;\,c(r,t)$	The concentration of air in the water at an arbitrary point and time
$c_\infty\,;\,c(\infty,t)$	The concentration of air in the water at infinity; also the concentration of air in water when the water is in equilibrium with air at pressure P_0 and temperature T_0
c_p	The specific heat for air under constant pressure
c_v	The specific heat for air under constant volume
D	The diffusion constant for air in water
F	$F = \frac{R_0}{r}\,e^{-V\left(\frac{r}{R_0}-1\right)}\cos V\left(\frac{r}{R_0}-1\right)$
G	$G = \frac{R_0}{r}\,e^{-V\left(\frac{r}{R_0}-1\right)}\sin V\left(\frac{r}{R_0}-1\right)$
$I_0\,;\,I_0(V)$	$I_0(V) = \int_1^\infty e^{-V(x-1)}\,x^{-3}\sin V(x-1)\,dx$
k	The Boltzmann constant
KE	The kinetic energy of the motion of the water
m	The average molecular mass for air
N	Number of molecules in the bubble
N_0	The initial number of molecules in the bubble

$\frac{d}{dt}N_L$	The rate at which molecules leave the bubble and enter the water
$\frac{d}{dt}N_G$	The rate at which molecules stream away from the water surface and enter the bubble
n	The molecular density of the air
n_0	The molecular density for air at pressure P_0 and temperature T_0
P_g	The pressure in the bubble
P_0	The undisturbed hydrostatic pressure; the average pressure at infinity
P_s	The sound pressure
P	The pressure in the water at infinity
p	The relative amplitude of the sound pressure; $P_s = pP_0 \sin \omega t$
R	The instantaneous radius of the bubble
$\dot{R}$	The time rate of change of R
R_0	The equilibrium radius of the bubble
R_ω	The resonant radius at angular frequency ω
r	The distance from the center of the bubble to a point in the water
T	The temperature of the air in bubble, Kelvin
T_0	The constant temperature of the water, Kelvin
t	Time
v	The radial velocity of the fluid at the point r
V	$V = \sqrt{\frac{\omega}{2D}}\,R_0$
$\frac{d}{dt}W$	The rate at which work is done upon the water

β	$\beta = (m/2kT)^{1/2}$
β_0	The value of β at $T = T_0$
γ	The ratio of specific heats $\frac{c_p}{c_v}$; approximately 4/3 for air at standard conditions
δ	Damping constant
κ	A constant of integration
λ	$\lambda = \left(\frac{P_0}{2\pi\rho_0}\right)^{\frac{1}{2}} \frac{n_0}{c_\infty} \frac{R_0}{D}$
ρ_w	The density of the water
ρ_0	The density of air at pressure P_0 and temperature T_0
τ	The period of the sound beam
ν	Frequency
ν_0	Natural frequency
ω	Angular frequency
ω_0	The angular frequency corresponding to natural frequency

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ABSTRACT

A calculation is made to determine what part might be played by the diffusion of air into existing bubbles as a consequence of the mechanical motions induced by a sound beam. The case of an isolated bubble in a weak sound beam of wave length considerably greater than the bubble radius is considered. The effects of surface tension, viscosity and energy dissipation are neglected. The diffusion problem is treated by means of a perturbation technique. The growth of the bubble is found to be of second order in the sound pressure, i.e., proportional to the sound intensity. Numerical results presented show that the effect is sufficient to account for significant bubble growth especially in the case of very small bubbles.

INTRODUCTION

The deaeration of water by an ultrasonic beam is a phenomenon which has become increasingly familiar to experimenters in ultrasonics. However, there is as yet no satisfactory account of the mechanism responsible for this effect.** The calculation that is made here lends support to the view that the deaeration results from the growth of small air bubbles which is caused by the mechanical motion that is forced by the sound beam. The motion of a bubble responding to a sound beam is unsymmetrical in such a manner that the average radius becomes slightly greater than the equilibrium radius. Moreover, because the diffusion of air into the bubble varies with the area of the boundary, the periods when the bubble is expanded have greater effect upon the diffusion of air into the bubble than the periods when the bubble is contracted. Hence it is to be expected that the mechanical motions of an

*A dissertation submitted to the faculty of the Graduate School of Arts and Sciences of the Catholic University of America in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

**After most of the work presented here had been accomplished it was discovered that a similar calculation had been made by F.G. Blake, Jr. See Reference 2 on page 20 of this report. A comparison of the two calculations is presented in Appendix I of this paper.

air bubble will result in the growth of the bubble. This mechanism for the growth of an air bubble in a sound field is very closely related to the mechanism for growth of an air bubble in a cavitating flow. The latter case has been treated in an earlier paper.[1] The main difference between these two phenomena are the much smaller scale and greater symmetry of the motion of the sound field. On the other hand a bubble can readily be exposed to a sound field for a relatively long time whereas the time spent in a cavitating flow is usually brief. The calculation that is presented here follows along the lines of the earlier calculation. However, because of the small scale of the motion it is now possible to treat the diffusion of air through the water in more detail.

A single small bubble is considered to be immersed in an infinite body of water through which a sound beam is passing. The radius of the bubble is assumed to be so small in comparison to the wave length of the sound beam that the beam can be represented by a periodic variation in the hydrostatic pressure at infinity. Consequently spherical symmetry is preserved in the motion. Only the case of a weak sound beam is considered so that the variation of the pressure at infinity may be considered small compared to the average hydrostatic pressure. Surface tension is neglected and it is assumed that all conditions are such that in the absence of the sound beam equilibrium would obtain. The conduction of heat is also neglected so that the temperature in the water remains constant and uniform and the compressions and expansions of the air in the bubble are adiabatic. In addition the presence of water vapor in the bubble is ignored.

The object of this calculation is to determine the average rate of growth of the bubble when conditions have become steady. It will be assumed that the change of the amount of air in the bubble during the time required to attain steady conditions and during a period of the sound field are both insignificant in comparison to the amount of air initially in the bubble. Therefore, the effect of the change of air content during one period on the rate of growth will be ignored. Thus the pressure of the air in the bubble will be assumed to be directly given by the adiabatic relationship without correction for the change in the air content in the bubble. Similarly the density of air in the bubble will be taken as inversely proportional to the volume.

The calculation is divided into three parts covering (1) the equation of motion of the boundary of the bubble, (2) the boundary conditions for the diffusion equation, (3) the solution of the diffusion equation.

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[1]References are listed on page 20.

THE EQUATION OF MOTION OF THE BOUNDARY OF THE BUBBLE

The equation of motion of an air bubble in water under the conditons specified above has been treated by many writers. The derivation is repeated here in order to obtain the equation in a form suitable for present purposes.

