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TITLE: Central Conics on Parabolic Dupin Cyclides

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TITLE: International Conference on Curves and Surfaces [4th], Saint-Malo, France, 1-7 July 1999. Proceedings, Volume 1. Curve and Surface Design

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Central Conics on Parabolic Dupin Cyclides

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Abstract. Hyperbolas, ellipses and degenerate conics on parabolic Dupin cyclides are investigated. These central conics are obtained as the intersections of parabolic cyclides and the planes perpendicular to the two planes of symmetry of the cyclides. They are also the images of central conics in the parametric space. Since the conics are planar curves, they are transformed into planar or spherical curves on Dupin cyclides via inversion. Lemniscates of Bernoulli on Dupin cyclides and Viviani's curves on right-circular cylinders are included in the inverted conics. Two intersecting lines on a parabolic ring cyclide, which are degenerate conics, are inverted into Villarceau circles on a ring cyclides.

§1. Introduction

It is known that any Dupin cyclide [1,2,5,6], which is a quartic surface is the image under inversion of a torus, circular cylinder or circular cone. Parabolic cyclides, which are cubic surfaces, are obtained as an inverse surface with its inversion center at a point on the Dupin cyclides. While quartic cyclides are closed surfaces, cubic cyclides are unbounded in extent.

Since both of them are expressed as biquadratic rational surfaces [4,5,6,7], the image of a rational curve on the parameter space of a Dupin cyclide is a rational curve on the cyclide. It is well known that the isoparametric curves on a Dupin cyclide, which are lines of curvature of the cyclides, are circular.

In this paper, non-circular rational curves of lower degree on Dupin cyclides, especially central conics on parabolic cyclides, are investigated. The other curves are derived from conics on cyclides via inversion.

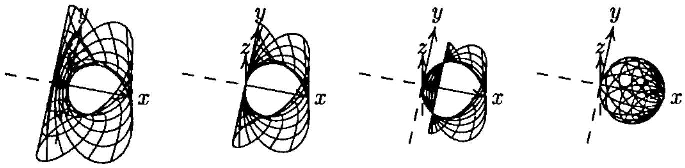
§2. Conics on Parabolic Dupin Cyclides

Let the curves $c_1(u)$ and $c_2(v)$ be the focal parabolas given by

$$c_1(u) = \frac{1}{2}\begin{pmatrix} q+(p-q)u^2 \\ 2(p-q)u \\ 0 \end{pmatrix}, \qquad c_2(v) = \frac{1}{2}\begin{pmatrix} p+(q-p)v^2 \\ 0 \\ 2(q-p)v \end{pmatrix}, \tag{1}$$

Curve and Surface Design: Saint-Malo 1999
Pierre-Jean Laurent, Paul Sablonnière, and Larry L. Schumaker (eds.), pp. 399–406.

ISBN 0-8265-1356-5.



(1) $q < 0 < p$ (2) $q = 0 < p$ (3) $0 < q < p$ (4) $0 < q = p$

Fig. 1. Parabolic cyclides and a sphere.

and the functions $r_1(u)$ and $r_2(v)$ be the radius functions given by

$$r_1(u) = \frac{q-(p-q)u^2}{2}, \qquad r_2(v) = \frac{p-(q-p)v^2}{2}. \tag{2}$$

Parabolic cyclides are parameterized with the parameters u and v and the shape parameters p and q as [4,6]

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{r_1(u)c_2(v) - r_2(v)c_1(u)}{r_1(u) - r_2(v)} = \frac{1}{u^2+v^2+1}\begin{pmatrix} pu^2+qv^2 \\ (p-(q-p)v^2)u \\ (q-(p-q)u^2)v \end{pmatrix}. \tag{3}$$

The isoparametric curves, which are circles, are the curvature lines of the parametric surface. Since the radius functions $r_1(u)$ and $r_2(v)$ represent the curvature radius, the principal curvatures κ_1 and κ_2 are their reciprocals, i.e., $\kappa_1 = 1/r_1(u)$ and $\kappa_2 = 1/r_2(v)$.

By eliminating the parameters u and v, we obtain the following implicit form of the parabolic cyclide (3):

$$(x-q)y^2 + (x-p)z^2 + x(x-p)(x-q) = 0. \tag{4}$$

Equation (4) implies that the cross section between the surface and the plane parallel to the yz-plane is a central conic.

