A unified approach to fast algorithms of discrete trigonometric transforms

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Abstract

We present a unified approach to fast algorithms of various discrete trigonometric transforms. With the help of so-called Euler formulas we describe an elegant and useful connection between Fourier matrices and trigonometric matrices. It is known that FFTs are closely related to the factorizations of the unitary Fourier matrix into a product of unitary sparse matrices. Using these Euler formulas and FFTs, we obtain fast algorithms of discrete trigonometric transforms. As a further consequence of these Euler formulas and Gaussian sums, we compute all eigenvalues of some trigonometric matrices.

1 Introduction.

The fast Fourier transform (FFT) and related algorithms for orthogonal trigonometric transforms are essential tools for practical computations. Special discrete trigonometric transforms are the discrete Hartley transforms (DHT), discrete cosine transforms (DCT), and the discrete sine transforms (DST) of various types. These transforms have found important applications in approximation methods with Chebyshev polynomials, quadrature methods of Clenshaw–Curtis type (see [3]), signal processing, and image compression (see [4, 6, 9]).

Euler formulas describe the algebraic connection between Fourier matrices of a certain type and corresponding cosine and sine matrices. Using these formulas, FFTs can be transformed into fast and stable algorithms for the DCT and DST. Further, from these Euler formulas the orthogonality of various trigonometric matrices follows immediately. For simplicity we consider only symmetric trigonometric matrices, i.e. Fourier and Hartley matrices of type I and IV as well as cosine and sine matrices of type I, IV, V and VIII.

This paper is organized as follows; first we introduce generalized Fourier matrices. New Euler formulas for these matrices describe a close connection with various orthogonal Hartley, cosine and sine matrices. These results simplify and extend former results of [9], pp. 83–96. Applying these Euler formulas and FFTs, we obtain fast algorithms of discrete trigonometric transforms. As a further consequence of these formulas and Gaussian sums, we can compute all eigenvalues of orthogonal symmetric trigonometric matrices.

2 Euler formulas for Fourier matrices of type I

Let $N \geq 2$ be a given integer. The *Fourier matrix of type* I is the classical Fourier matrix defined in unitary form

$$\boldsymbol{F}_N^{\mathrm{I}} := \frac{1}{\sqrt{N}} \left(\omega_N^{jk}\right)_{j,k=0}^{N-1}$$

with $\omega_N := \exp(-2\pi\mathrm{i}/N)$. Note that the Gaussian sum (see [5], pp. 326–330) yields the trace of $\boldsymbol{F}_N^{\mathrm{I}}$:

$$\operatorname{tr} \boldsymbol{F}_N^{\mathrm{I}} = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \omega_N^{j^2} = \frac{1+\mathrm{i}^N}{1+\mathrm{i}}. \tag{2.1}$$

Closely related with type I Fourier matrices are the *cosine* and *sine matrices of types* I and V:

$$\begin{aligned}
\boldsymbol{C}_{N+1}^{\mathrm{I}} &:= \sqrt{\frac{2}{N}} \left(\varepsilon_j^N \varepsilon_k^N \cos\frac{jk\pi}{N}\right)_{j,k=0}^{N}, \\
\boldsymbol{S}_{N-1}^{\mathrm{I}} &:= \sqrt{\frac{2}{N}} \left(\sin\frac{(j+1)(k+1)\pi}{N}\right)_{j,k=0}^{N-2}, \\
\boldsymbol{C}_{N+1}^{\mathrm{V}} &:= \frac{2}{\sqrt{2N+1}} \left(\varepsilon_j^{N+1} \varepsilon_k^{N+1} \cos\frac{2jk\pi}{2N+1}\right)_{j,k=0}^{N}, \\
\boldsymbol{S}_{N}^{\mathrm{V}} &:= \frac{2}{\sqrt{2N+1}} \left(\sin\frac{2(j+1)(k+1)\pi}{2N+1}\right)_{j,k=0}^{N-1}.
\end{aligned}$$

