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A Basis for Homogeneous Polynomial Solutions to Homogeneous Constant Coefficient PDE's: An Algorithmic Approach through Apolarity

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Abstract. Some recent methods of Computer Aided Geometric Design are related to the *apolar bilinear form*, an inner product on the space of homogeneous multivariate polynomials of a fixed degree, already known in 19th century invariant theory. A generalized version of this inner product was introduced in [8] to derive in a straightforward way some of the recent results in CAGD. Here we extend this work by applying it to compute solution spaces of homogeneous constant coefficient PDE's.

§1. The Homogeneous Apolar Bilinear Form

1.1. Review

In [8] we introduced a generalization of the *apolar bilinear form* defined on the space of homogeneous polynomials (of a certain degree, and with a fixed number of variables). This bilinear form, used extensively in the symbolic method of the classical theory of invariants, has been revitalized by Rota and his co-workers, cf [2] and [4]. In CAGD, a similar binary form on the space of *univariate* polynomials of a fixed degree has been studied by Goldman [3]. It is related to the blossoming approach introduced by Ramshaw [7].

In this section we review some of the properties of the apolar bilinear form. Then we extend [8] by studying constant coefficient partial differential equations of the form $p(\partial)f = 0$, where p is a fixed multivariate homogeneous polynomial. In particular, we derive an algorithm computing a basis for solution spaces consisting of homogeneous polynomials of a fixed degree.

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1.2 Vector spaces of forms

Let $e_1, \dots, e_s$ be the standard basis vectors on $\mathbb{R}^s$, and let $x = (x_1, \dots, x_s)$ be the standard coordinates on $\mathbb{R}^s$. The standard inner product on $\mathbb{R}^s$ is denoted by $(\cdot,\cdot)$, i.e., $(u,v) = u_1v_1 + \cdots + u_sv_s$, for $u, v \in \mathbb{R}^s$.

A central object in this paper is the space of real homogeneous polynomials of degree n on $\mathbb{R}^s$, denoted by $\mathcal{H}_n(\mathbb{R}^s)$. A polynomial in $\mathcal{H}_n(\mathbb{R}^s)$ is the sum of monomials of the form $c_\alpha x_1^{\alpha_1} \cdots x_s^{\alpha_s}$, where $c_\alpha \in \mathbb{R}$ and $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{Z}^s_{\geq 0}$ is a multi-index of weight $|\alpha| = \alpha_1 + \cdots + \alpha_s$. For convenience the monomial $x_1^{\alpha_1} \cdots x_s^{\alpha_s}$ is denoted by x^α. Linear homogeneous polynomials on $\mathbb{R}^s$ are of the form $f(x) = (u, x)$, for some $u \in \mathbb{R}^s$. We denote f by $(u, \cdot)$.

For multi-indices $\alpha = (\alpha_1, \dots, \alpha_s)$ and $\beta = (\beta_1, \dots, \beta_s)$ in $\mathbb{Z}^s_{\geq 0}$ we define $\alpha \mid \beta$ iff $\alpha_i \leq \beta_i$ for $i = 1, \cdots, s$. The relation $\mid$ is a partial order on $\mathbb{Z}^s_{\geq 0}$. Note that $\alpha \mid \beta$ iff there is a $\lambda \in \mathbb{Z}^s_{\geq 0}$ such that $\beta = \alpha + \lambda$. We shall write $\alpha \nmid \beta$ if it is not the case that $\alpha \mid \beta$. A monomial order is a linear ordering $\leq_{\mathsf{mon}}$ on $\mathbb{Z}^s_{\geq 0}$ such that (i) if $\alpha \leq_{\mathsf{mon}} \beta$ and $\gamma \in \mathbb{Z}^s_{\geq 0}$, then $\alpha + \gamma \leq_{\mathsf{mon}} \beta + \gamma$, and (ii) $\leq_{\mathsf{mon}}$ is a well-ordering on $\mathbb{Z}^s_{\geq 0}$, i.e., every non-empty subset of $\mathbb{Z}^s_{\geq 0}$ has a smallest element with respect to $\leq_{\mathsf{mon}}$. We use the notation $\alpha <_{\mathsf{mon}} \beta$ in case $\alpha \leq_{\mathsf{mon}} \beta$ and $\alpha \neq \beta$. Furthermore, we use the property that $\alpha \leq_{\mathsf{mon}} \beta$ whenever $\alpha \mid \beta$. Well known examples are the graded (reverse) lexicographic orders, defined by $\alpha \leq_{\mathsf{mon}} \beta$ if $|\alpha| < |\beta|$, or $|\alpha| = |\beta|$ and in $\alpha - \beta$ the left-most (right-most) non-zero entry is negative (positive). Monomial orders play a paramount role in algorithms for multivariate polynomials, especially with regard to termination conditions; See e.g. [1].

