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Parametric shape-preserving spatial interpolation and $\nu-$splines

Carla Manni

Department of Mathematics, University of Torino, Italy
manni@dm.unito.it

Abstract

In this paper we present a class of C^2 spatial interpolating curves depending on a set of tension parameters and we illustrate their ability to reproduce the shape of the data. The curves are constructed using cubic splines and basically reduce to classical ν-splines for particular values of the tension parameters.

1 Introduction

Shape-preserving interpolation via functional as well as parametric splines is a well studied topic for the planar case. On the other hand, shape-preserving interpolation for spaces curves is considerably more complex than for planar ones and the related literature is apparently limited. On this concern, a considerable part of the available schemes only ensures geometric continuity of the obtained curve (see [1, 8] and references quoted therein). Recently, C^2 and C^3 shape-preserving interpolating space curves have been obtained using polynomial splines of variable degree, [2, 3, 6]. However, working with low(fixed)-degree polynomial splines seems to be a standard choice in the CAD/CAM community. This motivates the careful investigation of shape preserving properties of cubic $\nu-$splines recently carried out in [7] and the present paper.

In this paper we present a method for constructing C^2 spatial interpolating curves reproducing the shape of the polygonal line which interpolates the given data. The curve is constructed via the so called "parametric approach", [10], using classical cubic splines. The shape of the curve is controlled by the amplitude of the tangent vectors at the data sites which play the role of tension parameters. It turns out that, for particular values of the tension parameters, the proposed scheme provides a new, geometrically evident, description of classical $C^1 - G^2$ cubic $\nu-$splines, [11]. Moreover, the method produces a suitable reparameterization for the above mentioned curves ensuring C^2 continuity. The reparameterization is a cubic polynomial involving the tension parameters (see (3.3)). Thus, the evaluation of the curve for a fixed value of the new parameter requires the solution of a cubic equation.

The geometric meaning of the tension parameters coupled with the powerful "shape-preserving" properties of the Bernstein-Bézier representation can be efficiently used to construct an iterative algorithm for C^2 shape-preserving interpolation. The algorithm converges in a finite number of iterations and requires at each iteration the solution of a diagonally dominant linear system.

The paper is organized as follows. In Section 2 we state the problem. In Section 3 we describe the construction of the required interpolant and we illustrate its dependence on the tension parameters. The asymptotic behavior and the shape-preserving properties of the obtained curve are briefly discussed in Section 4. We conclude in Section 5 with a graphical example.

2 The problem

In this section we introduce the problem of *shape-preserving* interpolation by curves in $\mathbb{R}^3$. The adopted notion of shape-preserving follows the definitions of [2] and [6]. Let

$$\mathbf{I}_i \in \mathbb{R}^3, \; i = 0, \dots, N,$$

be the interpolation points with $\mathbf{I}_i \neq \mathbf{I}_{i+1}$. Define, for all admissible indices,

$$\begin{aligned} \mathbf{L}_i &:= \mathbf{I}_{i+1} - \mathbf{I}_i, \\ \mathbf{N}_i &:= \begin{cases} \frac{\mathbf{L}_{i-1} \times \mathbf{L}_i}{\|\mathbf{L}_{i-1} \times \mathbf{L}_i\|}, & \text{if } \|\mathbf{L}_{i-1} \times \mathbf{L}_i\| > 0, \\ \mathbf{0}, & \text{elsewhere}, \end{cases} \\ \Delta_i &:= \begin{cases} \frac{|\mathbf{L}_{i-1} \;\; \mathbf{L}_i \;\; \mathbf{L}_{i+1}|}{\|\mathbf{L}_{i-1} \times \mathbf{L}_i\| \; \|\mathbf{L}_i \times \mathbf{L}_{i+1}\|}, & \text{if } \|\mathbf{L}_{i-1} \times \mathbf{L}_i\| \; \|\mathbf{L}_i \times \mathbf{L}_{i+1}\| > 0, \\ 0, & \text{elsewhere}, \end{cases} \end{aligned}$$

where $|\mathbf{a} \; \mathbf{b} \; \mathbf{c}|$ denotes the determinant of the matrix with columns $\mathbf{a}, \mathbf{b}, \mathbf{c}$. The vectors $\mathbf{N}_i$ and the scalars Δ_i are, respectively, the discrete binormals and the discrete torsions of the data.

