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TRUNCATION ERROR REDUCTION METHOD FOR POISSON EQUATION

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Abstract

A new so-called truncation error reduction method (TERM) is developed in this work. This is an iterative process which uses a coarse grid (2h) to estimate the truncation error and then reduces the error on the original grid (h). The purpose is to use multigrid and simple stencils for high-order accuracy.

1. Introduction

The driven force for this work is to develop a new method which can use simple stencils with multigrid method for high order accuracy. The multigrid (Brandt, 1984) was originally used to accelerate the convergence for elliptic systems. The work here is to use multigrid for high-order accuracy with a simple stencil which allows to achieve high-order accuracy with much fewer points than the traditional finite schemes. The high-order scheme is particularly important for direct numerical simulation (Orszag, S. A. & Patterson, G. S., 1972; Moine and Mahesh, 1998) and the large eddy simulation (LES) (Lilly, 1966; Leseiur & Metais, 1996).

Let us take a look at the problem with coarse grid DNS from the mathematical point of view. Here, the coarse grid DNS means to use a grid which is acceptable by currently available computers. The problem with the coarse grid DNS is really caused by the truncation error while the mesh size 'h' and time step 'k' are not small enough. There are resolution problems with coarse grid DNS as well, of course. Before we can reduce the truncation error, we need to give an estimation which is given by an iterative process including a coarse grid discretization and a coarse-to-fine grid interpolation.

We use the Poisson equation as a test case to check the TERM method and

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will use this method for CFD late.

2. Truncation errors reduction methods for Poisson equation

Let us take a look at the following Poisson equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -k^2\pi^2\left(\sin(k\pi x) + \cos(k\pi y)\right) \tag{1}$$

$$u(0, y) = \cos(k\pi y); \qquad u(1, y) = \cos(k\pi y); \tag{2}$$

$$u(x,0) = 1 + \sin(k\pi x); \qquad u(x,1) = (-1)^k + \sin(k\pi x)$$

The exact solution is $\sin(k\pi x) + \cos(k\pi y)$. We adopt a standard second

order central difference scheme with uniform grids, $\Delta x = \Delta y = h$, and obtain a

finite difference scheme:

$$\frac{u_{i-1,j}^h - 2u_{i,j}^h + u_{i+1,j}^h}{h^2} + \frac{u_{i,j-1}^h - 2u_{i,j}^h + u_{i,j+1}^h}{h^2} + \tau_{i,j}^h = s_{i,j} \tag{3}$$

where u is the exact solution and $s_{i,j} = -k^2\pi^2\sin(k\pi x_i) - k^2\pi^2\cos(k\pi x_i)$.

This finite difference scheme will give an exact solution if we can find the

exact truncation error, $\tau_{i,j}^h$, which has a second order, $O(h^2)$. We all know it is

difficult to find the exact truncation error, but we can try to give a more accurate
estimate by an iterative process which can be described as follows.

1. Assume $\tau_{i,j}^{h,0} = 0$, do one or two times Gauss-Seidel iterations, multigrid

   (Brandt, 1984) may be much faster, to get an approximation $u_{i,j}^{h,1}$.

2. Find the approximation on the coarse grid $u_{i2,j2}^{2h,1} = I_h^{2h} u_{i1,j1}^{h,1}$. In our case, a

   simple injection is used for I_h^{2h} (or $u_{i2,j2}^{2h,1} = u_{i1,j1}^{h,1}$).

3. Estimate the truncation error on the coarse grid:

$$\tau_{i,j}^{2h,1} = s_{i,j} - \frac{u_{i-1,j}^{2h} - 2u_{i,j}^{2h} + u_{i+1,j}^{2h}}{(2h)^2} - \frac{u_{i,j-1}^{2h} - 2u_{i,j}^{2h} + u_{i,j+1}^{2h}}{(2h)^2} \quad (4)$$

4. Estimate the truncation error on the fine grid:

$$\tau_{i1,j1}^{h,1} = \frac{1}{4} \times I_{2h}^{h} \tau_{i2,j2}^{2h,1} \quad (5)$$

where I_{2h}^{h} should have same or higher order than the finite difference scheme and we should use 1/16 instead of 1/4 in the formula if we use a fourth-order scheme.