The set of multi-indices in $\mathbb{Z}^s_{\geq 0}$ of weight n, denoted by $\Gamma_{s,n}$, is a finite set with $\#\Gamma_{s,n} = \binom{n+s-1}{n}$ elements. For $\alpha \in \Gamma_{s,n}$ the factorial function is defined by $\alpha! = \alpha_1! \cdots \alpha_s!$, and the multinomial coefficient $\binom{n}{\alpha}$ is defined by

$$\binom{n}{\alpha} = \frac{n!}{\alpha_1! \cdots \alpha_s!}.$$

With a polynomial $f(x) = \sum_{\alpha \in \Gamma_{s,m}} c_\alpha\, x^\alpha$, we associate the homogeneous differential operator $f(\partial) = \sum_{\alpha \in \Gamma_{s,m}} c_\alpha\, \partial^\alpha$, where $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_s^{\alpha_s}$. Here $\partial = (\partial_1, \dots, \partial_s)$, with $\partial_i = \partial / \partial x_i$. The directional derivative $D_u : \mathcal{H}_n(\mathbb{R}^s) \to \mathcal{H}_{n-1}(\mathbb{R}^s)$ with respect to $u \in \mathbb{R}^s$ is the differential operator (u, ∂), i.e., $D_u = u_1\, \partial_1 + \cdots + u_s\, \partial_s$. Note that $\partial_i = (e_i, \partial) = D_{e_i}$. Considering e_i as a multi-index of weight one, we also have $\partial_i = \partial^{e_i}$.

1.3. Apolar pairing

This subsection is concerned with a straightforward generalization of the rather well-known apolar inner product $[\,f\,,\,g\,] = f(\partial)g$, defined on the space of homogeneous polynomials $\mathcal{H}_n(\mathbb{R}^s)$. The main result concerns a characterization of this inner product in terms of three simple properties that will be the basis for the construction of special bases of $\mathcal{H}_n(\mathbb{R}^s)$ in later sections.

Definition 1. *For fixed integers m and n, with $0 \leq m \leq n$, the* apolar pairing *is the map*

$$[\cdot\,,\cdot]_{m,n} : \mathcal{H}_m(\mathbb{R}^s) \times \mathcal{H}_n(\mathbb{R}^s) \to \mathcal{H}_{n-m}(\mathbb{R}^s),$$

associating to the homogeneous polynomials $f \in \mathcal{H}_m(\mathbb{R}^s)$ and $g \in \mathcal{H}_n(\mathbb{R}^s)$ the homogeneous polynomial $[f\,, g]_{m,n}$ of degree $n-m$, defined by

$$[f\,, g]_{m,n} = \frac{(n-m)!}{n!} f(\partial)g.$$

Note that we have in fact a family of pairings, one for each pair of integers m and n with $0 \leq m \leq n$. In this paper, the term pairing refers to the whole family of bilinear maps. From now on we shall drop the subscripts m and n, since they are implicitly known as the degree of the first and second argument of the pairing operator.

Theorem 2. *The apolar pairing is the unique bilinear pairing with the following properties:*

1) (Apolar pairing with constants). *For $f \in \mathcal{H}_n(\mathbb{R}^s)$:*

$$[1\,, f] = f,$$

where $1 \in \mathcal{H}_0(\mathbb{R}^s)$ is the constant homogeneous polynomial of degree 0.

2) (Apolar pairing with linear forms). *For $f \in \mathcal{H}_n(\mathbb{R}^s)$ and $u \in \mathbb{R}^s$:*

$$[(u,\cdot)\,, f] = \frac{1}{n} D_u f.$$

3) (Transposition of a homogeneous factor). *For $f_1 \in \mathcal{H}_{m_1}(\mathbb{R}^s)$, $f_2 \in \mathcal{H}_{m_2}(\mathbb{R}^s)$, and $g \in \mathcal{H}_n(\mathbb{R}^s)$, with $m_1 + m_2 \leq n$:*

$$[f_1 f_2\,, g] = [f_1\,, [f_2\,, g]].$$

It is obvious that apolar pairing is a bilinear operator, satisfying these properties. For the proof of uniqueness, we refer to [8]. Identifying the space of zero degree polynomials with $\mathbb{R}$, we see that for $n = m$, apolar pairing corresponds to a real bilinear form on the space of homogeneous polynomials of degree m. The next result states that this bilinear form is even an inner product. Again, the proof is contained in [8].

