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Review of Theoretical Approaches to Nonlinear Supercavitating Flows

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Summary: The purpose of the paper is to present a brief review of basic theoretical approaches to two-dimensional (2D) nonlinear supercavitating flows in the framework of theory of jets in an ideal fluid. In this connection discussed are Kirchhoff and Zhukovsky methods, Chaplygin method of "singular points", method of integral equation, *etc.* A simple model problem of a supercavitating (SC) flat plate at zero cavitation number $\sigma = 0$ is chosen to illustrate the core of the methods and their comparative effectiveness. Some mathematical aspects of open and closed cavity closure schemes are studied as well with use of Chaplygin method applied to a SC plate with a spoiler at nonzero cavitation number. An influence is demonstrated of free and solid boundaries onto the cavity volume and hydrodynamic characteristics of the plate. *Mathematica 4.0* software is used as a main tool for the flow pattern visualization of the problems under consideration. An analytical exact solution is presented to the 2D nonlinear flow problem of an arbitrary supercavitating foil and numerical results are discussed.

1. Basic assumptions of theory of jets in an ideal fluid

Theory of jets in an ideal fluid is appeared to be one of the best-studied fields of theoretical hydrodynamics dealing with flows confined by free and solid boundaries, the pressure constancy condition being satisfied on the former. H. Helmholtz [8] and G. Kirchhoff [12] were the first to formulate and solve some relatively simple problems of the theory of jets. Nowadays one can find brilliant surveys of the advances and development in the theory in books by Birkhoff & Zarantanello [1], Gilbarg [5] and Gurevich [7] (see also English translation of the book [6]). The works by Tulin [23], Terentev [21] and Maklakov [16] should also be distinguished.

As it follows from the name of the theory itself, we suppose a fluid to be an ideal one. For simplicity we neglect the gravity influence and specify that the flow be 2D, steady and incompressible. The flow has a velocity potential φ if

$$\boldsymbol{v} = \operatorname{grad}\varphi \,,$$

where $\boldsymbol{v} = v_x + \mathrm{i}v_y$ is the total velocity vector in the flow. The harmonic function $\varphi(x, y)$ is the real part of an analytical function of complex potential (or characteristic function) $w(z) = \varphi + i\psi$, where $z = x + \mathrm{i}y$ and (x, y) denote axes of rectangular Cartesian coordinate system. The imaginary part of w is called stream function ψ, the velocity vector being given by

$$v_x = \frac{\partial\varphi}{\partial x} = \frac{\partial\psi}{\partial y} \,; \qquad v_y = \frac{\partial\varphi}{\partial y} = -\frac{\partial\psi}{\partial x} \,.$$

Function ψ has a constant value on each separate streamline including free surfaces while φ increases along a streamline in downstream direction. A so called conjugate velocity can be introduced as follows

$$\frac{\mathrm{d}w}{\mathrm{d}z} = v_x + \mathrm{i}v_y = v\,\mathrm{e}^{-\mathrm{i}\theta} \,, \tag{1}$$

which is a mirror image of velocity vector $\boldsymbol{v}$ with respect to y-axis. In formula (1) v denotes absolute value of the vector $\boldsymbol{v}$ and θ is an angle made by the vector to x-axis.

As a result, a problem in question is considered to be solved when the complex potential function $w(z) = w(x+\mathrm{i}y)$ is found. As is customary, on all the solid boundaries of a given topography the kinematic (flow tangency) condition is applied, which requires the fluid flow to be tangent to the surface of the boundary, that is derivative of φ with respect to normal vector $\boldsymbol{n}$ be equal to zero. On the other hand the free streamline condition of constant speed is satisfied on all the free surfaces. The main difficulty of such free surface flow problems is connected to nonlinearity of the boundary conditions which are, moreover, to be satisfied on the surfaces of unknown geometry.

Paper presented at the RTO AVT Lecture Series on "Supercavitating Flows", held at the von Kármán Institute (VKI) in Brussels, Belgium, 12-16 February 2001, and published in RTO EN-010.

2. Model problem – cavitating plate at zero cavitation number

Consider a 2D cavitating flat plate in a uniform flow with velocity absolute value at infinity v_∞ and angle of attack α, see figure 1. The origin of the Cartesian coordinate system is taken at the plate's trailing edge, x–axis being directed downstream and y upwards. There is an incident stream with speed v_∞ coming from the left. The region occupied by the fluid is bounded by the plate $[AB]$ and by two semi-infinite free surfaces (AC) and (BC).

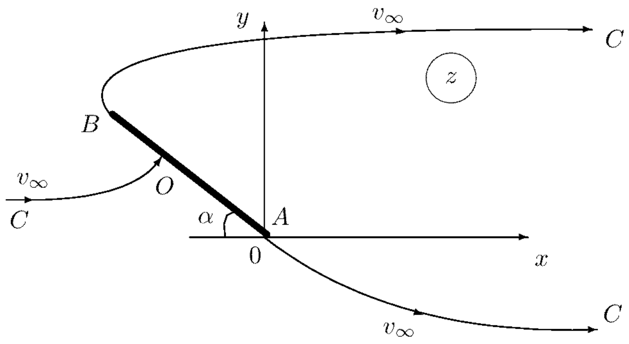

Figure 1: Flow pattern for the cavitating flat plate at zero cavitation number.

Assume the velocity on the streamlines v_0 to be equal to v_∞ and therefore the cavitation number

$$\sigma = \frac{p_\infty - p_0}{\rho v_\infty^2/2} = 0\,,$$

where p_∞ and p_0 denote the pressure at infinity and within the cavity correspondingly. It readily follows from the Bernoulli equation for an upstream reference point and free surface-boundary point

$$p_0 + \frac{1}{2}\rho v_0^2 = p_\infty + \frac{1}{2}\rho v_\infty^2\,.$$

We render all the parameters of the problem nondimensional by a suitable choice of scale so that $v_\infty = 1$ and plate length equals to unit $l = 1$.

2.1. Kirchhoff method

To solve the problem under consideration, it is convenient to determine the function

$$\zeta(w) = v_\infty \frac{\mathrm{d}z}{\mathrm{d}w} \tag{2}$$

rather than the complex potential $w(z) = \varphi + \mathrm{i}\psi$ itself. Indeed, if function $\zeta(w)$ is found then $z(w)$ is easily defined as

$$z = \frac{1}{v_\infty}\int \frac{v_\infty \mathrm{d}z}{\mathrm{d}w}\,\mathrm{d}w = \frac{1}{v_\infty}\int \zeta \mathrm{d}w\,. \tag{3}$$

Note that it's often no sense in reversing function $z(w)$ to find $w(z)$, all the more so that the operation is rather cumbersome.
So the erux of the problem is to determine w and ζ as functions of the same single variable so that equation (3) can be integrated. Since function $\zeta(w)$ defines a conformal mapping of w-plane onto the ζ-plane, one has to determine these regions and map one to another.
If one choses the stream function ψ to have the value zero on the dividing streamline $(COAC)$ and $(COBC)$ and since all the streamlines are by definition lines of constant ψ and therefore become lines parallel to the φ-axis, then the w-plane has the form shown in figure 2. The plate–cavity combination appears as a semi-infinite slit in this plane. The velocity potential φ at the edges of the plate (points A and B) has the values φ_A and φ_B. It is easy to see that

$$\zeta = \frac{v_\infty}{v}\mathrm{e}^{\mathrm{i}\,\theta}\,.$$

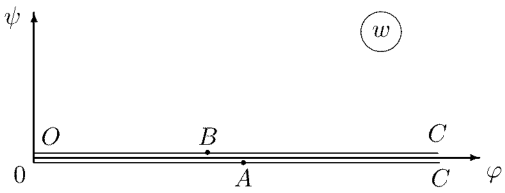

Figure 2: The complex velocity potential w.

On $[OA]$ one has $\theta = -\alpha$ and v_∞/v ratio varies from infinity at point O to unit value at point A. On the ζ-plane interval $[OA]$ corresponds to a semi-infinite line inclined at angle $-\alpha$ to the positive direction of x-axis and starting from the point $\zeta_A = \mathrm{e}^{-\mathrm{i}\alpha}$. A symmetrical line inclined at angle $\pi - \alpha$ and starting from the point $\zeta_B = -\mathrm{e}^{-\mathrm{i}\alpha}$ corresponds to another part of the plate $[BO]$. Angle θ varies from $-\alpha$ to zero and ratio $v_\infty/v = 1$ on streamline (AC). Further on, on streamline (CA) angle θ increases from zero to $\pi - \alpha$ and v_∞/v remains the same. Thus free surface on the z-plane becomes a half-circle of unit radius on the ζ-plane. The ζ-plane is depicted in figure 3. A stagnation point C on the z-plane corresponds to $\zeta_C = 1$.

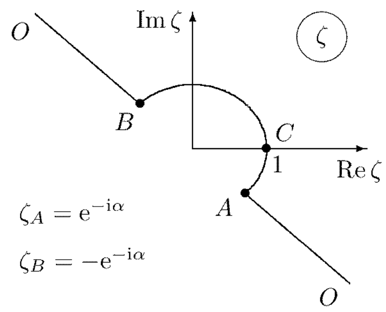

Figure 3: The ζ plane.

To obtain a solution to the problem under consideration it is sufficient to transform w-plane onto ζ-plane, both being given in figures 2 and 3. Such a problem is fairly simple and much easier than that of determining harmonic function φ in the region z with unknown free boundaries. The substitution of the complicated boundary value problem by a simple problem of conformal mapping is the core of the Kirchhoff method.
The appropriate $w - \zeta$ mapping can be found step by step.
First, one transforms ζ-plane to the first quadrant of the auxiliary τ-plane, see figure 4, by conformal mapping

$$\tau = a\,\frac{\zeta - \zeta_A}{\zeta + \zeta_A} = a\,\frac{\zeta - \mathrm{e}^{-\mathrm{i}\alpha}}{\zeta + \mathrm{e}^{-\mathrm{i}\alpha}}\,.$$

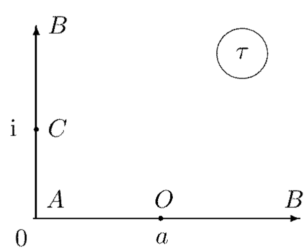

Figure 4: The τ plane.

With the correspondence between ζ and τ planes we have for point C

$$\mathrm{i} = a\,\frac{1 - \mathrm{e}^{-\mathrm{i}\alpha}}{1 + \mathrm{e}^{-\mathrm{i}\alpha}}\,, \quad \text{therefore} \quad a = \cot\frac{\alpha}{2}\,.$$

Second, a new variable $\tau_1 = \tau^2$ varies in the upper semi-plane, see figure 5.

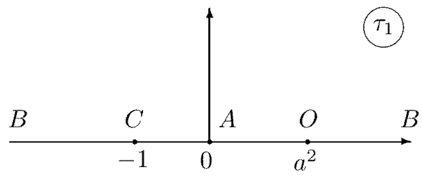

Figure 5: The τ_1 plane.

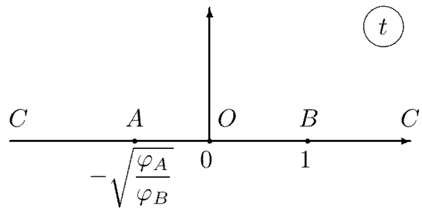

Figure 6: The t plane.

On the other hand, transformation $t = \sqrt{w/\varphi_B}$ transforms w-plane onto the upper semi-plane, see figure 6. Finally, all we need do is to transform t-plane onto τ_1-plane by using formula

$$\tau_1 = \frac{a^2 + t}{1 - t}\,,$$

which results in

$$\zeta = \zeta_A \frac{a\sqrt{\sqrt{\varphi_B} - \sqrt{w}} + \sqrt{\sqrt{\varphi_A} + \sqrt{w}}}{a\sqrt{\sqrt{\varphi_B} - \sqrt{w}} - \sqrt{\sqrt{\varphi_A} + \sqrt{w}}}\,, \tag{4}$$

where radical $\sqrt{w}$ is positive on the interval $[OB]$ and negative on $[OA]$, $\zeta_A = -\zeta_B = \mathrm{e}^{-\mathrm{i}\alpha}$ and $a = \cot(\alpha/2)$. Referring to figures 5 and 6, it is obvious that

$$\sqrt{\frac{\varphi_A}{\varphi_B}} = a^2 = \cot^2\frac{\alpha}{2}\,,$$

while parameter φ_A remains still unknown.
To determine the unknown parameter we have a condition connected with plate length, see figure 1:

$$z_B = l\,\mathrm{e}^{\mathrm{i}(\pi-\alpha)}\,.$$

That is why from equation (3) we find

$$z_B = \frac{1}{v_\infty}\int\limits_{\varphi_A}^{\varphi_B}\zeta\mathrm{d}w = \frac{\zeta_A}{v_\infty}\int\limits_{\varphi_A}^{\varphi_B}\frac{a\sqrt{\sqrt{\varphi_B} - \sqrt{w}} + \sqrt{\sqrt{\varphi_A} + \sqrt{w}}}{a\sqrt{\sqrt{\varphi_B} - \sqrt{w}} - \sqrt{\sqrt{\varphi_A} + \sqrt{w}}}\,\mathrm{d}w\,.$$

It is a complicated integration but it is seen that

$$\frac{1}{v_\infty}\int\limits_{\varphi_A}^{\varphi_B}\zeta\mathrm{d}w = \frac{1}{v_\infty}\int\limits_0^\infty \zeta(\tau)\frac{\mathrm{d}w}{\mathrm{d}t}(\tau)\frac{\mathrm{d}t}{\mathrm{d}\tau_1}(\tau)\frac{\mathrm{d}\tau_1}{\mathrm{d}\tau}(\tau)\,\mathrm{d}\tau\,,$$

where

$$\zeta(\tau) = \zeta_A\frac{a+\tau}{a-\tau}\,; \qquad \frac{\mathrm{d}w}{\mathrm{d}t}(\tau) = 2\varphi_B\frac{\tau^2 - a^2}{\tau^2+1}\,; \qquad \frac{\mathrm{d}t}{\mathrm{d}\tau_1}(\tau) = \frac{1+a^2}{(\tau^2+1)^2}\,; \qquad \frac{\mathrm{d}\tau_1}{\mathrm{d}\tau}(\tau) = 2\tau$$

and the following relation holds:

$$l\,\mathrm{e}^{\mathrm{i}(\pi-\alpha)} = -4\zeta_A\frac{\varphi_B}{v_\infty}(1+a^2)\int\limits_0^\infty\frac{\tau(\tau+a)^2}{(\tau^2+1)^3}\,\mathrm{d}\tau\,.$$

As a result one obtains

$$\varphi_B = l\, v_\infty \, \frac{2\sin^4 \dfrac{\alpha}{2}}{2 + \pi \sin \dfrac{\alpha}{2} \cos \dfrac{\alpha}{2}} \,. \tag{5}$$

The total force F acting on the cavitating flat plate is calculated by integration of pressure distribution coefficient C_p

$$F = -\frac{\mathrm{i}\rho v_\infty^2}{2} \int\limits_{z_A}^{z_B} C_p \, \mathrm{d}z \,, \tag{6}$$

where

$$C_p = \frac{p - p_\infty}{\rho v_\infty^2/2}$$

is given by Bernoulli equation

$$p - p_\infty = \frac{\rho}{2}\,(v_\infty^2 - v^2) = \frac{\rho v_\infty^2}{2}\left(1 - \frac{1}{|\zeta|^2}\right) .$$

Taking into account that $\zeta \mathrm{d}w = v_\infty \mathrm{d}z$ on the plate, we find

$$F = -\frac{\mathrm{i}\rho v_\infty}{2} \int\limits_{\varphi_A}^{\varphi_B} \left(1 - \frac{1}{|\zeta|^2}\right) \zeta \, \mathrm{d}w \,.$$

Using the same technique as above, we arrive at the following expression for the force coefficient

$$C_F = C_D + \mathrm{i}\, C_L = \frac{F}{\rho v_\infty^2 l/2} = \, = \mathrm{e}^{\mathrm{i}(\pi/2-\alpha)} \, \frac{2\pi \sin\alpha}{4 + \pi \sin\alpha} \,. \tag{7}$$

It is obvious from what was dealt with above that Kirchhoff method is quite complicated even for such a simple problem of a cavitating flat plate at zero cavitation number. Nevertheless it was the first to enable one to solve free surface flow problems. A new important step in this direction was made by N. Zhukovsky in 1890 [26]. He proposed a new approach to solution of theory of jets in an ideal fluid problem, which was significantly improved as compared to Kirchhoff one.