From the x-value of (3), we obtain another central conic in the parameter space of the surface:

$$(x-p)u^2 + (x-q)v^2 + x = 0. \tag{5}$$

The central conics of (4) on parabolic cyclides are the images of the central conics of (5) in the parametric space.

The x-value of (3) is also transformed into

$$x = \frac{u^2}{u^2+v^2+1}p + \frac{v^2}{u^2+v^2+1}q + \frac{1}{u^2+v^2+1}0. \tag{6}$$

Hence, the x-value is the convex combination of p, q and 0. From the symmetry of (3), we can restrict the domain of the shape parameters p and q within $0 < p$ and $q \leq p$ for the classification of the shape of the surface. Since one of the principal curvatures is always positive, i.e., $0 \leq \kappa_2$, under this restriction, the Gaussian curvature $K = \kappa_1\kappa_2$ has the same sign as that of another principal curvature κ_1. Since parabolic cyclides extend to infinity, Figure 1 illustrates several shapes of the parabolic cyclide for $(u,v) \in ([-\infty,\infty] \times [-1,1]) \cup ([-1,1] \times [-\infty,\infty])$.

2.1 Hyperbolas on parabolic ring cyclides ($q < 0 < p$)

x	(y, z)	κ_1	(u, v)
$x = q$	$z = \pm 0$	$\kappa_1 < 0$	$v = \pm\infty$
$q < x < 0$	$\frac{z^2}{-x(x-q)} - \frac{y^2}{-x(p-x)} = 1$	$\kappa_1 < 0$	$\frac{v^2}{\frac{-x}{x-q}} - \frac{u^2}{\frac{-x}{p-x}} = 1$
$x = 0$	$z = \pm\sqrt{\frac{-q}{p}}y$	$\kappa_1 < 0$	$v = \pm\sqrt{\frac{p}{-q}}u$
$0 < x < p$	$\frac{y^2}{x(p-x)} - \frac{z^2}{x(x-q)} = 1$	$\kappa_1 < 0$	$\frac{u^2}{\frac{x}{p-x}} - \frac{v^2}{\frac{x}{x-q}} = 1$
$x = p$	$y = \pm 0$	$\kappa_1 = 0$	$u = \pm\infty$

There are two straight lines at $x = 0$ on a parabolic ring cyclide.

2.2 Hyperbolas on thorn cyclides ($q = 0 < p$)

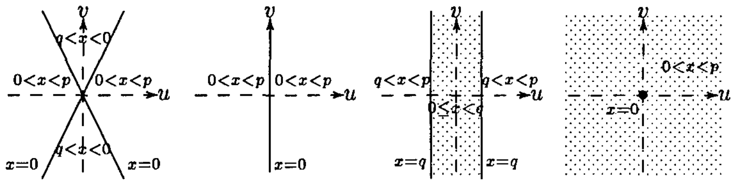
x	(y, z)	κ_1	(u, v)
$x = 0$	$z = \pm 0$	$\kappa_1 \leq 0$	$u = 0$ or $v = \pm\infty$
$0 < x < p$	$\frac{y^2}{x(p-x)} - \frac{z^2}{x^2} = 1$	$\kappa_1 < 0$	$\frac{u^2}{\frac{x}{p-x}} - v^2 = 1$
$x = p$	$y = \pm 0$	$\kappa_1 < 0$	$u = \pm\infty$

There is a singular point at the origin on a thorn cyclide.

2.3 Ellipses and hyperbolas on parabolic horn cyclides ($0 < q < p$)

x	(y, z)	κ_1	(u, v)
$x = 0$	$y = z = 0$	$\kappa_1 > 0$	$u = v = 0$
$0 < x < q$	$\frac{y^2}{x(p-x)} + \frac{z^2}{x(q-x)} = 1$	$\kappa_1 > 0$	$\frac{v^2}{\frac{x}{q-x}} + \frac{u^2}{\frac{x}{p-x}} = 1$
$x = q$	$z = \pm 0$	$\kappa_1 = \infty$	$u = \pm\sqrt{\frac{q}{p-q}}$
$q < x < p$	$\frac{y^2}{x(p-x)} - \frac{z^2}{x(x-q)} = 1$	$\kappa_1 < 0$	$\frac{u^2}{\frac{x}{p-x}} - \frac{v^2}{\frac{x}{x-q}} = 1$
$x = p$	$y = \pm 0$	$\kappa_1 = 0$	$u = \pm\infty$

There are two singular points $(q, \pm\sqrt{(p-q)q}, 0)$ on a parabolic horn cyclide.