Let the origin of coordinates be the center of the bubble. Let r be the distance from the origin to a point in the water and let v be the radial velocity of the fluid at this point. Let R be the radius of the bubble and $\dot{R}$ be the time rate of change of R. Then the incompressibility of the water requires

$$v = \frac{R^2}{r^2}\dot{R} \qquad [1]$$

and the rate of change of kinetic energy is

$$\frac{d}{dt}KE = \frac{d}{dt}\int_R^\infty \rho_w \frac{v^2}{2}4\pi r^2 dr = 4\pi\rho_w\left[\dot{R}\ddot{R}R^3 + \frac{3}{2}R^2\dot{R}^3\right] \qquad [2]$$

where ρ_w is the density of the water. If P_g is the pressure in the bubble and P_∞ is the pressure in the water at infinity, the rate at which work is done upon the water is given by

$$\frac{d}{dt}W = (P_g - P_\infty)\frac{d}{dt}\frac{4\pi R^3}{3} \qquad [3]$$

Writing $P_\infty = P_0 + P_s$ where P_0 is the undisturbed hydrostatic pressure and P_s is the sound pressure, and, since the motion is assumed to be adiabatic, $P_g = P_0(R_0/R)^{3\gamma}$, γ being the ratio of specific heats, c_p/c_v; for the air in the bubble, this equation becomes:

$$\frac{d}{dt}W = \left[P_0\left(\frac{R_0}{R}\right)^{3\gamma} - P_0 - P_s\right]4\pi R^2\dot{R} \qquad [4]$$

Since no damping effect is included $dKE/dt = dW/dt$ and the general equation of motion may be written

$$\rho_w R\ddot{R} + \frac{3}{2}\rho_w\dot{R}^2 + P_0\left[1 - \left(\frac{R_0}{R}\right)^{3\gamma}\right] + P_s = 0 \qquad [5]$$

For sonic motions we may write $R = R_0 + \Delta R$, where R_0 is the equilibrium radius of the bubble and $\Delta R << R_0$ whence to second order in ΔR the equation of motion [5] becomes:

$$\rho_w R_0\Delta\ddot{R} + 3\gamma P_0\frac{\Delta R}{R_0} + \rho_w\Delta R\,\Delta\ddot{R} + \frac{3}{2}\rho_w(\Delta\dot{R})^2 - \frac{3}{2}\gamma(3\gamma+1)P_0\left(\frac{\Delta R}{R_0}\right)^2 + P_s = 0 \qquad [6]$$

To first order for free sonic oscillation, i.e. $P_s = 0$ the familiar result is obtained

$$\omega_0^2 = 4\pi^2\nu_0^2 = \frac{3\gamma P_0}{\rho_w R_0^2} \qquad [7]$$

where ν_0 is the natural frequency and ω_0 the corresponding angular frequency.

Writing $P_s = p P_0 \sin\omega t$, where $p << 1$ the following solution, to the second order in p, is obtained by the standard perturbation technique. This solution can be directly verified by substitution in Equation (6)

$$\Delta R = R - R_0 = R_0\left[a_0(\sin\omega t)p + (a_1 + a_2\cos 2\,\omega t)\,p^2\right] \qquad [8]$$

where

$$a_0 = \frac{-1}{3\gamma\left(1-\frac{\omega^2}{\omega_0^2}\right)} \approx -\frac{1}{4} \qquad [9]$$

$$a_1 = \frac{3\gamma + 1 - \frac{\omega^2}{\omega_0^2}}{4(3\gamma)^2\left(1-\frac{\omega^2}{\omega_0^2}\right)^2} = a_0^2\left(\frac{3\gamma + 1 - \frac{\omega^2}{\omega_0^2}}{4}\right) \approx \frac{5}{64} \qquad [10]$$

$$a_2 = -\frac{3\gamma + 1 + 5\frac{\omega^2}{\omega_0^2}}{4(3\gamma)^2\left(1-\frac{\omega^2}{\omega_0^2}\right)^2\left(1-4\frac{\omega^2}{\omega_0^2}\right)} = -\frac{a_0^2\left(3\gamma + 1 + 5\frac{\omega^2}{\omega_0^2}\right)}{4\left(1-4\frac{\omega^2}{\omega_0^2}\right)} \approx -\frac{5}{64} \qquad [11]$$

The numerical approximations are obtained by using $\gamma = 4/3$ for air and ignoring ω/ω_0 when compared to unity in accordance with the hypothesis that we are dealing with a bubble whose radius is small compared to the radius for resonance.

Differentiation of Equation [8] gives

$$\dot{R} = R_0\left(a_0\,\omega p\cos\omega t - 2\,a_2\,\omega p^2\sin 2\,\omega t\right) \qquad [12]$$

THE BOUNDARY CONDITION FOR THE DIFFUSION EQUATION

The traffic of air molecules at the boundary of the bubble is studied by considering the rate dN_L/dt at which molecules leave the bubble and enter the water and then the rate dN_G/dt at which molecules stream away from the water surface and enter the bubble thus obtaining the net rate of transfer of molecules $dN/dt = dN_G/dt - dN_L/dt$ which is the rate of growth of the bubble. Because of the spherical symmetry it is only necessary to find the rate of transfer of molecules per unit area of the boundary $(dN/dt) \div 4\pi R^2$. This rate may be determined by considering just a small section of the boundary.

It is assumed that the mean free path of the molecules in the bubble is sufficiently small compared to the radius of the bubble so that the section of the spherical surface may be taken small enough to be virtually a plane. From kinetic theory the rate of flow of molecules per unit area through a plane surface element in the bubble is given by $n/2\sqrt{\pi}\,\beta$ where n is the molecular density and $\beta = \sqrt{m/2kT}$, m being the molecular mass, T the temperature (Kelvin) of the air in the bubble and k the Boltzmann constant.[3] Since the motion is assumed to be adiabatic $\beta = \beta_0(R/R_0)^{3/2(\gamma-1)}$ where β_0 is the value of β at equilibrium i.e. $\beta_0 = (m/2kT_0)^{1/2} = (\rho_a/2P_0)^{1/2}$ and ρ_a is the density of the air equilibrium. Also $n = n_0(R_0/R)^3$ where n_0 is the molecular density at equilibrium. Although some of the molecules that strike the surface of the water rebound into the gas, almost all remain in the water and it is sufficiently accurate for our purpose to assume that all the molecules that strike the boundary remain in the water.* Thus the specific rate of loss is

$$\frac{\frac{d}{dt}N_L}{4\pi R^2} = n_0\left(\frac{P_0}{2\pi\rho_a}\right)^{\frac{1}{2}}\left(\frac{R_0}{R}\right)^{\frac{3}{2}(\gamma+1)} \qquad [13]$$

