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# Approximated Planes in Parallel Coordinates

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**Abstract.** For the visualization of multivariate problems, a multidimensional system of *Parallel coordinates* is used which provides a one-to-one mapping between subsets of  $N$ -space and subsets of 2-space. A rigorous methodology for doing and *seeing*  $N$ -dimensional geometry emerges as well as several applications. Here an application to *Error Tolerancing* involving the visualization and characterization of “approximate coplanarity” is presented. The exact description of the neighborhood of an  $N$ -dimensional hyperplane in a parallel coordinate system is given.

## §1. Introduction

The parallel coordinate system serves as a tool for visualization of multi-dimensional objects and multivariate relations. It was shown ([1,2]) that this representation gives a simple and constructive geometrical description for subsets of points which are strictly coplanar (i.e. belong to a common  $p$ -flat in  $N$ -dimensional case). This allows the visualization of coplanar points and the existence of linear dependencies between variables. It leads to numerous applications in different fields, and also practical applications involving finite error tolerancing. Here exact descriptions of approximated hyperplanes in the parallel coordinate system are given, providing a methodology for their visualization.

In the next section a brief review of previous results is given. It is followed in Section [3] by an exact mathematical formulation of the problem in the general case. Sections [4] and [5] contain some auxiliary lemmas which make the main result more intuitive, as well as the main result itself.

Although we have a complete and precise proof of the main result in the general case of “approximated”  $p$ -flats in  $N$ -dimensional space, lack of space prevents us from presenting it here.

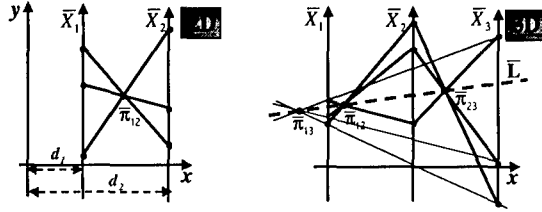


Fig. 1. Three collinear points in 2D (left) and 3D (right).

§2. Representation of Affine Subspaces in Parallel Coordinates

The parallel coordinate system is constructed in the following way: in the Euclidean plane  $\mathbb{R}^2$  (or more precisely in the 2D projective plane  $\mathbb{P}^2$ ) with  $xy$ -Cartesian coordinates,  $N$  copies of the axis  $y$  labeled  $\bar{X}_1, \dots, \bar{X}_N$  are placed equidistant (usually the distance between adjacent axes is taken as 1) and perpendicular to the  $x$ -axis. They are the axis of the Parallel Coordinate system.

A point with Euclidean coordinates  $(p_1, \dots, p_N)$  is represented by a polygonal line with  $N$  vertices  $\bar{\pi}_i = (i-1, p_i)$ , one on each axis. In this way, a 1-1 correspondence between points in  $\mathbb{R}^N$  and planar polygonal lines with vertices on the parallel axes is established.

In 2D, a point is represented by a line (usually just the segment between the axes is shown). It can be easily proved that the lines representing points of a line  $c_1x_1 + c_2x_2 = c_0$  (for  $c_1 + c_2 \neq 0$ ) intersect at the point  $\bar{\pi}_{12}$  with  $xy$ -coordinates  $(\frac{c_2}{c_1+c_2}, \frac{c_0}{c_1+c_2})$ , or more generally at the point

$$\bar{\pi}_{12} = \left( \frac{d_1c_1 + d_2c_2}{c_1 + c_2}, \frac{c_0}{c_1 + c_2} \right), \tag{1}$$

where  $d_1$  and  $d_2$  are distances between axis  $y$  and  $\bar{X}_1, \bar{X}_2$ , respectively. (Lines with slope 1 are mapped onto the ideal points of projective plane, but in what follows we will not consider any “degenerate” cases). Hence a fundamental point  $\leftrightarrow$  line duality is induced (see Fig. 1).

In 3D, a line can be fully described by any pair from its three projections on coordinate planes. Each such projection is a line in 2D-space of the corresponding coordinates, and so can be represented in parallel coordinates exactly as was described above. Hence, if  $c_i^{\{ij\}}x_i + c_j^{\{ij\}}x_j = c_0^{\{ij\}}$  - projection of the line on  $X_iX_j$  Euclidean plane ( $i, j \in \{1, 2, 3\}$ ), then it is represented in the parallel coordinate system by the point  $\bar{\pi}_{ij}$  whose coordinates may be computed from equation (1), and can be found geometrically as the intersection of corresponding lines. The three points  $\bar{\pi}_{12}, \bar{\pi}_{23}$  and  $\bar{\pi}_{13}$  are always collinear as a consequence of Desargue’s Theorem, and any two of them represent the line in parallel coordinates. We denote by  $\bar{L}$  the line on which the three points lie (see Fig. 1).