Let the parameter values σ_i, $i = 0, \dots, N$, with $\sigma_i < \sigma_{i+1}$ be given, and let

$$h_i := \sigma_{i+1} - \sigma_i, \; i = 0, 1, \dots, N-1$$

be the corresponding spacings. We wish to construct a curve $\mathbf{Q}(s)$, $s \in [\sigma_0, \sigma_N]$, which interpolates the data, $\mathbf{Q}(\sigma_i) = \mathbf{I}_i$, $i = 0, \dots, N$, such that $\mathbf{Q} \in C^2[\sigma_0, \sigma_N]$. In addition, we also require that $\mathbf{Q}(s)$ is shape-preserving, that is it reproduces the convexity and torsion of the polygonal line connecting the interpolation points. More specifically, denoting with dashes derivatives with respect to the parameter s, we define

$$\mathbf{K}(s) := \frac{\mathbf{Q}'(s) \times \mathbf{Q}''(s)}{\|\mathbf{Q}'(s)\|^3}, \text{ if } \mathbf{Q}'(s) \neq \mathbf{0}, \; \tau(s) := \frac{|\mathbf{Q}'(s) \;\; \mathbf{Q}''(s) \;\; \mathbf{Q}'''(s)|}{\|\mathbf{Q}'(s) \times \mathbf{Q}''(s)\|^2}, \text{ if } \mathbf{K}(s) \neq \mathbf{0} \tag{2.1}$$

as the *curvature vector* and the *torsion* of the curve respectively. $\mathbf{Q}(s)$ is shape-preserving if it satisfies the following criteria ([2, 6, 7]).

(i) *Convexity criteria:*

(i.1) if $\mathbf{N}_i \cdot \mathbf{N}_{i+1} > 0$, then $\mathbf{K}(s) \cdot \mathbf{N}_j > 0, j = i, i+1, \; s \in [\sigma_i, \sigma_{i+1}]$,

(i.2) if $\mathbf{N}_i \cdot \mathbf{N}_{i+1} < 0$, then $\mathbf{K}(s) \cdot \mathbf{N}_j$, $j = i, i+1$, has one change in sign in $[\sigma_i, \sigma_{i+1}]$,

(i.3) if $\mathbf{N}_i \cdot \mathbf{N}_j \neq 0$ then $(\mathbf{K}(\sigma_i) \cdot \mathbf{N}_j)(\mathbf{N}_i \cdot \mathbf{N}_j) > 0$, $j = i-1, i, i+1$.

(ii) *Torsion criteria:* if $\Delta_i \neq 0$ then $\tau(s)\Delta_i > 0, \; s \in [\sigma_i^+, \sigma_{i+1}^-]$.

For the sake of brevity we refer to [7] for the more technical *collinearity* and *coplanarity* criteria.

3 Constructing the interpolating curve

In order to construct the curve $\mathbf{Q}$ we consider, as a first step, a cubic curve $\mathbf{C}$ interpolating the data. We put

$$\mathbf{C}(t)_{|[\sigma_i,\sigma_{i+1}]} := \mathbf{C}_i(t;\lambda_i^{(0)},\lambda_i^{(1)}), \tag{3.1}$$

$$\mathbf{C}_i(t;\lambda_i^{(0)},\lambda_i^{(1)}) := \mathbf{I}_i H_0^{(0)}(u) + \mathbf{I}_{i+1} H_1^{(0)}(u) + \lambda_i^{(0)} h_i \mathbf{T}_i H_0^{(1)}(u) + \lambda_i^{(1)} h_i \mathbf{T}_{i+1} H_1^{(1)}(u),$$
$$t \in [\sigma_i,\sigma_{i+1}], \quad u := (t-\sigma_i)/h_i, \tag{3.2}$$

where $0 < \lambda_i^{(0)}, \lambda_i^{(1)} \leq 1$ are shape parameters, $\mathbf{T}_i$, $\mathbf{T}_{i+1}$ are vectors to be determined and $H_i^{(j)}(u)$ denote the elements of the cardinal basis for cubic Hermite interpolation, that is $H_i^{(j)}(u)$ are the polynomials of third degree such that

