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CAGD techniques for differentiable manifolds

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Abstract

The paper outlines procedures for extending the de Casteljau, de Boor and Aitken algorithms in such a way as to allow the construction on a Riemannian manifold of curves analogous to Bezier, B-spline, and Lagrange curves. These curves lie in the manifold and respect intrinsic geometry.

1 Introduction

Given a sequence of points in a Riemannian manifold M we describe methods for extending the de Casteljau, de Boor, and Aitken algorithms. These methods allow construction of corresponding interpolating or approximating curves that lie in the manifold and respect intrinsic geometry. In the case that the manifold is a sphere, opportunity for applications exist in the domain of geological and geographical mapping, for instance the creation of topographical contour lines or isotherms, and in the field of video production, where it is desirable to have smooth camera trajectories interpolating fixed camera positions. For higher dimensional manifolds there are applications in the field of data analysis. For the case of a sphere, there is an extensive literature dealing with the general problem of data fitting, and a superb review can be found in Fasshauer and Schumaker [2]. Shoemake [7] uses properties of quaternion arithmetic to describe curves on the unit quaternion sphere, and Levesley and Ragozin [4], using techniques different from those presented in this paper, describe methods for Lagrange interpolation in differentiable manifolds.

The techniques described in this paper come from the simple observation that in the de Casteljau, de Boor, and Aitken algorithms one may formally substitute appropriately parametrized geodesic arcs for straight line segments. These ideas are introduced in detail in the next section in the context of the blossoming paradigm, [6] and [3]. Unfortunately many of the useful properties of blossoms depend on the affine structure of Euclidean space which in general has no counter part in a Riemannian manifold. In particular, geodesic blossoms may be neither symmetric or multi-affine, and in general they do not possess uniqueness characteristics common to the Euclidean blossom.

For an arbitrary Riemannian manifold [1] or indeed an arbitrary differentiable 2-manifold embedded in $\mathbb{R}^3$, it may not be possible to construct unique shortest geodesic arcs between two points. However, if the manifold is compact or in the case that the two points lie in a sufficiently small neighborhood, such arcs are known to exist. But even then, there appears to be no general method that allows explicit construction. So, the task of constructing geodesic blossoms becomes a study of special cases in which specific methods can be set forth. For the general case, a discrete variational method can be used to obtain good approximations.

In Section 3 a few specific examples are discussed. The case in which the manifold is a sphere is given special attention. There we introduce a variation which allows the discussion of Archimedian curves which are constructed by substituting Archimedian spirals for geodesics. This variation allows the natural construction of curves that lie off the sphere. Although the spherical geodesic blossoms are neither symmetric or multi-affine, a simple reparametrization of geodesic arcs results in spherical blossoms that have all desirable characteristics. Section 3 also contains a brief discussion of the problem of finding geodesics in developable surfaces and in surfaces of revolution.

2 Preliminaries

Let M be a C^∞ Riemannian manifold. There is the following theorem that guarantees the existence locally of geodesics.

Theorem 2.1 *If M is a Riemannian manifold, $x_0 \in M$. Then there exists a neighborhood V of x_0 and $\varepsilon > 0$ so that if $x \in V$ and v is a non-zero tangent vector at x and $\|v_x\| < \varepsilon$, then there is a unique C^∞ geodesic $\alpha : (-2, 2) \to M$ defined on the open interval $(-2, 2)$ such that $\alpha(0) = x$ and $\left(\dfrac{d\alpha}{dt}\right)_{t=0} = v_x$.*

For compact Riemannian manifolds there is the Hopf-Rinow theorem that tells us that points can be connected by geodesic arcs.

Theorem 2.2 *(Hopf and Rinow) If a connected Riemannian manifold M is compact, then any pair of points x and y may be joined by a geodesic whose length corresponds to the distance in the manifold from x to y.*

We also need the notion of geodesic convexity and the result of J. H. C. Whitehead that geodesically convex neighborhoods exist for all $x \in M$.