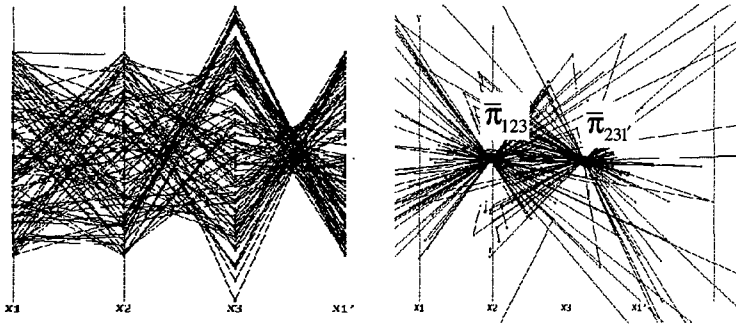


Fig. 2. Randomly sampled set of coplanar points in 3D and intersection of lines  $\bar{L}$  showing coplanarity.

Let us now consider a plane  $c_1x_1 + c_2x_2 + c_3x_3 = c_0$ . Every pair of points belonging to this plane define a line  $\bar{L}$  in parallel coordinates (which can be constructed geometrically from the representation of points themselves). It can be shown by direct computation that all such lines intersect at the common point with coordinates

$$\bar{\pi}_{123} = \left( \frac{c_1 + 2c_2}{c_1 + c_2 + c_3}, \frac{c_0}{c_1 + c_2 + c_3} \right). \tag{2}$$

This condition can be used to characterize coplanarity. Note, that one point is not sufficient in order to specify the plane. The solution is to introduce an additional axis  $\bar{X}'_1$  placed after  $X_3$  and at a unit distance from it, and consider the representation of points also in axes  $(\bar{X}_2, \bar{X}_3, \bar{X}'_1)$ . This leads to the additional point  $\bar{\pi}_{231}' = \left( \frac{c_2 + 2c_3 + 3c_1}{c_1 + c_2 + c_3}, \frac{c_0}{c_1 + c_2 + c_3} \right)$  (see Fig. 2).

This generalizes nicely to the  $N$ -dimensional case, and it can be shown that a representation of a hyperplane in parallel coordinates also can be recursively constructed by a simple geometric procedure, using affine subspaces of lower dimensions. A hyperplane is represented by  $N - 1$  indexed points; the "first" one has coordinates

$$\bar{\pi}_{12\dots N} = \left( \frac{c_2 + 2c_3 + \dots + (N - 1)c_N}{c_1 + c_2 + \dots + c_N}, \frac{c_0}{c_1 + c_2 + \dots + c_N} \right), \tag{3}$$

and the others have very similar formulas.

A  $p$ -flat in  $N$ -dimensional case can be described by  $N - p$  linearly independent equations, where each of them has the form  $\sum_{k=1}^{p+1} c_{i_k} x_{i_k} = c_0$ , and so corresponds to a hyperplane in axes  $(\bar{X}_{i_1}, \bar{X}_{i_2}, \dots, \bar{X}_{i_{p+1}})$ . It follows that a  $p$ -flat is represented by  $p(N - p)$  indexed points. The ensuing discussion is restricted to "approximated" hyperplanes. It is easy to show that the general case of an "approximate"  $p$ -flat in  $N$ -dimensional space can be reduced to the study of some "approximate" hyperplane.

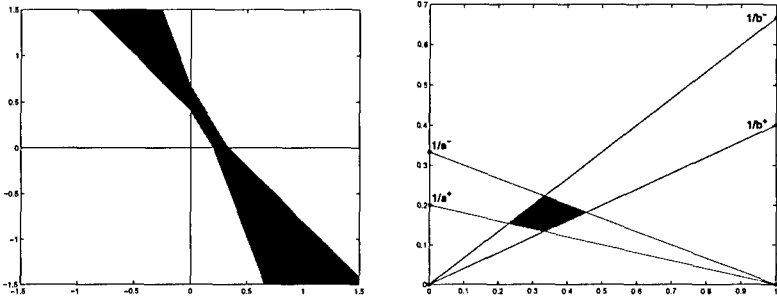


Fig. 3. Sampled proximate line  $ax + by = 1$ ,  $a \in [a^-, a^+]$ ,  $b \in [b^-, b^+]$  in orthogonal and parallel coordinates.