$$\frac{d^l H_i^{(j)}(r)}{du^l} = \delta_{lj}\delta_{ri}, \quad r,l = 0,1.$$

One can immediately verify that the curve (3.2) interpolates the points $\mathbf{I}_i$, $\mathbf{I}_{i+1}$ at the extremes of the interval $[\sigma_i,\sigma_{i+1}]$ and has tangent vectors $\lambda_i^{(0)}\mathbf{T}_i$, $\lambda_i^{(1)}\mathbf{T}_{i+1}$ at the same extremes. The parameters $\lambda_i^{(0)}, \lambda_i^{(1)}$ determine the amplitude of the tangent vectors of the curve at the two end points of the interval and they control the shape of the curve. To be more specific, since $H_0^{(0)}(u) + H_1^{(0)}(u) = 1$, we have that $\mathbf{C}_i(t;0,0)$ reduces to the line through $\mathbf{I}_i$, $\mathbf{I}_{i+1}$. Thus, the parameters $\lambda_i^{(0)}, \lambda_i^{(1)}$ act as *tension parameters* stretching the curve from the classical Hermite cubic interpolating $\mathbf{I}_i$, $\mathbf{I}_{i+1}$ with tangents $\mathbf{T}_i$, $\mathbf{T}_{i+1}$ $(\lambda_i^{(0)}, \lambda_i^{(1)} = 1)$ to the line segment $(\lambda_i^{(0)}, \lambda_i^{(1)} = 0)$. The curve (3.1) turns out to be of class G^1.

Let us consider now the new global parameter

$$s(t)_{|[\sigma_i,\sigma_{i+1}]} := s_i(t;\lambda_i^{(0)},\lambda_i^{(1)}) := \sigma_i H_0^{(0)}(u) + \sigma_{i+1} H_1^{(0)}(u) + \lambda_i^{(0)} h_i H_0^{(1)}(u) + \lambda_i^{(1)} h_i H_1^{(1)}(u). \tag{3.3}$$

It is not difficult to see that, if

$$0 < \lambda_i^{(0)}, \lambda_i^{(1)} \leq 1 \tag{3.4}$$

then

$$\frac{ds_i(t;\lambda_i^{(0)},\lambda_i^{(1)})}{dt} > 0, \quad t \in [\sigma_i,\sigma_{i+1}].$$

Thus (3.3) implicitly defines a function $t = t(s)$, which provides a reparameterization for (3.1). In the following we assume that conditions (3.4) hold and we define

$$\mathbf{Q}(s) := \mathbf{C}(t(s)). \tag{3.5}$$

Since $\mathbf{Q}'(\sigma_i) = \mathbf{T}_i$, $i = 0,\ldots,N$, $\mathbf{Q}$ is of class C^1. For each sequence of the tension parameters $\lambda_i^{(0)}, \lambda_i^{(1)}$ we will determine the tangent vectors $\mathbf{T}_i, \mathbf{T}_{i+1}$ so that $\mathbf{Q}$ is also of class C^2. Let us denote by dots derivatives with respect to the local parameter u. Imposing continuity of $\mathbf{Q}''(s)$ at σ_i, $i = 1, \ldots, N-1$, from (3.3), (3.5) and from the chain rule for derivatives, we obtain

$$\frac{\ddot{\mathbf{C}}_{i-1}(1^-)h_{i-1}\lambda_{i-1}^{(1)} - \ddot{s}_{i-1}(1^-)h_{i-1}\lambda_{i-1}^{(1)}\mathbf{T}_i}{(h_{i-1}\lambda_{i-1}^{(1)})^3} = \frac{\ddot{\mathbf{C}}_i(0^+)h_i\lambda_i^{(0)} - \ddot{s}_i(0^+)h_i\lambda_i^{(0)}\mathbf{T}_i}{(h_i\lambda_i^{(0)})^3}. \tag{3.6}$$