Proposition 3. *The apolar bilinear form $[\cdot\,,\cdot] : \mathcal{H}_m(\mathbb{R}^s) \times \mathcal{H}_m(\mathbb{R}^s) \to \mathbb{R}$ is an inner product on the space of homogeneous polynomials of degree m.*

1.3. Dual bases

First we recall the definition of a dual basis pair with respect to the apolar inner product $[\cdot\,,\,\cdot]$ on $\mathcal{H}_m(\mathbb{R}^s)$.

Definition 4. *The dual basis of a basis $\{f_\alpha \mid \alpha \in \Gamma_{s,m}\}$ of $\mathcal{H}_m(\mathbb{R}^s)$ is a collection $\{g_\alpha \mid \alpha \in \Gamma_{s,m}\}$ of polynomials in $\mathcal{H}_m(\mathbb{R}^s)$ such that, for $\alpha, \beta \in \Gamma_{s,m}$:*

$$[\,f_\alpha\,,\,g_\beta\,] = \delta_{\alpha,\beta}.$$

It is easy to prove the standard fact from linear algebra that a dual basis is indeed a basis. Given a dual basis pair, a polynomial $f \in \mathcal{H}_m(\mathbb{R}^s)$ can be expressed with respect to either basis in terms of coefficients depending on the other one:

$$f = \sum_{\alpha\in\Gamma_{s,m}} [\,g_\alpha\,,\,f\,]f_\alpha = \sum_{\alpha\in\Gamma_{s,m}} [\,f_\alpha\,,\,f\,]g_\alpha. \tag{1}$$

Example. (Dual of homogeneous Bernstein-Bézier basis). Let $\{x^1, \ldots, x^s\}$ be a basis of $\mathbb{R}^s$, and denote by $u_1(x), \ldots, u_s(x)$ the coordinates of any $x \in \mathbb{R}^s$ in this basis. The polynomials

$$B_\alpha(x) = \binom{n}{\alpha} u_1(x)^{\alpha_1} \cdots u_s(x)^{\alpha_s},$$

where $\alpha \in \Gamma_{s,n}$, form the homogeneous Bernstein-Bézier basis of $\mathcal{H}_n(\mathbb{R}^s)$ with respect to the basis $\{x^1, \ldots, x^s\}$ of $\mathbb{R}^s$. Its dual basis consists of the polynomials

$$l_\beta(y) = (x^1, y)^{\beta_1} \cdots (x^s, y)^{\beta_s},$$

i.e., $[\,B_\alpha\,,\,l_\beta\,] = \delta_{\alpha,\beta}$. For the proof, see [8].

§2. Solving Homogeneous Constant Coefficient PDE's

We now show how dual bases can be used for the efficient computation of a basis for the solution space of a homogeneous partial differential equation with constant coefficients, i.e., the space

$$\{f \in \mathcal{H}_n(\mathbb{R}^s) \mid p(\partial)f = 0\}. \tag{2}$$

Here $p \in \mathcal{H}_m(\mathbb{R}^s)$ is a polynomial, $p \neq 0$, that will be fixed throughout the paper. Furthermore, m and n denote fixed integers such that $0 \leq m \leq n$. Our approach is both an alternative and an algorithmic counterpart of Pedersen's work [5,6]. These papers deal with algebraic properties of the space of solutions. We continue Pedersen's work by presenting an optimal algorithm for the computation of a basis for the solution space. Our techniques are new, since they are based on properties of dual bases, together with some recursive properties of the apolar bilinear form introduced in [8].

2.1. Characterizing a basis for the space of solutions

From now on we consider a family of functions $\{f_\alpha \mid \alpha \in \mathbb{Z}^s_{\geq 0}\}$ such that (i) $f_\alpha \cdot f_\beta = f_{\alpha+\beta}$, and (ii) for all $n \geq 0$, the set $\{f_\alpha \mid \alpha \in \Gamma_{s,n}\}$ is a basis of $\mathcal{H}_n(\mathbb{R}^s)$. The dual basis of the latter set is denoted by $\{g_\alpha \mid \alpha \in \Gamma_{s,n}\}$. An example of such a pair of bases is formed by the Bernstein-Bézier basis, together with the lineal polynomials introduced in the example at the end of the preceding section.