### §3. Exact Problem Definition

We will use the following definition of “similarity”:

**Definition 1.** Proximate flats are defined here as flats with proximate equations. An approximate hyperplane is defined as a set of hyperplanes given by equations

$$c_1x_1 + c_2x_2 + \dots + c_Nx_N = c_0, \tag{4}$$

where every coefficient can vary:  $c_i \in [c_i^-, c_i^+]$ ,  $i = 0, \dots, N$ .

Such a slab of hyperplanes is extremely difficult to visualize in orthogonal coordinates, even for 3D. Another problem is that even in the 2D case, line neighborhoods are unbounded in orthogonal coordinates, so neighborhoods for different lines always overlap. Fig. 3 shows that samples of proximate lines (in 2D) form a cloud in the form of a very simple and nice convex quadrilateral.

In the  $N$ -dimensional case, a hyperplane is described by  $N - 1$  points which means that in the approximated case, we will get  $N - 1$  “clouds” of indexed points in parallel coordinates. In order to make things simpler, we will use the following

**Assumption 2.** Free coefficients of equations are not allowed to vary, and are supposed to be identically equal to 1.

(We have a complete analogue of the main result for the case when this assumption is not applied.)

**Lemma 3.** It is sufficient to study only the range of the first indexed point of a hyperplane. That is, the following mathematical problem should be considered: find the range of the function  $f : \mathbb{R}^N \rightarrow \mathbb{R}^2$  such that

$$f(c_1, \dots, c_N) = \left( \frac{\sum_{j=1}^N (j-1)c_j}{\sum_{j=1}^N c_j}, \frac{1}{\sum_{j=1}^N c_j} \right) \tag{5}$$

when  $c_j \in [c_j^-, c_j^+]$  ( $c_j^- < c_j^+$ ,  $j = 1, \dots, N$ ).

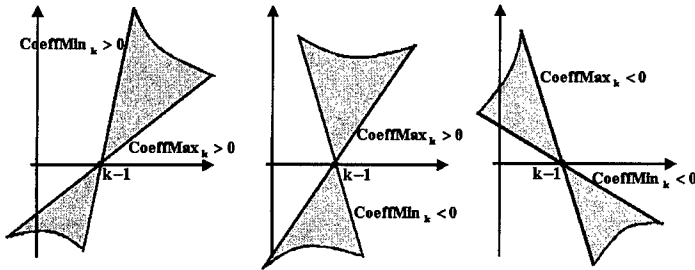


Fig. 4. Possible place for domain  $\Omega$  depending on the signs of  $CoeffMin_k$  and  $CoeffMax_k$ .

The notations used are:

- $x(c_1, \dots, c_N)$  and  $y(c_1, \dots, c_N)$  - the first and the second coordinates of  $f_N(c_1, \dots, c_N)$  in  $\mathbb{R}^2$  respectively;
- $B = [c_1^-, c_1^+] \times \dots \times [c_N^-, c_N^+]$  - box in the space of coefficients;
- $\Omega = f(B)$  - image of  $B$  in the parallel coordinate system.

We now show why Lemma 3 holds in the 3D case. Here studying the range of  $\bar{\pi}_{123}$  (equation (2)) is sufficient because  $\bar{\pi}_{231'}$  can be rewritten in the form

$$\bar{\pi}_{231'} = (1, 0) + \left( \frac{c_3 + 2c_1}{c_1 + c_2 + c_3}, \frac{1}{c_1 + c_2 + c_3} \right)$$

This implies that the equation of  $\bar{\pi}_{231'}$  can be obtained from the one of  $\bar{\pi}_{123}$  by shift and cyclical change of parameters  $c_1 \rightarrow c_2, c_2 \rightarrow c_3, c_3 \rightarrow c_1$ .

For the general case, this "reduction" lemma can be proved using simple combinatorics, and it can be shown that in order to describe the range of one of the indexed points corresponding to a  $p$ -flat in  $N$ -dimensional space, it is always sufficient to consider  $f_{2N}$  with some coefficients identically equal to zero.