2.2. Zhukovsky method

A method proposed by Zhukovsky can be applied to the 2D free surface flow problems with 1-connected flow region bounded by only straight solid boundaries. A new function (so-called Zhukovsky function)

$$\omega = \log \zeta = -\log \frac{\mathrm{d}w}{v_\infty \mathrm{d}z} = -\log \frac{v}{v_\infty} + \mathrm{i}\theta \tag{8}$$

is introduced instead of Kirchhoff's $\zeta = v_\infty \mathrm{d}z/\mathrm{d}w$ function. Another improvement, following to Zhukovsky method, is that one has to connect functions w and ω through variable t varying in the upper semi-plane rather than directly determine a relationship between the two functions. Being aware of dependences $w(t)$ and $\omega(t)$, one can eliminate variable t and hence get a Kirchhoff's solution to the problem. However, this operation is quite aimless because of the following relationships

$$\zeta = \mathrm{e}^{\omega(t)} \,, \quad z = \frac{1}{v_\infty} \int \zeta \mathrm{d}w = \frac{1}{v_\infty} \int \mathrm{e}^{\omega} \frac{\mathrm{d}w}{\mathrm{d}t} \, \mathrm{d}t \,, \tag{9}$$

enabling one to solve the problem.

The meaning of a novel approach is as follows. It was shown in the previous section that w-plane is bounded by horizontal lines ψ = const which are images of streamlines in the physical z-plane. On the other hand, ω-plane is bounded by vertical and horizontal lines in the case of straight solid walls in z-plane. Indeed, such solid boundaries are lines of constant θ, that is imaginary part of ω, while free surfaces, where pressure (speed) constancy condition is satisfied, are lines of constant real part of ω. Therefore w and ω planes are bounded by polygonal segments. As a result, both w and ω domains can be transformed onto the upper half of auxiliary t-plane by using Schwarz–Christoffel transformation.

Schwarz–Christoffel formula allows one to transform half-plane t into interiority of a polygon with n vertexes (n-gon) on the ω-plane. It is assumed that each vertex is of angle $\alpha_i \le 2\pi$, $i = 1, \dots, n$ and

$$\sum_{i=1}^{n} \alpha_i = \pi(n-2)\,.$$

It this case the formula is as follows:

$$\omega = C_1 \int \prod_{i=1}^{n} (t - t_i)^{\alpha_i/\pi - 1}\, \mathrm{d}t + C_2\,, \tag{10}$$

where points t_i correspond to the polygon vertexes. If $t_k = \infty$ then term $(t - t_k)^{\alpha_k/\pi - 1}$ in the integrand should be excluded. Right-hand side of formula (10) includes $2n+3$ unknowns, namely, n real parameters $t_1, \dots, t_n$, $n-1$ of n real parameters α_i, $i = 1, \dots, n$ and four real parameters in two complex ones C_1 and C_2. An n-gon is totally described by $2n$ coordinates of its vertexes on ω-plane what generates $2n$ conditions. Therefore one can arbitrarily chose three of $2n+3$ unknown parameters, say t_1, t_2 and t_3. This fact is a version of a well-known postulate that any conformal mapping is determined by three preassigned boundary points.
Consider the same model problem of cavitating flat plate at zero cavitation number formulated above.
We already have got the relationship between w and t planes, see figures 2 and 6:

$$w = \varphi_B t^2\,. \tag{11}$$

It is to be emphasized that in a neighborhood of point $t = 0$

$$\frac{\mathrm{d}w}{\mathrm{d}t} = 2\varphi_B t = \mathrm{O}(t)\,,$$

where $\mathrm{O}(t)$ denotes a quantity which is of the same order of magnitude as t.
On streamline (AC), $v = v_\infty$ wherefrom $\log(v_\infty/v) = 0$ while angle θ increases from $-\alpha$ up to value zero and therefore $\omega = -\log(v/v_\infty) + \mathrm{i}\theta$ varies from $-\mathrm{i}\alpha$ to 0. On streamline (BC) function ω varies from $\mathrm{i}(\pi - \alpha)$ to zero. On the interval $[OA]$ of the cavitating plate imaginary part of ω is a constant: $\mathrm{Im}\,\omega = -\alpha$ while its real part decreases from infinity down to zero. At point C angle θ varies in a jump–like manner from $-\alpha$ up to $\pi - \alpha$, see figure 7.

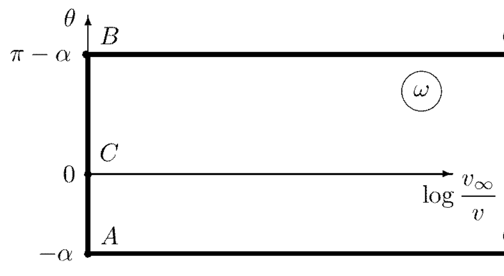

Figure 7: The ω plane.

Thus, domain on ω-plane corresponding to a region occupied by the fluid on z-plane is a triangle OAB with zero angle at infinite point O and right angles at points A and B. Assume that $t_O = 0$, $t_B = 1$ and $t_C = \infty$, see figure 6, (a postulate about three preassigned boundary points). Then Schwarz–Christoffel formula (10) reduces to

$$\omega(t) = C_1 \int_1^t \frac{\mathrm{d}t}{t\sqrt{(t-1)(t+a^2)}} + C_2\,,$$

where $a^2 = \sqrt{\varphi_A/\varphi_B}$. At point B, $\omega(1) = \mathrm{i}(\pi - \alpha)$, that is why $C_2 = \mathrm{i}(\pi - \alpha)$. The integral above yields

$$\omega(t) = C_1 \left(-\frac{1}{a} \arcsin \frac{(1-a^2)t + 2a^2}{(1+a^2)\,t} + \frac{\pi}{2a} \right) + \mathrm{i}(\pi - \alpha)$$

At point A, $\omega(-a^2) = -\mathrm{i}\alpha$, that is why $C_1 = -\mathrm{i}a$ and, finally,

$$\omega(t) = \mathrm{i}\,\arcsin\frac{(1-a^2)\,t + 2a^2}{(1+a^2)\,t} + \mathrm{i}\left(\frac{\pi}{2} - \alpha\right) . \tag{12}$$

Using a relationship between the arc sine function and natural logarithm, expression (12) can be re-written as

$$\omega(t) = \log\frac{(1+a^2)\,t}{\mathrm{i}a^2(t-2) - \mathrm{i}t + 2a\sqrt{(t-1)(t+a^2)}} + \mathrm{i}\left(\frac{\pi}{2} - \alpha\right) . \tag{13}$$

The system of two nonlinear conditions

$$\omega(\infty) = 0\,; \quad \text{and} \quad z_B = l\,\mathrm{e}^{\mathrm{i}(\pi-\alpha)}$$

allows the two unknowns φ_A and φ_B to be derived. The former gives

$$a = \sqrt[4]{\frac{\varphi_A}{\varphi_B}} = \cot\frac{\alpha}{2}\,,$$

while the latter merit some additional explanation. Equations (9), (11) and (13) can be combined to give

$$z_B = \frac{1}{v_\infty}\int\limits_{-a^2}^{1} \frac{v_\infty \mathrm{d}z}{\mathrm{d}w}\cdot\frac{\mathrm{d}w}{\mathrm{d}t}\,\mathrm{d}t\,,$$

wherefrom

$$l = \frac{\varphi_B}{v_\infty}\left(1 + \frac{\pi}{4}\sin\alpha\right)\sin^{-4}\frac{\alpha}{2}\,,$$

what coincides with expression (5).
The total force F acting on the cavitating flat plate is calculated in a manner of the previous section

$$F = -\frac{\mathrm{i}\rho v_\infty^2}{2}\int\limits_{z_A}^{z_B} C_p\,\mathrm{d}z = -\frac{\mathrm{i}\rho v_\infty^2}{2}\int\limits_{-a^2}^{1} C_p\,\frac{\mathrm{d}z}{\mathrm{d}w}\,\frac{\mathrm{d}w}{\mathrm{d}t}\,\mathrm{d}t\,,$$

where

$$C_p = 1 - \left|\mathrm{e}^{-2\omega(t)}\right| \quad \text{and} \quad \frac{\mathrm{d}z}{\mathrm{d}w} = \frac{1}{v_\infty}\,\mathrm{e}^{\omega(t)}$$

and give the same result just as above.
Nowadays some modifications of Zhukovsky method (so-called hodograph method) is widely used and, for instance, was applied to flow domains bounded by not only polygonal segments but by smooth curves as well [11]. Note that the first adaptation of the classical Zhukovsky (hodograph) method to flow past a body with a curved topography was considered by Levi-Civita [15] and Villat [24].

2.3. Mixed boundary value problem method

This method seems to be easy to understand and is very often used. The solution is derived in two steps. The first one allows us to find complex potential $w(\zeta)$ or its derivative $\mathrm{d}w/\mathrm{d}\zeta$ by mapping w-plane onto the preassigned auxiliary ζ-domain (in a manner of Zhukovsky method). In the case of a cavitating plate, using w-plane (depicted in figure 2) and auxiliary ζ upper half-plane, see figure 8, we get

$$\frac{\mathrm{d}w}{\mathrm{d}\zeta} = N\zeta\,, \tag{14}$$

where N denotes a real parameter to be determined.
The second step is formulation of a mixed boundary value problem for Levi-Civita [15] function

$$\omega_{\mathrm{LC}} = \mathrm{i}\log\frac{\mathrm{d}w}{v_\infty \mathrm{d}z} = \theta + \mathrm{i}\log\frac{v}{v_\infty} = \theta + \mathrm{i}\tau\,. \tag{15}$$

It is obvious that Levi-Civita function is a product of the imaginary unit $-\mathrm{i}$ and Zhukovsky function (8).

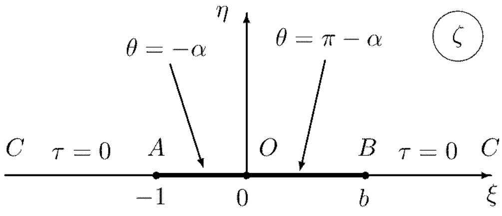

Figure 8: The ζ plane.

With the correspondence between physical z and auxiliary ζ planes the mixed boundary value problem, see figure 8, is:

$$\tau = 0 \quad \text{as} \quad \xi \le -1 \quad \text{and} \quad \xi \ge b$$

$$\theta = \begin{cases} -\alpha\,, & \text{as} \quad -1 \le \xi \le 0\,; \\ \pi - \alpha\,, & \text{as} \quad 0 \le \xi \le b\,. \end{cases}$$

where b is a coordinate (to be determined) of image of point B in ζ-plane.
The solution to the mixed boundary value problem (Riemann–Hilbert problem [17]) is obtained via formulae proposed by Keldysh and Sedov [10]. The formulae are extremely useful for both nonlinear and linearized cavitating flow problem and used also in many other fields of fluid mechanics and so merit some additional explanation.
Below presented are Keldysh–Sedov formulae in the case of the upper semi-plane. It can be readily re-written for another ζ-regions like circle, *etc.*
The Riemann–Hilbert problem is formulated for a function $f(z) = u(z) + \mathrm{i}v(z)$ which is holomorphic in upper semi-plane $z = x + \mathrm{i}y$, its real part $u(z)$ being given on a set of segments $[a_k\,, b_k]$, $k = 1, \ldots, N$ on the x-axis and its imaginary part $v(z)$ being given on the other portion of x-axis. Following to [10], three types of solution to the problem exist:
⋆ solution unbounded at all points a_k, b_k ($\infty - \infty$ class):

$$f(z) = \frac{1}{\pi \mathrm{i}\, R(z)} \left\{ \sum_{k=1}^{N} \int_{a_k}^{b_k} \frac{u(\zeta) R(\zeta)}{\zeta - z}\,\mathrm{d}\zeta + \mathrm{i} \sum_{k=1}^{N} \int_{b_k}^{a_{k+1}} \frac{v(\zeta) R(\zeta)}{\zeta - z}\,\mathrm{d}\zeta + P_N(x) \right\}, \tag{16}$$

where $a_1 = a_N + 1$, $P_N(x)$ is a polynomial of degree N or $N-1$ depending on behaviour of $f(z)$ at infinity, and

$$R(z) = \sqrt{\prod_{k=1}^{N} (z - a_k)(z - b_k)}\ ;$$

⋆ solution bounded at all points a_k and unbounded at all b_k ($0 - \infty$ class) as $f(\infty) = 0$:

$$f(z) = \frac{1}{\pi \mathrm{i}} \frac{R_a(z)}{R_b(z)} \left\{ \sum_{k=1}^{N} \int_{a_k}^{b_k} \frac{u(\zeta)}{\zeta - z} \frac{R_b(\zeta)}{R_a(\zeta)}\,\mathrm{d}\zeta + \mathrm{i} \sum_{k=1}^{N} \int_{b_k}^{a_{k+1}} \frac{v(\zeta)}{\zeta - z} \frac{R_b(\zeta)}{R_a(\zeta)}\,\mathrm{d}\zeta \right\}, \tag{17}$$

where

$$R_a(z) = \sqrt{\prod_{k=1}^{N} (z - a_k)}\ ,\ R_b(z) = \sqrt{\prod_{k=1}^{N} (z - b_k)}\ ;$$

