TITLE: On the q-Bernstein Polynomials

On the q-Bernstein polynomials

Halil Oruç and Necibe Tuncer

Department of Mathematics, Dokuz Eylül University, Tinaztepe Kampüsü
35160 Buca İzmir, Turkey

halil.oruc@deu.edu.tr, necibe.tuncer@deu.edu.tr

Abstract

We discuss here recent developments on the convergence of the q-Bernstein polynomials $B_n f$ which replaces the classical Bernstein polynomial with a one parameter family of polynomials. In addition, the convergence of iterates and iterated Boolean sum of q-Bernstein polynomial will be considered. Moreover a $q-$difference operator $\mathcal{D}_q f$ defined by $\mathcal{D}_q f = f[x, qx]$ is applied to q-Bernstein polynomials. This gives us some results which complement those concerning derivatives of Bernstein polynomials. It is shown that, with the parameter $0 < q \leq 1$, if $\Delta^k f_r \geq 0$ then $\mathcal{D}_q^k B_n f \geq 0$. If f is monotonic so is $\mathcal{D}_q B_n f$. If f is convex then $\mathcal{D}_q^2 B_n f \geq 0$.

1 Introduction

First we begin by introducing some notations to be used. For any fixed real number $q > 0$, the q-integer $[k]$ is defined as

$$[k] = \begin{cases} (1-q^k)/(1-q), & q \neq 1, \\ k, & q = 1, \end{cases}$$

for all positive integer k. The term Gaussian coefficient is also used, since they were first studied by Gauss (see Andrews [1]).

Let $p(N, M, n)$ denote the number of partitions of a positive integer n into at most M parts, each less than or equal to N. Then the Gaussian polynomial, $G(N, M, n)$, appears as the generating function

$$G(N, M, n) = \begin{bmatrix} N+M \\ M \end{bmatrix} = \sum_{n \geq 0} p(N, M, n) q^n.$$

Note that $\begin{bmatrix} n \\ k \end{bmatrix}$ defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \frac{[n]!}{[r]![n-k]!}, & n \geq k \geq 0, \\ 0, & \text{otherwise}, \end{cases}$$

where $[n]! = [n][n-1]\cdots[1]$ with $[0]! = 1$, is called Gaussian polynomial (or q-binomial coefficient) since it is a polynomial in q with the degree $(n-k)k$. The q-binomial coefficients satisfy the recurrence relations,

$$\begin{bmatrix} n+1 \\ k \end{bmatrix} = q^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix} + \begin{bmatrix} n \\ k \end{bmatrix} \tag{1.1}$$

and

$$\begin{bmatrix} n+1 \\ k \end{bmatrix} = \begin{bmatrix} n \\ k-1 \end{bmatrix} + q^k \begin{bmatrix} n \\ k \end{bmatrix}. \tag{1.2}$$

The following Euler identity can be verified using the recurrence relation (1.1) by induction that

$$(1+x)(1+qx)\cdots(1+q^{k-1}x) = \sum_{r=0}^{k} q^{r(r-1)/2} \begin{bmatrix} k \\ r \end{bmatrix} x^r. \tag{1.3}$$

Phillips [8] introduced a generalization of Bernstein polynomials (q-Bernstein polynomials) in terms of q-integers

$$B_n(f;x) = \sum_{r=0}^{n} f_r \begin{bmatrix} n \\ r \end{bmatrix} x^r \prod_{s=0}^{n-r-1} (1-q^s x), \tag{1.4}$$

where $f_r = f\left(\frac{[r]}{[n]}\right)$ and an empty product denotes 1. When $q = 1$ the (1.4) reduces the classical Bernstein polynomials. The $B_n(f;x)$ generalizes many properties of classical Bernstein polynomials. Firstly, generalized Bernstein polynomials satisfy the end point interpolation

$$B_n(f;0) = f(0), \quad B_n(f;1) = f(1).$$

Phillips [8] also states the generalization of well known forward difference form (see Davis [3]) of the classical Bernstein polynomials by the following theorem.

Theorem 1.1 *The generalized Bernstein polynomial, defined by (1.4), may be expressed in the q-difference form*

$$B_n(f;x) = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} \Delta^r f_0 x^r \tag{1.5}$$

where $\Delta^r f_i = \Delta^{r-1} f_{i+1} - q^{r-1}\Delta^{r-1} f_i$ *for* $r \geq 1$ *and* $\Delta^0 f_i = f_i$.