(1) $q < 0 < p$ (2) $q = 0 < p$ (3) $0 < q < p$ (4) $0 < q = p$

Fig. 2. Parameter space.

2.4 Circles on spheres $(0 < q = p)$

x	(y, z)	κ_1	(u, v)
$x = 0$	$y = z = 0$	$\kappa_1 = \frac{2}{p}$	$u = v = 0$
$0 < x < p$	$y^2 + z^2 = x(p - x)$	$\kappa_1 = \frac{2}{p}$	$u^2 + v^2 = \dfrac{x}{p - x}$
$x = p$	$y = z = 0$	$\kappa_1 = \frac{2}{p}$	$u = \pm\infty$ or $v = \pm\infty$

The implicit form (4) of parabolic cyclides becomes the quadratic form of $y^2 + z^2 + x(x - p) = 0$.

2.5 Central conics on parabolic cyclides

On cubic cyclides, there are hyperbolas on the region with negative Gaussian curvature, and ellipses on the region with positive Gaussian curvature. Since the central conics are the images of central conics in the parameter space, the parameter space is subdivided by straight lines into regions corresponding to the geometric properties of their images, as illustrated in Figure 2. The preimages of the singular points are the straight lines parallel to the v-axis in the parameter space. The shaded regions are the preimages of the surface regions with positive Gaussian curvature.

For $0 < x < p$, the centers of the central conics are at $(x, 0, 0)$, the vertices are at the points $(x, \pm\sqrt{(p - x)x}, 0)$, and the foci are at the points $(x, \pm\sqrt{(p - q)x}, 0)$. The foci lie on the parabola $y^2 = (p - q)x$.

The central conics at $x = (p + q)/2$ are rectangular hyperbolas

$$y^2 - z^2 = \frac{p^2 - q^2}{4}, \qquad \left(u^2 - v^2 = \frac{p + q}{p - q}\right). \tag{7}$$

The mean curvature $H = (\kappa_1 + \kappa_2)/2$ is zero at any point along the rectangular hyperbolas on parabolic cyclides.

§3. Inverse Surfaces of Parabolic Cyclides

The inverse surfaces of the parabolic cyclides with inversion radius 1 with respect to a point $(r, 0, 0)$, which is the center of the central conics, are expressed