Thus, after some manipulations, from (3.2) we have

$$u_i\mathbf{T}_{i-1} + \mathbf{T}_i + v_i\mathbf{T}_{i+1} = \mathbf{z}_i, \quad i = 1, \ldots, N-1, \tag{3.7}$$

$$\begin{aligned} u_i &= \frac{h_{i-1}\lambda_{i-1}^{(0)}(h_i\lambda_i^{(0)})^2}{w_i}, \\ v_i &= \frac{h_i\lambda_i^{(1)}(h_{i-1}\lambda_{i-1}^{(1)})^2}{w_i}, \\ w_i &= h_{i-1}(3 - \lambda_{i-1}^{(0)})(h_i\lambda_i^{(0)})^2 + h_i(3 - \lambda_i^{(1)})(h_{i-1}\lambda_{i-1}^{(1)})^2, \\ \mathbf{z}_i &= \frac{3}{w_i}\mathbf{L}_i(h_{i-1}\lambda_{i-1}^{(1)})^2 + \frac{3}{w_i}\mathbf{L}_{i-1}(h_i\lambda_i^{(0)})^2. \end{aligned} \tag{3.8}$$

In order to uniquely determine the vectors $\mathbf{T}_i$ we need two additional equations that will be obtained by imposing boundary conditions. Classical boundary conditions are *periodic conditions:*

$$u_0\mathbf{T}_{N-1} + \mathbf{T}_0 + v_0\mathbf{T}_1 = \mathbf{z}_0, \quad u_N\mathbf{T}_{N-1} + \mathbf{T}_N + v_N\mathbf{T}_1 = \mathbf{z}_N$$

(with $u_0, v_0, u_N, v_N, \mathbf{z}_0, \mathbf{z}_N$ defined according to (3.8) setting $h_{-1} = h_{N-1}$, $\lambda_{-1}^{(0)} = \lambda_{N-1}^{(0)}$, $\lambda_{-1}^{(1)} = \lambda_{N-1}^{(1)}$, $\mathbf{L}_{-1} = \mathbf{L}_{N-1}$, $h_N = h_0$, $\lambda_N^{(0)} = \lambda_0^{(0)}$, $\lambda_N^{(1)} = \lambda_0^{(1)}$, $\mathbf{L}_N = \mathbf{L}_0$) and *end tangent conditions*:

$$\mathbf{T}_0 = \mathbf{D}_0, \qquad \mathbf{T}_N = \mathbf{D}_N,$$

(where $\mathbf{D}_0, \mathbf{D}_N$ are given in input). In the following we will denote by $\mathcal{I}$ the set of indices $\{1, \ldots, N-1\}$ ($\{0, \ldots, N\}$) when end tangent (periodic) conditions are considered. It is not difficult to see that (3.7) for any choice of the above mentioned boundary conditions provide a diagonally dominant system

$$A\mathbf{T} = \mathbf{z}. \tag{3.9}$$

Thus we can state the following

Theorem 3.1 *For any sequence $\lambda_i^{(0)}, \lambda_i^{(1)}$, $i = 0, \ldots, N-1$, satisfying (3.4), there exists a unique $\mathbf{Q} \in C^2[\sigma_0, \sigma_N]$ defined via (3.1)–(3.3), (3.5) which interpolates the given data and satisfies periodic or end tangent conditions.*

We notice that for $\lambda_k^{(0)} = \lambda_k^{(1)} = 1$, system (3.9) reduces to the system for the computation of classical C^2 cubic splines. Moreover, if $\lambda_{k-1}^{(1)} = \lambda_k^{(0)} = \lambda_k$, $k \in \mathcal{I}$, the curve $\mathbf{C}$ is of class C^1 and equation (3.6) reads

$$\frac{d^2}{dt^2}\mathbf{C}_i(\sigma_i^+) - \frac{d^2}{dt^2}\mathbf{C}_{i-i}(\sigma_i^-) = \frac{h_i^{-2}\ddot{s}_i(0^+) - h_{i-1}^{-2}\ddot{s}_{i-1}(1^-)}{\lambda_i}\frac{d}{dt}\mathbf{C}_i(\sigma_i^+).$$

Then (3.6) is equivalent to impose that the cubic curve (3.1) is a C^1-G^2 cubic $\nu-spline$ [5, 7, 11] where, from (3.3), for $i \in \mathcal{I}$

$$\nu_i := \frac{h_i^{-2}\ddot{s}_i(0^+) - h_{i-1}^{-2}\ddot{s}_{i-1}(1^-)}{\lambda_i} = \frac{(6-4\lambda_i-2\lambda_{i+1})h_i^{-1} + (6-2\lambda_{i-1}-4\lambda_i)h_{i-1}^{-1}}{\lambda_i}. \tag{3.10}$$

4 Asymptotic behavior and shape-preservation

In this section we briefly discuss the asymptotic behavior and the resulting shape-preserving properties of the curve $\mathbf{Q}$, defined by (3.1)–(3.3), (3.5) and (3.9), as the tension parameters $\lambda_i^{(0)}, \lambda_i^{(1)}$ approach zero. The following lemma (see also [7]) concerns the asymptotic behavior of the tangents $\mathbf{T}_i$. We omit the details of the proof which are completely analogous to those of Theorem 3 in [9].

