Defense Technical Information Center Compilation Part Notice

ADP012526

TITLE: l = 1 Diocotron Instability of Single Charged Plasmas in a Cylindrical Penning Trap with Central Conductor

DISTRIBUTION: Approved for public release, distribution unlimited

This paper is part of the following report:

TITLE: Non-Neutral Plasma Physics 4. Workshop on Non-Neutral Plasmas [2001] Held in San Diego, California on 30 July-2 August 2001

To order the complete compilation report, use: ADA404831

$l = 1$ Diocotron Instability of Single Charged Plasmas in a Cylindrical Penning Trap with Central Conductor

M. Romé*, A. V. Arefiev†, I. A. Kotelnikov** and R. Pozzoli*

*I.N.F.M. and Dipartimento di Fisica, Università degli Studi di Milano, Italy
†Physics Department, University of Texas at Austin, TX, U.S.A.
**Budker Institute of Nuclear Physics and Physics Department, Novosibirsk State University, Russian Federation

Abstract. The linear stability analysis of the $l = 1$ diocotron perturbations in a single charged plasma confined in a cylindrical Penning trap is critically revisited. Particular attention is devoted to the instability due to the presence of stationary points in the radial profile of the azimuthal rotation frequency. The asymptotic analysis of Smith and Rosenbluth [1] for the case of a single-bounded plasma column (algebraic instability proportional to $t^{1/2}$) is extended to the case of a cylindrical Penning trap with an additional coaxial inner conductor, and it is shown that the algebraic instability found in the case of a single-bounded plasma column becomes exponential at longer times. The relevant linear growth rate is computed by a suitable inverse Laplace transform (contour integral in the complex plane). The analytical results are compared with the numerical solution of the linearized two-dimensional drift Poisson equations.

INTRODUCTION

The linear stability analysis of the $l = 1$ diocotron perturbations in a low density single charged plasma, radially bounded by two cylindrical conductors held at fixed potential, is critically reviewed. Using a model with a radial step density profile, Levy [2] showed that the plasma is neutrally stable when it is in contact with one or both conductors, or if the charge on the inner conductor is large enough. If the central conductor is absent, the $l = 1$ diocotron mode is neutrally stable, while lower $l \geq 2$ modes may be unstable. The effect of a central conductor on the stability of an hollow plasma column has been also studied experimentally [3].

Particular attention is devoted here to the instability due to the presence of one or several stationary points in the radial profile of the azimuthal rotation frequency of the plasma. The asymptotic analysis of Smith and Rosenbluth [1] for the case of a single-bounded plasma column (instability proportional to $t^{1/2}$) is extended to include algebraic instabilities growing as t^{α}, with $1/2 < \alpha \leq 1$. The asymptotic analysis is generalized to the case of a trap with a coaxial cylindrical inner conductor, and it is shown that the algebraic instability found in the previous case becomes exponential at longer times: the relevant linear growth rate are computed. Finite length and finite Larmor radius effects are neglected.

CP606, *Non-Neutral Plasma Physics IV*, edited by F. Anderegg et al.

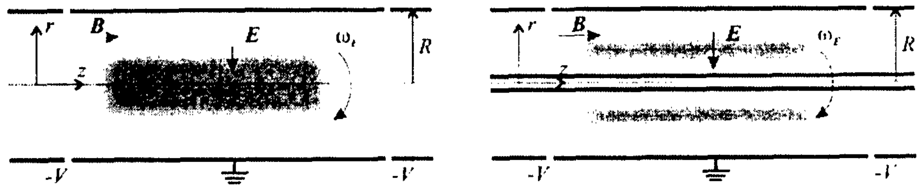

FIGURE 1. Scheme of a cylindrical Penning trap. **Left**: without central conductor; **Right**: with central conductor.