It is easily verified by induction that q-differences satisfy

$$\Delta^r f_i = \sum_{k=0}^{r} (-1)^k q^{k(k-1)/2} \begin{bmatrix} r \\ k \end{bmatrix} f_{r+i-k}. \tag{1.6}$$

Using the q-difference form of the q-Bernstein polynomials (1.5), one may show that q-Bernstein polynomials reproduce linear functions, since $B_n(1;x) = 1$; $\quad B_n(x;x) = x$.

2 Convergence

In the discussion of the uniform convergence of the q-Bernstein operator, the Bohman-Korovkin Theorem (see Cheney [2]) is used as in the classical case. The Bohman-Korovkin Theorem states that for a *linear monotone* operator $\mathcal{L}_n$, the convergence of

$\mathcal{L}_n f \to f$ for $f(x) = 1, x, x^2$ is sufficient for the sequence of operators $\mathcal{L}_n$ to have the uniform convergence property $\mathcal{L}_n f \to f$, $\forall f \in C[0,1]$. Observe that the q-Bernstein operator is a *monotone linear* operator for $0 < q \leq 1$. For a fixed value of q with $0 < q < 1$

$$[n] \to \frac{1}{1-q} \quad as \quad n \to \infty.$$

Notice that, since $B_n(x^2; x) = x^2 + \frac{x(1-x)}{[n]}$, $B_n(x^2; x)$ does not converge to x^2. Phillips [8] studies the uniform convergence of q-Bernstein polynomial.

Theorem 2.1 *Let $q = q_n$ satisfy $0 < q_n < 1$ and let $q_n \to 1$ as $n \to \infty$. Then,*

$$B_n(f; x) \to f(x), \quad \forall f(x) \in C[0,1].$$

The degree of q-Bernstein approximation to a bounded function on $[0,1]$ may be described in terms of the *modulus of continuity* with the following theorem.

Theorem 2.2 *If f is bounded on $[0,1]$ and $B_n f$ denotes the generalized Bernstein operator associated with f defined by (1.4), then*

$$\|f - B_n f\|_\infty \leq \frac{3}{2}\omega(1/[n]^{1/2}).$$

An error estimate for the convergence of q-Bernstein polynomials is given in Phillips [8] by the Voronvskaya type theorem.

Theorem 2.3 *Let f be bounded on $[0,1]$ and let x_0 be a point of $[0,1]$ at which $f''(x_0)$ exists. Further, let $q = q_n$ satisfy $0 < q_n < 1$ and let $q_n \to 1$ as $n \to \infty$. Then the rate of convergence of the sequence of generalized Bernstein polynomials is governed by*

$$\lim_{n\to\infty} [n](B_n(f; x_0) - f(x_0)) = \frac{1}{2}x_0(1 - x_0)f''(x_0).$$

It is well known that the classical Bernstein polynomials $B_n f$ provide simultaneous approximation of the function and its derivatives. That is if $f \in C^p[0,1]$, then

$$\lim_{n\to\infty} B_n^{(p)}(f; x) = f^{(p)}(x)$$

uniformly on $[0,1]$. It is worthwhile to examine if this property hold for q-Bernstein polynomials. Phillips [7] proved that the p^{th} derivative of q-Bernstein polynomials converges uniformly on $[0,1]$ to the p^{th} derivative of f under some restrictions of the parameter q. This property results from the generalization of the following theorem.

Theorem 2.4 *Let $f \in C^1[0,1]$ and let the sequence (q_n) be chosen so that the sequence (ϵ_n) converges to zero from above faster than $(1/3^n)$, where*

$$\epsilon_n = \frac{n}{1 + q_n + q_n^2 + \cdots + q_n^{n-1}} - 1.$$

Then the sequence of derivatives of the generalized Bernstein polynomials, $B_n' f$, converges uniformly on $[0,1]$ to $f'(x)$.

Up to now the convergence of q-Bernstein polynomials is examined by taking a sequence $q = q_n$ such that $q_n \to 1$ as $n \to \infty$. In the recent developments, the convergence of q-Bernstein polynomials is examined for fixed real q, $0 < q < 1$ and for $q \geq 1$. It is proved in Oruç and Tuncer [6] that for a fixed q, $0 < q < 1$, the uniform convergence holds if and only if f is linear on the interval $[0, 1]$. Moreover, if $q \geq 1$, $B_n f \to f$ as $n \to \infty$ if f is a polynomial.