⋆ solution bounded at all points a_k, b_k ($0 - 0$ class) as $f(\infty) = 0$:

$$f(z) = \frac{R(z)}{\pi \mathrm{i}} \left\{ \sum_{k=1}^{N} \int_{a_k}^{b_k} \frac{u(\zeta)}{\zeta - z} \frac{\mathrm{d}\zeta}{R(z)} + \mathrm{i} \sum_{k=1}^{N} \int_{b_k}^{a_{k+1}} \frac{v(\zeta)}{\zeta - z} \frac{\mathrm{d}\zeta}{R(z)} \right\}. \tag{18}$$

The latter solution exists if and only if the relationship holds

$$\sum_{k=1}^{N} \int_{a_k}^{b_k} \frac{u(\zeta)}{R(z)}\,\mathrm{d}\zeta + \mathrm{i}\sum_{k=1}^{N} \int_{b_k}^{a_{k+1}} \frac{v(\zeta)}{R(z)}\,\mathrm{d}\zeta = 0\,. \tag{19}$$

Coming back to solution of the model problem, one has to use the latter formula (18) along with additional condition (19) to derive $\omega_{\mathrm{LC}}(\zeta)$ for the free surfaces detaches smoothly from the edges of the plate A and B and therefore velocity absolute value is finite there. Thus, using boundary conditions, one arrives at the expression

$$\omega_{\mathrm{LC}}(\zeta) = -\frac{1}{\pi}\sqrt{(\zeta+1)(\zeta-b)} \int_{-1}^{b} \frac{\theta(t)}{\sqrt{(1+t)(b-t)}}\,\frac{\mathrm{d}t}{t-\zeta}$$

which yields

$$\omega_{\mathrm{LC}}(\zeta) = \mathrm{i}\log\frac{\sqrt{b-\zeta}\,-\sqrt{b}\,\sqrt{1+\zeta}}{\sqrt{b-\zeta}\,+\sqrt{b}\,\sqrt{1+\zeta}} - \alpha\,. \tag{20}$$

Condition (19) gives

$$-\alpha + \frac{\pi}{2} + \arcsin\frac{1-b}{1+b} = 0\,, \tag{21}$$

or $b = \tan^2(\alpha/2)$.
Expressions (14) and (20) give analytical solution to the problem. Conformal mapping is

$$z(\zeta) = \int_{-1}^{\zeta} \frac{\mathrm{d}w}{\mathrm{d}\zeta}\,\exp(\mathrm{i}\,\omega_{\mathrm{LC}}(\zeta))\,\mathrm{d}\zeta\,.$$

Condition $z_B = z(b) = l\,\mathrm{e}^{\mathrm{i}(\pi-\alpha)}$ enables one to find unknown parameter N in the form

$$N = \frac{8\,l\,v_\infty}{4+\pi\sin\alpha}\,\cos^4\frac{\alpha}{2}\,.$$

Taking into account that $\varphi_B = w(b) = Nb^2/2$, one obtains the relationship coinciding with (5).
Note that the method of mixed boundary value problem is effectively applicable to free streamline problems with curved topography of an obstacle. Actually in this case just $\theta(z)$ is given and function $\theta(\zeta) = \mathrm{Re}\,\omega_{\mathrm{LC}}$ is unknown on ξ-axis because it is not a constant and the relationship $z(\zeta)$ is yet to be found. A special integral equation is formulated is a manner presented below in section 5.

2.4. Chaplygin method of singular points.

Chaplygin method of singular points based on the idea of determining of a holomorphic function in flow domain and therefore in auxiliary complex plane, the function's zeros and poles being known and Liuville's theorem applied. The theorem guarantees uniqueness of a solution obtained. The auxiliary region, being the image of the flow region, is chosen to be bounded by straight lines, segments and circular arcs so that one could cover all the auxiliary complex plane by mirror images of the initial domain of auxiliary variable range. Quadrants, rectangles, disk sectors, *etc.* are usually used as such ranges depending on special features of a problem under consideration.
To get the desired solution to the problem, one has to determine two analytical functions, namely derivatives of the complex potential with respect of physical z and auxiliary u variables as functions of u:

$$\frac{\mathrm{d}w}{\mathrm{d}u}(u) \quad \text{and} \quad \chi(u) = \frac{\mathrm{d}w}{v_\infty \mathrm{d}z} = \frac{v}{v_\infty}\,\mathrm{e}^{-\mathrm{i}\theta}\,.$$

Conformal mapping which transform flow region into an auxiliary one, then, is

$$z(u) = \int \frac{\mathrm{d}z}{\mathrm{d}u}\,\mathrm{d}u = \int \frac{\mathrm{d}z}{\mathrm{d}w}\cdot\frac{\mathrm{d}w}{\mathrm{d}u}\,\mathrm{d}u\,.$$

Chaplygin practically introduced some basic postulates (though he had never published them) which of considerable assistance while finding a derivative of w. If $w(u)$ transforms w-region into u-region then:

- a third-order pole of function $\mathrm{d}w/\mathrm{d}u$ corresponds to a jet of infinite width;
- a second-order pole of function $\mathrm{d}w/\mathrm{d}u$ corresponds to a jet of semi-infinite width (that is, jet bounded by a free or rigid surface). Such a semi-infinite jet is called "ocean" in [1];
- a first order pole of function $\mathrm{d}w/\mathrm{d}u$ corresponds to a jet of a finite width;
- in points of auxiliary u-plane corresponding to a vertex of solid wall bounding flow region, function $\chi(u)$ has an exponential zero or singularity depending on angle of the vertex;
- function $\mathrm{d}w/\mathrm{d}u$ has a zero, more precisely behaves like $\mathrm{O}(t-t_0)$, in point t_0 corresponding to a point on physical z-plane where a streamline divides. It can be a stagnation point or leading edge of stagnation zone, *etc.*;
- conformality condition for derivative of complex potential $\mathrm{d}w/\mathrm{d}u$ often does not satisfied at points where free streamlines spring from the body (jet detachment points). To avoid this, one has to chose a form of auxiliary u-region so that its boundary in points corresponding to jet detachment points makes right angles.
- function $\mathrm{d}w/\mathrm{d}u$ has a zero or pole (depending on specific features of the problem) in points where conformality condition does not satisfied for $w-u$ transformation.

Now apply the method of singular point to the model problem considered above with Kirchhoff and Zhukovsky methods, see figure 1. The first quadrant of auxiliary u-plane is chosen as a region to be transformed into the flow domain, see figure 9. As is customary, the coordinates of three points O, B and C on the boundary of the first quadrant of u-plane are chosen arbitrarily and location of point A is to be found. Actually the figure coincides with figure 4 for τ-plane used in section 'Kirchhoff method'.

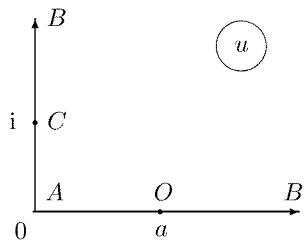

Figure 9: The auxiliary u-plane.

Following to the main postulate of Chaplygin method, find both derivatives of w with respect of z and u as functions of $u=\xi+\mathrm{i}\eta$.

Function $\chi(u)=\mathrm{d}w/(v_\infty \mathrm{d}z)$ is a bounded function in the flow region. It becomes zero in just one stagnation point O, where $u=a$. On the plate $[AOB]$ corresponding to real ξ-axis of u

$$\arg\chi(u)=-\theta(u)=\begin{cases}\alpha\,, & \text{as}\quad 0\le u\le a\,;\\ \alpha-\pi\,, & \text{as}\quad a\le u\le\infty\,,\end{cases}$$

while on free streamlines (AC) and (BC) (imaginary η-axis of u), $|\chi(u)|=1$. Due to the Schwarz principle of symmetry, all the zeros of $\chi(u)$ becomes zeros in corresponding mirror points with respect to the imaginary axis and poles in corresponding mirror points with respect to the real axis. That is why, being continued onto the whole u-plane, function $\chi(u)$ has a zero at point $u=a$ and a pole in $u=-a$. Then the following ratio

$$\left(\frac{u-a}{u+a}\Big/\chi(u)\right)$$

is a bounded and holomorphic function everywhere including infinite point and therefore is a constant due to Liouville theorem. Thus

$$\chi(u)=\frac{\mathrm{d}w}{v_\infty\mathrm{d}z}=\frac{v}{v_\infty}\,\mathrm{e}^{-\mathrm{i}\theta}=N_1\,\frac{u-a}{u+a}\,,$$

where N_1 is a constant defined by condition $\chi(0)=\mathrm{e}^{\mathrm{i}\alpha}$ which yields $N_1=\mathrm{e}^{\mathrm{i}(\alpha-\pi)}$ and finally

$$\chi(u)=\frac{\mathrm{d}w}{v_\infty\mathrm{d}z}=v\,\mathrm{e}^{-\mathrm{i}\theta}=\mathrm{e}^{\mathrm{i}(\alpha-\pi)}\,\frac{u-a}{u+a}\,. \tag{22}$$

Derivative $\mathrm{d}w/\mathrm{d}u$ has a third-order pole at infinite point C, where $u = \mathrm{i}$, and zeros at stagnation point O ($u = a$) and at point A ($u = 0$) where conformality condition does not satisfied. There is no other pole or zero in the first quadrant of u-plane for $\mathrm{d}w/\mathrm{d}u$. On continuation of the function into the whole u-plane by mirror mapping of the first quadrant with respect to real and imaginary axes of u (where $\psi = \operatorname{Im} w = \mathrm{const}$) we get additional third-order pole at point $u = -\mathrm{i}$ and zero at point $u = -a$. Just as above, the ratio

$$\left(\frac{u(u^2 - a^2)}{(u^2+1)^3} \Big/ \frac{\mathrm{d}w}{\mathrm{d}u} \right)$$

is a bounded and holomorphic function everywhere including infinite point and therefore is a constant due to Liouville theorem. Therefore

$$\frac{\mathrm{d}w}{\mathrm{d}u} = N\, \frac{u(u^2 - a^2)}{(u^2+1)^3}\,, \tag{23}$$

where N is a real constant.

Two expressions (22) and (23) allows one to solve the problem under consideration which has two unknown parameters a and N. With the correspondence between z-plane and u-plane, the conformal mapping transforming latter into former is as follows

$$z(u) = \int_0^u \frac{\mathrm{d}z}{\mathrm{d}w}\,\frac{\mathrm{d}w}{\mathrm{d}u}\,\mathrm{d}u = \frac{N}{v_\infty} \mathrm{e}^{\mathrm{i}(\pi-\alpha)} \int_0^u \frac{u\,(u+a)^2}{(u^2+1)^3}\,\mathrm{d}u\,. \tag{24}$$

Note that (23) and (24) can be readily integrated to give

$$w(u) = \frac{N}{4(1+a^2)} \left(\frac{u^2 - a^2}{u^2+1} \right)^2 ; \tag{25}$$

$$z(u) = \frac{N}{4v_\infty}\, \mathrm{e}^{\mathrm{i}(\pi-\alpha)} \left(a \arctan u + \frac{u\big(u^3 + a(u^2-1) + a^2 u(u^2+2)\big)}{(1+u^2)^2} \right). \tag{26}$$

Two conditions

$$\chi(\mathrm{i}) = 1\,, \quad \text{and} \quad z_B = z(\infty) = l\, \mathrm{e}^{\mathrm{i}(\pi-\alpha)}$$

generate the following nonlinear system in two unknowns

$$\mathrm{e}^{\mathrm{i}(\alpha-\pi)}\, \frac{\mathrm{i} - a}{\mathrm{i} + a} = 1\,; \quad \frac{N}{v_\infty} \int_0^u \frac{u\,(u+a)^2}{(u^2+1)^3}\,\mathrm{d}u = l\, \mathrm{e}^{\mathrm{i}(\pi-\alpha)}$$

which can be rewritten to yield

$$a = \cot\frac{\alpha}{2}\,; \quad N = v_\infty\, l\, \sin^2\frac{\alpha}{2}\, \frac{16}{4 + \pi \sin\alpha}\,. \tag{27}$$

It is seen from (25) that

$$\varphi_B = w(\infty) = \frac{1}{4} N \sin^2\frac{\alpha}{2}\,, \qquad \varphi_A = w(0) = \frac{1}{4} N \cos^2\frac{\alpha}{2}\, \cot^2\frac{\alpha}{2}\,,$$

what coincides with expression (5).

Taking into account that on the plate $[AB]$

$$C_p = 1 - \frac{v^2}{v_\infty^2} = 1 - |\chi(u)|^2 = 1 - \frac{1}{v_\infty^2}\,\frac{\mathrm{d}w}{\mathrm{d}z}\,\overline{\frac{\mathrm{d}w}{\mathrm{d}z}}\,, \tag{28}$$

the total force F can be derived (6):

$$F = -\frac{\mathrm{i}\rho v_\infty^2}{2} \int_{z_A}^{z_B} C_p\, \mathrm{d}z$$

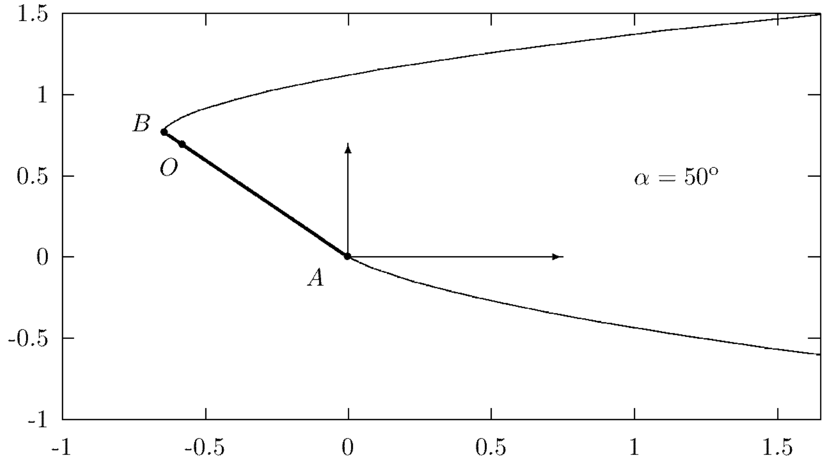

Figure 10: Flow pattern for the cavitating flat plate at zero cavitation number and $\alpha = 50^{\rm o}$.

to give the force coefficient in the form

$$C_F = -\frac{\rm i}{l}\int\limits_0^\infty \left(1 - \frac{1}{v_\infty^2}\frac{{\rm d}w}{{\rm d}z}\overline{\frac{{\rm d}w}{{\rm d}z}}\right)\frac{{\rm d}z}{{\rm d}w}\frac{{\rm d}w}{{\rm d}u}{\rm d}u = -\frac{\rm i}{l}\left(\int\limits_0^\infty \frac{{\rm d}z}{{\rm d}u}\,{\rm d}u - \frac{1}{v_\infty^2}\int\limits_0^\infty \frac{{\rm d}w}{{\rm d}u}\overline{\frac{{\rm d}w}{{\rm d}z}}\,{\rm d}u\right) = $$
$$= -\frac{\rm i}{l}\left(z_B - z_A - \frac{1}{v_\infty^2}\int\limits_0^\infty \frac{{\rm d}w}{{\rm d}u}\overline{\frac{{\rm d}w}{{\rm d}z}}\,{\rm d}u\right) . \tag{29}$$

Using expressions (22) and (23) one easily obtains coefficient C_F, see (7).