Lemma 5. *For $\alpha, \beta \in \mathbb{Z}^s_{\geq 0}$ with $|\alpha| \leq |\beta|$:*

$$[\, f_\alpha \,,\, g_\beta \,] = \begin{cases} g_{\beta-\alpha}, & \text{if } \alpha \mid \beta, \\ 0, & \text{if } \alpha \nmid \beta. \end{cases}$$

Proof: Let $m = |\alpha|$ and $n = |\beta|$, and let $f = [\, f_\alpha \,,\, g_\beta \,] \in \mathcal{H}_{n-m}(\mathbb{R}^s)$. Consider the apolar inner product $[\, f_\gamma \,,\, f \,]$, for $\gamma \in \Gamma_{s,n-m}$. Since $f_\alpha \cdot f_\gamma = f_{\alpha+\gamma}$, transposition of a factor (See Theorem 2, part 3) f_α yields: $[\, f_\gamma \,,\, f \,] = [\, f_{\alpha+\gamma} \,,\, g_\beta \,] = \delta_{\alpha+\gamma,\beta}$. First consider the case $\alpha \nmid \beta$. Then $\alpha + \gamma \neq \beta$, and hence $[\, f_\gamma \,,\, f \,] = 0$, for all $\gamma \in \Gamma_{s,n-m}$. Since the apolar pairing is an inner product and $\{f_\gamma \mid \gamma \in \Gamma_{s,n-m}\}$ is a basis of $\mathcal{H}_{n-m}(\mathbb{R}^s)$, it follows that $f = 0$ in this case. If $\alpha \mid \beta$ the previous derivation shows that $[\, f_\gamma \,,\, f \,] = \delta_{\gamma,\beta-\alpha}$, so identity (1) implies $f = \sum_{\gamma \in \Gamma_{s,n-m}} [\, f \,,\, f_\gamma \,]\, g_\gamma = g_{\beta-\alpha}$. □

In the following, our fixed polynomial p in (2) is given in the form

$$p = \sum_{\alpha \in \Gamma_{s,m}} c_\alpha \, f_\alpha, \text{ where } c_\alpha = [\, p \,,\, g_\alpha \,].$$

The following result characterizing the kernel of a polynomial differential operator is the key ingredient for the algorithm developed in the next section.

With p we associate the linear map $D_p : \mathcal{H}_n(\mathbb{R}^s) \to \mathcal{H}_{n-m}(\mathbb{R}^s)$ defined by $D_p(f) = [\, p \,,\, f \,]$, and the map $T_p : \mathcal{H}_{n-m}(\mathbb{R}^s) \to \mathcal{H}_n(\mathbb{R}^s)$ is multiplication by p, i.e., $T_p(f) = p \cdot f$. Given an integer k, and a subspace $U \subset \mathcal{H}_k(\mathbb{R}^s)$, we denote by $U^\perp$ the orthogonal complement of U with respect to the apolar inner product $[\,\cdot\,,\,\cdot\,]$ on $\mathcal{H}_k(\mathbb{R}^s)$.

Proposition 6.
1) *$KerD_p = (ImT_p)^\perp$.*
2) *The map D_p is onto.*

Proof: Theorem 2, part 3, implies that T_p and D_p are adjoint operators, i.e., $[\, T_p(f) \,,\, g \,] = [\, f \,,\, D_p(g) \,]$, for $f \in \mathcal{H}_{n-m}(\mathbb{R}^s)$ and $g \in \mathcal{H}_n(\mathbb{R}^s)$. The first claim follows from this identity. Now since T_p is injective, the result of the first part implies that $\mathit{dim}\, \mathrm{Ker} D_p = \mathit{dim}\, \mathcal{H}_n(\mathbb{R}^s) - \mathit{dim}\, \mathcal{H}_{n-m}(\mathbb{R}^s)$. Therefore, $\mathit{dim}\, \mathrm{Im} D_p = \mathit{dim}\, \mathcal{H}_{n-m}(\mathbb{R}^s)$, and hence D_p is onto. □

As a special case, consider the polynomial $p = f_{\alpha_0}$ for some $\alpha_0 \in \Gamma_{s,m}$. According to Lemma 5, $\mathrm{Ker} D_p$ contains g_β whenever $\beta \in \Gamma_{s,n}$ such that $\alpha_0 \nmid \beta$. Since $\alpha_0 \mid \beta$ iff β is of the form $\beta = \alpha_0 + \lambda$ for some $\lambda \in \Gamma_{s,n-m}$, it follows that $\#\{\beta \in \Gamma_{s,n} \mid \alpha_0 \dotplus \beta\} = \#\Gamma_{s,n} - \#\Gamma_{s,n-m} = dim\,\mathrm{Ker}D_p$. The last equality follows from Proposition 6, part 1. Therefore, a basis for the solution space $\mathrm{Ker}D_p$ is the collection $\{g_\beta \mid \beta \in \Gamma_{s,n}$ and $\alpha_0 \dotplus \beta\}$. The following result generalizes this special case.