The rate at which molecules stream into the bubble from the water surface depends upon the concentration of the dissolved air and the temperature of the water. The density of air in equilibrium with water when both are at the same fixed temperature is proportional to the concentration of dissolved gas in accordance with Henry's Law. The effect of temperature is to vary the constant of proportionality. Let $c(r,t)$ be the molecular concentration of air in water. Then $c_\infty = c(\infty, t)$ is the molecular concentration of air in water when the water is in equilibrium with air at pressure P_0 and temperature T_0. Since the temperature of the water is everywhere fixed at T_0, when the concentration of air in the water at the boundary of the bubble is c_∞ the rate at which molecules leave a unit area of the water surface is $n_0(P_0/2\pi\rho_a)^{1/2}$ When the concentration of air in the water at the boundary is $c(R,t)$ the water would be in equilibrium with air at temperature T_0 and molecular density $\left(c(R,t)/c_\infty\right)n_0$ so that the specific rate at which molecules stream away from a

*This may not be the case when impurities have collected at the boundary of the bubble. It has been observed for example that the deaeration of tap water that has been quiescent for a long period of time will not occur until the sonic pressure is increased beyond a minimum threshold level. The suggestion has been made by F. E. Fox and K. F. Herzfeld that the accumulation of an organic "skin" at the boundary of the bubble may be responsible for this phenomena. Such a skin would stifle the diffusion process. The calculation made here presupposes a fresh surface at bubble or water free of impurities with the resul[illegible] threshold pressure is obtained.

unit area of the water surface is

$$\frac{\frac{d}{dt} N_G}{4\pi R^2} = n_0 \left(\frac{P_0}{2\pi\rho_a}\right)^{\frac{1}{2}} \frac{c(R,t)}{c_\infty}$$

The net rate of transfer is thus given by

$$\frac{d}{dt} N = 4\pi R^2 \left(\frac{P_0}{2\pi\rho_a}\right)^{\frac{1}{2}} n_0 \left[\frac{c(R,t)}{c_\infty} - \left(\frac{R_0}{R}\right)^{\frac{3\gamma+3}{2}}\right] \quad [15]$$

The net rate of transfer of molecules through the boundary is also

$$\frac{d}{dt} N = 4\pi R^2 D \left(\frac{\partial c}{\partial r}\right)_{r=R}$$

where D is the diffusion constant for air in water and c is given in molecular concentration. Hence the boundary condition at $r = R$ can be

$$\frac{R_0}{c_\infty}\left(\frac{\partial c}{\partial r}\right)_{r=R} = \lambda\left[\frac{c(R,t)}{c_\infty} - \left(\frac{R_0}{R}\right)^{\frac{3\gamma+3}{2}}\right]$$

where

$$\lambda = \left(\frac{P_0}{2\pi\rho_a}\right)^{\frac{1}{2}} \left(\frac{n_0}{c_\infty}\right)\left(\frac{R_0}{D}\right)$$

Under normal conditions λ is exceedingly large. For standard conditions of temperature and pressure the following values may be used:

$$P_0 = 1.01 \times 10^6 \, dyne\ cm^{-2}\ ; \ D = 1.9 \times 10^{-5} \, cm^2\ sec^{-1}\ ; \ c_\infty = 5.02 \times 10^{17} \, cm^{-3}$$
$$n_0 = 2.50 \times 10^{19} \, cm^{-3}\ ; \ \rho_a = 1.20 \times 10^{-3} \, gm\ cm^{-3}.$$

Then for a bubble whose equilibrium radius is 10^{-3} cm, $\lambda \approx 3 \times 10^6$. For all cases of present interest the value of the left hand side of Equation [17] is always small relative to λ. Hence little error is made if the equation is divided by λ and set equal to zero so that the boundary condition becomes

$$c(R,t) = c_\infty \left(\frac{R_0}{R}\right)^{\frac{3\gamma+3}{2}} \quad [18]$$

This condition is simply the statement that the concentration of dissolved air in the water at the boundary of the bubble instantaneously assumes the value required for equilibrium with [illegible]. If the boundary condition is used in the form of Equation [18] it is essential that the rate of growth of the bubble be computed with the formula given by Equation [16] and not with that of Equation [15] since the latter would involve the product of a very large number and a very small number which has been set equal to

zero in Equation [18].

THE SOLUTION OF THE DIFFUSION EQUATION

The diffusion of the dissolved air through the water is described by the partial differential equation:

$$(D\nabla^2 - \vec{v}\cdot\nabla)c = \frac{D}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial c}{\partial r}\right) - \frac{R^2}{r^2}\dot{R}\frac{\partial c}{\partial r} = \frac{\partial c}{\partial t} \quad [19]$$

where R and $\dot{R}$ are given as functions of time by Equations [8] and (12] respectively. The boundary conditions are imposed by the fixed concentration at infinity, i.e. $c(\infty, t) \equiv c_\infty$ and either Equation [17] or Equation [18]. Although there is no inherent difficulty in employing the more precise Equation [17], the labor involved in using Equation [18] is considerably less and in view of the approximate nature of this calculation in other respects, little is lost by the use of the Equation [18]. In either form the boundary condition at $r = R$ is in general awkward to handle because the condition is imposed at a moving boundary.* However, this difficulty can be avoided in the present case because the motion of the boundary is small so that equivalent conditions at a fixed boundary i.e. $r = R_0$ can be extrapolated from the condition at the moving boundary.

A solution is sought which is a perturbation from the undisturbed state. Thus $c(r,t)$ is expanded in powers of p:

$$c(r,t) = c_\infty + c_1(r,t)p + c_2(r,t)p^2 + \cdots \quad [20]$$

and at $r = R$ expansions of the following type are employed: (see Appendix II).

$$c(R,t) = c(R_0,t) + (R - R_0)\frac{\partial c}{\partial r}(R_0,t) + \frac{1}{2}(R - R_0)^2 \frac{\partial^2 c}{\partial r^2}(R_0,t) + \cdots \quad [21]$$

*In connection with a similar heat diffusion problem, M. S. Plesset and S. A. Zwick[7], have employed a technique for treating the condition at the moving boundary that could be adapted to the present problem. This technique is based upon the fact that the diffusion takes place mainly in the water immediately surrounding the bubble. Successive approximations can be made each effectively extending the diffusion to a larger volume of the fluid. However in the present case general evaluation of the explicit solutions which are obtained for these approximations would have to be predicated upon the smallness of the motion. The method employed in this paper uses the fact that only small motions are to be considered to avoid any approximation in regard to the region in which the diffusion takes place.