Lemma 4.1 *The vectors $\mathbf{T}_i$, $i = 0, \ldots, N$, obtained from (3.9) are bounded independently of $\lambda_j^{(0)}, \lambda_j^{(1)}$, $j = 0, \ldots, N-1$. Moreover,*

$$\begin{aligned}\lim_{\lambda_{i-1}^{(0)},\ \lambda_i^{(1)} \to 0} \mathbf{T}_i &= \frac{h_i(\lambda_i^{(0)})^2}{h_i(\lambda_i^{(0)})^2 + h_{i-1}(\lambda_{i-1}^{(1)})^2}\frac{\mathbf{L}_{i-1}}{h_{i-1}} + \frac{h_{i-1}(\lambda_{i-1}^{(1)})^2}{h_{i-1}(\lambda_{i-1}^{(1)})^2 + h_i(\lambda_i^{(0)})^2}\frac{\mathbf{L}_i}{h_i} \\ &=: (1-\alpha_i)\frac{\mathbf{L}_{i-1}}{h_{i-1}} + \alpha_i\frac{\mathbf{L}_i}{h_i}, \quad i \in \mathcal{I}.\end{aligned} \tag{4.1}$$

Since the tangents are bounded independently on the tension parameters, from the previous section we have that $\mathbf{Q}$ approaches the piecewise linear function interpolating the data as the tension parameters tend to zero. Moreover, each tangent $\mathbf{T}_i$ determined by (3.9) tends to a strictly convex combination of $\mathbf{L}_{i-1}/h_{i-1}$ and $\mathbf{L}_i/h_i$ as the tension parameters $\lambda_{i-1}^{(0)}, \lambda_i^{(1)}$ tend to zero while $\lambda_{i-1}^{(1)}/\lambda_i^{(0)}$ remains bounded and strictly positive. Due to these two main facts, we are able to easy control the shape of the curve $\mathbf{Q}$ and to ensure that it reproduces the shape of the data as the tension parameters approach zero as we will discuss briefly in the following.

Since $\mathbf{C}$ and $\mathbf{Q}$ only differ for a reparameterization they have the same image. Thus, as far as the shape-preserving properties are concerned, we can consider the expression of $\mathbf{C}$. As noticed in Section 3, if $\lambda_{i-1}^{(1)} = \lambda_i^{(0)}$, $i \in \mathcal{I}$, the curve $\mathbf{C}$ with $\mathbf{T}_j$ obtained by (3.9), is a C^1-G^2 cubic ν-spline. In such a case, using (3.10), the careful shape analysis carried out in [7] and the resulting algorithm can be considered. However, the simple geometric meaning of the tension parameters $\lambda_i^{(0)}, \lambda_i^{(1)}$ coupled with the "shape-preserving" properties of the Bézier-Bernstein representation, allow us to more easily establish the shape-preserving results also for completely general configurations of $\lambda_{i-1}^{(1)}$, $\lambda_i^{(0)}$. Thus, we express the curve segment $\mathbf{C}_i(t;\lambda_i^{(0)},\lambda_i^{(1)})$ in Bézier-Bernstein form:

$$\mathbf{C}_i(t;\lambda_i^{(0)},\lambda_i^{(1)}) = \sum_{l=0}^{3}\mathbf{C}_{i,l}\binom{3}{l}t^l(1-t)^{3-l},$$

$$\mathbf{C}_{i,0} := \mathbf{I}_i,\ \mathbf{C}_{i,1} := \mathbf{I}_i + \tfrac{1}{3}h_i\lambda_i^{(0)}\mathbf{T}_i,\ \mathbf{C}_{i,2} := \mathbf{I}_{i+1} - \tfrac{1}{3}h_i\lambda_i^{(1)}\mathbf{T}_{i+1},\ \mathbf{C}_{i,3} := \mathbf{I}_{i+1}.$$

Let us consider at the beginning the convexity criteria.

