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DISCRETE FILTERING ON UNSTRUCTURED GRIDS BASED ON LEAST-SQUARES GRADIENT RECONSTRUCTION

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1. Introduction

The equations for Large-Eddy Simulation (LES) of turbulent flows are formally derived by applying a low-pass filter to the Navier-Stokes equations. In doing so, it is often tacitly assumed that the filtering and differentiation operations commute. This assumption is invalid if the filter width is not uniform—as is the case if wall-bounded flows are computed—unless special filter operators are constructed, see, e.g, Vasilyev *et al.* (1998).

Recent work by Marsden *et al.* (2000) resulted in a framework for the construction of filters on unstructured grids which commute with differentiation to a potentially arbitrarily high order. They also demonstrated a filter operator with a second-order commutation error. However, their method appears to be quite complicated in its construction, particularly in three dimensions and near boundaries. Furthermore, it is dependent on geometric comparisons with user-specified parameters.

The goal of the present work is to develop a simpler filtering method than that of Marsden *et al.* (2000). The new filtering method is based on the following observation: The conditions for filtering a function to a given order of commutation error derived by Vasilyev *et al.* (1998) are formally identical to the conditions for reconstructing the gradient of a function to a given order of truncation error. In other words, the construction of filtering operators may be reinterpreted as the construction of—suitably reformulated—gradient-reconstruction methods. This apparently trivial observation has important consequences because the reconstruction of gradients is central to many flow-solution methods on unstructured grids and is well understood, see Haselbacher and Blažek (2000).

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2. Least-Squares Gradient Reconstruction

The least-squares gradient-reconstruction procedure originally developed by Barth (1991) is based on approximating the variation along an edge linking vertices 0 and i by a truncated Taylor series, e.g., for a linear approximation,

$$\phi_i = \phi_0 + (\nabla\phi)_0 \cdot \Delta\mathbf{r}_{0i}, \tag{1}$$

where $\Delta\mathbf{r}_{0i} = \mathbf{r}_i - \mathbf{r}_0$ and $\mathbf{r} = \{x, y\}^t$. The application of Eq. (1), or corresponding higher-order approximations, to all edges incident to vertex 0 gives a system of linear equations for the derivatives at vertex 0,

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \tag{2}$$

where $\mathbf{A}$ is a $d_0 \times n_0$ matrix of geometrical terms, $\mathbf{x}$ is an n_0–vector containing derivatives, and $\mathbf{b}$ is a d_0–vector of function values, with n_0 being the number of derivatives reconstructed and d_0 denoting the degree of vertex 0. Since there are usually more incident edges than derivatives, Eq. (2) is solved for $\mathbf{x}$ in a least-squares fashion.

A general closed-form solution of Eq. (2) can be derived through the QR-decomposition of $\mathbf{A}$ using the Gram-Schmidt process. In the following, we denote by $\mathbf{a}_i$ and $\mathbf{q}_i$ the ith column vector of the matrices $\mathbf{A}$ and $\mathbf{Q}$, respectively, and by r_{ij} the ijth element of the upper triangular matrix $\mathbf{R}$. There is no summation over repeated indices. The general solution is

$$\mathbf{x} = \mathbf{W}^t\mathbf{b}, \tag{3}$$

where $\mathbf{W}$ is a $d_0 \times n_0$ matrix with column vectors $\mathbf{w}_i$ given by

$$\mathbf{w}_i = c_{ii}\mathbf{q}_i + \sum_{k=i+1}^{n_0} c_{ik} c_{kk} \mathbf{q}_k, \tag{4}$$

with

$$\mathbf{q}_i = c_{ii}\left(\mathbf{a}_i + \sum_{j=1}^{j<i} c_{ji} \mathbf{a}_j\right). \tag{5}$$

The geometrical quantities c_{ij} are defined as

$$c_{ii} = r_{ii}^{-1} \tag{6}$$

$$c_{ij} = -\left(c_{ii} r_{ij} + \sum_{k=i+1}^{j-1} c_{ik} c_{kk} r_{kj}\right) \quad \text{for} \quad j > i \le n_0, \tag{7}$$

where

$$r_{ij} = \frac{1}{r_{ii}}\left(\mathbf{a}_i \cdot \mathbf{a}_j - \sum_{k=1}^{i-1} r_{ki} r_{kj}\right) \quad \text{for} \quad j \ge i \le n_0. \tag{8}$$

The general closed-form solution allows the reconstruction of derivatives to an arbitrarily high order of accuracy on unstructured grids.