Definition 2.3 *Given a subset X of M and a point $x_0 \in X$, X is star shaped with respect to the point x_0, if for every $x \in X$ there is a unique shortest geodesic connecting x_0 with x which lies in X.*

Definition 2.4 *A subset X of M is geodesically convex if it is star shaped with respect to each of its points.*

Definition 2.5 *Given a subset A of a geodesically convex set X the geodesic convex hull of A is the smallest convex set which contains A.*

Theorem 2.6 *(J. H. C. Whitehead) Let V be an open subset of a Riemannian manifold M and let $x \in M$, then there is a geodesically convex open neighborhood U of x such that $U \subset V$.*

Let M be a Riemannian manifold and let X be a geodesically convex subset of M. Given points P_i in M we describe extensions of the de Castlejau, de Boor, and Aitken algorithms.

2.1 Riemannian Lagrange curves

Let M be a Riemannian manifold, and let $A = \{P_0, P_1, \cdots, P_n\}$ be a subset of a geodesically convex subset X. Given parameter points, $t_0 < t_1 < \cdots < t_n$, assume that A is contained in a sufficiently small neighborhood in which specified geodesics exist. For $0 \le i \le n-1$, define $\gamma_i^1 : [t_0, t_n] \to X$ to be the unique geodesic parametrized so that $\gamma_i^1(t_i) = P_i$ and $\gamma_i^1(t_{i+1}) = P_{i+1}$. For $1 < r \le n$ and $0 \le i \le n-r$ define $\gamma_i^r : [t_0, t_n]^r \to X$ so that $\gamma_i^r(u_1, u_2, \cdots, u_{r-1}, \cdot)$ is the unique geodesic parametrized so that $\gamma_i^r(u_1, u_2, \cdots, u_{r-1}, t_i) = \gamma_i^{r-1}(u_1, u_2, \cdots, u_{r-1})$ and $\gamma_i^r(u_1, u_2, \cdots, u_{r-1}, t_{i+r}) = \gamma_{i+1}^{r-1}(u_1, u_2, \cdots, u_{r-1})$. The function $\gamma_0^n : [t_0, t_n]^n \to X$ is called the *geodesic Aitken blossom* associated with the points $P_i \in X$, $0 \le i \le n$ and the parameter points, $t_0 < t_1 < \cdots < t_n$. If $\Delta : [t_0, t_n] \to [t_0, t_n]^n$ is the *diagonal map* defined by $\Delta(u) = \underbrace{(u, u, \cdots, u)}_{n}$, the *geodesic Lagrange curve* associated with X and the points P_i is the function $\Gamma_0^n = \gamma_0^n \circ \Delta$.

Theorem 2.7 *If $\Gamma_0^n : [t_0, t_n] \to M$ is the geodesic Lagrange curve associated with the points $P_i \in M$, $0 \le i \le n$, as defined above, then $\Gamma_0^n(t_i) = P_i$.*

Proof: Observe that for $1 \le r \le n$ and $0 \le i \le n-r$, γ_i^r depends for its definition only on the points, P_j, where $i \le j \le i+r$. If $n = 1$, and we are given points, P_0 and P_1, the result follows from the definition of γ_0^1 . Inductively assume it is true for $k < n$. For $k = n$, if $i = 0$, by definition

$$\Gamma_0^n(t_0) = \gamma_0^n\underbrace{(t_0, t_0, \cdots, t_0)}_{n} = \gamma_0^{n-1}\underbrace{(t_0, t_0, \cdots, t_0)}_{n-1} = \cdots = \gamma_0^1(t_0) = P_0$$