Here we set $\varepsilon_j^N := \sqrt{2}/2$ for $j \in \{0, N\}$ and $\varepsilon_j^N := 1$ for $j \in \{1, \ldots, N-1\}$. In this notation a subscript of a matrix denotes the order, while a superscript signifies the type of the matrix. In the following, $\boldsymbol{I}_N$ denotes the identity matrix and $\boldsymbol{J}_N$ the counteridentity matrix, which has the columns of $\boldsymbol{I}_N$ in reverse order. Blanks in a block matrix indicate blocks of zeros. The direct sum of matrices $\boldsymbol{A}, \boldsymbol{B}$ will be denoted by $\boldsymbol{A} \oplus \boldsymbol{B}$. Defining the orthogonal matrices

$$\boldsymbol{P}_{2N}^{\mathrm{I}} := \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & & 0 & \\ & \boldsymbol{I}_{N-1} & & \boldsymbol{I}_{N-1} \\ 0 & & \sqrt{2} & \\ & \boldsymbol{J}_{N-1} & & -\boldsymbol{J}_{N-1} \end{pmatrix}, \quad \boldsymbol{P}_{2N+1}^{\mathrm{V}} := \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & & \\ & \boldsymbol{I}_N & \boldsymbol{I}_N \\ & \boldsymbol{J}_N & -\boldsymbol{J}_N \end{pmatrix},$$

we obtain for Fourier matrices of type I the following Euler formulas:

Theorem 2.1 *Depending on whether the order of the Fourier matrix of type* I *is even or odd, we have*

$$(\boldsymbol{P}_{2N}^{\mathrm{I}})^{\mathrm{T}} \boldsymbol{F}_{2N}^{\mathrm{I}} \boldsymbol{P}_{2N}^{\mathrm{I}} = \boldsymbol{C}_{N+1}^{\mathrm{I}} \oplus (-\mathrm{i}) \boldsymbol{S}_{N-1}^{\mathrm{I}}, \tag{2.2}$$

$$(\boldsymbol{P}_{2N+1}^{\mathrm{V}})^{\mathrm{T}}\boldsymbol{F}_{2N+1}^{\mathrm{I}}\boldsymbol{P}_{2N+1}^{\mathrm{V}} = \boldsymbol{C}_{N+1}^{\mathrm{V}} \oplus (-\mathrm{i})\boldsymbol{S}_N^{\mathrm{V}}. \tag{2.3}$$

Proof: It is obvious that $(\boldsymbol{P}_{2N}^{\mathrm{I}})^{\mathrm{T}}\boldsymbol{P}_{2N}^{\mathrm{I}} = \boldsymbol{I}_{2N}$. Splitting $\boldsymbol{F}_{2N}^{\mathrm{I}}$ into four blocks

$$\boldsymbol{F}_{2N}^{\mathrm{I}} = \frac{1}{\sqrt{2N}} \begin{pmatrix} \left(\omega_{2N}^{jk}\right)_{j,k=0}^{N} & \left(\omega_{2N}^{j(N+k+1)}\right)_{j,k=0}^{N,N-2} \\ \left(\omega_{2N}^{(N+j+1)k}\right)_{j,k=0}^{N-2,N} & \left(\omega_{2N}^{(N+j+1)(N+k+1)}\right)_{j,k=0}^{N-2} \end{pmatrix}$$

and using the classical Euler formula $\exp(-\mathrm{i}x) = \cos x - \mathrm{i}\sin x$, we obtain (2.2) by blockwise computation of $(\boldsymbol{P}_{2N}^{\mathrm{I}})^{\mathrm{T}}\boldsymbol{F}_{2N}^{\mathrm{I}}\boldsymbol{P}_{2N}^{\mathrm{I}}$. The proof of (2.3) is similar. □

Remark 2.2 An analogous result to (2.2) can be found in [9], pp. 85–90, but with a complex matrix instead of $\boldsymbol{P}_{2N}^{\mathrm{I}}$. Compare also with [1]. The Euler formula (2.3) is new. Note that the results and their proofs are simpler than in [9], pp. 85–90 and [1].