Theorem 7. (Basis for solution space of PDE). *Let $\leq_{\mathrm{mon}}$ be a monomial order on $\mathbb{Z}^s_{\geq 0}$ and $\alpha_0 \in \Gamma_{s,m}$ be defined by $\alpha_0 = min_{\leq_{\mathrm{mon}}}\{\alpha \in \Gamma_{s,m} \mid [\, p\,,\, g_\alpha\,] \neq 0\}$. Furthermore, for any $\lambda \in \Gamma_{s,n-m}$, let $p_\lambda \in \mathcal{H}_{n-m}(\mathbb{R}^s)$ be the polynomial defined by $p_\lambda = [\, p\,,\, g_{\alpha_0+\lambda}\,]$. Then*

1) *The set $\mathcal{P}_{n-m} = \{p_\lambda \mid \lambda \in \Gamma_{s,n-m}\}$ is a basis of $\mathcal{H}_{n-m}(\mathbb{R}^s)$.*
2) *Let $\mathcal{Q}_{n-m} = \{q_\lambda \mid \lambda \in \Gamma_{s,n-m}\}$ be the dual of the basis $\mathcal{P}_{n-m}$ of $\mathcal{H}_{n-m}(\mathbb{R}^s)$, i.e., $[\, p_\lambda\,,\, q_\mu\,] = \delta_{\lambda,\mu}$. A basis for the solution space $KerD_p = \{f \in \mathcal{H}_n(\mathbb{R}^s) \mid p(\partial)f = 0\}$ is the set*

$$\{\overline{g}_\beta \mid \beta \in \Gamma_{s,n} \text{ with } \alpha_0 \dotplus \beta\}, \tag{3}$$

where $\overline{g}_\beta \in \mathcal{H}_n(\mathbb{R}^s)$ is defined by

$$\overline{g}_\beta = g_\beta - \sum_{\lambda \in \Gamma_{s,n-m}} [\, p \cdot q_\lambda\,,\, g_\beta\,]\, g_{\alpha_0+\lambda}.$$

Remark. The first claim of Theorem 7 is not necessarily true for other choices of α_0. Consider e.g. the polynomial $p(x) = 2x_1^2 + 2x_1x_2 + x_2^2$, and let $\alpha_0 = (1,1) \in \Gamma_{2,2}$. Here we take the monomial basis for the space of polynomials of degree n on $\mathbb{R}^2$, i.e., we take $f_\beta(x) = x^\beta$, for $x \in \mathbb{R}^2$ and $|\beta| = \beta_1 + \beta_2 = n$. The dual basis consists of the functions g_β, where $g_\beta(x) = \binom{n}{\beta}x^\beta$. For $\lambda \in \Gamma_{2,2}$ we have $g_{\alpha_0+\lambda}(x) = \binom{n}{\alpha_0+\lambda}x_1^{\lambda_1+1}x_2^{\lambda_2+1}$. Take $q(x) = 2x_1^2 - 2x_1x_2 + x_2^2$, then $p(x) \cdot q(x) = 4x_1^4 + x_2^4$, and hence, for all $\lambda \in \Gamma_{2,2}$:

$$[\, p_\lambda\,,\, q\,] = [\,[\, p\,,\, g_{\alpha_0+\lambda}\,]\,,\, q\,] = [\, p \cdot q\,,\, g_{\alpha_0+\lambda}\,] = 0,$$

yet $q \neq 0$. Hence the functions p_λ, where λ ranges over $\Gamma_{2,2}$, do not constitute a basis for $\mathcal{H}_2(\mathbb{R}^2)$.

To prove Theorem 7 we need the following two lemmas.

Lemma 8. *For $\lambda, \mu \in \Gamma_{s,n-m}$ we have*

$$[\, p_\lambda\,,\, f_\mu\,] = \begin{cases} c_{\alpha_0+\lambda-\mu}, & \text{if } \mu \mid \alpha_0 + \lambda, \\ 0, & \text{otherwise.} \end{cases}$$

Proof: The proof consists of a straightforward calculation:

$$\begin{aligned}[\, p_\lambda\,,\, f_\mu\,] &= [\, f_\mu\,,\, [\, p\,,\, g_{\alpha_0+\lambda}\,]\,] = [\, p\,,\, [\, f_\mu\,,\, g_{\alpha_0+\lambda}\,]\,] \\ &= \begin{cases} [\, p\,,\, g_{\alpha_0+\lambda-\mu}\,], & \text{if } \mu \mid \alpha_0 + \lambda, \\ 0, & \text{otherwise.} \end{cases}\end{aligned}$$

The last identity is justified by applying Lemma 5. □

Lemma 9. *Let $\lambda_0 \in \Gamma_{s,n-m}$. For $f \in \mathcal{H}_{n-m}(\mathbb{R}^s)$, the following statements are equivalent:*

1) $[\,g_\lambda\,,\,f\,] = 0$, *for all* $\lambda \leq_{\text{mon}} \lambda_0$,
2) $[\,p_\lambda\,,\,f\,] = 0$, *for all* $\lambda \leq_{\text{mon}} \lambda_0$.