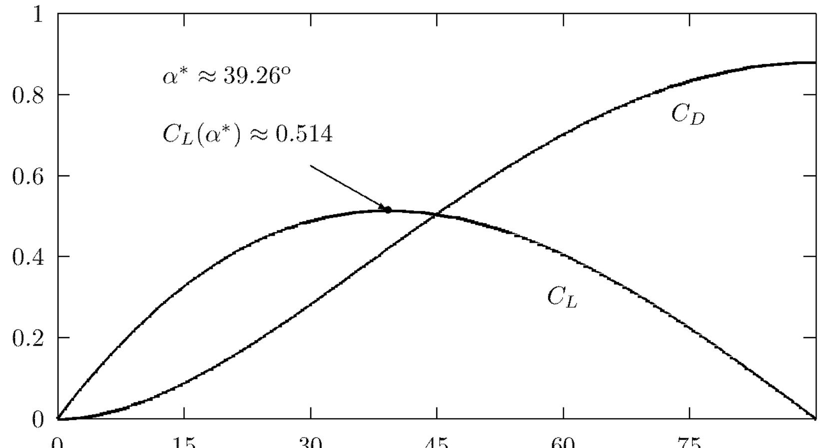

Figure 11: Lift and drag coefficients C_L and C_D versus angle of attack α.

Flow pattern $z(u) = x + {\rm i}y$ can be calculated in a parametric form from expression (24) or (26). The coordinates of the stagnation point O are as follows

$$z_O = z(a) = l\,{\rm e}^{{\rm i}(\pi-\alpha)}\,\frac{\sin\alpha}{4+\pi\sin\alpha}\,\left(\pi - \alpha + 2\cot\frac{\alpha}{2} + \sin 2\alpha\right) .$$

Flow pattern for the cavitating plate at $\alpha = 50^{\rm o}$ and zero cavitation number is depicted in figure 10. Stagnation point O is shown as well. Figure 11 illustrates behaviour of lift and drag coefficients C_L and C_D versus angle of attack. It is obvious that the lift equals to zero for $\alpha = 0$ and $\pi/2$. The corresponding curve attains its maximum value $C_L \approx 0.514$ at $\alpha^* \approx 39.26^o$. Hydrodynamic fineness

$$Q = \frac{C_L}{C_D + C_f}$$

where C_f denotes friction coefficient is shown in figure 12 for $C_f = 0.003$ and 0.008. The former curve attains its maximum (11.237) at $\alpha_{\max 1} = 2.5^o$ while the latter one (6.798) at $\alpha_{\max 2} \approx 4.07^o$.

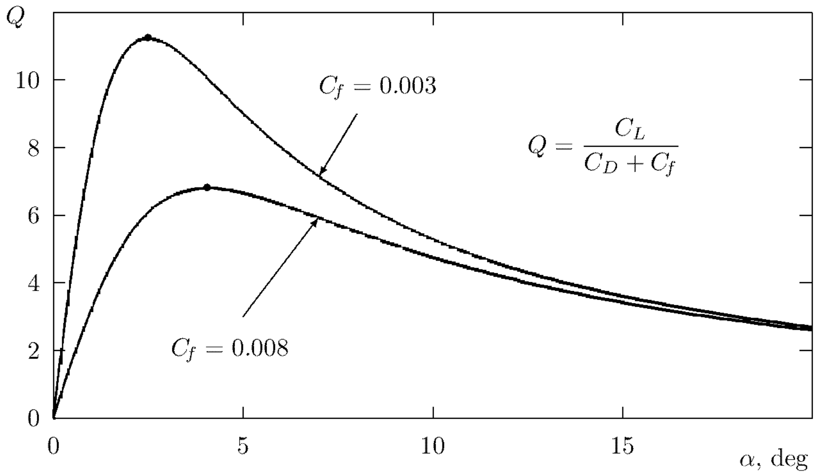

Figure 12: Hydrodynamic fineness versus angle of attack α for friction coefficient $C_f = 0.003$ and 0.008.

It is to be underlined that Chaplygin method of singular point appears to be the most effective while solving 2D nonlinear problems of theory of jets in an ideal fluid. It can be readily applied to considerably more complicated problems than that discussed above and gives quite satisfactory results.

3. A variety of cavity closure schemes

Consider a flat plate with attached cavity, the pressure within the cavity being $p = p_0$. In the flow region the following dynamic condition is satisfied $p \geq p_0$, or using Bernoulli integral

$$v = v_0 \quad \text{on the cavity boundary;} \qquad v < v_0 \quad \text{in the ambient fluid.}$$

Then the cavitation number σ can be rewritten as

$$\sigma = \frac{p_\infty - p_0}{\rho v_\infty^2/2} = \frac{v_0^2}{v_\infty^2} - 1 > 0\,,$$

where v_0 is an absolute velocity value on the cavity boundary. As a result, the cavity length becomes finite because of the cavity pressure is less than ambient, including that at infinity. The smaller the cavitation number, the larger the cavity extent and in a limiting case, as $\sigma \to 0$, the cavitation flow coincides with a streamline one considered above.

The Brillouin paradox [2] is well-known: cavity of finite length with closed continuous boundary is mathematically impossible. Indeed, if the upper and lower parts of the cavity make a closed contour then a stagnation point should appear. That is impossible due to the requirement of a constant absolute velocity value on the cavity boundary. That is why a set of cavity closure models has been developed, for the definitions and properties of which reference may be made to [1, 5, 7, 23]. Each scheme has its own advantages and disadvantages from the mathematical and physical viewpoint and should be chosen depending on specific features of the problem at hand.

The section contains a set of analytical solutions to the nonlinear problem of a supercavitating flow past a flat plate of chord l, with angle of attack α and cavitation number $\sigma > 0$, the cavity closure scheme being varied. The schemes considered are as follows: Tulin–Terentev (single spiral vortex termination), Efros–Kreisel–Gilbarg (re-entrant jet termination), Riabouchinsky (symmetrical plate termination), Tulin (double spiral vortex termination), Zhukovsky–Roshko (horizontal plates termination). All the analytical solution were derived via Chaplygin method of singular points. Numerical results were obtained in *Mathematica for Windows* computer mathematical environment.

3.1. Tulin–Terentev scheme: single spiral vortex termination

Consider a cavitating plate, see figure 13. A cavity closure scheme involving single spiral vortex is accepted as a model of cavity termination. Tulin [23] was the first to propose the scheme which was later thoroughly analysed by Terentev [7]. In the point of cavity 'collapse' the model requires

$$\omega(w) \sim -\frac{\mathcal{A}}{\sqrt{w}}\,,$$

where $\mathcal{A} > 0$.

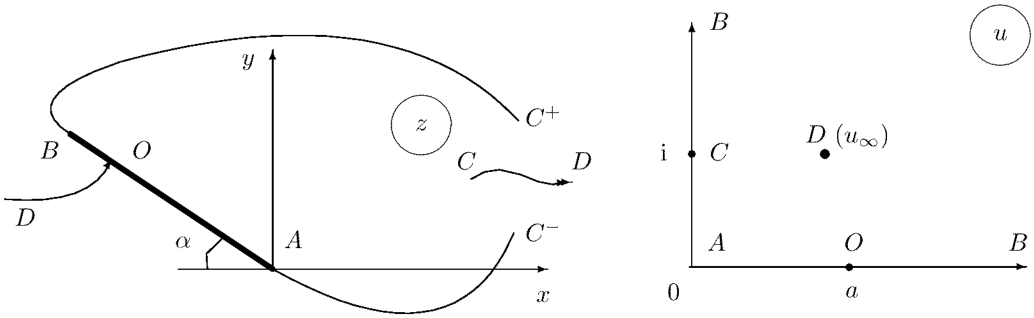

Figure 13: Flow pattern (z-plane) and auxiliary u-plane for the cavitating flat plate, Tulin–Terentev scheme.

On the dividing ('zero') streamline, where $\psi = 0$, the following relationship holds in a neighborhood of point C

$$\lim\left(\arg\frac{\mathrm{d}w}{\mathrm{d}z}\right) = \pm\infty\,,$$

what corresponds to two spiral streamlines with centers at points C^+ and C^-. Sketch of the flow in the region of the cavity termination is shown in figure 14.

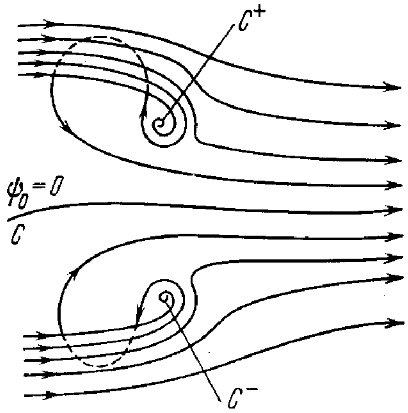

Figure 14: Schematics of the flow in the vicinity of the cavity termination, from [7].

Following to the main postulates of Chaplygin method, with the correspondence between physical z-plane and first quadrant of auxiliary u-plane, see figure 13, and applying Schwarz principle of symmetry to cover all the u-plane, one obtains derivatives of complex potential w in the form

$$\begin{aligned}\frac{\mathrm{d}w}{\mathrm{d}z} &= v_0\,\mathrm{e}^{\mathrm{i}(\alpha-\pi)}\,\frac{u-a}{u+a}\,\exp\left(\frac{2bu}{u^2+1}\right)\,;\\ \frac{\mathrm{d}w}{\mathrm{d}u} &= N\,\frac{u(u^2+1)(u^2-a^2)}{(u^2-u_\infty^2)^2(u^2-\overline{u_\infty}^2)^2}\,,\end{aligned} \tag{30}$$

wherefrom

$$z(u) = \frac{N}{v_0} \mathrm{e}^{\mathrm{i}(\pi-\alpha)} \int\limits_0^u \frac{u(u^2+1)(u+a)^2}{(u^2-u_\infty^2)^2(u^2-\overline{u_\infty}^2)^2} \times \exp\left(-\frac{2bu}{u^2+1}\right) \mathrm{d}u\,. \tag{31}$$

It is easy to see that stagnation point O generates zeros at point $u = a$ for both the derivatives, point at infinity D corresponding to $u = u_infty = c + \mathrm{i}d$ generates second-order pole of $\mathrm{d}w/\mathrm{d}u$ and at point $u = \mathrm{i}$ (image of points C, C^+ and C^-), $\mathrm{d}w/\mathrm{d}u = \mathrm{O}(u-i)$ and $\omega = \mathrm{O}(u-\mathrm{i})^{-1}$.
Three conditions

$$\frac{\mathrm{d}w}{\mathrm{d}z}(u_\infty) = v_\infty\,; \qquad z_B = z(\infty) = l\mathrm{e}^{\mathrm{i}(\pi-\alpha)}\,; \qquad \oint\limits_{u_\infty} \frac{\mathrm{d}z}{\mathrm{d}u}\,\mathrm{d}u = 0 \tag{32}$$

give a nonlinear system of five equations in five unknown parameters of the problem a, b, c, d and N to be derived. The latter condition implies an existence of a contour completely surrounding the plate and the cavity on the first Riemann sheet and actually is a cavity closure condition. Terentev was the first to propose such a relationship [20] which can be used to tell so-called closed cavity closure schemes from 'open' ones. All the calculations are effectively accomplished with use of *Mathematica for Windows* software.

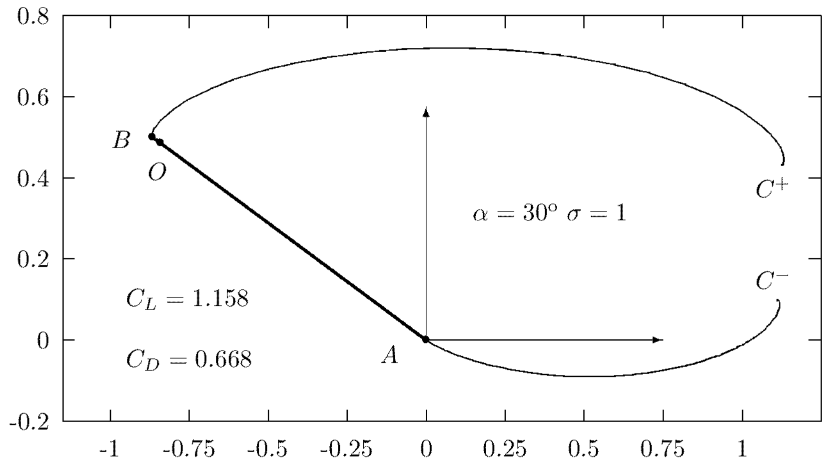

Figure 15: Flow pattern for the cavitating flat plate at $\sigma = 1$ and $\alpha = 30^\mathrm{o}$.

Force coefficient C_F is derived in a similar manner of formula (29):

$$C_F = -\mathrm{i}\frac{1+\sigma}{l}\left(z_B - z_A - \frac{1}{v_0^2}\int\limits_0^\infty \frac{\mathrm{d}w}{\mathrm{d}u}\overline{\frac{\mathrm{d}w}{\mathrm{d}z}}\mathrm{d}u\right) \tag{33}$$

Figure 15 illustrates the flow patter for $\alpha = 30^\mathrm{o}$ and $\sigma = 1$. It is to be emphasized that single spirals at point C shrink very rapidly and so the region of double-sheeted flow is quite small. The flow region in the vicinity of point C^- is shown in figure 16. Note that vertical distance δ between C^+ and C^- is directly connected with drag coefficient C_D:

$$C_D = (1+\sigma)\,\frac{\delta}{l}\,.$$

3.2. Efros–Kreisel–Gilbarg model: re-entrant jet termination

The flow pattern is depicted in figure 17. The model proposed almost simultaneously by Efros, Kreisel and Gilbarg [1] involves a re-entrant jet of unknown width δ and direction μ at infinity point C which lies on the second Riemann sheet. Gurevich [7] was the first to successfully apply the scheme to the problem of a supercavitating plate which is perpendicular to an inflow. An additional stagnation point E is appeared in the vicinity of the cavity trailing edge.

