BUILDING TREFFTZ FINITE ELEMENTS FOR ELECTROMAGNETIC PROBLEMS

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Generalized algorithm to build descriptor matrices of convex politope Trefftz finite elements is introduced in the paper. A finite number of plane waves is used to expand electromagnetic field inside the elements. Projections of the intra-element field on an additional set of basis functions defined on the element surface are used to build admittance matrix descriptors of the element.

INTRODUCTION

As a generalization of the method of minimum autonomous blocks, introduced by V.V. Nikol'skii and T.I. Lavrova in the late 70s [1], Trefftz Finite Element method (TFEM) has been recently introduced into the computational electromagnetics [2], [3]. As the conventional finite element method (FEM), the TFEM is based on a division or decomposition of a boundary value problem for Maxwell's equations into a set of elements. Though, instead of polynomial functions, plane-wave solutions of the Maxwell's equations are used as the intra-element basis functions to expand electric and magnetic fields inside the elements. Though, the basic concepts of the method are quite general and are outlined in [2], [3], there is no formalized procedure to build descriptors of complex polytope structures such as polygonal prisms, tetrahedrons and so on. This paper introduces such formalized procedure to build the admittance matrices of convex polytope elements.

BUILDING DESCRIPTORS OF 3-D ELEMENTS

Let us consider a 3-D boundary value electromagnetic problem in the frequency domain. The problem is described by the Maxwell's equations and boundary conditions in a Cartesian coordinate system. The problem is subdivided into a set of small convex polytope elements. All external and internal boundaries of the problem are mapped on the boundaries of the elements. An element can be represented as a convex polytope Ω_p in three-dimensional Euclidian space with N_{face} polygonal faces F_n, $n = 1,...,N_{face}$. The element is uniformly filled with an isotropic medium. Let us expand the polytope element interior field using N_{int} pairs of plane wave solutions of the Maxwell's equations. The field distribution inside the element can be expressed as

$$\begin{pmatrix} \vec{E}(\bar{r}) \\ \vec{H}(\bar{r}) \end{pmatrix} = \sum_{m-1}^{N_{int}} \left[A_m^+ \cdot \begin{pmatrix} \overline{E}_{0m} \\ \overline{H}_{0m} \end{pmatrix} \cdot e^{ik_0 \bar{k}_m * \bar{r}} + A_m^- \cdot \begin{pmatrix} \overline{E}_{0m} \\ \overline{H}_{0m} \end{pmatrix} \cdot e^{-ik_0 \bar{k}_m * \bar{r}} \right], \ \bar{r} \in \Omega_p,\ \bar{k}_m * \overline{E}_{0m} = 0,\ \overline{H}_{0m} = \frac{\bar{k}_m \times \overline{E}_{0m}}{Z_0} \quad (1)$$

where symbol $*$ denotes scalar products, symbol $\times$ denotes vector product, $\bar{k}_m$ is the unit vector of propagation direction of the plane wave number m, $\bar{r}$ is a radius vector of a point inside the polytope or on its boundary, $\overline{E}_{0m}$ is the unit electric field vector of the plane wave, $\overline{H}_{0m}$ is the magnetic field vector of the wave, A_m^+ and A_m^- are unknown magnitudes of the waves or expansion coefficients, k_0 is a propagation constant of the plane wave, and Z_0 is a characteristic impedance of the plane wave:

$$k = \omega\sqrt{\varepsilon\mu},\ Z_0 = \sqrt{\mu/\varepsilon}\,. \tag{2}$$

$N_{\rm int}$ magnitudes of the plane waves in (1) are considered to be linearly independent.

To provide connectivity of the element interior with the surrounding elements and to impose the inter-element boundary conditions, an additional set of basis functions is defined on the faces of the polytope element. Let us choose L_n vector basis functions defined on the polygonal face F_n for the electric field expansion and L_n basis functions on the same face to expand the magnetic field. The total number of pairs of the surface field expansion functions is

$$N_{surf} = \sum_{n=1}^{N_{face}} L_n \tag{3}$$

Designating the surface basis functions defined on the face F_n as $\bar{e}_{n(l)}$ and $\bar{h}_{n(l)}$, the electric and magnetic fields on the surface of the polytope element can be expressed as follows:

$$\vec{E}_{surf} = \sum_{n=1}^{N_{face}}\sum_{l=1}^{L_n} v_{n(l)}\cdot\bar{e}_{n(l)} \quad , \quad \vec{H}_{surf} = \sum_{n=1}^{N_{face}}\sum_{l=1}^{L_n} i_{n(l)}\cdot\bar{h}_{n(l)} \tag{4}$$

where $v_{n(l)}$, $i_{n(l)}$ are unknown expansion coefficients. The total number of the boundary field expansion coefficients in (4) is $2N_{surf}$. It is in addition to the $N_{\rm int}$ independent interior expansion coefficients (1). The number of the interior basis functions $N_{\rm int}$ can be chosen equal to the number of the electric or magnetic field surface basis functions N_{surf}. It provides a possibility to uniquely define a matrix descriptor of the element. To do so, we can project the interior field on the surface basis functions [2], [3]:

$$v_{n(l)} = \mathrm{P}_n\left\{\vec{E},\bar{e}_{n(l)}\right\},\quad i_{n(l)} = \mathrm{P}_n\left\{\vec{H},\bar{h}_{n(l)}\right\},\quad l=1,\ldots,L_n,\ n=1,\ldots,N_{face} \tag{5}$$