BASIC EQUATIONS

In the model, a one component plasma is assumed to be contained within two infinitely long cylindrical conducting walls, of radii a and R, respectively. The external conductor is grounded (see Fig. 1). The case where the central conductor is absent is treated simply by setting $a = 0$. The system is immersed in a static and uniform magnetic field $\vec{B} = B\hat{z}$, directed along the axis of the trap. In the model considered here, the evolution of the system is described by the two-dimensional drift Poisson equations [4], written in polar coordinates (r, θ),

$$\frac{\partial n}{\partial t} + (\nabla\varphi \times \nabla n)\cdot\hat{z} = 0, \qquad \nabla^2\varphi = n. \tag{1}$$

Adimensional quantities are used. The density, n, is normalized over an arbitrary reference density, $\hat{n}$; the lengths over the radius of the outer conductor, $\hat{r} = R$ (an explicit notation for R is kept in the following); the frequencies over $\hat{\omega} = 4\pi e^2\hat{n}/m\omega_c$, where $\omega_c = -eB/mc$ (e and m being the charge and the mass of the particles, respectively, and c the velocity of light in vacuo); the potential, φ, over $\hat{\varphi} = 4\pi e\hat{n}\hat{r}^2$; and the electric charge per unit length on the central conductor, Q, over $\hat{Q} = \pi e\hat{n}\hat{r}^2$.

Linearizing Eqs. (1) for perturbations $\delta\varphi = \phi_l(r,t)e^{il\theta}$ and $\delta n = n_l(r,t)e^{il\theta}$ with a given azimuthal number l, yields the following second-order differential equation for the potential amplitude ϕ_l (see, e.g., Ref. [4]):

$$\left[\frac{\partial}{\partial t} + il\omega_E(r)\right]\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} - \frac{l^2}{r^2}\right]\phi_l - \frac{il}{r}n_0'(r)\phi_l = 0; \tag{2}$$

$$\omega_E(r) = \frac{1}{r^2}\int_a^r n_0(r)\,r\,dr + \frac{Q}{2r^2},$$

where $n_0(r)$ and $\omega_E(r)$ are the unperturbed density and the unperturbed azimuthal frequency of the plasma, respectively, and a prime denotes the radial derivative. Eq. (2) has to be supplemented with the boundary conditions $|\phi_l(0)| < +\infty$, $\phi_l(R) = 0$ in the case without central conductor, and $\phi_l(a) = 0$, $\phi_l(R) = 0$ in the case with central conductor, respectively.

In the following, the analysis is restricted to the case $l = 1$. The Laplace transforms for the perturbed potential and density can be written in this case as

$$\phi_p(r) = r[p + i\omega_E(r)] \int_r^R \frac{h(x) - \bar{h}(p)}{x^3[p + i\omega_E(x)]^2} dx; \tag{3a}$$

$$n_p(r) = \frac{n_1(r,0)}{p + i\omega_E(r)} - i n_0'(r) \int_r^R \frac{h(x) - \bar{h}(p)}{x^3[p + i\omega_E(x)]^2} dx, \tag{3b}$$

where

$$h(r) = \int_a^r x^2 n_1(x,0)\, dx; \qquad \bar{h}(p) = \int_a^R \frac{h(x)\, dx}{x^3\, [p + i\omega_E(x)]^2} \Bigg/ \int_a^R \frac{dx}{x^3\, [p + i\omega_E(x)]^2}. \tag{4}$$

The inversion of the Laplace transforms, e.g., $\phi_1(r,t) = \int_{Br} \frac{dp}{2\pi i} \phi_p(r) e^{pt}$, involves an integration in the complex plane along a Bromwich contour, Br, which goes to the right of all the transform's singular points and branch cuts.

TRAP WITHOUT CENTRAL CONDUCTOR

The inverse Laplace transformation can be performed in a closed form if $a = 0$, recovering the result of Ref. [1]:

$$\phi_1(r,t) = r \int_r^R [h(x)/x^3]\, [1 + i\,\omega_E(r)t - i\,\omega_E(x)t]\, e^{-i\omega_E(x)t}\, dx; \tag{5a}$$

$$n_1(r,t) = n_1(r,0) e^{-i\omega_E(r)t} - i t\, n_0'(r) \int_r^R [h(x)/x^3] e^{-i\omega_E(x)t}\, dx. \tag{5b}$$

It is readily seen that an unstable perturbation can not grow faster than t, as the integrals in Eqs. (5) are decreasing functions of time. An asymptotically non-decaying solution can originate from the density jumps, from the two ends of the interval of integration in Eqs. (5), or from the stationary points inside the same interval.