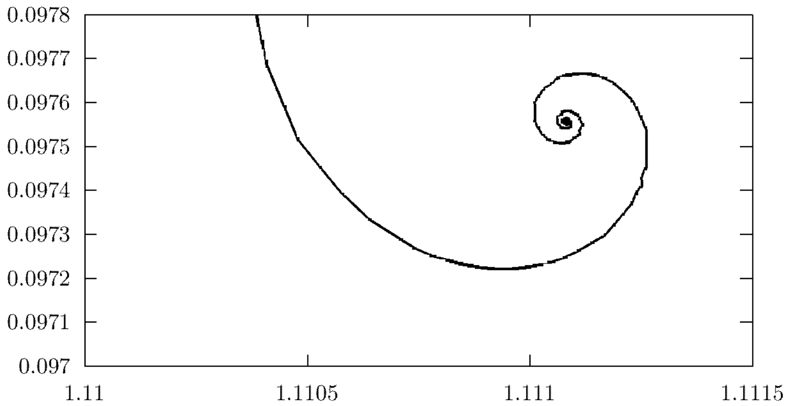

Figure 16: Flow region in the vicinity of lower single spiral vortex.

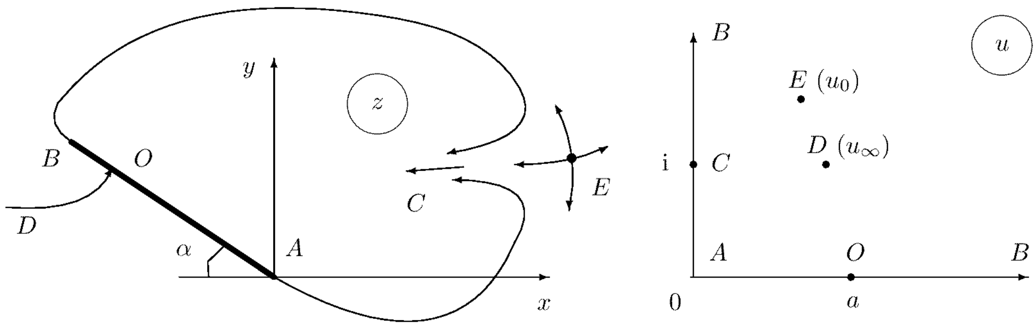

Figure 17: Flow pattern (z-plane) and auxiliary u-plane for the cavitating flat plate, Efros–Gilbarg scheme.

Adopting first quadrant of auxiliary u-plane as an image of the flow region, with the correspondence between physical z and auxiliary u planes (see figure 17) solution to the problem is as follows:

$$\begin{aligned} \frac{\mathrm{d}w}{\mathrm{d}z} &= v_0\,\mathrm{e}^{\mathrm{i}(\alpha-\pi)}\,\frac{u-a}{u+a}\,\frac{(u-u_0)(u-\overline{u_0})}{(u+u_0)(u+\overline{u_0})}\,; \\ \frac{\mathrm{d}w}{\mathrm{d}u} &= N\,\frac{u(u^2-a^2)(u^2-u_0^2)(u^2-\overline{u_0}^2)}{(u^2+1)(u^2-u_\infty^2)^2(u^2-\overline{u_\infty}^2)^2}\,, \end{aligned} \tag{34}$$

where $u_0 = b + \mathrm{i}c$ and $u_\infty = d + \mathrm{i}f$. These two expressions can be combined to give

$$z(u) = \frac{N}{v_0}\,\mathrm{e}^{\mathrm{i}(\pi-\alpha)} \int_0^u \frac{u(u+a)^2}{(u^2+1)} \times \frac{(u+u_0)^2(u+\overline{u_0})^2}{(u^2-u_\infty^2)^2(u^2-\overline{u_\infty}^2)^2}\,\mathrm{d}u\,. \tag{35}$$

The same conditions (32) yield a nonlinear system of five equations in six unknowns a, b, c, d, f and N. It is easy to see that the solution is not unique for the number of unknowns is greater than the number of conditions. Nevertheless, one can use an additional condition connected with the direction of the re-entrant jet at the point of infinity C, see figure 18. Both the asymptotic analysis and numerical results for the cavitating flat plate have shown angle μ not to significantly affect the hydrodynamic coefficients and even most of the flow pattern. That is why it seems reasonable to consider the parameter as a given one. Thus, an additional condition is as follows

$$\frac{\mathrm{d}w}{\mathrm{d}z}(\mathrm{i}) = v_0\,\mathrm{e}^{-\mathrm{i}\mu}\,.$$

Using residue theory, one arrives at the following relationships (which are correct for an arbitrary cavitating hydrofoil):

$$C_D = \frac{2q}{v_\infty l}\left(1 - \frac{v_0}{v_\infty}\cos\mu\right)\,; \qquad C_L = \frac{2q}{v_\infty l}\left(\frac{\Gamma}{q} - \frac{v_0}{v_\infty}\sin\mu\right)\,, \tag{36}$$

where q denotes the flow rate in the re-entrant jet C and Γ is circulation along a large contour completely surrounding the cavitating foil and the cavity and enclosing most of the flow. Note that

$$\mathrm{res} = \oint\limits_{u_\infty} \frac{\mathrm{d}F}{\mathrm{d}\zeta}\mathrm{d}\zeta = \Gamma + iq$$

and, moreover,

$$q = \pi N \frac{(1+a^2)\left((1+b^2-c^2)^2+4b^2c^2\right)}{2\left((1+d^2-f^2)^2+4d^2f^2\right)^2}\ .$$

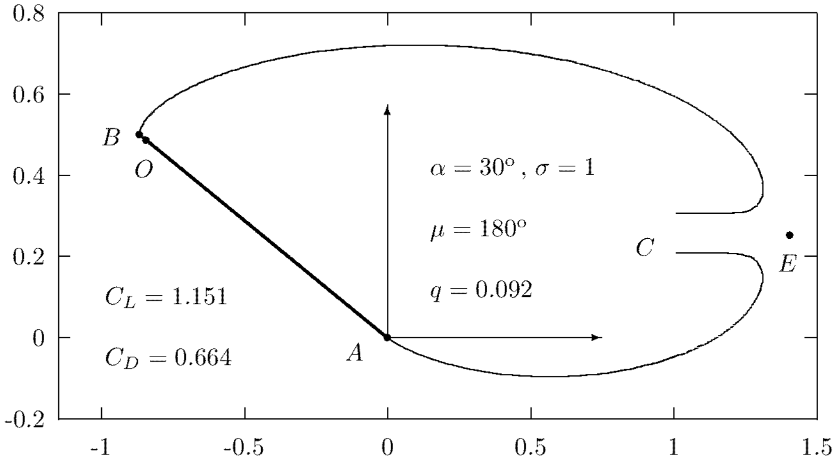

Figure 18: Flow pattern for the cavitating flat plate at $\sigma = 1$, $\alpha = 30^\circ$ and $\mu = 180^\circ$.

Another form of the force coefficient is

$$C_F = \frac{2}{v_\infty l}\left(-\sqrt{1+\sigma}\,\mathrm{e}^{\mathrm{i}\mu}\mathrm{Im}\,(\mathrm{res}) + \mathrm{i}\,\overline{\mathrm{res}}\right). \tag{37}$$

Flow pattern for the cavitating flat plate at $\sigma = 1$ and $\alpha = 30^\circ$ is shown in figure 18 along with hydrodynamic coefficients and position of stagnation points O and E. The direction of re-entrant jet is chosen to be $\mu = 180^\circ$.

3.3. Riabouchinsky model: symmetrical plate termination

The model introduced by D. Riabouchinsky [19] assumes the flow region to be symmetric and so the cavity terminates on the symmetrical plate. Another version of the scheme was proposed [21] with central symmetry, see figure 19, which is more convenient from the viewpoint of mathematical analysis. Adopting this scheme, transform flow region into the rectangular $\pi/2 \times \pi|\tau|/2$, where τ is purely imaginary, on the auxiliary u-plane, see figure 19.

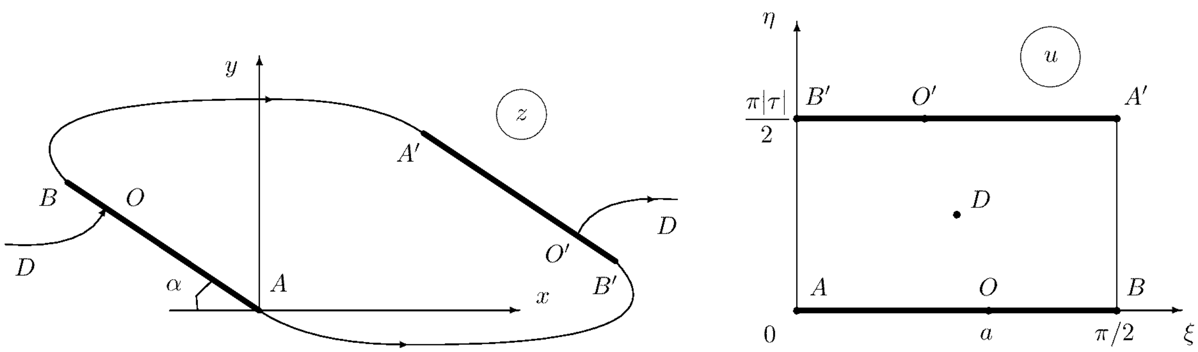

Figure 19: Flow pattern (z-plane) and auxiliary u-plane for the cavitating flat plate, Riabouchinsky scheme.

Conjugate velocity $\mathrm{d}w/\mathrm{d}z$ has constant absolute value v_0 of both vertical sides of the rectangular, while on the horizontal sides its argument is a step function with a jump-like behaviour in stagnation points O and O' (argument varies by π). That is why $\mathrm{d}w/\mathrm{d}z$ has zeros at points $u = a$ and $u = \pi - a + \pi\tau/2$. Using Schwarz principle of symmetry, one arrives at the conclusion that $\mathrm{d}w/\mathrm{d}z$ is a doubly periodic elliptic function with periods π and $\pi\tau$. Basing on the Liouville theorem and accepting notation of Whittaker & Watson [25] one can write down such a functions as a ratio of elliptic theta-function ϑ_i, $i = 1, \ldots, 4$ which *nome* $0 < q < 1$ is real.
Function $\mathrm{d}w/\mathrm{d}u$ is real on horizontal sides of the rectangular and must be purely imaginary on vertical sides. That is why this derivative is also a doubly periodic elliptic function with periods π and $\pi\tau$. The point D at infinity becomes the rectangular center $u = \pi/4 + \pi\tau/4$ where $\mathrm{d}w/\mathrm{d}u$ has a second order pole. Stagnation points generates simple zeros and so do points where conformality condition does not satisfied, that is all the rectangle vertexes.

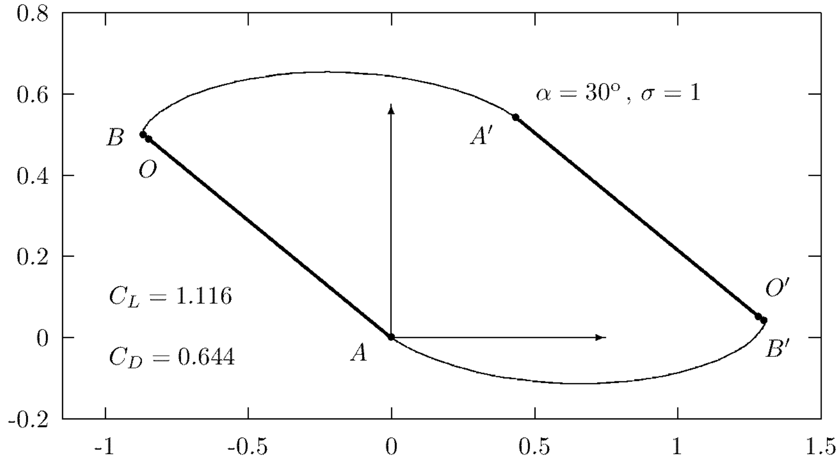

Figure 20: Flow pattern for the cavitating flat plate at $\sigma = 1$, $\alpha = 30^{\circ}$.

The solution to the problem is as follows

$$\frac{\mathrm{d}w}{\mathrm{d}z} = v_0\, \mathrm{e}^{\mathrm{i}(\alpha-\pi)}\, \frac{\vartheta_1(u-a)\vartheta_3(u+a)}{\vartheta_1(u+a)\vartheta_3(u-a)}\,.$$

$$\frac{\mathrm{d}w}{\mathrm{d}u} = N\, \frac{\vartheta_1(2u)}{\vartheta_3^2(2u)} \times \vartheta_1(u-a)\vartheta_1(u+a)\vartheta_3(u-a)\vartheta_3(u+a)\,. \tag{38}$$

wherefrom

$$z(u) = \frac{N}{v_0}\, \mathrm{e}^{\mathrm{i}(\pi-\alpha)} \int\limits_0^u \frac{\vartheta_1(2u)\vartheta_1^2(u+a)\vartheta_3^2(u-a)}{\vartheta_3^2(2u)}\, \mathrm{d}u\,. \tag{39}$$

An advantage of the model proposed is that it always involves less number of unknown parameters then the other schemes. Indeed, one has to determine just three unknowns a, τ and N in the problem under consideration instead of $5-6$ in other cases. Two former out of three conditions (32) give the nonlinear system of three equations in three unknowns, where $u_\infty = \pi/4 + \pi\tau/4$, the latter condition (32) being satisfied due to the central symmetry of the flow region.
Force coefficient and flow pattern are calculated using expressions (33) and (39) correspondingly. Figure 20 illustrates some numerical results obtained in *Mathematica* package for the same set of the flow parameters which was considered in the previous sections.

3.4. Tulin model: double spiral vortex termination

The scheme was proposed by M. Tulin [23] and involves two double spiral-like streamlines at the trailing edge of the cavity. Such a double spiral vortex flow is known to be a unique opportunity to conjugate smoothly two streamlines with different absolute velocity values [1], see figure 21. The scheme is appeared to be especially efficient for confined streamline problems with one/two free surfaces which bounded the flow region [14, 22].

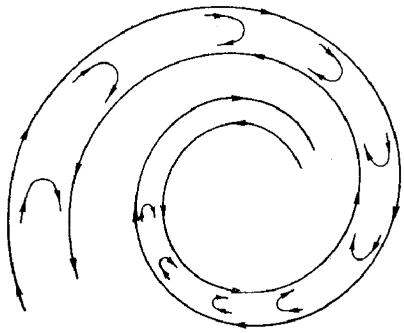

Figure 21: Flow pattern in the vicinity of a double spiral streamline, from [1].

In the case of the cavitating flat plate, first quadrant of auxiliary u-plane is chosen as image of the region occupied by the fluid in z-plane, see figure 22. The wake behind the cavity begins at points C and E and thins continuously in the downstream direction.
Using Chaplygin method of singular points one obtains that function $\mathrm{d}w/\mathrm{d}u$ is real on ξ-axis and purely imaginary on η-axis. It has zeros at stagnation point O ($u = a$) and at point A where conformality condition does not satisfied and three-order pole at point at infinity D ($u = \mathrm{i}$).