Lemma 4.2 *If* $\mathbf{N}_i \cdot \mathbf{N}_j \neq 0$ *and* $\frac{\lambda_{i-1}^{(1)}}{\lambda_i^{(0)}} \to c > 0$, *then*

$$\lim_{\lambda_{i-1}^{(0)},\lambda_i^{(1)}\to 0} (\mathbf{K}(\sigma_i)\cdot\mathbf{N}_j)(\mathbf{N}_i\cdot\mathbf{N}_j) > 0.$$

Proof: From the properties of Bézier curves (see [5]) and from (2.1) and (3.5)

$$\begin{aligned}\operatorname{sgn}(\mathbf{K}(\sigma_i)\cdot\mathbf{N}_j) &= \operatorname{sgn}((\mathbf{C}_{i,1}-\mathbf{C}_{i,0})\times(\mathbf{C}_{i,2}-\mathbf{C}_{i,1}))\cdot\mathbf{N}_j \\ & \operatorname{sgn}\left(\left[\mathbf{T}_i\times\left(\mathbf{L}_i - \frac{\lambda_i^{(0)}h_i}{3}\mathbf{T}_i - \frac{\lambda_i^{(1)}h_i}{3}\mathbf{T}_{i+1}\right)\right]\cdot\mathbf{N}_j\right)\end{aligned}$$

where $\operatorname{sgn}(y)$ denotes the sign of y. Moreover, from (4.1)

$$\lim_{\lambda_{i-1}^{(0)},\lambda_i^{(1)}\to 0}(\mathbf{T}_i\times\mathbf{L}_i)\cdot\mathbf{N}_j = \left(\alpha_i\frac{\mathbf{L}_i}{h_i}\times\mathbf{L}_i + (1-\alpha_i)\frac{\mathbf{L}_{i-1}}{h_{i-1}}\times\mathbf{L}_i\right)\cdot\mathbf{N}_j = \frac{(1-\alpha_i)}{h_{i-1}}\mathbf{N}_i\cdot\mathbf{N}_j.$$

Hence, we obtain the assertion if $\mathbf{N}_i\cdot\mathbf{N}_j \neq 0$. □

The previous lemma ensures that, if $\lambda_{i-1}^{(0)}, \lambda_i^{(1)}$ are small enough the third convexity criterion, (i.3), stated in Section 2 is satisfied. In addition, the sign of $\mathbf{K}(\sigma_k)\cdot\mathbf{N}_j,\ k = i, i+1$ can be checked considering the Bézier coefficients $\mathbf{C}_{i,l},\ l = 0,1,2,3$, of $\mathbf{C}_i$. Furthermore, thanks to the shape-preserving properties of totally positive bases, for small values of the tension parameters, (see [4]) the number of changes in sign of $\mathbf{K}(s)\cdot\mathbf{N}_j,\ s\in[\sigma_i,\sigma_{i+1}]$ is bounded by the number of changes of sign in the pair $\mathbf{K}(\sigma_k)\cdot\mathbf{N}_j, k = i, i+1$. Thus, also the first and the second convexity criteria (i.1) and (i.2) are satisfied if the tension parameters are small enough.

As far as the torsion is concerned, we recall that the sign of the torsion of a cubic curve coincides with the sign of the discrete torsion of its Bézier control polygon (see for example [5]) thus it is not difficult to obtain the following

Lemma 4.3 *If* $\Delta_i \neq 0$ *and* $\frac{\lambda_{j-1}^{(1)}}{\lambda_j^{(0)}} \to c > 0,\ j = i, i+1$, *then*

$$\lim_{\lambda_{i-1}^{(0)},\lambda_{i-1}^{(1)},\lambda_i^{(0)},\lambda_i^{(1)},\lambda_{i+1}^{(0)}\lambda_{i+1}^{(1)}\to 0} \tau(s)\Delta_i > 0,\ s\in[\sigma_i^+,\sigma_{i+1}^-].$$

With similar arguments it is not difficult to prove that also the collinearity and the coplanarity criteria stated in [7] are fulfilled as the tension parameters approach zero. We omit the details for the sake of brevity.