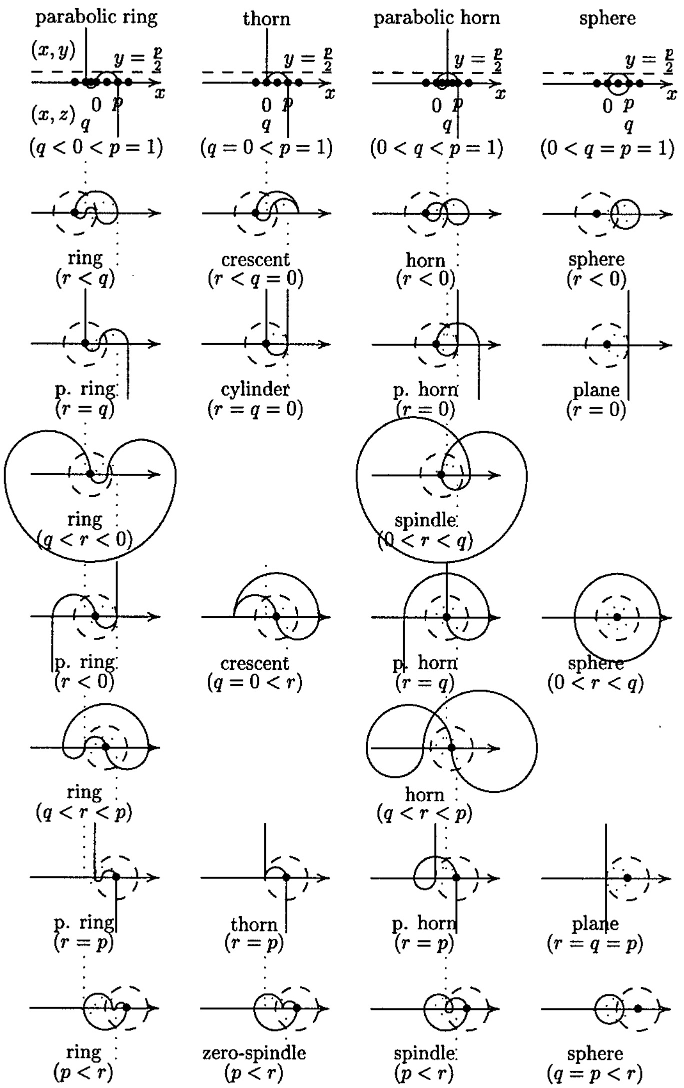


Fig. 3. Inversion of parabolic cyclides.

as

$$
\begin{aligned}
X &= \frac{x-r}{(x-r)^2+y^2+z^2} + r = \frac{(p-r)u^2+(q-r)v^2-r}{(p-r)^2u^2+(p-q)^2u^2v^2+(q-r)^2v^2+r^2} + r, \\
Y &= \frac{y}{(x-r)^2+y^2+z^2} = \frac{(p-q)v^2+p}{(p-r)^2u^2+(p-q)^2u^2v^2+(q-r)^2v^2+r^2}\,u, \qquad (8) \\
Z &= \frac{z}{(x-r)^2+y^2+z^2} = \frac{(q-p)u^2+q}{(p-r)^2u^2+(p-q)^2u^2v^2+(q-r)^2v^2+r^2}\,v.
\end{aligned}
$$

Since the inversion center $(r, 0, 0)$ is on both planes of symmetry, the xy- and xz-planes, the inverse surfaces have the same planes of symmetry. The inverse surfaces are illustrated in Figure 3 with the cross sections between the surfaces and their planes of symmetry. The points are the inversion centers and the dashed curves are the inversion spheres in the figure. Equation (8) is a parameterization of Dupin cyclides with the shape parameters p, q and r.

The two intersecting lines on parabolic ring cyclides are inverted to two circles on ring cyclides. Since the circles are called Villarceau circles, we may call the two straight lines Villarceau lines.

While right circular cylinders are obtained as the inverse surfaces of thorn cyclides with the inversion center at the origin, as illustrated in Figure 3, right circular cones are obtained via inversion of parabolic horn cyclides in a circle with the center at one of the pinch points $(q, \pm\sqrt{(p-q)q}, 0)$. The resultant circular cones are obtained with a half-vertex angle of $\arctan\sqrt{q/(p-q)}$.

Various cyclides are obtained via inversion by specifying various points as the inversion center.

§4. Rational Curves on Dupin Cyclides

Special rational curves [3] on Dupin cyclides are shown in this section. Since the curves are inverse curves of conics, they are planar or spherical. Figure 3 will be useful in imaging the rational curves. The rational curves include the inverse curves of conics on right circular cones, because the inverse surfaces of right circular cones are also Dupin cyclides.

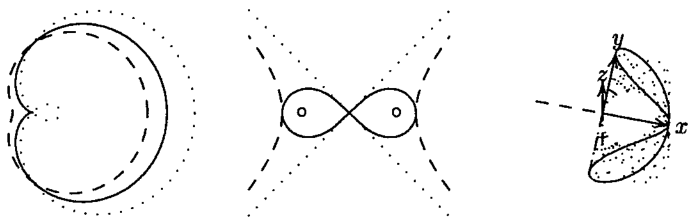
4.1 Rational quartics on Dupin cyclides

The polar equation of a conic, of which the focus is at the pole, is given by $r = l/(1 + e\cos\theta)$, where e is the eccentricity and l is a constant. The conic is a parabola ($e = 1$), an ellipse ($0 \le e < 1$), a circle ($0 = e$), a hyperbola ($1 < e$) or a rectangular hyperbola ($e = \sqrt{2}$).

The inverse curve $r = (1 + e\cos\theta)/l$ is called a limaçon of Pascal, and is expressed as a rational quartic. Hence, there are limaçons of Pascal on Dupin cyclides as the inverse curves of a conic with the inversion center at their foci. Some limaçons of Pascal are illustrated in Figure 4 (1). In the case of $e = 1$, a limaçon of Pascal is called a cardioid. There are cardioids on horn cyclides inverted from a circular cone with its inversion center at the focus of a parabola on the cone.