If $n(r)$ is non-monotonic, $\omega_E(r)$ may also be non-monotonic with its extrema being the points of stationary phase for the integrand functions in Eqs. (5). If $r = r_0$ is a nondegenerate stationary point for ω_E ($\omega_E'(r_0) = 0$, $\omega_E''(r_0) \neq 0$), then

$$\phi_1 \approx \frac{i\sqrt{2\pi}\, r\, h(r_0)}{\sqrt{|\omega_E''(r_0)|}\, r_0^3} [\omega_E(r) - \omega_E(r_0)]\, H(r_0 - r)\, \sqrt{t} \exp\left[-i\omega_E(r_0)t - \frac{i\pi}{4} \mathrm{sign}(\omega_E'')\right], \tag{6}$$

where $H(x)$ is the Heaviside's step function. The perturbation grows in this case proportionally to $\sqrt{t}$ [1] (SR-instability).

In the presence of an inflection point, r_*, of the ω_E profile, different situations may occur, according to the value of $\omega_E'(r_*)/\omega_E'''(r_*)$. If $\omega_E'(r_*)/\omega_E'''(r_*) < 0$, $\omega_E(r)$ has two extrema at $r_\pm = r_* \pm \sqrt{-2\omega_E'/\omega_E'''}$. At large times, $t \gg |\Gamma|^{-1}$, with $\Gamma \equiv \sqrt{8\omega_E'^3/9\omega_E'''}$,

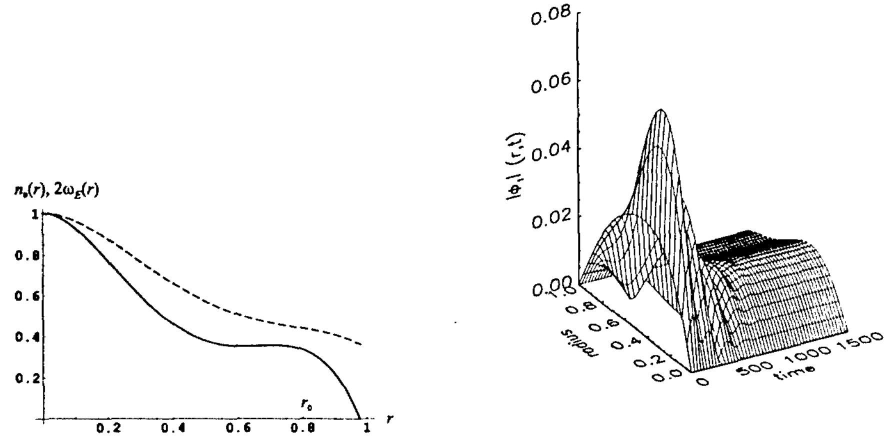

FIGURE 2. **Left**: Smooth n_0 profile (solid line) and corresponding ω_E profile (dashed line), which give rise to exponentially damping initial perturbations. **Right**: Amplitude of the perturbed potential, $|\phi_1|(r,t)$, vs. r and t.

these extrema contribute separately to ϕ_1, and the initial perturbation grows as $\sqrt{t}$. If $\omega_E'(r_*) = 0$, these points merge at $r = r_*$, and the perturbation grows proportionally to $t^{2/3}$:

$$\phi_1 \approx \frac{i2^{1/3}\Gamma(1/3)\, r h(r_*)}{3^{1/6}|\omega_E'''|^{1/3}\, r_*^3}\left[\omega_E(r) - \omega_E(r_*)\right] H(r_* - r)\, t^{2/3} \mathrm{e}^{-i\omega_E t}. \tag{7}$$

Finally, if $\omega_E'(r_*)/\omega_E'''(r_*) > 0$, the extremum points move into the complex plane, while ϕ_1 grows algebraically according to Eq. (7) in the initial time evolution, $t \ll |\Gamma|^{-1}$, and then decays exponentially at late times (this situation is illustrated in Fig. 2):

$$\phi_1 \approx \frac{i\sqrt{2\pi}\, r h(r_*)}{(2\omega_E'\omega_E''')^{1/4}\, r_*^3}\left[\omega_E(r) - \omega_E(r_*)\right] H(r_* - r)\, \sqrt{t}\, \exp[-i\omega_E t - \Gamma t]. \tag{8}$$