Theorem 2.5 *Let $q \geq 1$ be a fixed real number. Then, for any polynomial p,*

$$\lim_{n\to\infty} B_n(p; x) = p(x).$$

For any fixed integer i, the q-Bernstein polynomials of monomials (see Goodman *et.al.* [4]) can be written explicitly as

$$B_n(x^i; x) = \sum_{j=0}^{i} \lambda_j \, [n]^{j-i} S_q(i, j) x^j, \tag{2.1}$$

where

$$\lambda_j = \prod_{r=0}^{j-1} \left(1 - \frac{[r]}{[n]}\right),$$

an empty product denotes 1, and

$$S_q(i, j) = \frac{1}{[j]! \, q^{j(j-1)/2}} \sum_{r=0}^{j} (-1)^r q^{r(r-1)/2} \begin{bmatrix} j \\ r \end{bmatrix} [j - r]^i, \quad 0 \leq i \leq j, \tag{2.2}$$

is the Stirling polynomial of second kind. Thus for any polynomial p of degree m, one may write

$$B_n(p; x) = \mathbf{a}^T \mathbf{A} \mathbf{x}, \tag{2.3}$$

where $\mathbf{a}$ is the vector whose elements are the coefficients of p, $\mathbf{A}$ is an $(m+1) \times (m+1)$ lower triangular matrix with the elements

$$a_{i,j} = \begin{cases} \lambda_j [n]^{j-i} S_q(i, j), & 0 \leq j \leq i, \\ 0, & i < j, \end{cases} \tag{2.4}$$

and $\mathbf{x}$ is the vector whose elements form the standard basis for the space of polynomials P_m of degree m.

Lemma 2.1 *Let $0 < q < 1$ be a fixed real number. Then*

$$\lim_{n\to\infty} B_n(p; x) = p(x)$$

if and only if $p(x)$ is linear.

This lemma can be generalized for any function $f \in C[0, 1]$.

Theorem 2.6 *Let $0 < q < 1$ be a fixed real number and $f \in C[0, 1]$. Then*

$$\lim_{n\to\infty} B_n(f; x) = f(x)$$

if and only if $f(x)$ is linear.

3 The iterates

The iterates of classical Bernstein polynomials were first studied by Kelisky and Rivlin [5]. The authors proved that iterates of Bernstein polynomials converge to linear end point interpolants on $[0,1]$. Several generalization of the result due to Kelisky and Rivlin has been considered by many authors; see Sevy [9] and Wenz [10]. The recent result is the convergence of iterates of generalized Bernstein polynomials. It is proved in Oruç and Tuncer [6] that the q-Bernstein polynomials do preserve the convergence property of iterates of classical Bernstein polynomial. The iterates of generalized Bernstein polynomial are defined by

$$B_n^{M+1}(f;x) = B_n(B_n^M(f;x);x), \quad M = 1,2,\ldots, \tag{3.1}$$

where $B_n^1(f;x) = B_n(f;x)$.

Theorem 3.1 *Let $q \geq 0$ be a fixed real number. Then*

$$\lim_{M\to\infty} B_n^M(f;x) = f(0) + (f(1) - f(0))x. \tag{3.2}$$

Let $\mathcal{A}$ and B be operators then the Boolean sum of $\mathcal{A}$ and B is defined to be

$$\mathcal{A} \oplus B = \mathcal{A} + B - \mathcal{A} \circ B.$$

We will be concerned with iterated Boolean sums of the generalized Bernstein polynomials in the form $B_n \oplus B_n \oplus \cdots \oplus B_n$ and will denote such an M-fold Boolean sum of the generalized Bernstein operators by $\oplus^M B_n$. Sevy [9] and Wenz [10] proved that the limit of iterated Boolean sums of Bernstein polynomials is the interpolation polynomial with respect to the nodes $(\frac{i}{n}, f(\frac{i}{n}))$ $i = 0,\ldots,n$ as $M \to \infty$. The second theorem of this section will give a result for the convergence of iterates of Boolean sums of generalized Bernstein polynomials. It is proved in Oruç and Tuncer [6] that the iterates of Boolean sums of q-Bernstein polynomials converge to the interpolating polynomial at the nodes $\left(\frac{[i]}{[n]}, f\left(\frac{[i]}{[n]}\right)\right)$.