Corollary 2.3 *The matrices* $\boldsymbol{C}_{N+1}^{\mathrm{I}}, \boldsymbol{S}_{N-1}^{\mathrm{I}}, \boldsymbol{C}_{N+1}^{\mathrm{V}}, \boldsymbol{S}_N^{\mathrm{V}}$ *are orthogonal.*

Proof: Since $\boldsymbol{F}_{2N}^{\mathrm{I}}$ is unitary and $\boldsymbol{P}_{2N}^{I}$ is orthogonal, $\boldsymbol{C}_{N+1}^{\mathrm{I}} \oplus (-\mathrm{i})\boldsymbol{S}_{N-1}^{\mathrm{I}}$ is unitary by (2.2). Hence the real matrices $\boldsymbol{C}_{N+1}^{\mathrm{I}}$ and $\boldsymbol{S}_{N-1}^{\mathrm{I}}$ are orthogonal. Other proofs can be found in [4], pp. 12–16 and [6].

The proof for the type V matrices uses (2.3) and follows similar lines. □

Remark 2.4 Results analogous to (2.2) and (2.3) are true for the *Hartley matrix of type* I (see [9], pp. 77–80 and [8], pp. 224–227)

$$\boldsymbol{H}_N^{\mathrm{I}} := \frac{1}{\sqrt{N}}\left(\mathrm{cas}\,\frac{jk\pi}{N}\right)_{j,k=0}^{N-1}$$

with $\mathrm{cas}\,x := \cos x + \sin x$. Then we obtain the formulas

$$(\boldsymbol{P}_{2N}^{\mathrm{I}})^{\mathrm{T}}\boldsymbol{H}_{2N}^{\mathrm{I}}\boldsymbol{P}_{2N}^{\mathrm{I}} = \boldsymbol{C}_{N+1}^{\mathrm{I}} \oplus \boldsymbol{S}_{N-1}^{\mathrm{I}}, \tag{2.4}$$

$$(\boldsymbol{P}_{2N+1}^{\mathrm{V}})^{\mathrm{T}}\boldsymbol{H}_{2N+1}^{\mathrm{I}}\boldsymbol{P}_{2N+1}^{\mathrm{V}} = \boldsymbol{C}_{N+1}^{\mathrm{V}} \oplus \boldsymbol{S}_N^{\mathrm{V}}. \tag{2.5}$$

The Euler formula (2.2) can be used for fast and numerically stable computations of DCTs and DSTs of type I: Let $\boldsymbol{x} \in \mathbb{R}^{N+1}$ and $\boldsymbol{y} \in \mathbb{R}^{N-1}$ with $N = 2^t$ $(t \geq 2)$ and set $\boldsymbol{z} := \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} \in \mathbb{R}^{2N}$. Since $\boldsymbol{P}_{2N}^{\mathrm{I}}\boldsymbol{z}$ is real, we can apply Edson's algorithm for the FFT of *real* data (see [8], pp. 215–223 and [7]). The output of the conjugate even result is in the form $\boldsymbol{U}_{2N}\boldsymbol{F}_{2N}^{\mathrm{I}}(\boldsymbol{P}_{2N}^{\mathrm{I}}\boldsymbol{z})$ where $\boldsymbol{U}_{2N} := (\boldsymbol{I}_{N+1} \oplus (-\mathrm{i})\boldsymbol{I}_{N-1})\,(\boldsymbol{P}_{2N}^{\mathrm{I}})^{\mathrm{T}}$. Therefore by

$$\boldsymbol{U}_{2N}\boldsymbol{F}_{2N}^{\mathrm{I}}\boldsymbol{P}_{2N}^{\mathrm{I}}\boldsymbol{z} = (\boldsymbol{C}_{N+1}^{\mathrm{I}} \oplus (-1)\boldsymbol{S}_{N-1}^{\mathrm{I}})\boldsymbol{z} = \begin{pmatrix} \boldsymbol{C}_{N+1}^{\mathrm{I}}\boldsymbol{x} \\ -\boldsymbol{S}_{N-1}^{\mathrm{I}}\boldsymbol{y} \end{pmatrix}$$

we have calculated $\boldsymbol{C}_{N+1}^{\mathrm{I}}\boldsymbol{x}$ and $\boldsymbol{S}_{N-1}^{\mathrm{I}}\boldsymbol{y}$ simultaneously using $5Nt$ flops.