Proof: It follows from (1) that

$$[\,p_\lambda\,,\,f\,] = \sum_{\mu \in \Gamma_{s,n-m}} [\,p_\lambda\,,\,f_\mu\,][\,g_\mu\,,\,f\,]. \tag{4}$$

Consider $\mu \in \Gamma_{s,n-m}$ such that $\lambda <_{\text{mon}} \mu$, then $\alpha_0 + \lambda - \mu <_{\text{mon}} \alpha_0$. Therefore, the definition of α_0 implies $c_{\alpha_0+\lambda-\mu} = 0$. In view of Lemma 8, we know that $[\,p_\lambda\,,\,f_\mu\,]$ is equal either to $c_{\alpha_0+\lambda-\mu}$ or to 0, so in any case we have $[\,p_\lambda\,,\,f_\mu\,] = 0$. This observation allows us to write (4) as

$$[\,p_\lambda\,,\,f\,] = c_{\alpha_0}[\,g_\lambda\,,\,f\,] + \sum_{\mu : \mu <_{\text{mon}} \lambda} [\,p_\lambda\,,\,f_\mu\,][\,g_\mu\,,\,f\,]. \tag{5}$$

This identity shows that the first statement implies the second one. So assume that statement 2 holds. We may assume that $f \neq 0$, otherwise there is nothing to prove. Let λ_1 be the least multi-index with respect to the monomial order $\leq_{\text{mon}}$ such that $[\,g_{\lambda_1}\,,\,f\,] \neq 0$. Then (5) implies $[\,p_{\lambda_1}\,,\,f\,] = c_{\alpha_0}[\,g_{\lambda_1}\,,\,f\,]$ is nonzero. Hence $\lambda_0 <_{\text{mon}} \lambda_1$. Consequently $[\,g_\lambda\,,\,f\,] = 0$ for all $\lambda \leq_{\text{mon}} \lambda_0$, which is statement 1. □

Proof of Theorem 7: Let $U \subset \mathcal{H}_{n-m}(\mathbb{R}^s)$ be the space spanned by the p_λ, $\lambda \in \Gamma_{s,n-m}$. Since $\#\Gamma_{s,n-m} = \mathit{dim}\,\mathcal{H}_{n-m}(\mathbb{R}^s)$, it is sufficient to prove that $U = \mathcal{H}_{n-m}(\mathbb{R}^s)$, or, equivalently, that $U^\perp = \{0\}$. Thus, if $f \in U^\perp$, then $[\,f\,,\,p_\lambda\,] = 0$, for all $\lambda \in \Gamma_{s,n-m}$. According to Lemma 9 this implies $[\,f\,,\,p_\lambda\,] = 0$, for all $\lambda \in \Gamma_{s,n-m}$, so $f = 0$. This proves 1). Now in view of Proposition 6, the space $\mathrm{Ker}D_p$ is of dimension $\#\Gamma_{s,n} - \#\Gamma_{s,n-m}$, i.e., of dimension $\#\{\beta \in \Gamma_{s,n} \mid \alpha_0 \nmid \beta\}$. On the other hand, it is straightforward to see that the polynomials $\overline{g}_\beta$, $\beta \in \Gamma_{s,n}$ with $\alpha_0 \nmid \beta$, are linearly independent. Therefore, in order to prove that they form a basis of $\mathrm{Ker}D_p$ we just have to prove that they belong to $\mathrm{Ker}D_p$. Taking Proposition 6, part 1, into account, we actually have to check that $[\,p \cdot q_\mu\,,\,\overline{g}_\beta\,] = 0$, for all $\beta \in \Gamma_{s,n}$ with $\alpha_0 \nmid \beta$. Since

$$\begin{aligned}
p \cdot q_\mu &= \sum_{\gamma \in \Gamma_{s,n}} [\,p \cdot q_\mu\,,\,g_\gamma\,]\,f_\gamma \\
&= \sum_{\lambda \in \Gamma_{s,n-m}} [\,[\,p\,,\,g_{\alpha_0+\lambda}\,]\,,\,q_\mu\,]\,f_{\alpha_0+\lambda} + \sum_{\substack{\gamma \in \Gamma_{s,n} \\ \alpha_0 \nmid \gamma}} [\,p \cdot q_\mu\,,\,g_\gamma\,]\,f_\gamma \\
&= f_{\alpha_0+\mu} + \sum_{\substack{\gamma \in \Gamma_{s,n} \\ \alpha_0 \nmid \gamma}} [\,p \cdot q_\mu\,,\,g_\gamma\,]\,f_\gamma,
\end{aligned}$$

it follows that $[\,p \cdot q_\mu\,,\,\overline{g}_\beta\,] = 0$, for all $\beta \in \Gamma_{s,n}$ with $\alpha_0 \nmid \beta$. □