5. Get a revised finite difference scheme on the fine grid:

$$\frac{u_{i-1,j}^{h} - 2u_{i,j}^{h} + u_{i-1,j}^{h}}{h^2} + \frac{u_{i,j-1}^{h} - 2u_{i,j}^{h} + u_{i,j+1}^{h}}{h^2} + \tau_{i,j}^{h,1} = s_{i,j} \quad (6)$$

6. Go to step 1 and start a new loop until $\| \tau^{h,n+1} - \tau^{h,n} \| <$ tolerance

This method can be extended with some changes to other flow problems which are usually non-linear and time-dependent.

3. Numerical test for Poisson equation with different grids

A number of numerical tests by using this truncation error reduction method (TERM) for different grids (8x8, 16x16, 32x32, 64x64, 128x128) and different k (1, 2, 3, 4, 8, 16) have been conducted.

Table 1 compares the numerical results with the exact solution. The comparison shows the error ratio of coarse grid 2h over fine grid h for standard second order central difference scheme, $\|u - u^{2h}\| / \|u - u^{h}\|$, is near 4, which shows the numerical scheme has a second order accuracy (Burden & Faires, 1996). By using the TERM method, the accuracy has been significant increased. First we find the error ratio is nearly 16, which shows we got a fourth order accuracy with second order stencils. From Table 1, we can find a solution with a grid of 16x16 by the new method can get same results as one obtained with a grid of 64x64 by the standard central difference scheme. It shows we can save the grids or the computer memory by 16 times for this 2-d Poisson equation. The

computation time can be saved almost 100 times. Similar conclusion can be found with other grids and different wave number k. More encouraging, the method shows a same achievement when the wave number becomes higher. It still can improve the results when we have only three points for one wave (see the cases of 8x8 for k=4 and 16X16 for k=8). It provides a high potential which could be used for coarse grid DNS for a more accurate numerical simulation for transitional turbulent flow.

Figure 1-4 show the L_2 norm of errors between the numerical solution and exact solution against the standard central difference method and the TERM method for different grids. It clearly shows that the new method increases the accuracy significantly and has potential for more accurate coarse grid simulation. For 2-D, the new method saves at least 16 times in grid point numbers and much more in CPU time. We can anticipate it will save at least 64 times in the number of grid points for 3-D problems.

4. Conclusion

The new so-called truncation error reduction method (TERM) can significant increase the accuracy of numerical solution for Poisson equation for different wave numbers. It shows TERM can use coarse grids to achieve much more accurate numerical solution than the standard finite difference scheme for Poisson equations. It can save hundreds times in memory and computational time. Potentially, the method may be used for fluid dynamics and for coarse grid DNS.

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Analytic Solution $u(x, y) = \sin k\pi x + \cos k\pi y$		IMAX = 8 JMAX = 8	IMAX = 16 JMAX = 16	IMAX=32 JMAX=32	IMAX=64 JMAX=64	IMAX=128 JMAX=128
k=1	ERROR (central difference)	3.8297E-003	1.0188E-003	2.6273E-004	6.6712E-005	1.6809E-005
	ERROR (TERM)	7.8555E-004	6.1889E-005	4.3033E-006	2.8284E-007	1.8116E-008
k=2	ERROR (central difference)	3.5781E-002	9.3823E-003	2.4108E-003	6.1159E-004	1.5406E-004
	ERROR (TERM)	9.9472E-003	8.4777E-004	6.0961E-005	4.0573E-006	2.6110E-007
k=3	ERROR (central difference)	9.3301E-002	2.3893E-002	6.1033E-003	1.5461E-003	3.8931E-004
	ERROR (TERM)	4.2942E-002	3.8647E-003	2.8834E-004	1.9441E-005	1.2565E-006
k=4	ERROR (central difference)	1.9922E-001	4.8970E-002	1.2387E-002	3.1303E-003	7.8776E-004
	ERROR (TERM)	1.3625E-001	1.1337E-002	8.7461E-004	5.9764E-005	3.8789E-006
k=8	ERROR (central difference)	1.3273	2.3419E-001	5.5260E-002	1.3733E-002	3.4418E-03
	ERROR (TERM)	6.7739	1.6488E-001	1.2195E-002	8.9128E-004	5.9201E-05
k=16	ERROR (central difference)	91.4400	1.5034	2.5141E-001	5.8164E-002	1.4329E-02
	ERROR (TERM)	92.3567	31.4476	1.8054E-001	1.2681E-002	9.0120E-04