$$\frac{\partial c(R,t)}{\partial r} = \frac{\partial c(R_0,t)}{\partial r} + (R-R_0)\frac{\partial^2 c}{\partial r^2}(R_0,t) + \frac{1}{2}(R-R_0)^2\frac{\partial^3 c}{\partial r^3}(R_0,t) + \cdots \qquad [22]$$

The assumption that equilibrium exists in the absence of the sound beam requires that the first term in the expansion of c in Equation [20] be identically c_∞ as written and it is easily verified that $c(r,t) \equiv c_\infty$ satifies the differential equation and the boundary conditions with the exclusion of terms of and beyond first order in p. Using Equations [8] and [20] in Equations [21] and [22] one obtains to second order in p

$$c(R,t) = c_\infty + c_1(R,t)p + \left[R_0 a_0(\sin\omega t)\frac{\partial c_1}{\partial r}(R_0,t) + c_2(R_0,t)\right]p^2 \qquad [23]$$

$$\frac{\partial c(R,t)}{\partial r} = \frac{\partial c_1(R_0,t)}{\partial r}p + \left[R_0 a_0(\sin\omega t)\frac{\partial^2 c_1}{\partial r^2}(R_0,t) + \frac{\partial c_2}{\partial r}(R_0,t)\right]p^2 \qquad [24]$$

Substitution from Equations [8] and [24] into Equation [16] and the taking of time averages results in the following expression for the average rate of growth of the bubble to second order in p

$$\frac{dN}{dt} = 4\pi R_0^2 D\left\{\overline{\frac{\partial c_1}{\partial r}}p + \left[2a_0\overline{(\sin\omega t)\frac{\partial c_1}{\partial r}} + R_0 a_0\overline{(\sin\omega t)\frac{\partial^2 c_1}{\partial r^2}} + \overline{\frac{\partial c_2}{\partial r}}\right]p^2\right\} \qquad [25]$$

where all derivatives with respect to r are to be evaluated at $r = R_0$. It is apparent from Equation [25] that although the time dependence of c_1 is required for the determination of the average rate of growth to second order in p only the time average of c_2 is needed.

The differential equation for c_1 is found by applying Equations [8], [12] and [20] to Equation [19], expanding in powers of p and equating the coefficients of the first power of p. Thus

$$\frac{D}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial c_1}{\partial r}\right) = \frac{\partial c_1}{\partial t} \qquad [26]$$

or the equivalent

$$D\frac{\partial^2}{\partial r^2}(rc_1) = \frac{\partial}{\partial t}(rc_1) \qquad [27]$$

The boundary condition at $r = R_0$ is likewise obtained using Equations [8] and [23] in [18]. Thus

$$\frac{c_1(R_0,t)}{c_\infty} = -\frac{3\gamma+3}{2}a_0\sin\omega t \qquad [28]$$

and the boundary condition at infinity is simply $c_1(\infty,t) = 0$. Hence the determination of c_1 is mathematically identical to the one-dimensional problem of heat conduction in a semi-infinite medium with prescribed time-dependence of the temperature at the boundary. The solution of this heat conduction prolem is well-known.* Application of this known solution gives

$$\frac{c_1(r,t)}{c_\infty} = -\left(\frac{3\gamma+3}{2}\right)a_0\left[F\sin\omega t - G\cos\omega t\right] \qquad [29]$$

where

$$F = \frac{R_0}{r}e^{-V\left(\frac{r}{R_0}-1\right)}\cos V\left(\frac{r}{R_0}-1\right)$$

$$G = \frac{R_0}{r}e^{-V\left(\frac{r}{R_0}-1\right)}\sin V\left(\frac{r}{R_0}-1\right)$$

and

$$V = \left(\frac{\omega}{2D}R_0^2\right)^{\frac{1}{2}}.$$

This solution can also be readily obtained by expansion of $c_1(r,t)$ in a Fourier-series in t.

The following evaluations can now be made:

$$\overline{\frac{\partial c_1}{\partial r}} = 0\ ;\quad \overline{\sin\omega t\,\frac{\partial c_1}{\partial r}} = \frac{3\gamma+3}{4}a_0 c_\infty\frac{(1+V)}{R_0}\ ;\quad \overline{\sin\omega t\,\frac{\partial^2 c_1}{\partial r^2}} = 0$$

The differential equation for $\overline{c_2}$ is obtained by applying Equations [8], [12] and [20] to Equation [19], expanding in powers of p, equating the coefficients of the second power of p, substituting the result just obtained for c_1 and taking time averages. Thus one obtains

$$\frac{d}{dr}\left(r^2\frac{d\overline{c_2}}{dr}\right) = c_\infty a_0^2\left(\frac{3\gamma+3}{2}\right)V^2R_0\frac{dG}{dr} \qquad [30]$$

The boundary conditions at $r = R_0$ is likewise obtained using Equations [8] and [23] in [18].

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*According to the interpretation of diffusion as a consequence of the random wandering of molecules as presented by Einstein, we may write $D = \overline{x^2}/2\tau$ where $\overline{x^2}$ is the average square of the displacement of an air molecule in water after the time τ which may be taken as the period of the sound beam. Then, since $\omega = 2\pi/\tau$, $V = \omega^{1/2}R_0/(2D)^{1/2} = (2\pi)^{1/2}R_0/(\overline{x^2})^{1/2}$ so that V may be interpreted as a multiple of the ratio of R_0 to the root-mean-square displacement in a period of the sound beam.

Thus

$$\frac{\overline{c_2}(R_0)}{c_\infty} = -\left(\frac{3\gamma+3}{2}\right)\left[\left(\frac{V+1}{2} - \frac{3\gamma+5}{8}\right)a_0^2 + a_1\right] \quad [31]$$