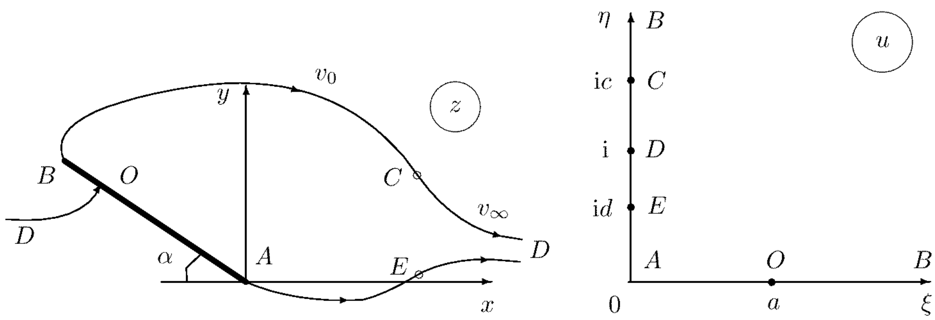

Figure 22: Flow pattern (z-plane) and auxiliary u-plane for the cavitating flat plate, Tulin scheme.

Another derivative $\mathrm{d}w/\mathrm{d}z$ has a zero at stagnation point O and behaves in a special manner at points C and E. Consider a function

$$\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^{\mathrm{i}} = \mathrm{e}^{\theta + \mathrm{i}\log v}$$

which is a step function on ξ-axis. Tracing small semi-circles described about points C ($u = \mathrm{i}c$ and E ($u = \mathrm{i}d$) in the counter-clockwise direction, one arrives at the conclusion that the argument of the function has an increment $(1/\pi)(\log(v_0/v_\infty)$ and $(1/\pi)(\log(v_\infty/v_0)$ correspondingly. That is why

$$\begin{aligned} \frac{\mathrm{d}w}{\mathrm{d}z} &= v_0\, \mathrm{e}^{\mathrm{i}(\alpha-\pi)} \frac{u-a}{u+a} \left(\frac{u-\mathrm{i}c}{u+\mathrm{i}c}\frac{u+\mathrm{i}d}{u-\mathrm{i}d}\right)^{\kappa} \\ \frac{\mathrm{d}w}{\mathrm{d}u} &= N\,\frac{u(u^2-a^2)}{(u^2+1)^3}\,, \end{aligned} \tag{40}$$

where

$$\kappa = \frac{\mathrm{i}}{\pi}\log\frac{v_\infty}{v_0}$$

and

$$z(u) = \int_0^u \frac{\mathrm{d}z}{\mathrm{d}w}\frac{\mathrm{d}w}{\mathrm{d}u}\,\mathrm{d}u = \frac{N}{v_0}\,\mathrm{e}^{\mathrm{i}(\pi-\alpha)} \int_0^u \frac{u(u+a)^2}{(u^2+1)^3}\left(\frac{u+\mathrm{i}c}{u-\mathrm{i}c}\frac{u-\mathrm{i}d}{u+\mathrm{i}d}\right)^{\kappa}\mathrm{d}u\,.$$

The problem involves four unknown parameters a, c, d and N. On the other hand, just two 'direct' conditions can be imposed:

$$\frac{\mathrm{d}w}{\mathrm{d}z}(u_\infty) = v_\infty \quad \text{and} \quad z_B = z(\infty) = l\mathrm{e}^{\mathrm{i}(\pi-\alpha)} . \tag{41}$$

Cavity closure condition applied in the previous sections

$$\oint\limits_{u_\infty} \frac{\mathrm{d}z}{\mathrm{d}u}\,\mathrm{d}u = 0$$

does not satisfied.
Thus this scheme is appeared to be the most indeterminate cavity closure model. Nevertheless, two additional conditions should be formulated:

$$\varphi_C = \varphi_E \quad \text{and} \quad \left.\frac{\mathrm{d}\theta}{\mathrm{d}\eta}\right|_{\eta=1} = 0 \tag{42}$$

to close the problem. The former implies the velocity potential to be the same at the end of the cavity at upper and lower boundaries. The latter specifies the angle θ (direction of the velocity vector) along the wake boundaries be minimal at point D. Note that both conditions are arbitrarily chosen and can be substituted by a pair of others, for instance $y_C = y_E$ and $\mathrm{d}^2w/\mathrm{d}\eta^2 = 0$ as $\eta = 1$.
Numerical results obtained with use of conditions (41) and (42) are shown in figure 23.

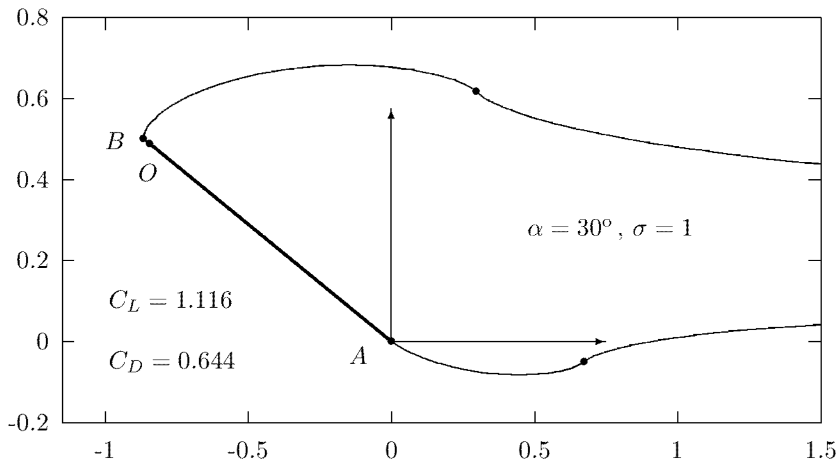

Figure 23: Flow pattern for the cavitating flat plate at $\sigma = 1$, $\alpha = 30^o$.

3.5. Zhukovsky–Eppler–Roshko model: horizontal plates termination

The model is often called 'open' one and assumes that free streamlines conjugate smoothly with two solid semi-infinite plates which are parallel to the inflow. The absolute velocity value monotonically decreases from v_0 down to v_∞ along this plate. This cavity closure model is very effective for streamline problems either under gravity or with one/two solid boundaries [7]. Zhukovsky [26] was the first to introduce the model.
Following to Chaplygin method, transform flow region into the rectangular $\pi/2 \times \pi|\tau|/2$, where τ is purely imaginary, on the auxiliary u-plane, see figure 24. Determining zeros and poles, one arrives at the following solution to the problem

$$\begin{gathered} \frac{\mathrm{d}w}{\mathrm{d}z} = v_0\,\mathrm{e}^{\mathrm{i}(\alpha-\pi)}\,\frac{\vartheta_1(u-a)}{\vartheta_1(u+a)}\,; \\ \frac{\mathrm{d}w}{\mathrm{d}u} = N\,\frac{\vartheta_1(2u)\,\vartheta_1(u-a)\,\vartheta_1(u+a)}{\vartheta_4^3(u-b)\,\vartheta_4^3(u+b)}\,, \end{gathered} \tag{43}$$

wherefrom

$$z(u) = \frac{N}{v_0}\,\mathrm{e}^{\mathrm{i}(\pi-\alpha)} \int\limits_0^u \frac{\vartheta_1(2u)\,\vartheta_1^2(u+a)}{\vartheta_4^3(u-b)\,\vartheta_4^3(u+b)}\,\mathrm{d}u\,. \tag{44}$$

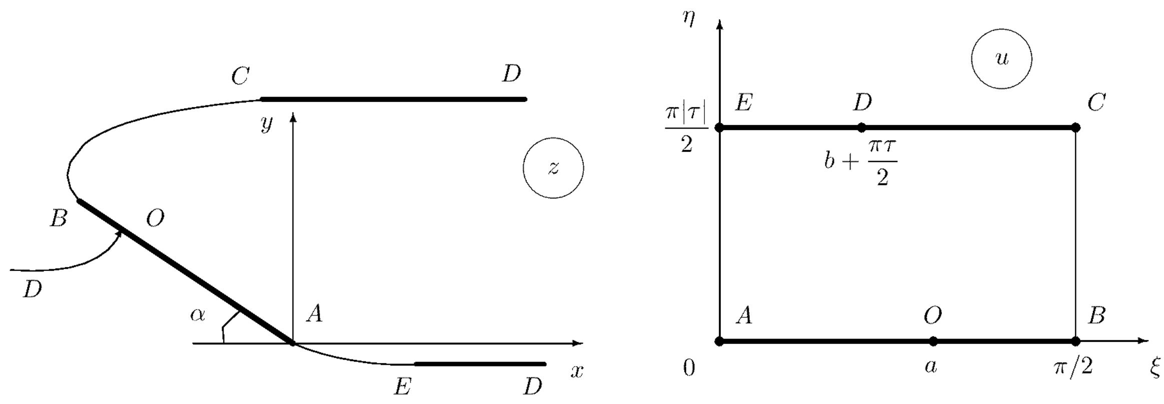

Figure 24: Flow pattern (z-plane) and auxiliary u-plane for the cavitating flat plate, Zhukovsky–Roshko–Eppler scheme.

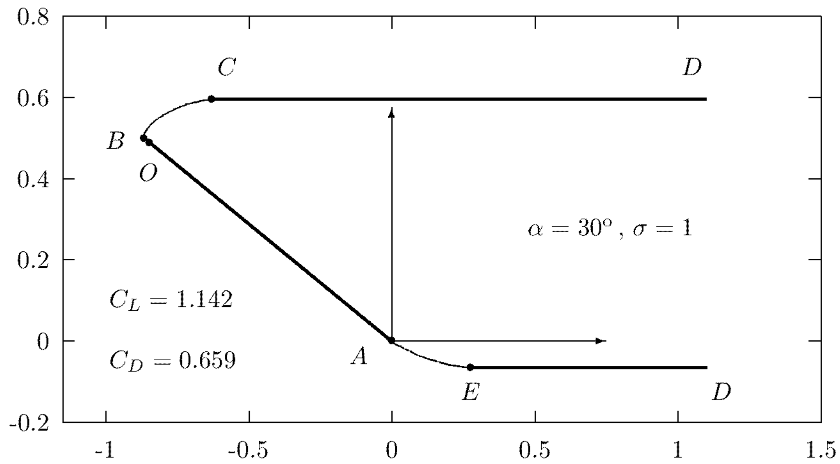

Figure 25: Flow pattern for the cavitating flat plate at $\sigma = 1$, $\alpha = 30^o$.

First two out of three conditions (32) where $u_\infty = b + \pi\tau/2$, along with two additional conditions

$$\frac{\mathrm{d}w}{\mathrm{d}z}\left(\frac{\pi}{2}+\frac{\pi\tau}{2}\right) = v_\infty \quad \text{and} \quad \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)'_u (u_\infty) = 0$$

allow four unknowns a, b, τ and N to be derived. The former of the additional conditions implies that the semi-infinite plates are horizontal and the latter specify the velocity absolute value be a monotone decreasing function having its minimum at point D ($u = u_\infty$).

The corresponding flow pattern and hydrodynamic coefficients are shown in figure 25. Note that this open cavity closure scheme give significantly smaller cavity length than so called closed models, such as Riabouchinsky, Tulin–Terentev, Efros–Kreisel–Gilbarg. As a result, the hydrodynamic coefficients become somewhat larger.

4. Free and solid boundaries

All the previous section has demonstrated an effectiveness of Chaplygin method of singular points applied to nonlinear cavitating problems of theory of jets in an ideal fluid. This section contains another documentation of the fact dealing with two somewhat more complicated problems: first one being that of the flow past a cavitating flat plate in the channel and second one – cavitating plate in the uniform jet of finite width. It was shown by Terentev [21] that Zhukovsky–Roshko and Tulin (double spiral vortex) schemes are to be used for the problems. We shall not dwell on the solution procedure and just describe the final results and some limiting cases.

4.1. Cavitating plate in a channel

The flow pattern $z = x + \mathrm{i}y$ and the auxiliary u-plane are depicted in figure 26. Given are the following parameters: angle of attack α, plate length l, channel width H, cavitation number σ and a distance between the lower wall of the channel and point A (trailing edge of the plate). It is also assumed that velocity v_1 at points D_1 and D_3 is the same (though it is an unknown value).

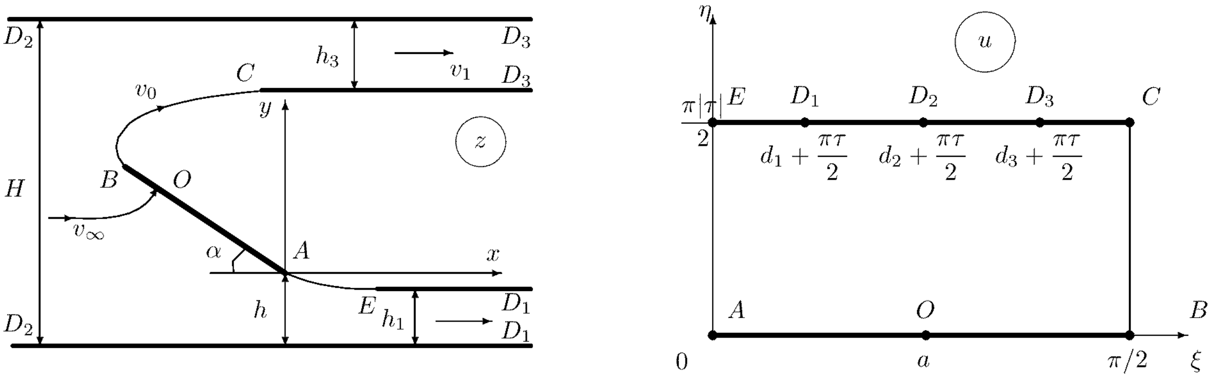

Figure 26: Flow pattern (z-plane) and auxiliary u-plane for the cavitating flat plate in the channel.

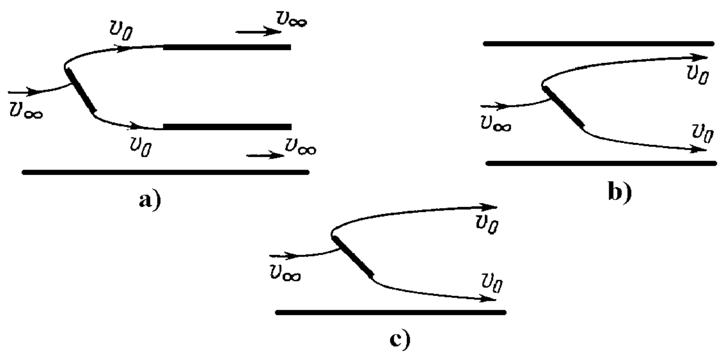

Figure 27: Some special cases of the problem for the cavitating plate in the channel.