Summarizing, from the previous discussion it follows that if the tension parameters are small enough then the Bézier control polygon of $\mathbf{C}$ reproduces the shape of the data and the curve **C** does the same thanks to the properties of Bézier-Bernstein representation. Thus, to obtain an automatic algorithm to compute the C^2 interpolant **Q** defined by (3.5), satisfying convexity and torsion criteria, basically we have to perform the following steps:

(a) for a given sequence of the tension parameters solve the system (3.9) and compute the Bézier coefficients of the resulting curve **C**;

(b) check if the control polygon of each segment $\mathbf{C}_i$ satisfies the convexity and torsion criteria;

(c) if this is not the case reduce the values of the related tension parameters according to a given rule and go to step (a).

5 A graphical example

To illustrate the performance of the presented scheme we consider the data proposed in [7], Example 2, consisting of 20 points with uniform parameterization in $[0,1]$. End tangent boundary conditions have been used (see Table 2 in [7]). Figures 1–3 show the behavior of the obtained C^2 curve **Q** compared with the classical C^2 cubic spline. The shape-preserving curve **Q** is defined by the following sequence of tension parameters

$\lambda_i^{(0)}$:	.6	.6	1	.9	.9	1	1	1	1	1	1	1	1	1	.75	1	1	1	1
$\lambda_i^{(1)}$:	.9	.6	.6	1	.9	.9	1	1	1	1	1	1	1	1	1	1	1	.75	1.

FIG. 1. C^2 cubic spline (left) and **Q** (right).

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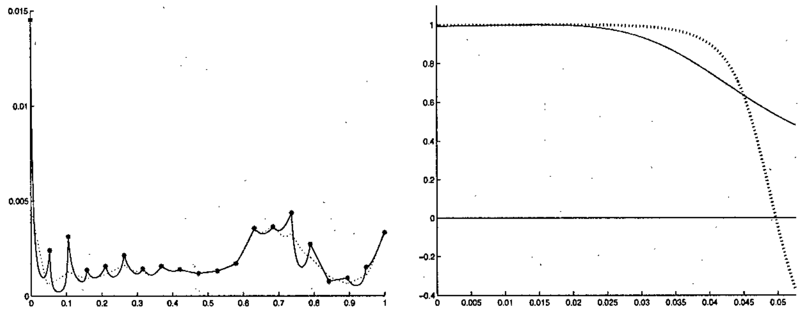

FIG. 2. Left: $\|\mathbf{K}(s)\|$ for the C^2 cubic spline (dotted line) and for $\mathbf{Q}$. Right: convexity ratio $\frac{\mathbf{K}(s)\cdot\mathbf{N}_0}{\|\mathbf{K}(s)\|}$ in $[\sigma_0, \sigma_1]$ (with $\mathbf{N}_0 := \frac{\mathbf{T}_0 \times \mathbf{L}_0}{\|\mathbf{T}_0 \times \mathbf{L}_0\|}$) for the C^2 cubic spline (dotted line) and for $\mathbf{Q}$.

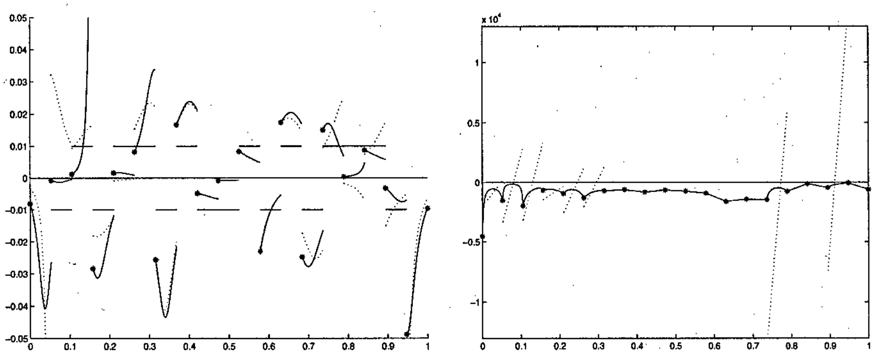

FIG. 3. Left: torsion of the C^2 cubic spline (dotted line) and of $\mathbf{Q}$ (the horizontal lines depict the sign of the discrete torsion). Right: first component of $d^2\mathbf{C}/dt^2$ (dotted line) and of $d^2\mathbf{Q}/ds^2$.

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