Equation [30] is easily integrated to obtain

$$\frac{\overline{c_2}}{c_\infty} = -\,a_0^2\left(\frac{3\gamma+3}{2}\right)V^2\int_r^\infty \frac{R_0^2 G}{r^2}\,\frac{dr}{R_0} + \frac{\kappa R_0}{r} \quad [32]$$

where κ is a constant of integration; a second constant of integration vanishes because of the boundary condition at infinity, $c_2(\infty) \equiv 0$. Substitution from Equation [32] into Equation [31] permits the evaluation of κ:

$$\kappa = -\left(\frac{3\gamma+3}{2}\right)\left[\left(\frac{V+1}{2} - \frac{3\gamma+5}{8} - V^2 I_0\right)a_0^2 + a_1\right]$$

$$= -\left(\frac{3\gamma+3}{2}\right)a_0^2\left[\frac{1}{8} + \frac{3\gamma}{8} + \frac{V}{2} - \frac{\omega^2}{4\omega_0^2} - V^2 I_0\right] \quad [33]$$

where $$I_0 = \int_1^\infty \frac{e^{-V(x-1)}\sin V(x-1)}{x^3}\,dx$$

Substitution of the results obtained for c_1 and $\overline{c_2}$ in Equation [25] gives the following expressions for dN/dt :

$$\frac{dN}{dt} = 4\pi R_0 D\left(\frac{3\gamma+3}{2}\right)c_\infty\left[\left(\frac{3}{2}V + \frac{3}{2} - \frac{3\gamma+5}{8} - V^2 I_0\right)a_0^2 + a_1\right]p^2 \quad [34a]$$

$$= 4\pi R_0 D\left(\frac{3\gamma+3}{2}\right)c_\infty\left[\frac{9}{8} + \frac{3\gamma}{8} + \frac{3V}{2} - V^2 I_0 - \frac{\omega^2}{4\omega_0^2}\right]a_0^2 p^2 \quad [34b]$$

or

$$\frac{1}{N_0}\frac{dN}{dt} = 3D\frac{c_\infty}{n_0}\left(\frac{3\gamma+3}{2}\right)\left[\frac{9}{8} + \frac{3\gamma}{8} + \frac{3V}{2} - V^2 I_0 - \frac{\omega^2}{4\omega_0^2}\right]\frac{a_0^2 p^2}{R_0^2} \quad [34c]$$

The integral I_0 is always less than V^{-1} since this latter value is obtained upon elimination of the $x^{-3}\sin V(x-1)$ factor of the integrand, this factor being less than unity while the remaining factor is always positive throughout the range of integration. Moreover as seen in Figure 4, I_0 , the evaluation of which is discussed in Appendix III, never exceeds 0.1 and is $(2V)^{-1}$ for large V. Hence it is apparent that dN/dt is always positive.

EVALUATION AND INTERPRETATION OF THE SOLUTION

The result for dN/dt is illustrated by the curves shown in Figure 1. Values of $1/N_0\, dN/dt$ are shown as a function of R_0 for various frequencies. For all of these curves the values previously given for P_0, D, c_∞, n_0 and ρ_a were used. Also used were $\gamma = 4/3$; $\rho_w = 1$ and p was taken as 1/3. When R_0 and ω/ω_0 are both extremely small then

$$V = \left(\frac{\omega}{2D}R_0^2\right)^{\frac{1}{2}} = \left(\frac{3\gamma P_0}{4\rho_w D^2}\right)^{\frac{1}{4}}\left(\frac{\omega}{\omega_0}R_0\right)^{\frac{1}{2}} = 7260\left(\frac{\omega}{\omega_0}R_0\right)^{\frac{1}{2}} cm^{-\frac{1}{2}}$$

is also small and approximately

$$\frac{1}{N_0}\frac{dN}{dt} = 3D\frac{c_\infty}{n_0}\left(\frac{3\gamma+3}{2}\right)\left(\frac{3\gamma+9}{8}\right)\left(\frac{1}{9\gamma^2}\right)\frac{p^2}{R_0^2} = \frac{4.52 \times 10^{-8}}{R_0^2}\,cm^2\,sec^{-1} \quad [35]$$

Since for any fixed frequency ω/ω_0 becomes small when R_0 is sufficiently small all of the curves are asymptotic to this expression when going towards small R_0 . However it should be noted that when the radius of the bubble is much smaller than $10^{-3}\,cm$, the accuracy of the calculation suffers from the neglect of the surface tension and for low frequencies from the use of the adiabatic rather than the isothermal assumption. Nevertheless the results clearly show that extremely small bubbles grow rapidly by diffusion. In fact the estimated rate of growth of bubbles of small size is so great that for low frequencies the growth per cycle is no longer a negligible fraction of the initial bubble size.*

*It is clear that the assumption that the growth of the bubble during a period of the sound beam is negligible will impose a lower limit upon the frequencies for which the calculation is accurate. The assumption as it has been applied refers not only to the change in the air content due to the average rate of growth, but also and more restricting, to the amplitude of the alternating component of the air content. In particular it has been assumed that this amplitude is much smaller than the amplitude of the volume variation i.e. that in amplitude $(N-N_0)/N_0 << [3(R-R_0)]/R_0$. An estimate of a bound for the amplitude of $(N-N_0)/N_0$ can be obtained by considering the growth that would occur in a half period if the conditions most favorable to growth i.e. maximum expansion where constantly maintained. Then at the boundary, from Equation (18), taking $\gamma = 4/3$; $c(R_{max}) = c_\infty(R_{max}/R_0)^{-7/2}$ where R_{max} is the maximum radius. Since steady conditions are assumed $c(r) = c_\infty[(R_{max}/R_0)^{-7/2} - 1](R_{max}/r) + c_\infty$. Hence from Equation (16), $dN/dt = 4\pi R_{max} D[1-(R_{max}/R_0)^{-7/2}]c_\infty$ and the relative growth in a half-period, which is taken as an estimate of the amplitude of the alternating component of the air content is:

$$\frac{1}{2}\frac{1}{N_0}\frac{dN}{dt}\frac{1}{\nu} = \frac{2\pi R_{max}}{N_0}D\left[1-\left(\frac{R_{max}}{R_0}\right)^{-\frac{7}{2}}\right]\frac{c_\infty}{\nu} \approx \frac{21}{4\nu}\frac{c_\infty}{n_0}\frac{D}{R_0^2}\frac{\Delta R_{max}}{R_0}$$

where $\Delta R_{max} = R_{max} - R_0$ and is assumed to be small. Hence using the values quoted in the text:

$$\frac{1}{2}\frac{1}{N_0}\frac{dN}{dt}\frac{1}{\nu}\bigg/\frac{3\Delta R_{max}}{R_0} \approx \frac{7}{4}\frac{c_\infty}{n_0}\frac{D}{R_0^2}\frac{1}{\nu} \approx \frac{6.7 \times 10^{-}\,cm^2 sec^{-1}}{R_0^2\,\nu}$$

If a ratio of 1/100 is considered acceptable, then the assumption is satisfied for frequencies such that $\nu > 6.7 \times 10^{-6} cm^2 sec^{-1}/R_0^2$. This is a rather conservative estimate. It is probable that the assumption applies quite well to somewhat lower frequencies. Also it should be noted that the failure of the assumption for low frequencies does not imply that the growth by diffusion would be less than that calculated; the reverse may well be the case.