The analytical solution to the problem is readily given by Chaplygin method [22]:

$$\frac{\mathrm{d}w}{\mathrm{d}z} = v_0\, \mathrm{e}^{\mathrm{i}(\alpha-\pi)}\, \frac{\vartheta_1(u-a)}{\vartheta_1(u+a)}\,;$$
$$\frac{\mathrm{d}w}{\mathrm{d}u} = N\, \frac{\vartheta_1(2u)\,\vartheta_1(u-a)\,\vartheta_1(u+a)}{\prod\limits_{i=1}^{3} \vartheta_4(u-d_i)\,\vartheta_4(u+b_i)}\,, \tag{45}$$

wherefrom

$$z(u) = \frac{N}{v_0}\, \mathrm{e}^{\mathrm{i}(\pi-\alpha)} \int\limits_0^u \frac{\vartheta_1(2u)\,\vartheta_1^2(u+a)}{\prod\limits_{i=1}^{3} \vartheta_4(u-d_i)\,\vartheta_4(u+b_i)}\, \mathrm{d}u\,. \tag{46}$$

Unknown parameters of the problem a, b_1, b_2, b_3, N, q, velocity v_1, widths h_1 and h_3 are derived from a quite complicated system of transcendental equations obtained on the base of the following conditions

$$\frac{\mathrm{d}w}{\mathrm{d}z}(u_\infty) = v_\infty\,; \quad \frac{\mathrm{d}w}{\mathrm{d}z}(u_1) = v_1\,; \quad \frac{\mathrm{d}w}{\mathrm{d}z}(u_2) = v_1\,; \quad Hv_\infty = v_1(h_1+h_3)\,; \quad z_B = z(\infty) = l\mathrm{e}^{\mathrm{i}(\pi-\alpha)}\,;$$
$$\frac{1}{2}\oint\limits_{u_\infty} \frac{\mathrm{d}w}{\mathrm{d}u}\,\mathrm{d}u = \mathrm{i}v_\infty H\,; \quad \frac{1}{2}\oint\limits_{u_1} \frac{\mathrm{d}w}{\mathrm{d}u}\,\mathrm{d}u = \mathrm{i}v_1 h_1\,; \quad \frac{\mathrm{d}w}{\mathrm{d}z}\left(\frac{\pi}{2} + \frac{\pi\tau}{2}\right) = v_0\,; \quad \mathrm{Im}\, z_E = \mathrm{i}(h_1 - h)\,, \tag{47}$$

where $u_\infty = u_2 = d_2 + \pi\tau/2$, $u_i = d_i + \pi\tau/2$, $i = 1, 3$.
After some algebra the number of unknowns (and equations) can be reduced from nine down to six and even to four, but the system remains complicated.
The general problem has lots of interesting special cases as h_1 or h_3 tends to infinity (one solid wall), $d_1 = 0$ and $d_3 = \pi/2$ (Kirchhoff scheme in the channel, when point C coincides with D_3 and E with D_1), *etc.*, see figure 27.

4.1. Cavitating plate in a jet of finite width

The flow region and the corresponding auxiliary u-plane are shown in figure 28. Tulin cavity closure model with double spiral vortex is adopted for it seems to be the most efficient from the mathematical viewpoint. The jet width is assumed to be H and submergence of the plate is h. Actually submergence is a distance between a dividing streamline and the upper jet boundary far upstream. The velocity on the jet boundary is v_∞.

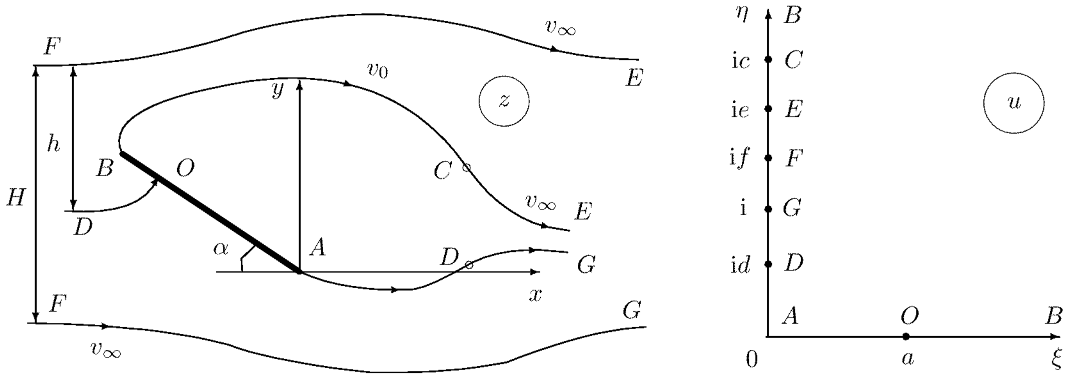

Figure 28: Flow pattern (z-plane) and auxiliary u-plane for the cavitating flat plate in the jet of finite width.

If one choses a first quadrant of the auxiliary u-plane as an image of the flow region, then solution to the problem is found through Chaplygin method in the form:

$$
\begin{aligned}
\frac{\mathrm{d}w}{\mathrm{d}z} &= v_0\,\mathrm{e}^{\mathrm{i}(\alpha-\pi)}\,\frac{u-a}{u+a}\left(\frac{u-\mathrm{i}c}{u+\mathrm{i}c}\,\frac{u+\mathrm{i}d}{u-\mathrm{i}d}\right)^{\kappa}\,; \\
\frac{\mathrm{d}w}{\mathrm{d}u} &= N\,\frac{u(u^2-a^2)}{(u^2+1)(u^2+e^2)(u^2+f^2)}\,,
\end{aligned}
\tag{48}
$$

where, as beforehand

$$\kappa = \frac{\mathrm{i}}{\pi}\,\log\frac{v_\infty}{v_0}$$

and

$$z(u) = \int\limits_0^u \frac{\mathrm{d}z}{\mathrm{d}u}\,\mathrm{d}u = \int\limits_0^u \frac{\mathrm{d}z}{\mathrm{d}w}\cdot\frac{\mathrm{d}w}{\mathrm{d}u}\,\mathrm{d}u\,.$$

Conditions

$$\frac{\mathrm{d}w}{\mathrm{d}z}(u_\infty) = v_\infty\,;\quad z_B = z(\infty) = l\mathrm{e}^{\mathrm{i}(\pi-\alpha)}\,;\quad \arg\frac{\mathrm{d}w}{\mathrm{d}z}(\mathrm{i}) = \arg\frac{\mathrm{d}w}{\mathrm{d}z}(\mathrm{i}e)\,;$$

$$w(\mathrm{i}d) = w(\mathrm{i}c)\,;\quad \frac{1}{2}\oint\limits_{u_\infty}\frac{\mathrm{d}w}{\mathrm{d}u}\,\mathrm{d}u = \mathrm{i}Hv_\infty\,;\quad \frac{1}{2}\oint\limits_{\mathrm{i}e}\frac{\mathrm{d}w}{\mathrm{d}u}\,\mathrm{d}u = \mathrm{i}hv_\infty$$

allow the unknowns a, c, d, e, f and N/V_∞ to be derived. Again, the third and fourth conditions (which specify the velocity vector be of the same direction at points E and G at infinity downstream and complex potential be of the same value at the ends of the cavity C and D correspondingly) are quite artificial and can be substituted by other reasonable relationships.

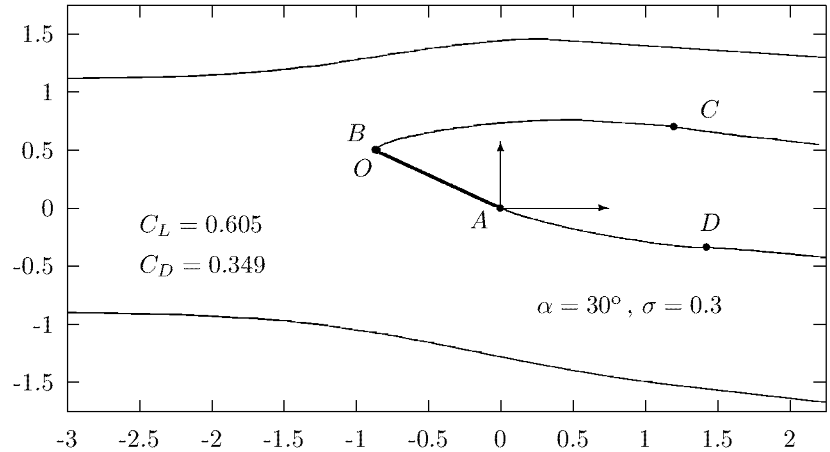

Figure 29: Flow pattern for the cavitating flat plate in the jet.

Note that solution (48) produces a solution by Larock and Street [14] for the cavitating plate beneath the free surface as its special case as $H \to \infty$ and $f \to 1^+$. Solution for the cavitating plate in unbounded inflow (40) is also a special case if $H\,, h \to \infty$ and $d\,, e \to 1^+$ as well.

Figure 29 illustrates numerical results (flow pattern in the vicinity of the plate and hydrodynamic coefficients) obtained with use of *Mathematica* for the cavitating plate with a following set of given parameters: $H = 2$, $h = 0.75$, $\alpha = 30^{\mathrm{o}}$, $\sigma = 0.3$.

5. Separated free streamline flow around a body with a curved boundary

5.1. Levi-Civita approach

Levi-Civita [15] was the first to propose an approach to solution of the plane free surface flow problems for obstacles of a curved topography. Such problems are significantly more complicated than those with flow regions bounded by polygonal segments. For the latter the imaginary part $\theta(u)$ of the Zhukovsky function $\omega(u)$ is given on the portion of the auxiliary u-domain boundary corresponding to the straight solid walls and its real part (velocity absolute value) is given on the other portion of the boundary. In the case of a curved obstacle all information one possesses is just the function $\theta(z)$ on the wetted portion on the body (which is sometimes also unknown) and velocity absolute value on free surfaces. The main difficulty of the problem is that it is impossible to set $\theta(u)$ without conformal mapping $z(u)$, if $\theta(z)$ is not a step function.

Following to Levi-Civita approach, flow region z is transformed into the semi-circle in auxiliary u-plane (one can also chose another domain as well) and Levi-Civita function (15) is written down in the form

$$\omega_{\mathrm{LC}} = \omega_0 + \Omega\,,$$

where ω_0 denotes Levi-Civita function for a polygonal body. Additional term is

$$\Omega(u) = \sum_{k=-n}^{\infty} A_k u^k\,,$$

where A_k are real coefficients to be found from the imposed conditions including Brillouin one (smooth detachment condition).

Brodetsky [3] calculated a flow around a circular and elliptic cylinder for $\sigma = 0$, three coefficients A_k being taken into account. In the case of circular cylinder the relationship holds

$$\frac{\mathrm{d}\theta}{\mathrm{d}s} = \kappa\,,$$

where s denotes an arc coordinate of the body boundary and κ is its curvature. Collocation method can be used to derive the coefficients A_k. Significantly more detailed numerical analysis of the problem was done in [7] by Terentev and Dmitrieva for an arbitrary cavitation number σ, more than 30 terms being taken into account. Two points were determined satisfying the Brillouin condition. The points correspond to two maxima on curves C_x versus detachment angle γ for a given σ, see figure 30.

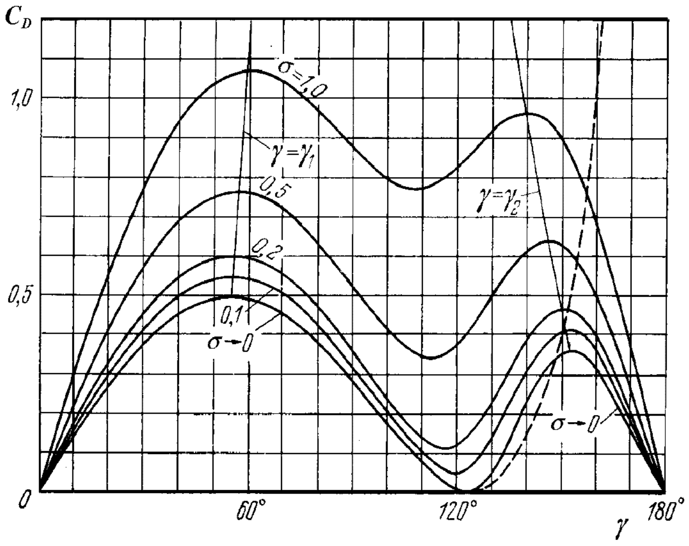

Figure 30: Drag coefficient of a circular cylinder versus detachment angle γ, from [7].

5.2. Method of integral equation

This method is an adaptation of mixed boundary value problem method (see section 2.3) to flow around an obstacle of a curved topography. By a conformal mapping of the flow region into the semi-plane (or semi-circle, *etc.*), the solution to the problem is reduced to an integral (or integro-differential) equation of Villat type [24] for a function $\lambda(u)$ characterizing the angle made by the fluid velocity vector. Using another form of such an integral equation, Nekrasov [18] was the first to prove both existence and uniqueness of flows around a small circular arc.

Consider the method applied to a free streamline problem for an arched contour, see figure 31, and transform the region in z plane occupied by the fluid into the upper semi-plane ζ, see figure 8.

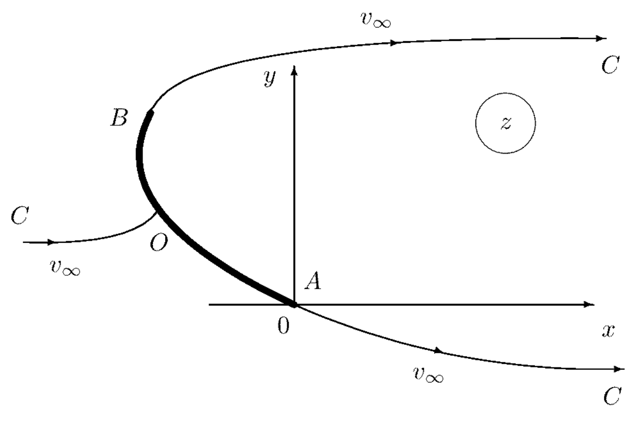

Figure 31: Flow pattern for the cavitating arc contour at zero cavitation number.