and likewise if $i = n$, $\Gamma_0^n(t_n) = \gamma_0^n\underbrace{(t_n, t_n, \cdots, t_n)}_{n} = \gamma_0^{n-1}\underbrace{(t_n, t_n, \cdots, t_n)}_{n-1} = \cdots = \gamma_0^1(t_n) = P_n$. For $i \ne 0$ and $i \ne n$, observe that the geodesics used in the construction of γ_0^{n-1} and γ_1^{n-1} may be restricted respectively to the intervals $[t_0, t_{n-1}]$ and $[t_1, t_n]$ so that γ_0^{n-1}becomes the geodesic Aitken blossom associated with the points $P_0, P_1, \cdots, P_{n-1}$and the parameter points $t_0 < t_1 < \cdots < t_{n-1}$, and γ_1^{n-1} becomes geodesic Aitken blossom associated with the points $P_1, P_2, \cdots, P_n$ and the parameter points $t_1 < t_2 < \cdots < t_n$. By the deductive assumption, $\gamma_0^{n-1}\underbrace{(t_i, t_i, \cdots, t_i)}_{n-1} = P_i = \gamma_1^{n-1}\underbrace{(t_i, t_i, \cdots, t_i)}_{n-1}$, and consequently $\gamma_0^n(\underbrace{t_i, t_i, \cdots, t_i}_{n-1}, \cdot)$ is the geodesic connecting $\gamma_0^{n-1}\underbrace{(t_i, t_i, \cdots, t_i)}_{n-1}$ with $\gamma_1^{n-1}\underbrace{(t_i, t_i, \cdots, t_i)}_{n-1}$, and is thus the constant function, $\gamma_0^n(\underbrace{t_i, t_i, \cdots, t_i}_{n}, u) = P_i$ for all $u \in [t_0, t_n]$. Thus in particular, $\gamma_0^n\underbrace{(t_i, t_i, \cdots, t_i)}_{n} = \Gamma_0^n(t_i) = P_i$. □

2.2 Riemannian Bézier curves

Following the previous format we introduce a Riemannian version of the de Casteljau algorithm. Accordingly, let X be a geodesically convex subset of a Riemannian manifold M. Let $A = \{P_0, P_1, \cdots, P_n\}$ be a subset of X. Define $\gamma_i^0 : [0, 1] \to X$ by $\gamma_i^0(u) = P_i$. For $1 \le r \le n$ and $0 \le i \le n-r$ define $\gamma_i^r : [0, 1]^r \to X$ to be the unique geodesic with the property that $\gamma_i^r(u_1, u_2, \cdots, u_{r-1}, 0) = \gamma_i^{r-1}(u_1, u_2, \cdots, u_{r-1})$ and $\gamma_i^r(u_1, u_2, \cdots, u_{r-1}, 1) = \gamma_{i+1}^{r-1}(u_1, u_2, \cdots, u_{r-1})$. The function $\gamma_0^n : [0, 1]^n \to X$ is called the *geodesic de Casteljau blossom* associated with the set A. If $\Delta : [0, 1] \to [0, 1]^n$ is the diagonal map, the *geodesic Bézier curve* associated with X and the set A is the function $\Gamma_0^n = \gamma_0^n \circ \Delta$.

2.3 Riemannian B-Spline curves

Given $A = \{P_0, P_1, \cdots, P_n\}$ contained in a geodesically convex subset X of a Riemannian manifold M, and given knots $t_1 < t_2 < \cdots < t_{2n}$, define $\gamma_i^0 : [t_1, t_{2n}] \to X$ by $\gamma_i^0(t) = P_i$, for $0 \le i \le n$. For $1 \le r \le n$ and $r \le i \le n$, define $\gamma_i^r : [t_i, t_{i+n+1-r}]^r \to X$ to be the unique geodesic with the property that $\gamma_i^r(u_1, u_2, \cdots, u_{r-1}, t_i) = \gamma_{i-1}^{r-1}(u_1, u_2, \cdots, u_{r-1})$ and $\gamma_i^r(u_1, u_2, \cdots, u_{r-1}, t_{i+n+1-r}) = \gamma_i^{r-1}(u_1, u_2, \cdots, u_{r-1})$. The function $\gamma_n^n : [t_n, t_{n+1}]^n \to X$ is called the *geodesic de Boor blossom* associated the set A. If $\Delta : [t_n, t_{n+1}] \to [t_n, t_{n+1}]^n$ is the diagonal map, the *geodesic B-Spline curve* associated with X and the points P_i is the function $\Gamma_n^n = \gamma_n^n \circ \Delta$.

We have the following results, which follow from the fact that both the geodesic de Casteljau and the geodesic de Boor blossoms are constructed from successive geodesic combinations beginning with the set $A = \{P_0, P_1, \cdots, P_n\}$.