The inverse of rectangular hyperbolas $x^2 - y^2 = a^2$ with respect to the center are obtained as the curves $a^2(x^2+y^2)^2 = x^2 - y^2$. The curves are called lemniscates of Bernoulli, and are illustrated in Figure 4 (2). The lemniscates are also rational quartic curves on Dupin cyclides.

The inverse surface of the thorn cyclide in the sphere $x^2 + y^2 + z^2 = p^2$ is the right circular cylinder $(x - p/2)^2 + z^2 = (p/2)^2$, as illustrated in Figure 3. The intersection of the sphere and the cylinder is called a Viviani's curve. As the intersection is fixed under the inversion, the Viviani's curve is also on the



(1) limaçons of Pascal (2) lemniscate of Bernoulli (3) Viviani's curve

Fig. 4. Rational quartics on Dupin cyclides.

thorn cyclide, and is expressed as

$$\frac{p}{(1+t^2)^2}\begin{pmatrix}(1-t^2)^2\\ 2t(1+t^2)\\ -2t(1-t^2)\end{pmatrix}, \tag{9}$$

which is the image of the curve in the parameter space

$$u^2 - \frac{1}{v^2} = 1, \qquad (u,v) = \left(\frac{1+t^2}{2t}, \frac{2t}{1-t^2}\right). \tag{10}$$

Figure 4 (3) illustrates the Viviani's curve. The Viviani's curve is inverted into a rectangular hyperbola with the inversion center at the knot of the curve.

4.2 Rational cubics on parabolic cyclides

The intersection between a parabolic cyclide and the plane $y = p/2$ is expressed as $(x-p)z^2+(x-q)(x-p/2)^2 = 0$ from (4). The curve is parameterized with a rational cubic as

$$x(t) = \frac{q+pt^2}{1+t^2}, \qquad z(t) = \frac{t(q-\frac{p}{2}(1-t^2))}{1+t^2}. \tag{11}$$

The curve is obtained as the pedal curve [3] of the parabola $z^2 = -2(x-q)$ with respect to the point $(x,z) = (p/2, 0)$. A pedal curve is the locus of the foot of the perpendicular from a fixed point to a variable tangent to a given curve. The curve has an asymptote $x = p$ and a singular point (node, cusp or isolated point) at $(p/2, 0)$. The curve may be called a trisectrix of Maclaurin ($q = -p$), a right strophoid ($q = 0$), or a cissoid of Diocles ($q = p/2$). Figure 5 illustrates the curve for various values of the shape parameter q.

The inverse curve in the circle $(x-p/2)^2 + z^2 = 1$ becomes

$$X = \frac{x-\frac{p}{2}}{(x-\frac{p}{2})^2+z^2} = \frac{1}{q+\frac{p}{2}(t^2-1)}, \quad Z = \frac{z}{(x-\frac{p}{2})^2+z^2} = \frac{t}{q+\frac{p}{2}(t^2-1)}, \tag{12}$$

and is implicitized into the conic $(p-2q)X^2 + 2X - pZ^2 = 0$. Therefore, the cubic curve is an inverse curve of a conic on a Dupin cyclide.

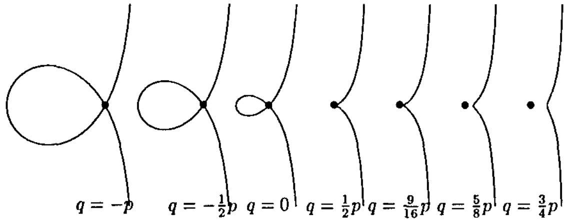


Fig. 5. Rational cubics on parabolic cyclides.

§5. Conclusion

It has been shown that there are central conics on parabolic Dupin cyclides. The central conics are cross sections between parabolic cyclides and the plane perpendicular to the two planes of symmetry of the cyclides. The preimage curves in the parameter space are also central conics.

Special cubic or quartic curves, which have been investigated for a long time, are also found on Dupin cyclides. They are planar or spherical rational curves and the images of conics on Dupin cyclides under inversion.

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