The frequency of the "saturated" mode is $\omega_E(R)$ (Levy's mode). In general, if the ω_E profile presents a stationary point, r_*, of order m ($\omega_E^{(j)}(r_*) = 0$ for $j < m$, and $\omega_E^{(m)}(r_*) \neq 0$), the following asymptotic behavior of the perturbed potential is obtained:

$$\phi_1 \approx \frac{2}{m}\left|\frac{m!}{\omega_E^{(m)}(r_*)}\right|^{1/m} \Gamma\left(\frac{1}{m}\right) t^{1-1/m}\, ir\, \frac{h(r_*)}{r_*^3}\left[\omega_E(r) - \omega_E(r_*)\right] H(r_* - r)\, \mathrm{e}^{-i\omega_E(r_*)t} \times$$
$$\left\{\cos\left(\frac{\pi}{2m}\right) + i\left[\frac{(-1)^{m-1} - 1}{2}\right] \mathrm{sign}(\omega_E^{(m)}(r_*))\, \sin\left(\frac{\pi}{2m}\right)\right\}. \tag{9}$$

For $m = 2$, this formula reduces to Eq. (6), and for $m = 3$ to Eq. (7).

Finally, a linear growth of the amplitude of an initial perturbation is found if an interval exists where $\omega_E(r)$ is constant, and if this interval does not contain the whole plasma column. This situation can be realized by means of a two-column plasma, where the densities of the internal ($0 \leq r \leq b_0$) and external ($a_1 \leq r \leq b_1$) columns are related by $n_0(b_0)\,b_0^2 = n_0(a_1)\,a_1^2$. In this case:

$$\phi_1 \approx it\,r\,[\omega_E(r) - \omega_E(a_1)]\,H(a_1 - r)\,\mathrm{e}^{-i\omega_E(a_1)t} \int_{a_1}^{b_1} [h(x)/x^3]\,dx. \tag{10}$$

TRAP WITH CENTRAL CONDUCTOR

The SR-instability comes from the neighborhood of a nondegenerate stationary point r_0 of ω_E. This instability can be interpreted as the contribution of the branching point $p = -i\omega_E(r_0)$ in the complex p-plane. When a central conductor is present in the trap, it results

$$\bar{h}(p) \simeq \frac{\mathcal{J}(p + i\,\omega_E(r_0))/r_0^3}{\mathcal{J}(p + i\,\omega_E(r_0))/r_0^3 + \beta(p)}\,h(r_0)\,; \tag{11}$$

$$\mathcal{J}(\sigma) = \frac{\pi\,\mathrm{e}^{-i\pi\mathrm{sign}[\omega_E''(r_0)]/4}}{\sqrt{|2\omega_E''(r_0)|}\,\sigma^{3/2}}\,, \qquad \beta(p) = \frac{1}{2a^2[p + i\Delta\omega_E][p + i\omega_E(a)]}\,,$$

where $\Delta\omega_E = n_0(a)/2$ and $\omega_E(a) = Q/2a^2$, respectively. The function $\mathcal{J}(\sigma)$ is two-valued since it contains the rational power $3/2$ of $\sigma = p + i\,\omega_E(r_0)$. The "physical sheet" of the Riemann surface, $\arg[\sigma] \in (-\pi, \pi)$, corresponds to the complex σ-plane with the branch cut $\sigma \in (-\infty, 0]$. The effect of the central conductor turns out to be negligible at small times, $t \ll T \approx |\omega_E''|^{1/3}\omega_E^{-4/3} r_0^2 a^{-4/3}$, while at larger times, $t \gg T$, the algebraic SR-instability disappears. The inverse Laplace transform can be performed (asymptotically for $t \to \infty$), by means of a suitable deformation of the inversion (Bromwich) contour inside the "physical sheet". In particular, an exponential instability is found if $[\omega_E(r_0) - \Delta\omega_E][\omega_E(r_0) - \omega_E(a)] > 0$, with a growth rate (in the limit $a \to 0$)

$$\gamma = \frac{3^{1/2}\pi^{2/3}\,a^{4/3}}{2^{2/3} r_0^2\,|\omega_E''|^{1/3}}\,[\omega_E(r_0) - \omega_E(a)]^{2/3}\,[\omega_E(r_0) - \Delta\omega_E]^{2/3}\,. \tag{12}$$

Typical results of the analysis and the numerical simulations of Eqs. (2) in the case with central conductor are shown in Figs. 3-4.