Theorem 2.8 *Given $A = \{P_0, P_1, \cdots, P_n\}$ contained in a geodesically convex subset of a Riemannian manifold, if $\gamma_0^n : [0, 1]^n \to X$ is the geodesic de Casteljau blossom of A, then $\gamma_0^n([0, 1]^n)$ is contained in the geodesic convex hull of the set A.*

Theorem 2.9 *Given $A = \{P_0, P_1, \cdots, P_n\}$ contained in a geodesically convex subset of a Riemannian manifold, if $\gamma_0^n : [t_n, t_{n+1}]^n \to X$ is the geodesic de Boor blossom of A relative to a knot sequence $t_1 < t_2 < \cdots < t_{2n}$, then $\gamma_0^n([t_n, t_{n+1}]^n)$ is contained in the geodesic convex hull of the set A.*

Since each of the three blossoms are constructed successively from C^∞ geodesics, it follows that the blossoms and their restrictions to the diagonal are also of class C^∞.

Theorem 2.10 *The geodesic Lagrange, Bézier, B-spline curves are of class C^∞ as are each of their corresponding blossoms.*

3 Examples

The impediments to implementation of these ideas depend on the manifold in question. In all cases it is necessary that the points P_i should lie in a region in which it is possible to construct geodesic arcs between points. The problem then reduces to that of finding methods for such constructions. Even in cases for which this is possible, there is the additional problem that many of the desirable properties associated with B-spline or Bézier curves in $\mathbb{R}^3$ may have no direct analogs. Many properties such as the ability to subdivide a curve depend on the blossom being symmetric or multi-affine, and for the generalizations presented here, this is seldom true. For the case of an orientable 2-manifold embedded in $\mathbb{R}^3$, there are in many cases good solutions to the problem of finding geodesics, but different classes of surfaces lead to different solution. In this section we mention a few. In the case that the manifold M is the 2-sphere S^2 a preliminary version of our results is reported in [5].

3.1 The sphere

In the case that $M = S^2$, a small alteration to methods presented so far allows the consideration of curves that lie off the sphere. Given points P and Q that lie off the sphere consider radial projections to points $\tilde{P}$ and $\tilde{Q}$ and let $\tilde{\gamma} : [a,b] \to S^2$ be a geodesic with the property that $\tilde{\gamma}(a) = \tilde{P}$ and $\tilde{\gamma}(b) = \tilde{Q}$. The curve $\gamma : [a,b] \to \mathbb{R}^3$ defined by

$$\gamma(t) = \left(\tfrac{b-t}{b-a} \cdot \|P\| + \tfrac{t-a}{b-a} \cdot \|Q\|\right) \cdot \tilde{\gamma}(t)$$

is called the *Archimedian spiral* connecting the points P and Q. To explicitly describe the curve $\tilde{\gamma}$, set $\tilde{P} = v_1$, $\tilde{Q} = v_2$ and for simplicity consider the parameter interval $[a,b]$ to be the unit interval $[0,1]$. For $< \cdot, \cdot >$ the standard inner product on $\mathbb{R}^3$ set

$$v_3 = (< v_1, v_2 > v_1 - v_2)/(\|< v_1, v_2 > v_1 - v_2\|)$$

so that v_3 is orthogonal to v_1 and in the plane containing v_1 and v_2. Letting $\theta =< v_1, v_2 >$ denote the angle between v_1 and v_2, the geodesic $\tilde{\gamma}$ connecting v_1 with v_2 is defined by

$$\begin{aligned}\tilde{\gamma}(t) &= \cos(t\theta)v_1 + \sin(t\theta)v_3 \\ &= \left(\cos(t\theta) + \frac{\sin(t\theta) < v_1, v_2 >}{\|< v_1, v_2 > v_1 - v_2\|}\right) v_1 - \frac{\sin(t\theta)}{\|< v_1, v_2 > v_1 - v_2\|} v_2.\end{aligned}$$

The corresponding Archimedian Lagrange, Bézier and B-spline curves may now be constructed with the general algorithms of Section 2.