where $\vec{E}$ and $\vec{H}$ are values of the electric and magnetic fields defined by (1) taken on the face F_n where basis functions $\bar{e}_{n(l)}$ and $\bar{h}_{n(l)}$ are defined. Either point matching or Galerkin projectors can be used in (5). Constant vector basis functions may be used with the point matching projectors defined as

$$\mathrm{P}_n\left\{\vec{D},\bar{b}_{n(l)}\right\} = \bar{D}(\bar{r}_{n(l)}) * \bar{b}_{n(l)}(\bar{r}_{n(l)}),\quad l=1,\ldots,L_n,\quad n=1,\ldots,N_{face} \tag{6}$$

where $\vec{D}$ is either $\vec{E}$ or $\vec{H}$, $\bar{b}_{n(l)}$ is either $\bar{e}_{n(l)}$ or $\bar{h}_{n(l)}$, $\bar{r}_{n(l)}$ is the radius vector of a matching point on the face F_n for the basis function l. The matching points can be defined as centroids of the polygonal areas where the corresponding face basis functions are defined. Galerkin or averaging projectors can be defined as

$$\mathrm{P}_n\left\{\vec{D},\bar{b}_{n(l)}\right\} = \frac{1}{\left|N_{n(l)}\right|}\int_{F_n}\vec{D} * \bar{b}^{*}_{n(l)}\cdot ds,\quad l=1,\ldots,L_n,\ n=1,\ldots,N_{face} \tag{7}$$

where the integral is taken over the surface of F_n, and $\left|N_{n(l)}\right|$ are the norms of the expansion functions. Substituting (1) into (5) we can obtain the following relations between the interior field expansion coefficients and the surface expansion coefficients:

$$\bar{v} = M_e^{+}\cdot\bar{A}^{+} + M_e^{-}\cdot\bar{A}^{-},\qquad \bar{i} = M_h^{+}\cdot\bar{A}^{+} + M_h^{-}\cdot\bar{A}^{-}, \tag{8}$$

where $\overline{A}^+$ and $\overline{A}^-$ are vectors with N_{int} coefficients A_m^+ and A_m^- (1), $\bar{i}$ and $\bar{v}$ are vectors with N_{surf} components defined as

$$\bar{i} = \left[\bar{i}_1, ..., \bar{i}_{N_{face}}\right]^t,\ \bar{i}_n = \left[i_{n(1)}, ..., i_{n(L_n)}\right]^t,\ \bar{v} = \left[\bar{v}_1, ..., \bar{v}_{N_{face}}\right]^t,\ \bar{v}_n = \left[v_{n(1)}, ..., v_{n(L_n)}\right]^t, \quad (9)$$

where symbol ^{t} denotes transposition.

Matrices $M_e^\pm$ and $M_h^\pm$ are N_{surf} by N_{int} complex matrices of projections of the interior basis functions (1) on the boundary basis functions with the elements defined with either projectors (6) or (7) as

$$\left(M_e^\pm\right)_{n(l),m} = \mathrm{P}_n\left\{\overline{E}_{0m}^\pm \cdot e^{\pm ik_0\bar{k}_m * \bar{r}}, \bar{e}_{n(l)}\right\},\ \left(M_h^\pm\right)_{n(l),m} = \mathrm{P}_n\left\{\overline{H}_{0m}^\pm \cdot e^{\pm ik_0\bar{k}_m * \bar{r}}, \bar{h}_{n(l)}\right\}$$
$$m = 1, ..., N_{\text{int}},\ l = 1, ...L_n,\ n = 1, ..., N_{face} \quad (10)$$

A descriptor matrix of the polygonal element can be deduced by eliminating unknown interior field expansion coefficients $\overline{A}^\pm$ from (8). From here on we assume that $N_{\text{int}} = N_{surf} = N$, which leads to square N by N matrices $M_e^\pm$ and $M_h^\pm$. An assumption of the equality of two terms in the magnetic field projection sum or alternatively in the electric field projection sum (8) leads to two alternative additional expressions to construct the descriptor matrix:

$$\overline{A}^- = T_h \cdot \overline{A}^+,\ T_h = (M_h^-)^{-1} \cdot M_h^+ \quad (11)$$

$$\overline{A}^- = T_e \cdot \overline{A}^+,\ T_e = (M_e^-)^{-1} \cdot M_e^+ \quad (12)$$

Now, an admittance matrix descriptor of the element relating the unknown boundary expansion coefficients can be defined as

$$\bar{i} = Y \cdot \bar{v},\ Y = M_h \cdot M_e^{-1},\ Y \in C^{N\times N}, \quad (13)$$

where $M_e = M_e^+ + M_e^- \cdot T_{e/h}$, $M_h = M_h^+ + M_h^- \cdot T_{e/h}$.

The linear independency of the element interior basis functions is the necessary condition of existence of a non-degenerate descriptor of a polytope element. Plane waves propagating in the directions perpendicular to the sides of a convex polytope provide such a system of functions. Assembling of the admittance matrices (13) into a global admittance matrix is a simple and straightforward procedure and is described in [2].

CONCLUSION

Trefftz finite element method has been generalized in the paper on the problems subdivided into a set of convex polytope elements. The boundary value problem has been reduced to a building and re-composition of admittance matrices of the polytope elements. Generalized matrix formulas are derived to build the admittance matrix descriptors of the elements.

REFERENCES

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