2.2. Computing a basis for the space of solutions

We now present a simple, efficient algorithm for computing the dual basis $\mathcal{Q}_{n-m}$, as well as an example showing how the algorithm works. Recall that for $\alpha = (\alpha_1, \ldots, \alpha_s) \in \Gamma_{s,m}$, the number c_α is equal to $[p, g_\alpha]$. We extend this definition to $\alpha \in \mathbb{Z}^s$ by putting $c_\alpha = 0$ in case at least one of the entries $\alpha_1, \ldots, \alpha_s$ is negative.

Corollary 10.

1) *The dual basis $\mathcal{Q}_{n-m} = \{q_\mu \mid \mu \in \Gamma_{s,n-m}\}$ of $\mathcal{P}_{n-m}$ is defined recursively by*

$$q_\mu = \frac{1}{c_{\alpha_0}}\bigl(f_\mu - \sum_{\substack{\nu \in \Gamma_{s,n-m} \\ \mu <_{\text{mon}} \nu}} c_{\alpha_0+\nu-\mu} q_\nu\bigr).$$

2) *For $\beta \in \Gamma_{s,n}$, with $\alpha_0 \dagger \beta$, the basis function $\overline{g}_\beta \in \mathcal{H}_n(\mathbb{R}^s)$, is of the form*

$$\overline{g}_\beta = g_\beta - \sum_{\mu \in \Gamma_{s,n-m}} a_{\mu\beta}\, g_{\alpha_0+\mu}.$$

where the coefficients $a_{\mu\beta}$ are defined recursively, for $\mu \in \Gamma_{s,n-m}$, by

$$a_{\mu\beta} = \frac{1}{c_{\alpha_0}}\bigl(c_{\beta-\mu} - \sum_{\substack{\nu \in \Gamma_{s,n-m} \\ \mu <_{\text{mon}} \nu}} c_{\alpha_0+\nu-\mu} a_{\nu\beta}\bigr).$$

Proof: Recall that we are looking for a set of functions $\mathcal{Q}_{n-m} = \{q_\mu \mid \mu \in \Gamma_{s,n-m}\}$, such that $[p_\nu, q_\mu] = \delta_{\nu,\mu}$. In particular, according to Lemma 9 the functions q_μ satisfy $[g_\nu, q_\mu] = 0$, for $\nu <_{\text{mon}} \mu$. Therefore, $q_\mu \in \text{Span}\{f_\nu \mid \nu \in \Gamma_{s,n-m}$ and $\mu \leq_{\text{mon}} \nu\}$, or, equivalently:

$$q_\mu \in \text{Span}(\{q_\nu \mid \nu \in \Gamma_{s,n-m} \text{ and } \mu <_{\text{mon}} \nu\} \cup \{f_\mu\}). \tag{6}$$

Assume we have determined q_ν for $\nu \in \Gamma_{s,n-m}$ with $\mu <_{\text{mon}} \nu$. To compute q_μ satisfying (6), we have to determine constants $d_{\mu\nu}$, for $\mu, \nu \in \Gamma_{s,n-m}$ with $\mu \leq_{\text{mon}} \nu$, such that

$$q_\mu = d_{\mu\mu} f_\mu + \sum_{\substack{\lambda \in \Gamma_{s,n-m} \\ \mu <_{\text{mon}} \lambda}} d_{\mu\lambda} q_\lambda.$$

Since $\mathcal{P}_{n-m}$ and $\mathcal{Q}_{n-m}$ are dual bases, the constants $d_{\mu\nu}$ are uniquely determined by the condition $[p_\nu, q_\mu] = \delta_{\nu,\mu}$. Combining the last two identities we see that

$$[p_\nu, q_\mu] = d_{\mu\mu}[p_\nu, f_\mu] + \sum_{\substack{\lambda \in \Gamma_{s,n-m} \\ \mu <_{\text{mon}} \lambda}} d_{\mu\lambda} \delta_{\lambda,\nu}.$$