One of the difficulties that arise with Archimedian curves is that geodesic blossoms are not necessarily symmetric or multi-affine. It is even not clear what these concepts might mean in a geodesic context. Consequently, certain results that hold for normal Bézier or B-spline curves that depend on these properties are no longer valid. In particular analogs of the subdivision algorithms that allow one to determine control points of a portion of a given Bézier or B-spline are not valid. However, it can be shown that a simple non-linear change in the parametrization of the geodesic arcs, makes it possible to recapture most of what is needed.

Definition 3.1 *Given two points A and B on the sphere. Let C be the smaller arc of the spherical geodesic joining A with B. The barycentric parametrization of C on the parameter interval $[a,b]$ is the function $\alpha : [a,b] \to C$ defined by*

$$\alpha(t) = q(x(t)),$$

where $x(t) = \frac{(b-t)}{b-a}A + \frac{(t-a)}{b-a}B$ and $q : \mathbb{R}^3 \to S^2$ is the radial projection $q(x) = \dfrac{x}{\|x\|}$.

In the following we prove a spherical version of the Menelaus theorem.

Theorem 3.2 *Given 3 points P_0, P_1, P_2 on S^2 let $\gamma : [0,1] \times [0,1] \to \mathbb{R}^3$ be the geodesic de Casteljau blossom in which all geodesic arcs are given the barycentric parametrization. Then $\gamma(s,t) = \gamma(t,s)$.*

Proof: Observe that an elementary geometric argument tells us that:

$$\begin{aligned}\gamma(s,t) = \gamma_0^2(s,t) &= q\left((1-t)\gamma_0^1(s) + t\gamma_1^1(s)\right) \\ &= q(\ (1-t)[(1-s)P_0 + sP_1] + t[(1-s)P_1 + sP_2]\ )\end{aligned}$$

and

$$\begin{aligned}\gamma(t,s) = \gamma_0^2(t,s) &= q\left((1-s)\gamma_0^1(t) + s\gamma_1^1(t)\right) \\ &= q(\ (1-s)[(1-t)P_0 + tP_1] + s[(1-t)P_1 + tP_2]\ ).\end{aligned}$$

And the result follows from the affine properties of $\mathbb{R}^3$. □

As an immediate consequence we have

Theorem 3.3 *Given points $P_0, P_1, \cdots, P_n$ on S^2, the associated de Casteljau blossom, in which geodesic arcs are given barycentric parametrization, is symmetric.*

The conventional blossoming description of subdivision can now be employed. From the blossom construction we can conclude that $\gamma_0^n(0,0,\cdots,0,\underbrace{1,1,\cdots,1}_{i}) = P_i$. In particular, it follows that, for $0 < u < 1$, the points $Q_i = \gamma_0^n(0,0,\cdots,0,\underbrace{u,u,\cdots,u}_{i})$ describe a geodesic de Casteljau blossom which is parametrized to the interval $[0,u]$ and which, because of the uniqueness of geodesic arcs, equals the restriction of γ_0^n to $[0,u]^n$. Likewise, for the interval $[u,1]$ the points $R_i = \gamma_0^n(\underbrace{u,u,\cdots,u}_{i}, 1,1,\cdots,1)$ determine a geodesic de Casteljau blossom which is parametrized to the interval $[u,1]$ and which equals the restriction of γ_0^n to $[u,1]^n$ Therefore, if $g : [0,1] \to S^2$ is the geodesic Bézier curve determined by $P_0, P_1, \cdots, P_n$ and if $g = \gamma_0^n \circ \Delta$, it follows that, $g\,|_{[0,u]} : t \mapsto \gamma_0^n(t,t,\cdots,t,\underbrace{u,u,\cdots,u}_{i})$ and $g\,|_{[1,u]} : t \mapsto \gamma_0^n(\underbrace{u,u,\cdots,u}_{i}, t,t,\cdots,t)$, for $0 < u < 1$.

More generally and along the lines of the proof above, we have the following theorem which allows all familiar properties of both Bézier and B-spline curves which have descriptions in terms of their corresponding blossoms to carry over to the spherical case.