3. Least-Squares Filtering

The least-squares gradient-reconstruction method can be turned into a filtering method by modifying Eq. (1), so that ϕ_0 is no longer a point value, but represents a filtered value $\overline{\phi}_0$,

$$\phi_i = \overline{\phi}_0 + (\nabla \phi)_0 \cdot \Delta \mathbf{r}_{0i}. \tag{9}$$

The effect of this modification is that the filtered value $\overline{\phi}_0$ is appended to the vector of unknowns $\mathbf{x}$. The resulting system of equations can be solved using the method described in Section 2. This leads to an expression for the filtered value in the form of a weighted sum,

$$\overline{\phi}_0 = \sum_{i=1}^{d_0} \omega_{0i} \phi_i. \tag{10}$$

The accuracy of the filtering operation is determined by the order of the derivatives included in the gradient reconstruction. For example, linear gradient reconstruction leads to a second-order accurate expression for the filtered value, with weights given by

$$\omega_{0i} = \frac{1}{r_{33}^2} \left(1 - \frac{r_{23}}{r_{22}} \Delta y_{0i} + \frac{r_{12}r_{23} - r_{13}r_{22}}{r_{11}r_{22}} \Delta x_{0i} \right). \tag{11}$$

It is easily verified that Eq. (11) leads to two vanishing moments,

$$\sum_{i=1}^{d_0} \omega_{0i} \Delta \mathbf{r}_{0i} = \mathbf{0},$$

which is merely a consequence of its second-order accuracy.

The spectral behaviour of Eq. (10) can be improved if the unfiltered value ϕ_0 is also included in the stencil,

$$\overline{\phi}_0 = \omega_{00} \phi_0 + (1 - \omega_{00}) \sum_{i=1}^{d_0} \omega_{0i} \phi_i. \tag{12}$$

This modification does not degrade the accuracy of Eq. (10).

The remainder of this work is based upon Eq. (12) and investigates linear and quadratic filter functions. For both functions, the stencil is extended to include an additional layer of vertices beyond the nearest neighbours. In

the interior, this gives a stencil of 18 vertices for $\omega_{00} = 0$ or 19 vertices for $\omega_{00} \neq 0$. The motivation for extending the support is twofold. First, having two layers of vertices allows the introduction of two parameters β and γ, which can be used to weight the contributions to Eq. (10) of the inner and outer layers, respectively. Together with ω_{00}, these additional degrees of freedom may be used to optimize damping of high-wavenumber components or to achieve a specified filter width. Second, the nearest-neighbour stencil leads to a singular matrix $\mathbf{A}$ for the quadratic filter function on uniform grids, thus necessitating the use of additional points in the stencil.

One advantage of the new filtering method is that it allows reusing data structures and geometric weights already employed to compute gradients in the flow-solution method. Furthermore, it is easily extended to three dimensions and does not require special treatment at boundaries beyond ensuring—as for interior vertices—that the degree of a given vertex is greater than the number of derivatives reconstructed at that vertex.

4. Determination of Filter Width

In the present work, the filter width is determined from the polar moment of the filter transfer function,

$$
J_{xy} = \int\limits_{0}^{\pi/\Delta} \int\limits_{0}^{\pi/\Delta} \left(k_x^2 + k_y^2 \right) G(k_x, k_y) \, dk_x \, dk_y, \tag{13}
$$

where $\mathbf{k} = \{k_x, k_y\}^t$ is the wave-number vector, $G(k_x, k_y)$ is the filter transfer function, and Δ is the grid spacing. In one dimension, this reduces to the second moment of the filter transfer function, whose use in determining the filter width was originally suggested by Lund (1997). In this section, we assume the grid spacing to be uniform.

The ratio of the filter width Δ_f to the grid spacing may be computed from the relation

$$
\alpha \equiv \frac{\Delta_f}{\Delta} = \left(\frac{\pi^5}{8 J_{xy} \Delta^4} \right)^{\frac{1}{4}}. \tag{14}
$$

The constants in Eq. (14) were chosen such that it gives the correct width for the Fourier cut-off filter.

Figure 1(a) depicts the transfer function for the linear filter function with $\omega_{00} = 1/5$ and $\beta = \gamma = 1$. While high wave-numbers are damped well, the transfer function deviates quickly from unity for low to moderate wave-numbers, and, as such, is a poor representation of the Fourier cut-off filter. Numerical evaluation of Eq. (13) gives $\alpha = 1.40$.