From this identity, which holds for all $\nu \in \Gamma_{s,n-m}$ with $\mu \leq_{\text{mon}} \nu$, we derive

$$d_{\mu\mu} = \frac{1}{c_{\alpha_0}},$$

$$d_{\mu\nu} = -\frac{c_{\alpha_0-\mu+\nu}}{c_{\alpha_0}}, \text{ for } \mu <_{\text{mon}} \nu,$$

which proves the first part. Now put $a_{\mu\beta} = [\,p\cdot q_\mu\,,\,g_\beta\,]$ in (3). Then, according to part 1,

$$a_{\mu\beta} = \frac{1}{c_{\alpha_0}}\Big([\,p\cdot f_\mu\,,\,g_\beta\,] - \sum_{\substack{\nu\in\Gamma_{s,n-m}\\ \mu<_{\mathrm{mon}}\nu}} c_{\alpha_0+\nu-\mu}[\,p\cdot q_\nu\,,\,g_\beta\,]\Big).$$

Since $[\,p\cdot f_\mu\,,\,g_\beta\,] = [\,p\,,\,[\,f_\mu\,,\,g_\beta\,]\,] = [\,p\,,\,g_{\beta-\mu}\,]$, the proof is complete. $\square$

The algorithm for computing a basis for the solution space of the partial differential equation $p(\partial)f = 0$ is now simple:

Algorithm (for computing a basis for KerD_p).

> **forall** $\mu \in \Gamma_{s,n-m}$ in decreasing $<_{\mathrm{mon}}$-order **do**
> **forall** $\beta \in \Gamma_{s,n}$ with $\alpha_0 + \beta$ **do**
> $$a_{\mu\beta} \leftarrow \frac{1}{c_{\alpha_0}}\Big(c_{\beta-\mu} - \sum_{\substack{\nu\in\Gamma_{s,n-m}\\ \mu<_{\mathrm{mon}}\nu}} c_{\alpha_0+\nu-\mu}a_{\nu\beta}\Big).$$

Example. Consider on $\mathbb{R}^3$ the homogeneous constant coefficient PDE

$$\frac{\partial^2 f}{\partial x_1\partial x_2} - \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_2\partial x_3} = 0,$$

corresponding to the homogeneous polynomial $p(x) = x_1x_2 - x_2^2 + x_2x_3$. In particular, the setting of this example corresponds to $s = 3$ and $m = 2$. We determine a basis for the solution space in $\mathcal{H}_3(\mathbb{R}^3)$, i.e., we take $n = 3$. To this end, consider the graded reverse lexicographical order on $\mathbb{Z}^3_{\geq 0}$. Let $f_\beta(x) = x^\beta$, and let $g_\beta(x) = \binom{n}{\beta}x^\beta$, where $n = |\beta|$. In this example we denote functions indexed by $\alpha = (i,j,k) \in \mathbb{Z}^3_{\geq 0}$, like f_α, by f_{ijk}.

The sets $\{f_\beta \mid \beta \in \Gamma_{s,n}\}$ and $\{g_\beta \mid \beta \in \Gamma_{s,n}\}$ are dual bases, and moreover $f_\alpha \cdot f_\beta = f_{\alpha+\beta}$, so the conditions for applying Theorem 7 and the algorithm from this section are satisfied. In the notation of Theorem 7, we have $\alpha_0 = (1,1,0)$. Note that $p = f_{110} - f_{020} + f_{011}$, so $c_{110} = 1$, $c_{020} = -1$ and $c_{011} = 1$, whereas all other coefficients c_{ijk}, with $i+j+k = 2$, are zero. Now the coefficients $a_{\mu\beta}$ are computed according to the algorithm above, in other words we successively determine the rows in the following table (computing for each row the entries in arbitrary order):

$a_{\mu\beta}$	$\beta = 300$	201	102	030	021	012	003
$\mu = 001$	0	0	0	0	-1	1	0
010	0	0	0	-1	1	0	0
100	0	0	0	-1	2	-1	0

This table corresponds to the following seven basis functions of KerD_p: g_{300}, g_{201}, g_{102}, $g_{030} + g_{120} + g_{210}$, $g_{021} + g_{111} - g_{120} - 2g_{210}$, $g_{012} - g_{111} + g_{210}$, and g_{003}.

These functions can be turned into monomial form by straightforward substitution, yielding the following explicit basis for the solution subspace of $\mathcal{H}_3(\mathbb{R}^3)$: x_1^3, $3x_1^2x_3$, $3x_1x_3^2$, $3x_1^2x_2 + 3x_1x_2^2 + x_2^3$, $-6x_1^2x_2 - 3x_1x_2^2 + 6x_1x_2x_3 + 3x_2^2x_3$, $3x_1^2x_2 - 6x_1x_2x_3 + 3x_2x_3^2$, x_3^3.

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