As R_0 increases towards resonant size the terms in V enter. Also a_0^2 increases. The curves then depart from the asymptote on the positive side. Naturally this occurs earlier for the higher frequencies. As R_0 attains resonant size $\overline{dN}/dt$ diverges to $+\infty$ because no damping effect has been included in the treatment of bubble motion so that a_0 becomes infinite. For bubbles of near-resonant size V is large, especially in the case of the lower frequencies. Then terms in V in the bracket on the left-hand-side of Equation [34] dominate Since for large V as shown in Appendix II $I_0 \approx (2V)^{-1}$, an approximation for $1/N_0\ \overline{dN}/dt$ for the near-resonant condition is

$$\frac{1}{N_0}\frac{\overline{dN}}{dt} = 3D\left(\frac{c_\infty}{n_0}\right)\left(\frac{3\gamma+3}{2}\right)\left(\frac{3\gamma P_0}{4\rho_\omega D^2}\right)^{\frac{1}{4}}\left(\frac{R_0}{R_\omega}\right)^{\frac{1}{2}}\left(\frac{1}{9\gamma^2}\right)\frac{p^2}{\left(1-\frac{R_0^2}{R_\omega^2}\right)^2}\frac{1}{R_0^{\frac{3}{2}}}$$

$$= 0.001815\left(\frac{R_0}{R_\omega}\right)^{\frac{1}{2}}\frac{p^2}{\left(1-\frac{R_0^2}{R_\omega^2}\right)^2}\frac{cm^{\frac{3}{2}}}{R_0^{\frac{3}{2}}} \qquad [36]$$

where R_ω is the radius at resonance.

For the lower frequencies the damping is very slight and it is to be expected that the curves will in fact rise very sharply as resonance is approached. However it is difficult to estimate the height of the peaks that will be reached at resonance. According to elementary treatment of the effect of a parallel sound beam[5] on an air bubble, the resonant motion of the bubble is a sinusoidal motion of amplitude $pR_0/3\gamma\delta$, where δ is the so-called damping constant and is a function of frequency. In the following table the values of δ are approximate empirical values read from Figure 2, page 446 of Reference 5.

Frequency, cycles per sec	1,000	10,000	40,000
Damping constant, δ	0.025	0.11	0.28
Nominal ratio of the amplitude of the first harmonic to the equilibrium radius; $p/3\gamma\delta$ for $p = 1/3$	3.3	0.7	0.3

The relative amplitudes show that the elementary theory fails at low frequencies even for relatively small sound pressures, and the motion of the bubble is then not sinusoidal but very asymmetrical.* The curves shown in Figure 1 have been extended only to the bubble radii at which $\omega/\omega_0 = 1/2$. In this case $a_0 = -1/3$ and for $p = 1/3$ the relative amplitude of the first harmonic of the bubble motion is 1/9 so that at this point the approximations made in the calculation should still be reasonable.** The greater damping that is

*Under such conditions one would expect to find a nonlinear amplitude effect upon the damping.

**When $\omega/\omega_0 = 1/2$ the second harmonic term diverges, i.e., a_2 becomes infinite. However, to second order in p , a_2 does not affect the diffusion and the inclusion of damping effects would probably limit a_2 to reasonably small values.

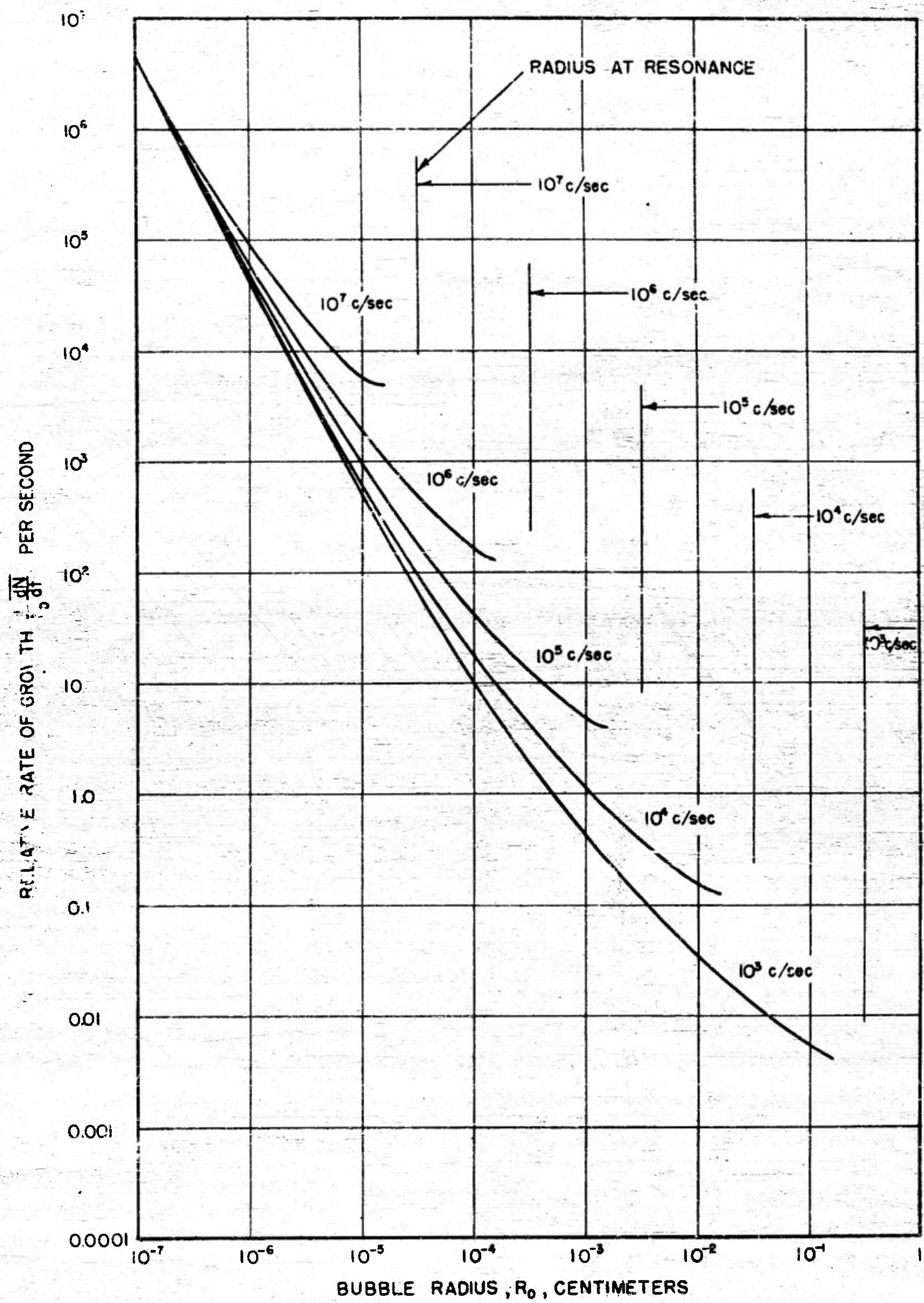

Figure 1 - Relative Rate of Growth as a Function of Frequency and Bubble Radius

observed for the higher frequencies would indicate that the curves for frequencies greater than 10^5 will not exhibit much of a peak at resonance. In fact because of the neglect of the damping, the rate of growth predicted for the higher frequencies even in the off-resonant condition may be somewhat over-estimated. It should be noted that the increasing asymmetry of the motion, whether produced by resonant conditions or by the application of sound beams of greater intensity, will contribute further to the growth of the bubble by diffusion.