Theorem 3.4 *Let $f : [0,1]^n \to \mathbb{R}^3$ be the Euclidean blossom generated by the de Casteljau algorithm using points $P_i \in S^2, 0 \leq i \leq n$. Then $\gamma_0^n = q \circ f$.*

3.2 Other surfaces

We briefly discuss two examples in which explicit descriptions of geodesics between points are possible.

A developable surface S [4], described as the image of a function $f : U \to \mathbb{R}^3$ for U an open subset of $\mathbb{R}^2$, possess the characteristic, among others, that distances are preserved by the function f. Therefore, a geodesic in the surface $f(U)$ may be considered as the image of a straight line in the plane. If $P_0, P_1, \cdots, P_n$ are points in S, let $Q_i = f^{-1}(P_i)$, $0 \leq i \leq n$. If $C \subset U$ is the Lagrange, Bézier, or B-spline curve obtained from the standard Euclidean versions of the algorithms, then it follows that $f(C)$ is the corresponding geodesic curve in S that would have been obtained using geodesic versions of the algorithms that we have described.

For surfaces of revolution the description of geodesics between two points is rather more involved. Let C be a curve in the yz-plane described implicitly by

$$\begin{cases} f(y) = z \\ x = 0 \end{cases},$$

for (y, z) belonging to some open set U contained in the upper half of the yz-plane. The surface S obtained by rotating C about the z-axis may be expressed as $g^{-1}(0)$ where $g : \mathbb{R} \times U \to \mathbb{R}$ is defined by $g(x, y, z) = f(\sqrt{x^2 + y^2}) - z = 0$. In polar coordinates letting $u = \sqrt{x^2 + y^2}$, we express S in the form

$$\begin{cases} x = u \cos\theta \\ y = u \sin\theta \\ z = f(u) \end{cases}.$$

Let $P = (u_1 \cos\theta_1, u_1 \sin\theta_1, f(u_1))$ and $Q = (u_2 \cos\theta_2, u_2 \sin\theta_2, f(u_2))$ be two points on S. Then it may be shown that the geodesic connecting P with Q is the function $\alpha : [u_1, u_2] \to S$ such that $\alpha(u) = (u \cos\theta(u), \sin\theta(u), f(u))$, where for fixed u_0,

$$\theta(u) = \int_{u_0}^{u} \sqrt{\frac{1 + (f'(t))^2}{\frac{1}{c^2}t^4 - t^2}}\, dt + c',$$

and constants c and c' satisfy the following equations:

$$\theta_2 - \theta_1 = \int_{u_1}^{u_2} \sqrt{\frac{1 + (f'(u))^2}{\frac{1}{c^2}u^4 - u^2}}\, du$$

$$c' = \theta_1 - \int_{u_0}^{u_1} \sqrt{\frac{1 + (f'(u))^2}{\frac{1}{c^2}u^4 - u^2}}\, du.$$

For complete details see [6].

4 Conclusion and future research

We have outlined a procedure by which conventional computer aided design constructions may be extended to arbitrary Riemannian manifolds. In practice, there are difficulties. In a given manifold points to be interpolated or approximated must lie in a region in which it is possible to construct necessary geodesic arcs. Supposing this the case, one then needs to find explicit descriptions of the geodesics. And then there is the question of the additional characteristics which the curves might possess. The paper raises more questions than it answers. In the case of a sphere, good results are obtained, and it is also possible to add variation that allows consideration of curves off the sphere but which project radially to geodesic Lagrange, Bézier, or B-spline curves. It is also shown, in the spherical case, that a change parametrization of geodesics results in blossoms that retain the desirable characteristics associated with Euclidean blossoms. For surfaces of revolution and developable surfaces, we know that geodesics can be found between points so the geodesic blossom constructions will always exist. It is however unlikely that these blossoms will be either symmetric or multi-affine; these characteristics depend on the affine structure of $\mathbb{R}^3$. Thus, in the case of a general Riemannian manifold, although the constructions may be valid, it is not clear that we will be able to employ fundamental operations such as subdivision which depend on the symmetry of the blossom. We have outlined three different methods of blossom construction, one for each of the algorithms considered. In the Euclidean case, we know that there is a unique symmetric, multi-affine polynomial that restricts to a given polynomial on the diagonal. This may not be true in our more general setting.

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