If we have to use an FFT with *complex* data, we combine real data vectors $\boldsymbol{x}, \boldsymbol{x}' \in \mathbb{R}^{N+1}$ resp. $\boldsymbol{y}, \boldsymbol{y}' \in \mathbb{R}^{N-1}$ into the complex vector $\boldsymbol{z}' := \begin{pmatrix} \boldsymbol{x} + \mathrm{i}\boldsymbol{x}' \\ \boldsymbol{y} + \mathrm{i}\boldsymbol{y}' \end{pmatrix}$. Then we can compute two DCTs $\boldsymbol{C}_{N+1}^{\mathrm{I}}\boldsymbol{x}, \boldsymbol{C}_{N+1}^{\mathrm{I}}\boldsymbol{x}'$ and two DSTs $\boldsymbol{S}_{N-1}^{\mathrm{I}}\boldsymbol{y}, \boldsymbol{S}_{N-1}^{\mathrm{I}}\boldsymbol{y}'$ simultaneously via an FFT of length $2N$ applied to the complex input vector $\boldsymbol{P}_{2N}^{\mathrm{I}}\boldsymbol{z}'$.

In a similar way, the Euler formula (2.3) can be used for fast computations of DCTs and DSTs of type V: For given $\boldsymbol{x}, \boldsymbol{x}' \in \mathbb{R}^{N+1}$ and $\boldsymbol{y}, \boldsymbol{y}' \in \mathbb{R}^N$ the transformed vectors

$C_{N+1}^{\mathrm{V}}\boldsymbol{x}, C_{N+1}^{\mathrm{V}}\boldsymbol{x}', S_N^{\mathrm{V}}\boldsymbol{y}$, and $S_N^{\mathrm{V}}\boldsymbol{y}'$ can be calculated at the same time as components of $(\boldsymbol{P}_{2N+1}^{\mathrm{V}})^{\mathrm{T}}\boldsymbol{F}_{2N+1}^{\mathrm{I}}\boldsymbol{P}_{2N+1}^{\mathrm{V}}\boldsymbol{z}' = (\boldsymbol{C}_{N+1}^{\mathrm{V}} \oplus (-\mathrm{i})\boldsymbol{S}_N^{\mathrm{V}})\boldsymbol{z}'$ where we use an FFT of length $2N+1$ with complex data $\boldsymbol{P}_{2N+1}^{\mathrm{V}}\boldsymbol{z}'$. If $2N+1=3^t$ or more generally, if $2N+1$ is a product of small primes (see [8], pp. 76–101 and [7]) the FFT of length $2N+1$ can be computed very efficiently.

3 Euler formulas for Fourier matrices of type IV

The *Fourier matrix of type* IV, defined by

$$\boldsymbol{F}_N^{\mathrm{IV}} := \frac{1}{\sqrt{N}}\left(\omega_{4N}^{(2j+1)(2k+1)}\right)_{j,k=0}^{N-1}$$

is related to the Fourier matrix of type I by the formula

$$\boldsymbol{F}_N^{\mathrm{IV}} = \omega_{4N}\boldsymbol{W}_N\boldsymbol{F}_N^{\mathrm{I}}\boldsymbol{W}_N \tag{3.1}$$

with $\boldsymbol{W}_N := \mathrm{diag}(\omega_{2N}^k)_{k=0}^{N-1}$ and is therefore unitary. If N is a power of 2 or 3, then $\boldsymbol{F}_N^{\mathrm{IV}}$ can be factorized into a product of sparse unitary matrices.