It is hard to predict what happens when the radius of the bubble grows beyond resonant size. If the bubble becomes substantially greater than resonant size the distortion produced by the motion may cause the bubble to split and thereafter the pattern of growth for small bubbles would again apply. For very high frequencies the resonant bubble size is so small that if the bubbles do not grow much beyond resonant size they would not be evident on visual inspection. Also very small bubbles may not have time to rise out of the water so that the water would not be deaerated.

The time required for a bubble to grow from one radius to another can be determined by integration of the values of $1/N_0 \, d\overline{N}/dt$ using the relationship $dR_0/dt = R_0/3(1/N_0 \, d\overline{N}/dt)$. Figure 2 shows the result of numerical integration applied to the values for $\nu = 10^4$ cycles per second. Because the growing bubble spends only a very small part of its life at small radii it would be very difficult to determine the original radius of the bubble from data on the time after application of a sound beam that is required for a bubble to be seen and the size of the bubble when it becomes visible.

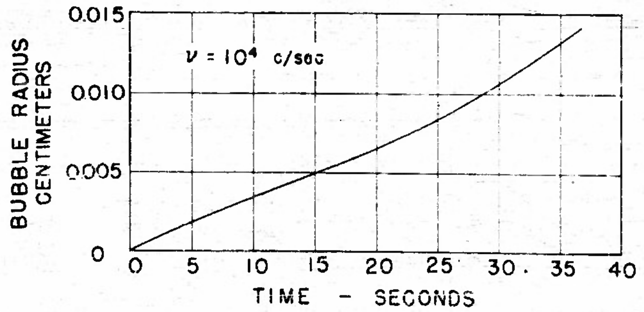

Figure 2 - Time History of Bubble Radius

This figure shows the variation of the radius of a bubble in a sound beam of 10^4 cycles per second. The initial radius was taken as 10^{-7} centimeters.

ACKNOWLEDGEMENT

The author wishes to express his appreciation to Professor K. F. Herzfeld of the Catholic University of America for the invaluable guidance and encouragement which he gave to this work.

APPENDIX I

COMPARISON WITH BLAKE'S CALCULATION

Since Blake[2] includes the effect of surface tension and uses an isothermal rather than an adiabatic relationship between P_g and R his basic assumptions are more appropriate for very small bubbles than the assumptions used for the present calculation, the main interest of which is centered on bubbles of the order of 10^{-3} *cm* and greater. A consequence of the inclusion of surface tension is that equilibrium with respect to diffusion is not obtained in the absence of the sound beam. Therefore Blake obtains, as would be expected, a threshold sound pressure above which the bubble grows but below which the bubble dissolves. However if one is mainly concerned in obtaining a realistic value for the threshold sound pressure required for the growth of nuclei, it would seem necessary to include in the treatment of the problem, the influence of whatever factors are responsible for the stability of the nuclei with respect to diffusion. Unfortunately definite information as to the nature of such factors is not available.

It is believed that the present calculation is more detailed and rigorous. In the treatment of the diffusion problem Blake neglects the motion of the boundary and also omits the convection term in the differential equation. Apart from the difference in the motion of the bubble due to the use of the isothermal assumption, it appears that Blake treats only the off-resonant case since no frequency effect appears in his equation for the bubble motion. Also Blake uses for the most part only first order effects. However the ultimate result shows that the growth of the bubble is a second order effect in the sound pressure. It is therefore advisable that second order terms be retained throughout.

Whereas Blake reports a slight reduction of the rate of growth of the bubble when terms beyond the first order in bubble motion are included, the results of the present calculation show that such terms, which reflect the asymmetry of the motion, make an important contribution. In the off-resonant case where V is small, the contribution of the second order term as represented by the presence of a_1, in Equation [34], is 2.5 times as great as the contribution of the first order. Terms of even higher order are undoubtedly required to properly represent the effects of asymmetry when the resonant size is approached.

When the terms that reflect the lack of equilibrium in the absence of the sound beam are dropped, and the surface tension is taken as zero then

Blake's result expressed in the nomenclature of this paper is:

$$\frac{1}{N_0}\frac{\overline{dN}}{dt} = \frac{1}{2}\frac{c_\infty}{n_0}D[1+2V]\frac{p^2}{R_0^2}$$

For comparison Equation [34] reduces in the off-resonant case to

$$\frac{1}{N_0}\frac{\overline{dN}}{dt} = \frac{21}{32}\frac{c_\infty}{n_0}D\left[\frac{13}{8}+\frac{3}{2}V\right]\frac{p^2}{R_0^2} = \frac{1}{2}\frac{c_\infty}{n_0}D[2.13+1.97V]\frac{p^2}{R_0^2}$$

where only the first order term in V is retained.

APPENDIX II

THE EXTRAPOLATION OF THE BOUNDARY CONDITON TO R_0

The meaning of the expansions illustrated by Equations [21] and [22] may require some clarification. Suppose that at time t_1 the radius of the bubble R is smaller than the equilibrium radius R_0 as shown in Figure 3. Then since the concentration of air molecules in the water, can be assumed to be analytic for $r > R$ the extrapolation from R to R_0, i.e. P_1 to P_2, is clearly justified. However at some later time t_2 when the radius of the bubble is greater than R_0, there is no water at R_0 so that physically the function, c does not exist at this point, P_4. The final justification for these expansions is simply that by applying them a solution, $c(r, t)$, is obtained that is analytic for all values of $r > 0$. It is certainly permissible to apply this solution for values of $r \geq R$ and it is apparent that the setting of the boundary conditions at R_0 in the manner described makes certain that the solution will satisfy the proper boundary conditions at R.