Then, again,

$$\frac{\mathrm{d}w}{\mathrm{d}\zeta} = N\zeta\,, \tag{49}$$

where N is a real parameter to be determined and the mixed boundary value problem for Levi-Civita function (15) arising in ζ-plane:

$$\tau = 0 \quad \text{as} \quad \xi \le -1 \quad \text{and} \quad \xi \ge b$$

$$\theta(\xi) = \begin{cases} -\alpha(\xi)\,, & \text{as} \quad -1 \le \xi \le 0\,; \\ \pi - \alpha(\xi)\,, & \text{as} \quad 0 \le \xi \le b\,. \end{cases}$$

where b is a coordinate (to be determined) of image of point B in ζ-plane and $\alpha(\xi)$ is the body angle at point $\zeta = \xi$, relationship $\alpha(s)$ being given, where s is an arc coordinate of the contour, $s = 0$ at point A and $s = S$ at point B.
The Keldysh–Sedov formula gives the solution to this problem in the $0 - 0$ class

$$\omega_{\mathrm{LC}}(\zeta) = -\frac{1}{\pi}\sqrt{(\zeta+1)(\zeta-b)} \times \left\{ -\int_{-1}^{b} \frac{\alpha(t)}{\sqrt{(1+t)(b-t)}}\frac{\mathrm{d}t}{t-\zeta} + \int_{0}^{b} \frac{\pi}{\sqrt{(1+t)(b-t)}}\frac{\mathrm{d}t}{t-\zeta} \right\}$$

which exists if and only if

$$-\frac{1}{\pi}\int_{-1}^{b} \frac{\alpha(t)\mathrm{d}t}{\sqrt{(1+t)(b-t)}} + \frac{\pi}{2} - \arcsin\frac{1-b}{1+b} = 0\,. \qquad (\star)$$

It is seen that

$$\alpha = \arctan\left(\frac{\mathrm{d}y/\mathrm{d}s}{\mathrm{d}x/\mathrm{d}s}\right)$$

This equation can be used in conjunction with the equality

$$\frac{\mathrm{d}z}{\mathrm{d}s} = \exp(\mathrm{i}\,\alpha)$$

to give

$$s(\xi) = \int_{0}^{z} \exp(-\mathrm{i}\,\alpha)\,\mathrm{d}z = \frac{N}{v_\infty}\int_{-1}^{\xi} \exp\left\{-\mathrm{i}\alpha(t)\right\}\,\exp\left\{\mathrm{i}\,\omega_{\mathrm{LC}}(t)\right\} t\,\mathrm{d}t\,.$$

Substituting the solution for ω_{LC} into this equation finally gives

$$s(\xi) = \frac{N}{v_\infty}\int_{-1}^{\xi} G(t,b)\,\mathrm{d}t\,, \qquad (50)$$

where

$$G(t,b) = \frac{2\sqrt{b(1+t)(b-t)} + t(b-1) + 2b}{1+b} \times \exp\left\{ \frac{\sqrt{(1+t)(b-t)}}{\pi}\,\mathrm{v.p.}\int_{-1}^{b} \frac{\alpha(t)}{\sqrt{(1+t)(b-t)}}\frac{dt}{t-\xi} \right\}$$

This integral equation along with relation $(\star)$ and an obvious condition

$$s(b) = S$$

allows unknown parameters b, N and function $\alpha(\xi)$ to be determined.

5.3. Arbitrary 2D supercavitating hydrofoil: analytical solution

Both the Levi-Civita and integral equation methods are known to connect with significant difficulties while considering a flow past an obstacle with a boundary of large curvature κ. At the same time, hydrofoils have such a region in the vicinity of the leading edge. One can find some theoretical analysis of the nonlinear problems of an arbitrary supercavitating hydrofoil and numerical results as well in [9, 4].
That is why an approach proposed by Maklakov [16] for arbitrary cavitating hydrofoil, see figure 32, is of interest as an improvement of Levi-Civita method.

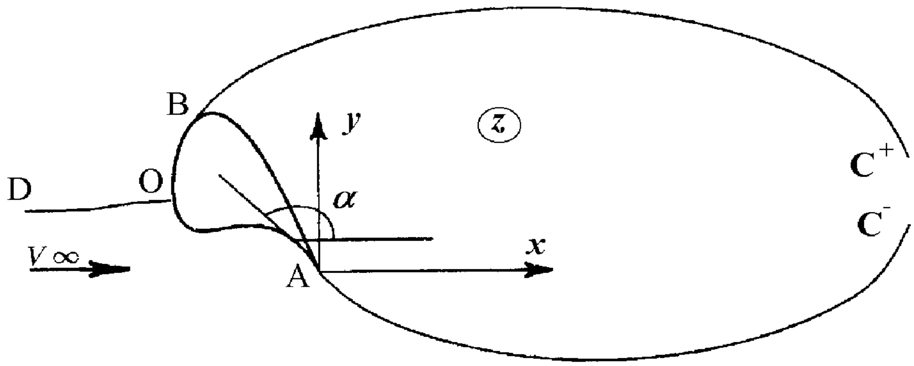

Figure 32: Flow region around a cavitating hydrofoil (Tulin–Terentev cavity closure scheme).

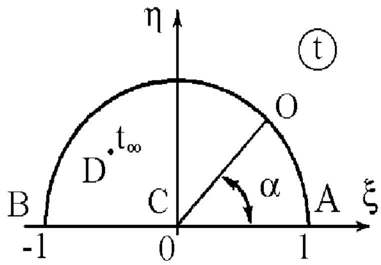

Figure 33: Auxiliary ζ-plane for cavitating hydrofoil.

Consider a cavitating hydrofoil which boundary is given by relation $\alpha = F(s)$, where s – arc coordinate ($s = 0$ at point A) and α is the tangential angle to the foil. Tulin–Terentev cavity closure scheme with a single spiral vortex model of cavity termination is adopted (subsection 3.1). Following to Maklakov, transform flow region z into the upper semi-circle $|t| \leq 1$, Im ≥ 0 on auxiliary plane $t = \xi + \mathrm{i}\eta$, see figure 33. Stagnation point O corresponds to $t_0 = \exp(\mathrm{i}\delta_0)$, point C – to the origin of the coordinate system in t-plane and point at infinity D has the image at $t_\infty = r_\infty \exp(\mathrm{i}\delta_\infty)$.
Chaplygin method of singular points gives the derivative

$$\frac{\mathrm{d}w}{\mathrm{d}t} = kv_0 f(t) = \frac{kv_0}{r_\infty} \, \frac{t(t^2-1)(t^2-2t\cos\delta_0+1)}{(t-t_\infty)^2(t-\overline{t_\infty})^2(t-1/t_\infty)^2(t-1/\overline{t_\infty})^2}\,, \tag{51}$$

where k is a real parameter to be determined.
Levi-Civita function is

$$\omega_{\mathrm{LC}}(t) = \omega_0(t) + M\,\frac{t^2-1}{t} + \Omega(t)\,, \tag{52}$$

where M denotes a real constant, $\omega_0(t)$ is Levi-Civita function for a flat plate at zero cavitation number

$$\omega_0(t) = \alpha_0 - \pi + \mathrm{i}\,\log\frac{t-\exp(\mathrm{i}\,\delta_0)}{1-t\,\exp(\mathrm{i}\,\delta_0)}\,,$$

and $\Omega(t)$ is analytic and continuous function in upper semi-circle. That is why conformal mapping can be written as

$$z(t) = \int\limits_1^t \frac{\mathrm{d}z}{\mathrm{d}w}\,\frac{\mathrm{d}w}{\mathrm{d}t}\,\mathrm{d}t = \int\limits_1^t \exp\left\{\mathrm{i}\,\omega_0(t) + \mathrm{i}\,M\,\frac{t^2-1}{t} + \mathrm{i}\Omega(t)\right\} f(t)\,\mathrm{d}t\,.$$

Levi-Civita method requires the expansion

$$\Omega(t) = \sum_{k=1}^{\infty} A_k t^k\,,$$

where A_k are real coefficients to be found. Maklakov [16] proposed another representation of Ω. If

$$\Omega(\mathrm{e}^{\mathrm{i}\,\delta}) = \lambda(\delta) + \mathrm{i}\,\mu(\delta)$$

then Schwarz formula for a circle gives a relation between Ω and its real part λ given on the semi-circle boundary

$$\Omega(t) = \frac{1-t^2}{\pi}\int\limits_0^{\pi} \frac{\lambda(\delta)\,\mathrm{d}\delta}{1-2t\cos\delta+t^2}\,.$$

Imaginary part of $\Omega(t)$ is

$$\mu(\gamma) = \boldsymbol{C}\,\lambda = \frac{1}{2\pi}\int\limits_0^{\pi}\big(\lambda(\varepsilon)-\lambda(\gamma)\big)\left(\cot\frac{\varepsilon+\gamma}{2} - \cot\frac{\varepsilon-\gamma}{2}\right)\mathrm{d}\varepsilon\,.$$

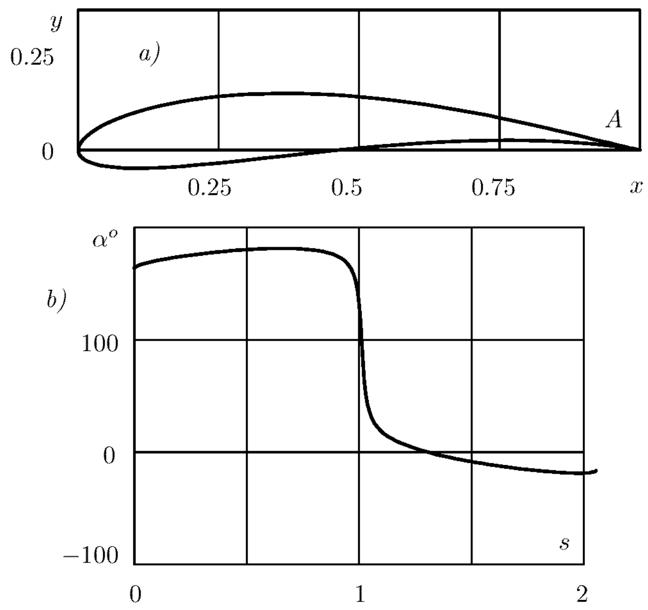

Figure 34: Zhukovsky hydrofoil (a) and relation $\alpha = F(s)$.

Using expression $\mathrm{d}s/\mathrm{d}\delta = |\mathrm{d}z/\mathrm{d}t|$ and conformal mapping $z(t)$ write down the relation between arc coordinate s and polar angle δ:

$$s(\delta) = k\int\limits_0^{\delta} g(\gamma)\,\exp\big(-\boldsymbol{C}\lambda - 2M\sin\gamma\big)\,\mathrm{d}\gamma\,,$$

where

$$g(\gamma) = 4\sin\gamma\,(1-\cos(\gamma+\delta_0)) \times \frac{1}{(1+r_\infty^2 - 2r_\infty\cos(\gamma-\delta_\infty))^2} \times \frac{1}{(1+r_\infty^2 - 2r_\infty\cos(\gamma+\delta_\infty))^2}\,.$$

It is easy to see that $\lambda = \alpha - \alpha_0$ and therefore, finally, one arrives at the following integral equation

$$\lambda(\delta) = -\alpha_0 + F\left(k\int\limits_0^{\delta} g(\gamma)\exp\big(-\boldsymbol{C}\lambda - 2M\sin\gamma\big)\mathrm{d}\gamma\right) \tag{53}$$

Five unknown parameters k, M, δ_0, r_∞, δ_∞ are derived from the conditions

$$\oint\limits_{t_\infty} \frac{\mathrm{d}z}{\mathrm{d}t}\,\mathrm{d}t = 0\,, \quad \frac{\mathrm{d}w}{\mathrm{d}z}(t_\infty) = v_\infty\,, \quad s(-1) = S\,,$$

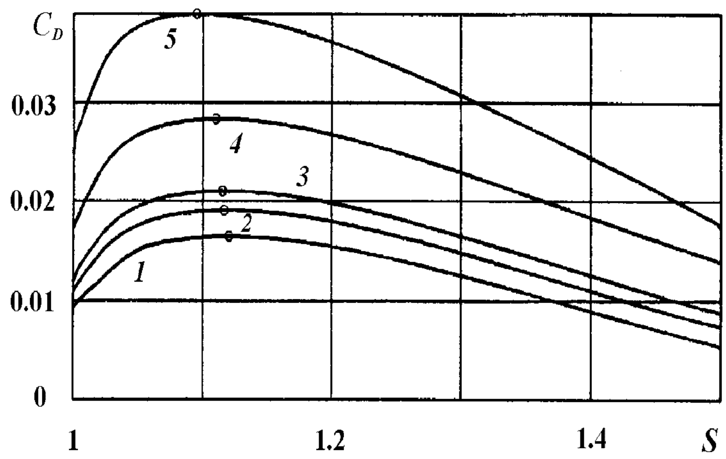

Figure 35: Drag coefficient C_D versus wetted length S.

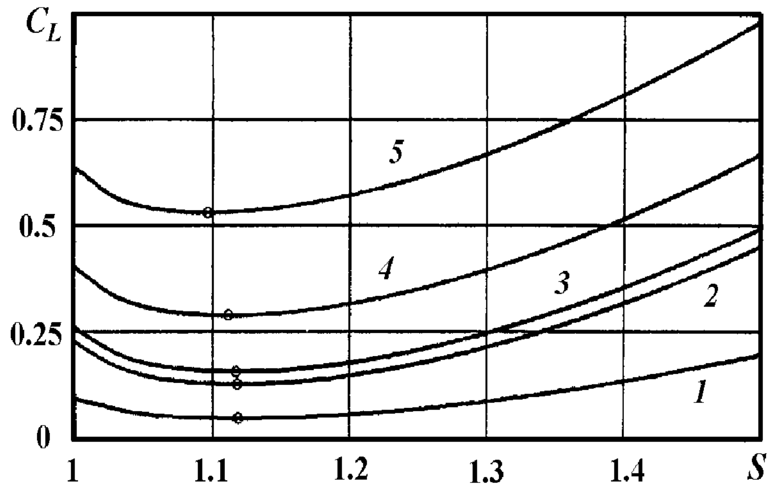

Figure 36: Lift coefficient C_L versus wetted length S.

where S denotes the whole (given) wetted length of the hydrofoil. The latter condition can be substituted by Brillouin condition at point B which specify the curvature of the cavity and hydrofoil be the same at detachment point B ($t = -1$). It was demonstrated [1] that this condition is equivalent to $\Omega'(-1) = 0$ which yields [16]

$$\pi \tan \frac{\delta_0}{2} + \int_0^{\pi} \lambda'(\gamma) \tan \frac{\gamma}{2} \mathrm{d}\gamma = 2\pi M \, .$$

Naturally, in this case the wetted length S is unknown because z-coordinates of point B are unknown as well. Some numerical results for a supercavitating Zhukovsky hydrofoil are shown in figures 34–36. Zhukovsky hydrofoil of unit chord length is chosen to have maximum thickness 11.82%, perimeter $P = 2.053$ and curvature radius at the leading edge $\rho = 0.016$, see figure 34, where the foil geometry and relation $\alpha = F(s)$ for the tangential angle to the foil versus arc coordinate s are shown. Figures 35 and 36 demonstrate drag and lift coefficient versus wetted length S (Brilllouin condition is not imposed) in the case of angle of attack $\alpha = 5^{\mathrm{o}}$ for a set of cavitation number $\sigma = 0$, 0.1, 0.15, 0.3 and 0.5 (curves '1' corresponds to $\sigma = 0$). It was found that drag (lift) coefficient attains its maximum (minimum) value at points where Brilllouin condition was satisfied.

Conclusion

A brief review of the main theoretical approaches to problems of the theory of jets in an ideal fluid was conveyed, including hodograph (Kirchhoff and Zhukovsky) methods, Chaplygin method of singular points, mixed boundary value problem method, some modifications of Levi-Civita approach and method of integral equations. Every analytical solution is illustrated by numerical results, including flow patterns.

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