Theorem 3.2 *The iterated Boolean sum of the q-Bernstein operator $\oplus^M B_n(f;x)$ associated with the function $f(x) \in C[0,1]$ converges to the interpolating polynomial $L_n f$ of degree n of $f(x)$ at the points $x_i = [i]/[n]$, $\quad i = 0,1,\ldots,n$.*

4 A difference operator $\mathcal{D}_q$ on generalized Bernstein polynomials

Given any function $f(x)$ and $q \in R$ we define the operator $\mathcal{D}_q$

$$\mathcal{D}_q f(x) = \frac{f(qx) - f(x)}{qx - x}. \tag{4.1}$$

Thus $\mathcal{D}_q f(x)$ is simply a divided difference, $\mathcal{D}_q f(x) = f[x, qx]$. Note that, for a function f and non-negative integer k

$$f[x, qx, \ldots, q^k x] = \frac{1}{[k]!}\mathcal{D}_q^k f(x).$$

Theorem 4.1 *For any integer* $0 \le k \le n$,

$$\mathcal{D}_q^k B_n(f;x) = [n] \cdots [n-k+1] \sum_{r=0}^{n-k} \Delta^k f_r \begin{bmatrix} n-k \\ r \end{bmatrix} x^r \prod_{s=k}^{n-r-1} (1-q^s x).$$

Proof: Recall the q-difference form of generalized Bernstein polynomials (1.5) and apply the operator $\mathcal{D}_q$ to $B_n(f;x)$ repeatedly k times to get,

$$\mathcal{D}_q^k B_n(f;x) = \sum_{r=0}^{n-k} \frac{[n]!}{[n-k-r]![r]!} \Delta^{k+r} f_0 x^r. \tag{4.2}$$

It will be useful to express Δ^{k+r} in terms of Δ^k. One may prove by induction on m that, for $0 \le m \le n-k$ we may write

$$\Delta^{m+k} f_i = \sum_{t=0}^{m} (-1)^t q^{t(t+2t-1)/2} \begin{bmatrix} m \\ t \end{bmatrix} \Delta^k f_{m+i-t}.$$

Now applying the latter identity to (4.2) gives

$$\mathcal{D}_q^k B_n(f;x) = \sum_{r=0}^{n-k} \sum_{t=0}^{r} (-1)^t q^{t(t+2k-1)/2} \frac{[n]!}{[n-k-r]![r]!} \begin{bmatrix} r \\ t \end{bmatrix} \Delta^k f_{r-t} x^r. \tag{4.3}$$

Writing $m = r - t$

$$\frac{[n]!}{[n-k-m-t]![m+t]!} \begin{bmatrix} m+t \\ t \end{bmatrix} = \frac{[n]!}{[n-k-m]![m]!} \begin{bmatrix} n-k-m \\ t \end{bmatrix} \tag{4.4}$$

and putting (4.4) in (4.3) we obtain

$$\mathcal{D}_q^k B_n(f;x) = \sum_{m=0}^{n-k} \frac{[n]!}{[n-k-m]![m]!} \Delta^k f_m x^m \sum_{t=0}^{n-k-m} (-1)^t q^{t(t+2k-1)/2} \begin{bmatrix} n-k-m \\ t \end{bmatrix} x^t.$$

Now, it can be easily derived from generalized binomial expansion (1.3), on replacing x by $q^k x$, that

$$\prod_{t=k}^{n-m-1} (1-q^t x) = \sum_{t=0}^{n-k-m} (-1)^t q^{t(t+2k-1)/2} \begin{bmatrix} n-k-m \\ t \end{bmatrix} x^t.$$

This completes the proof. □

From Theorem 4.1 we see that, with $0 < q \le 1$, if $\Delta^k f_r \ge 0$ for $0 \le r \le n-k$ then $\mathcal{D}_q^k B_n(f;x) \ge 0$. If f is convex on $0 \le x \le 1$ then $\mathcal{D}_q^2 B_n(f;x) \ge 0$ for $0 < q \le 1$. If f is increasing then $\mathcal{D}_q B_n(f;x) \ge 0$, for $0 < q \le 1$.

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