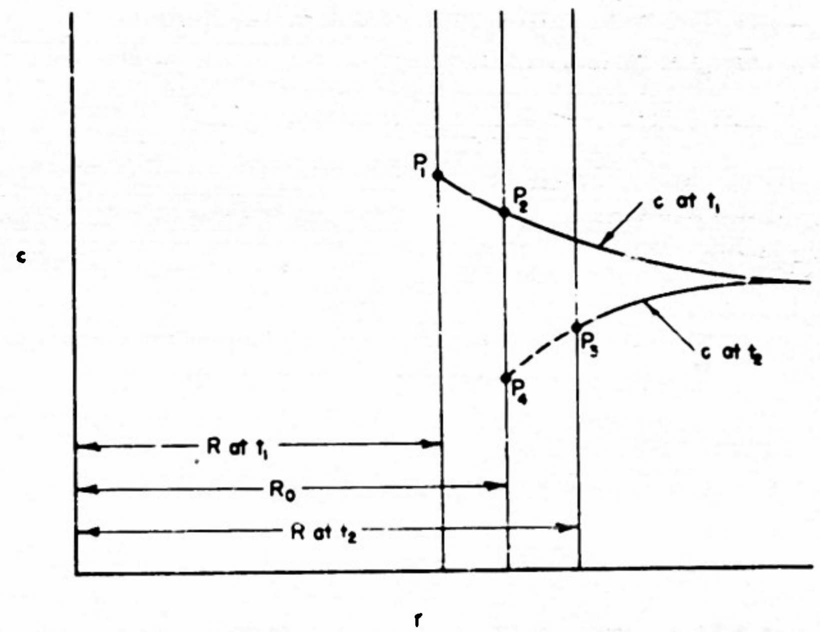

Figure 3 - The Extrapolation of Boundary Conditions from R to R_0

APPENDIX III

EVALUATION OF I_0

The integral $I_0(V) = \int_1^\infty e^{-V(x-1)} \frac{\sin V(x-1)}{x^3} dx$ can be expressed in terms of the generalized exponential integrals. Integrating by parts so as to reduce the exponent of the denominator one obtains:

$$I(V) = \frac{V}{2} - e^V V\left\{\left[\frac{\pi}{4} - E_s(0,V)\right] \sin V + \left[E_c(0,V) - \gamma - \ln V - \frac{1}{2}\ln 2\right] \cos V\right\}$$

where $\gamma = .5772157 \ldots$ is Euler's constant and $E_s(0,V)$, $E_c(0,V)$ are the generalized exponential integral functions. These integrals have been tabulated[6]. For values of V smaller than the tabulated range the following series expansions can be applied.

$$E_c(0,V) = V - \frac{V^3}{9} + \frac{V^4}{24} + \cdots$$

$$E_s(0,V) = V - \frac{V^2}{2} + \frac{V^3}{9}$$

For values of V greater than the tabulated range the integral I_0 can be evaluated directly using the expansion

$$I_0(V) = \frac{1}{2}\left[\frac{1}{V} - \frac{3}{V^2} + \frac{6}{V^3} + \cdots\right]$$

which can be obtained either by integrating by parts so as to increase the exponent of the denominator of the integrand, or by expanding the denominator $1/x^3$ in a power series about $x = 1$. For values of V that are integral multiples of π, the integral can be conveniently evaluated by use of the rapidly converging series

$$I_0(m\pi) = V^2 \sum_{n=0}^{\infty} K[(n+m)\pi]\, e^{-n\pi}$$

where values of $K(y) = \int_0^\pi \frac{e^{-x} \sin x}{(y+x)^3} dx$ are readily obtained by numerical integration. In this manner the following values were obtained.

V	I_0	V	I_0
π	0.07393	7π	0.01991
2π	0.05186	8π	0.01771
3π	0.03944	9π	0.01594
4π	0.03172	10π	0.01449
5π	0.02649	11π	0.01328
6π	0.02274	12π	0.01227

A plot of $I_0(V)$ over a considerable range of the variable is shown in Figure 4.

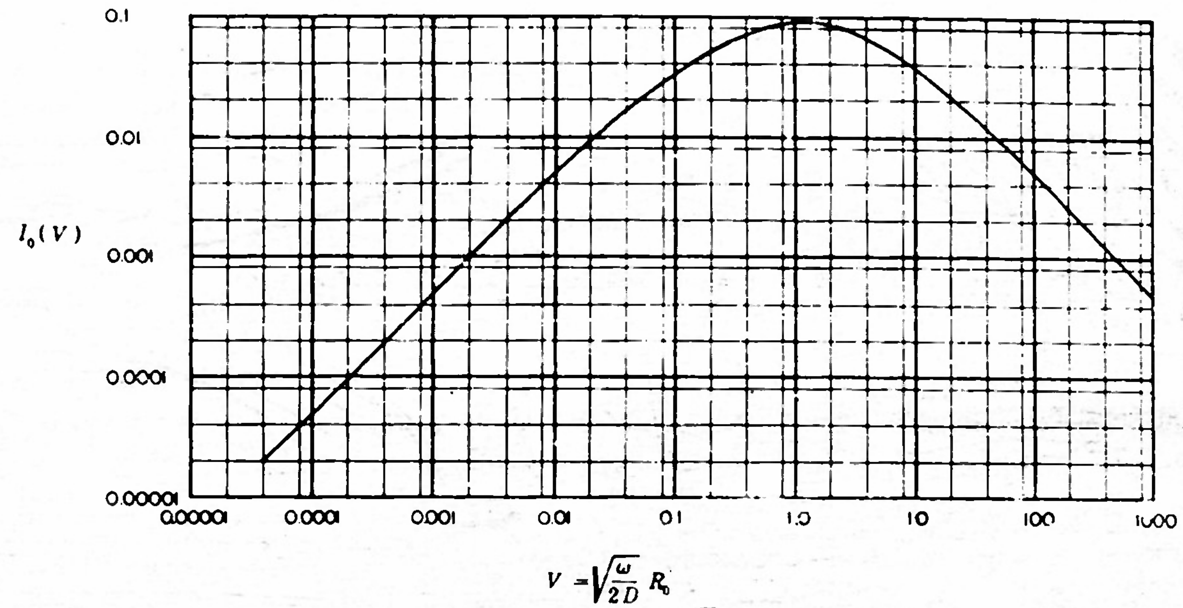

Figure 4 - Plot of $I_0(V) = \int_1^\infty e^{-V(x-1)} \frac{\sin V(x-1)}{x^3} dx$

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David W. Taylor Model Basin. Rept. 854.
THE DEAERATION OF WATER BY A SOUND BEAM, by Leonard Pode. Washington, May 1953.
iv, 50 p. incl. tables, figs., refs.

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