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Table 1. Error comparison between central difference scheme and TERM with different grids and wave numbers
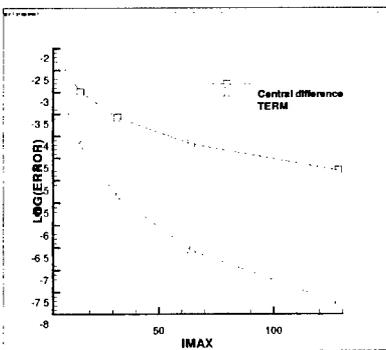
Figure 1: Comparison of errors between standard Central difference scheme and TERM scheme, k=1.
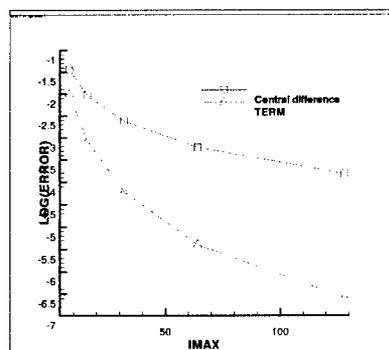
Figure 2: Comparison of errors between standard Central difference scheme and TERM scheme, k=2.
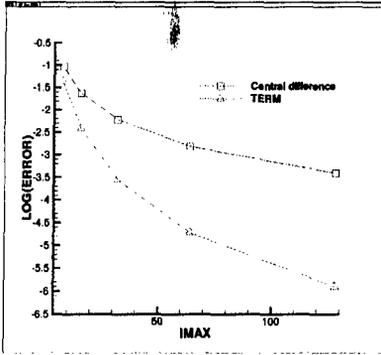
Figure 3: Comparison of errors between standard Central difference scheme and TERM scheme, k=3.
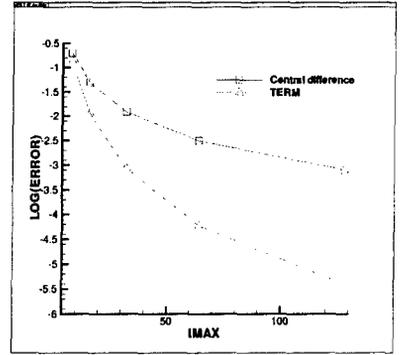
Figure 4: Comparison of errors between standard Central difference scheme and TERM scheme, k=4.

Appendix

We can prove the above TERM method for Poisson equation has fourth order for interior points. For simplicity, we consider the one dimensional case:

$$\frac{\partial^2 u}{\partial x^2} = f \tag{7}$$

Similarly to (3), the standard second order central difference scheme with uniform grids, $\Delta x = h$, can be written as

$$\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} + \tau_i = f_i \tag{8}$$
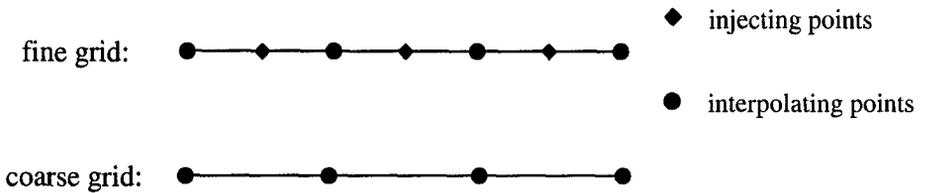
Figure 5   Injecting points and interpolating points

1. Injecting points where the fine grid point coincides with the coarse grid point ($u_i^{2h} = u_{2i}^h$):

On coarse grid

$$\frac{u_{i-2} - 2u_i + u_{i+2}}{(2h)^2} + \bar{\tau}_i = f_i \tag{9}$$

Here, $\bar{\tau}_i$ is the truncation error on coarse grid. We have used the following

relation between τ_i and $\bar{\tau}_i$

$$\tau_i = \frac{1}{4}\bar{\tau}_i + \tilde{\tau}_i = \frac{1}{4}\left(f_i - \frac{u_{i-2} - 2u_i + u_{i+2}}{4h^2} \right) + \tilde{\tau}_i \tag{10}$$