Lemma 3.1 *The trace of the Fourier matrix of type* IV *is equal to*

$$\mathrm{tr}\,\boldsymbol{F}_N^{\mathrm{IV}} = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\omega_{4N}^{(2k+1)^2} = \frac{1-\mathrm{i}^N}{1+\mathrm{i}}. \tag{3.2}$$

Proof: We begin with the generalized Gaussian sum (see [5], p. 330)

$$\frac{1}{\sqrt{N}}\sum_{j=0}^{2N-1}\omega_{4N}^{j^2} = 1-\mathrm{i}$$

which we split into two sums containing even and odd j respectively. Then

$$\frac{1}{\sqrt{N}}\sum_{j=0}^{2N-1}\omega_{4N}^{j^2} = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\omega_N^{k^2} + \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\omega_{4N}^{(2k+1)^2} = \mathrm{tr}\,\boldsymbol{F}_N^{\mathrm{I}} + \mathrm{tr}\,\boldsymbol{F}_N^{\mathrm{IV}},$$

and the results follows by (2.1). □

Now we introduce *cosine* and *sine matrices of type* IV and VIII which are closely related with the Fourier matrix of type IV:

$$\begin{aligned}
\boldsymbol{C}_N^{\mathrm{IV}} &:= \sqrt{\frac{2}{N}}\left(\cos\frac{(2j+1)(2k+1)\pi}{4N}\right)_{j,k=0}^{N-1},\\
\boldsymbol{S}_N^{\mathrm{IV}} &:= \sqrt{\frac{2}{N}}\left(\sin\frac{(2j+1)(2k+1)\pi}{4N}\right)_{j,k=0}^{N-1},\\
\boldsymbol{C}_N^{\mathrm{VIII}} &:= \frac{2}{\sqrt{2N+1}}\left(\cos\frac{(2j+1)(2k+1)\pi}{2(2N+1)}\right)_{j,k=0}^{N-1},\\
\boldsymbol{S}_{N+1}^{\mathrm{VIII}} &:= \frac{2}{\sqrt{2N+1}}\left(\varepsilon_{j+1}^{N+1}\varepsilon_{k+1}^{N+1}\sin\frac{(2j+1)(2k+1)\pi}{2(2N+1)}\right)_{j,k=0}^{N}.
\end{aligned}$$

As above we define orthogonal matrices

$$P_{2N}^{\mathrm{IV}} := \frac{1}{\sqrt{2}}\begin{pmatrix} I_N & I_N \\ -J_N & J_N \end{pmatrix}, \quad P_{2N+1}^{\mathrm{VIII}} := \frac{1}{\sqrt{2}}\begin{pmatrix} I_N & I_N & \\ & & \sqrt{2} \\ -J_N & J_N & \end{pmatrix}.$$

Theorem 3.2 *For the Fourier matrix of type* IV *and even resp. odd order, we obtain the following Euler formulas:*

$$\left(P_{2N}^{\mathrm{IV}}\right)^{\mathrm{T}} F_{2N}^{\mathrm{IV}} P_{2N}^{\mathrm{IV}} = C_N^{\mathrm{IV}} \oplus (-\mathrm{i}) S_N^{\mathrm{IV}}, \tag{3.3}$$

$$\left(P_{2N+1}^{\mathrm{VIII}}\right)^{\mathrm{T}} F_{2N+1}^{\mathrm{IV}} P_{2N+1}^{\mathrm{VIII}} = C_N^{\mathrm{VIII}} \oplus (-\mathrm{i}) S_{N+1}^{\mathrm{VIII}}. \tag{3.4}$$

Proof: Similar to that of Theorem 2.1. □

Corollary 3.3 *The matrices $C_N^{\mathrm{IV}}, S_N^{\mathrm{IV}}, C_N^{\mathrm{VIII}}$ and S_{N+1}^{VIII} are orthogonal.*

Remark 3.4 An analogous result to (3.3) can be found in [9], pp. 94–96. Compare also with [1]. Formula (3.4) is new. A different proof of the orthogonality of C_N^{IV} and S_N^{IV} can be found in [6].