The transfer function for the quadratic filter function with $\omega_{00} = 1/2$ and $\beta = \gamma = 1$ is shown in Fig. 1(b). Compared to the linear filter function, the quadratic filter function is a good approximation to the Fourier cut-off filter for low to moderate wave-numbers. For higher wave-numbers, preferred directions can be discerned which are aligned with the edges in the grid. For the quadratic filter function, Eq. (13) gives $\alpha = 1.12$.
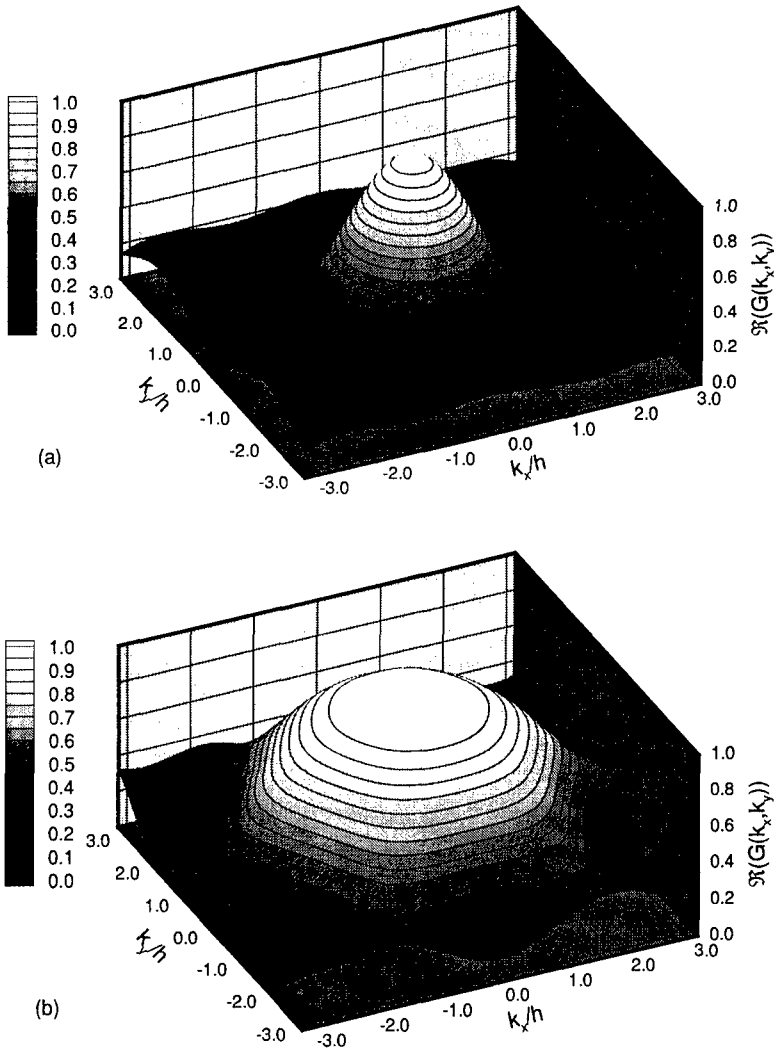

Figure 1. Transfer functions for filter functions on uniform grids and $\beta = \gamma = 1$. (a) Linear filter function with $\omega_{00} = 1/5$ and (b) quadratic filter function with $\omega_{00} = 1/2$.

5. Commutation Error

Marsden *et al.* (2000) proved that a filter with $p - 1$ vanishing moments is needed to achieve a commutation error of order p for smoothly varying filter widths in one dimension. An equivalent statement is that the filter must be accurate to order p to achieve a commutation error of order p for smoothly varying filter widths. For grids with arbitrarily varying filter widths, the commutation error will drop below p. It is thus necessary to construct filter operators of order $p + 1$ to obtain commutation errors of order p on arbitrary unstructured grids. We are interested in this general case since unstructured grids rarely satisfy smoothness constraints. To obtain a second-order commutation error, we thus require quadratic filtering.

The order of the commutation error is computed by carrying out a grid-refinement study using an analytic function for the unfiltered field. Five uniform triangular grids were generated for a hexagonal domain, containing 271, 1141, 4681, 18961, and 76321 vertices. The interior vertices were subsequently distorted by random amounts of a given fraction of the grid spacing in both coordinate directions. In the results presented below, this fraction was taken to be 0.35. The distorted grid with 1141 vertices is shown in Fig. 2.
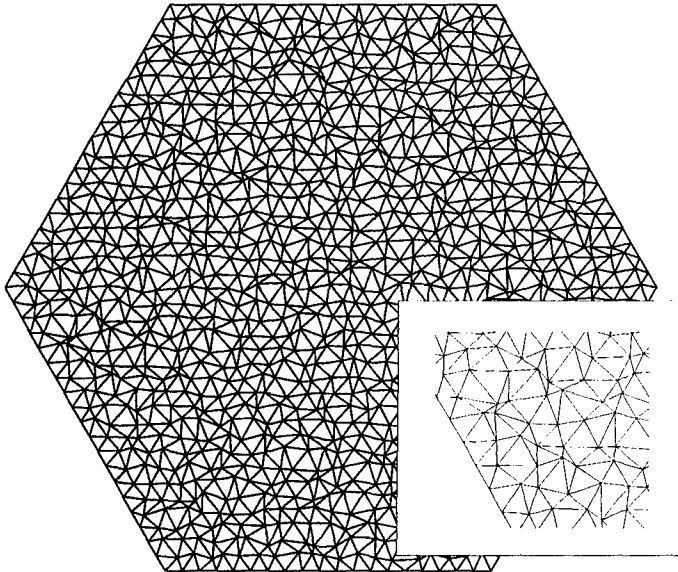

Figure 2. Distorted grid with 1141 vertices. Inset shows detail of distorted grid.