Here, $\tilde{\tau}_i$ is the truncation error for the new scheme. Substitute (10) into (8), we

have

$$\frac{-u_{i-2} + 16u_{i-1} - 30u_i + 16u_{i+1} - u_{i+2}}{12h^2} + \tilde{\tau}_i = f_i \tag{11}$$

Using Taylor series, we can find that $\tilde{\tau}_i \sim O(h^4)$ and (11) is a fourth order

scheme.

2. Interpolating points where the fine grid point is located between two coarse
   grid points.

On coarse grid

$$\frac{u_{i-3} - 2u_{i-1} + u_{i+1}}{(2h)^2} + \bar{\tau}_{i-1} = f_{i-1} \tag{12}$$

$$\frac{u_{i-1} - 2u_{i+1} + u_{i+3}}{(2h)^2} + \bar{\tau}_{i+1} = f_{i+1} \tag{13}$$

Here, $\bar{\tau}_{i-1}$ and $\bar{\tau}_{i+1}$ are the truncation errors on coarse grid. We have used the

following relation for $\bar{\tau}_{i-1}, \bar{\tau}_{i+1}$ and τ_i

$$\tau_i = \frac{1}{4} \times \frac{1}{2} \times (\bar{\tau}_{i-1} + \bar{\tau}_{i+1}) + \tilde{\tau}_i = \frac{1}{8}\left(f_{i-1} + f_{i+1} - \frac{u_{i-3} - u_{i-1} - u_{i+1} + u_{i+3}}{4h^2} \right) + \tilde{\tau}_i$$

$$(14)$$

Here, $\tilde{\tau}_i$ is the truncation error for the new scheme. Substitute (16) into (8), we have

$$\frac{-u_{i-3} + 33u_{i-1} - 64u_i + 33u_{i+1} - u_{i+3}}{24h^2} + \frac{(f_{i-1} - 2f_i + f_{i+1})}{6} + \tilde{\tau}_i = f_i \quad (15)$$

We have following relations on the fine grid:

$$\frac{u_{i-2} - 2u_{i-1} + u_i}{h^2} - \frac{h^2}{12}u^{(4)}(\xi_{i-1}) = f_{i-1} \quad (16)$$

$$\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} - \frac{h^2}{12}u^{(4)}(\xi_i) = f_i \quad (17)$$

$$\frac{u_i - 2u_{i+1} + u_{i+2}}{h^2} - \frac{h^2}{12}u^{(4)}(\xi_{i+1}) = f_{i+1} \quad (18)$$

Here $x_{i-2} \le \xi_{i-1} \le x_i$, $x_{i-1} \le \xi_i \le x_{i+1}$ and $x_i \le \xi_{i+1} \le x_{i+2}$. Substitute (16), (17), (18) into (15), we have

$$\frac{-u_{i-3} + 4u_{i-2} + 17u_{i-1} - 40u_i + 17u_{i+1} + 4u_{i+2} - u_{i+3}}{24h^2} + \tilde{\tau}_i -$$

$$\frac{h^2}{72}(u^{(4)}(\xi_{i-1}) - 2u^{(4)}(\xi_i) + u^{(4)}(\xi_{i+1})) = f_i \quad (19)$$

Because $\quad u^{(4)}(\xi_{i-1}) - 2u^{(4)}(\xi_i) + u^{(4)}(\xi_{i+1}) \approx u^{(6)}(\eta_i)h^2 \quad (20)$

here $x_{i-2} \le \eta_i \le x_{i+2}$. Therefore

$$\frac{h^2}{72}(u^{(4)}(\xi_{i-1}) - 2u^{(4)}(\xi_i) + u^{(4)}(\xi_{i+1})) \sim O(h^4) \quad (21)$$

Using Taylor series, we can find that $\tilde{\tau}_i \sim O(h^4)$ and (15) is a fourth order scheme.