Remark 3.5 Similar formulas as in Theorem 3.2 are true for the *Hartley matrix of type* IV (see [1, 2])

$$H_N^{\mathrm{IV}} := \frac{1}{\sqrt{N}}\left(\operatorname{cas}\frac{(2j+1)(2k+1)\pi}{2N}\right)_{j,k=0}^{N-1}.$$

Then we have

$$(P_{2N}^{\mathrm{IV}})^{\mathrm{T}} H_{2N}^{\mathrm{IV}} P_{2N}^{\mathrm{IV}} = C_N^{\mathrm{IV}} \oplus S_N^{\mathrm{IV}}, \tag{3.5}$$

$$(P_{2N+1}^{\mathrm{VIII}})^{\mathrm{T}} H_{2N+1}^{\mathrm{IV}} P_{2N+1}^{\mathrm{VIII}} = C_N^{\mathrm{VIII}} \oplus S_{N+1}^{\mathrm{VIII}}. \tag{3.6}$$

The Euler formulas can be used for a fast and numerically stable computation of DCT and DST of types IV and VIII:

Using (3.3) and (3.1), for arbitrary $x, x', y, y' \in \mathbb{R}^N$ the DCTs $C_N^{\mathrm{IV}} x$, $C_N^{\mathrm{IV}} x'$ and DSTs $S_N^{\mathrm{IV}} y$ and $S_N^{\mathrm{IV}} y'$ can be calculated via one FFT of length $2N$ with complex data $P_{2N}^{\mathrm{IV}} z'$ and $z' := \begin{pmatrix} x + \mathrm{i}x' \\ y + \mathrm{i}y' \end{pmatrix}$. If $N = 2^t$, this procedure requires about $10Nt$ operations. Likewise by (3.4), for $x, x' \in \mathbb{R}^N, y, y' \in \mathbb{R}^{N+1}$ the DCTs of type VIII, $C_N^{\mathrm{VIII}} x$, $C_N^{\mathrm{VIII}} x'$ and the DSTs $S_{N+1}^{\mathrm{VIII}} y$, $S_{N+1}^{\mathrm{VIII}} y'$ can be calculated via one FFT of length $2N+1$ with complex data $P_{2N+1}^{\mathrm{VIII}} z'$.

Remark 3.6 The sine, cosine, Hartley, and Fourier matrices considered above enjoy the interesting intertwining relations (see [2]):

$$\begin{aligned} C_{N+1}^{\mathrm{I}} J_{N+1} &= \Sigma_{N+1} C_{N+1}^{\mathrm{I}}, \quad S_{N-1}^{\mathrm{I}} J_{N-1} = \Sigma_{N-1} S_{N-1}^{\mathrm{I}}, \\ C_N^{\mathrm{IV}} J_N &= \Sigma_N S_N^{\mathrm{IV}}, \\ H_N^{\mathrm{I}} J_N' &= J_N' H_N^{\mathrm{I}}, \qquad H_N^{\mathrm{IV}} J_N = J_N H_N^{\mathrm{IV}}, \\ F_N^{\mathrm{I}} J_N' &= J_N' F_N^{\mathrm{I}}, \qquad \mathrm{F}_N^{\mathrm{IV}} J_N = J_N \mathrm{F}_N^{\mathrm{IV}}, \end{aligned} \tag{3.7}$$

with the diagonal matrix $\Sigma_{N+1} := \operatorname{diag}((-1)^k)_{k=0}^N$ and the reflection matrix $J_N' := 1 \oplus J_{N-1}$. Therefore applying (3.7) in the above algorithm, it is also possible to compute *four* DCTs (or four DSTs) of type IV and order N via one FFT of length $2N$ with complex data.