The commutation error is defined in terms of the discrete divergence,

$$E_{\mathrm{c}} = \left(\overline{\frac{\delta u}{\delta x}} + \overline{\frac{\delta v}{\delta y}} \right) - \left(\frac{\delta \overline{u}}{\delta x} + \frac{\delta \overline{v}}{\delta y} \right), \tag{15}$$

where $\delta(\cdot)/\delta x$ represents the discrete gradient operator in the x–coordinate direction. The function chosen in the present study is

$$\left\{ \begin{array}{c} u \\ v \end{array} \right\} = \left\{ \begin{array}{c} \cos(\pi x)\sin(\pi y) \\ \sin(\pi x)\cos(\pi y) \end{array} \right\}. \tag{16}$$

In computing the commutation-error norms, only the vertices were included whose gradients or filtered values were not affected by boundary effects. Effects of one-sided stencils arising from the presence of boundaries will be studied in future work. It was verified that commutation errors were identically zero for arbitrary functions on uniform grids.

The variation of the L_2-norms of the commutation error with grid refinement is shown in Fig. 3. Note that the commutation error is about an order of magnitude smaller than the truncation error of the divergence operator. The order of accuracy of the filtering operator, the divergence operator, and the order of the commutation error were computed from a linear least-squares curve fit for the finest four grids. The slopes were determined to be 3.05, 1.96, and 2.17, respectively. It is thus verified that the commutation error obtained with the new quadratic filtering method on a randomly distorted unstructured grid is of second order.

The ultimate test for commutation will be to specify a uniform filter width on a randomly distorted grid using the parameters ω_{00}, β, and γ and to check for zero commutation error. This is an objective of future work.

6. Conclusions

A new filtering method for unstructured grids was presented. Closed-form expressions were given which allow the construction of filtering operators of arbitrarily high order. The new filtering method is easily constructed, does not require special treatment at boundaries, and allows reusing data structures and geometric terms needed by the flow-solution method without filtering. Linear and quadratic filter functions were studied. A grid-refinement study on randomly distorted grids demonstrated a commutation error of second order.
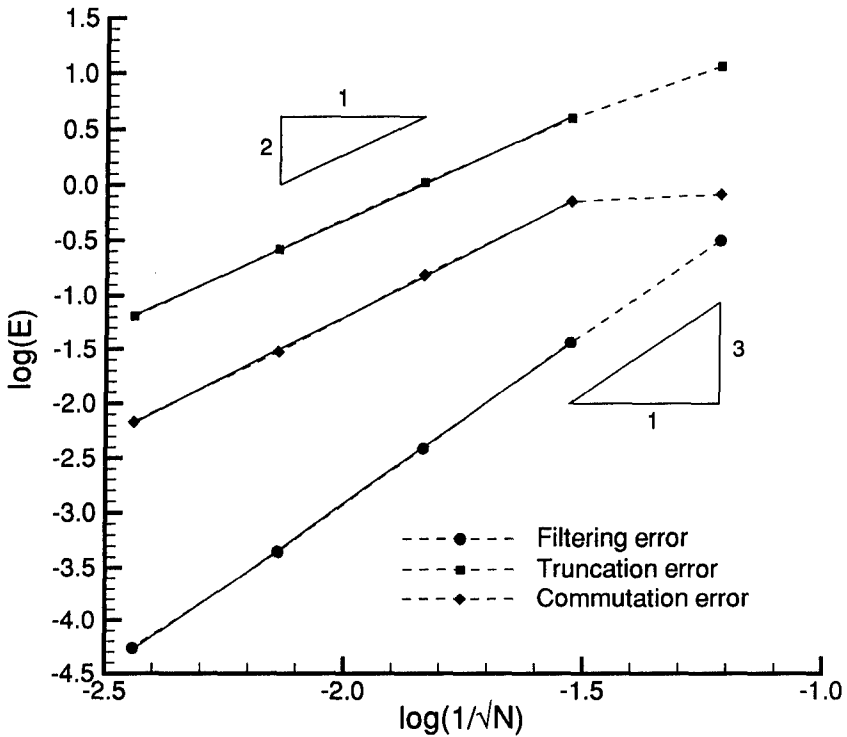
Acknowledgement

Figure 3. Variation of L_2–norm of errors with grid refinement. Solid lines represent linear curve fits to data, and N denotes the number of vertices.

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