CONCLUSIONS

The existence of $l = 1$ diocotron instabilities in a charged plasma confined in a cylindrical Penning trap, growing with time faster than the SR-instability ($\propto t^{\alpha}$, with $1/2 < \alpha \leq 1$), has been pointed out. In addition, it has been shown that the presence of an inner conductor (even very thin and uncharged) can transform the algebraic instability into an exponential one at late times. A criterion for the occurrence of the exponential instability

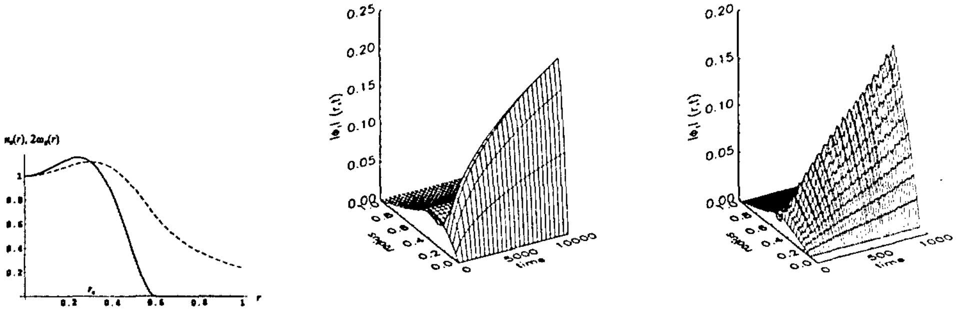

FIGURE 3. **Left**: $n_0(r)$ (solid line) and $\omega_E(r)$ (dashed line) profiles, which determine a SR algebraic instability [1]: $n_0 = \left[1+(r/r_p)^2/\Delta\right]\left[1-(r/r_p)^2\right]^2$ for $r < r_p$, $n_0 = 0$ for $r > r_p$, with $\Delta = 0.25$, $r_p = 0.6$. **Center**: Amplitude of the perturbed potential, $|\phi_1|(r,t)$, vs. r and t. **Right**: Amplitude of the perturbed potential, $|\phi_1|(r,t)$, vs. r and t, for the density profile truncated at the radius $a = 0.05$ of the central conductor. Q corresponds to the total charge of the particles lying within $r = 0.05$ in the previous case.

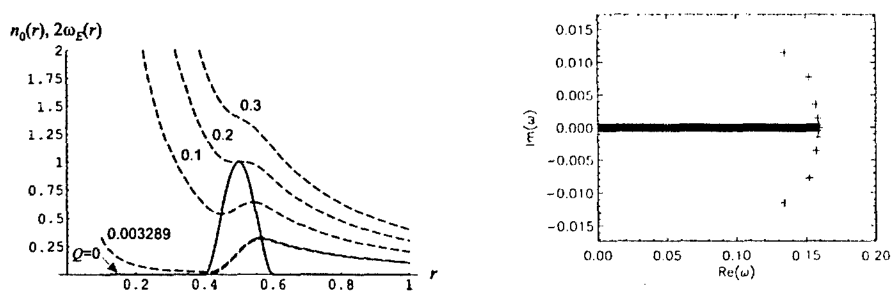

FIGURE 4. **Left**: Generalized parabolic density profile (solid line) used in the computations: $n_0 = [1-(r-r_c)^2/r_p^2]^2$ for $|r-r_c| < r_p$ and $n_c = 0$ otherwise (with $r_c = 0.5$ and $r_p = 0.1$) and ω_E profiles (dotted lines), plotted for $a = 0.1$ and different values of Q. **Right**: Example of frequency spectrum, for $Q = 0$ and $a = 0.2, 0.1, 0.05, 0.025$, and 0.001. The continuum spectrum lies on the Re[ω] axis from 0 to $\omega_E(r_0)$. For a given a, there are two complex conjugate discrete frequencies, which converge to $\omega_E(r_0)$ as $a \to 0$. Stable solutions (Im[ω] < 0) do not belong to the "physical" sheet of the Riemann surface.

and the growth rate have been computed analytically in the limit of thin inner conductor. This work has been supported by the Italian Ministry of Education and Scientific Research.

REFERENCES

1. R. A. Smith and M. N. Rosenbluth, Phys. Rev. Lett. **64**, 649 (1990).
2. R. H. Levy, Phys. Fluids **8**, 1288 (1965).
3. G. Rosenthal, G. Dimonte, and A. Y. Wong, Phys. Fluids **30**, 3257 (1987).
4. R. C. Davidson, *Physics of Nonneutral Plasmas*, (Addison Wesley, Redwood City, California, 1990).