4 Eigenvalues of trigonometric matrices

Finally we determine the eigenvalues of trigonometric matrices introduced above. Since the cosine and sine matrices of type I, IV, V and VIII, and the Hartley matrices of type I and IV are real, symmetric and orthogonal, only 1 and -1 are possible eigenvalues. For $x \in \mathbb{R}$ we denote by $\lfloor x \rfloor$ resp. $\lceil x \rceil$ the integer $k \in \mathbb{Z}$ with $k \le x < k+1$ resp. $k-1 < x \le k$.

Theorem 4.1 *The sine and cosine matrices* $C_N^{\mathrm{I}}, S_N^{\mathrm{I}}, C_N^{\mathrm{IV}}, S_N^{\mathrm{IV}}, C_N^{\mathrm{V}}, S_N^{\mathrm{V}}, C_N^{\mathrm{VIII}}$ *and* S_N^{VIII} *of order* $N \ge 2$ *possess the eigenvalues* 1 *and* -1 *with multiplicities*

$$m(1) = \lceil N/2 \rceil, \quad m(-1) = \lfloor N/2 \rfloor.$$

Proof: Since C_N^{I} is symmetric and orthogonal, only 1 and -1 can be eigenvalues. Their multiplicities fulfil

$$m(1) + m(-1) = N.$$

On the other hand, since C_N^{I} and S_{N-2}^{I} are real, it follows from (2.2) and the trace formula (2.1) that

$$m(1) - m(-1) = \operatorname{tr} C_N^{\mathrm{I}} = \operatorname{Re}\big(\operatorname{tr} F_{2N-2}^{\mathrm{I}}\big) = \operatorname{Re} \frac{1 + \mathrm{i}^{2N-2}}{1+\mathrm{i}} = \begin{cases} 1 & \text{for odd } N, \\ 0 & \text{for even } N. \end{cases}$$

From these two linear equations we obtain $m(1) = \lceil N/2 \rceil$ and $m(-1) = \lfloor N/2 \rfloor$. In the other cases, the proof is similar. □

From Theorem 4.1 and the Euler formulas (2.2)–(2.3) and (3.3)–(3.4) it follows immediately:

Corollary 4.2 *The Fourier matrices of type* I *and* IV *have only eigenvalues* $1, -1, \mathrm{i}, -\mathrm{i}$ *with multiplicities:*

	F_{2N}^{I}	F_{2N+1}^{I}	F_{2N}^{IV}	F_{2N+1}^{IV}
$m(1)$	$\lfloor N/2 \rfloor + 1$	$\lfloor N/2 \rfloor + 1$	$\lceil N/2 \rceil$	$\lceil N/2 \rceil$
$m(-1)$	$\lceil N/2 \rceil$	$\lceil N/2 \rceil$	$\lfloor N/2 \rfloor$	$\lfloor N/2 \rfloor$
$m(\mathrm{i})$	$\lceil N/2 \rceil - 1$	$\lfloor N/2 \rfloor$	$\lfloor N/2 \rfloor$	$\lceil N/2 \rceil$
$m(-\mathrm{i})$	$\lfloor N/2 \rfloor$	$\lceil N/2 \rceil$	$\lceil N/2 \rceil$	$\lfloor N/2 \rfloor + 1$

From Theorem 4.1 and formulas (2.4)–(2.5) and (3.5)–(3.6) it follows:

Corollary 4.3 *The Hartley matrices of type* I *and* IV *have only eigenvalues* 1 *and* -1 *with the following multiplicities:*

	H_{2N}^{I}	H_{2N+1}^{I}	H_{2N}^{IV}	H_{2N+1}^{IV}
$m(1)$	$2\lfloor N/2 \rfloor + 1$	$N+1$	$2\lceil N/2 \rceil$	$N+1$
$m(-1)$	$2\lceil N/2 \rceil - 1$	N	$2\lfloor N/2 \rfloor$	N

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