#### §4. Some Notes on the Domain $\Omega$

To get an intuitive feel about the structure of the domain  $\Omega$ , we consider the possible location of  $\Omega$  and what it looks like. Note that for every  $k = 1, \dots, N$  the following representation takes place:

$$x(c_1, \dots, c_N) = k - 1 + \frac{\sum_{j=1}^N (j - k)c_j}{\sum_{j=1}^N c_j} = k - 1 + Coeff_k y(c_1, \dots, c_N), \quad (6)$$

where  $Coeff_k = \sum_{j=1}^{k-1} \underbrace{(j - k)}_{<0} c_j + \sum_{j=k+1}^N \underbrace{(j - k)}_{>0} c_j$  does not depend on  $c_k$ .

If only  $c_k$  varies, while the other coefficients are fixed, then  $(x, y)$  lie on the straight line which passes through the point  $(k - 1, 0)$  and has slope  $1/Coeff_k$ . Here  $CoeffMin_k \leq Coeff_k(c_1, \dots, c_{k-1}, c_{k+1}, \dots, c_N) \leq CoeffMax_k$

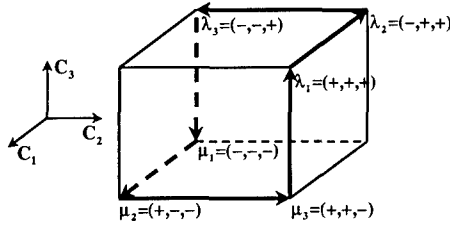


Fig. 5. Important vertices and edges in the space of parameters.

for every choice of  $(c_1, \dots, c_{k-1}, c_{k+1}, \dots, c_N)$ , where

$$\begin{aligned} \text{CoeffMin}_k &= \text{Coeff}_k(c_1^+, \dots, c_{k-1}^+, c_{k+1}^-, \dots, c_N^-), \\ \text{CoeffMax}_k &= \text{Coeff}_k(c_1^-, \dots, c_{k-1}^-, c_{k+1}^+, \dots, c_N^+). \end{aligned} \tag{7}$$

The domain  $\Omega$  lies between lines with maximal and minimal slopes. More precisely

**Lemma 4.**  $\Omega$  lies above the line corresponding to  $\text{CoeffMax}_k$  iff

$\text{CoeffMax}_k > 0$  (otherwise it lies below the line).  $\Omega$  lies above the line corresponding to  $\text{CoeffMin}_k$  iff  $\text{CoeffMin}_k < 0$  (see Fig. 4).

Let us now introduce the following notations for some important vertices and edges of box  $B$ . There are  $2N$  (among  $2^N$ ) important vertices:

$$\begin{aligned} \lambda_k &= (c_1^+, c_2^+, \dots, c_{k-1}^+, c_k^-, c_{k+1}^-, \dots, c_N^-), \\ \mu_k &= (c_1^-, c_2^-, \dots, c_{k-1}^-, c_k^+, c_{k+1}^+, \dots, c_N^+). \end{aligned} \tag{8}$$

for  $k = 1, \dots, N$  and  $2N$  important edges connecting these vertices (see Fig. 5):

$$\begin{aligned} \alpha_k(c_k) &= (c_1^-, \dots, c_{k-1}^-, c_k, c_{k+1}^+, \dots, c_N^+), & c_k \in [c_k^+, c_k^-], \\ \beta_k(c_k) &= (c_1^+, \dots, c_{k-1}^+, c_k, c_{k+1}^-, \dots, c_N^-), & c_k \in [c_k^-, c_k^+]. \end{aligned} \tag{9}$$

Edge  $\alpha_k$  connects vertices  $\lambda_k$  and  $\lambda_{k+1}$ , edge  $\beta_k$  - vertices  $\mu_k$  and  $\mu_{k+1}$ . (Here  $\lambda_{N+1} = \mu_1$  and  $\mu_{N+1} = \lambda_1$ ).

We also introduce the notation

$$\text{sum}(c_1, \dots, c_N) = c_1 + \dots + c_N \tag{10}$$

which will be useful in what follows.

As explained above, it is clear that  $\alpha_k$  is mapped onto the boundary line of  $\Omega$  corresponding to  $\text{CoeffMax}_k$ , and  $\beta_k$  is mapped onto the boundary line corresponding to  $\text{CoeffMin}_k$ . More precisely, we have

**Lemma 5.** The image of  $\alpha_k$  is the segment between  $f_N(\lambda_k)$  and  $f_N(\lambda_{k+1})$  if  $y = 1/\text{sum}(\alpha(c_k))$  does not change sign while  $c_k \in [c_k^+, c_k^-]$ , and the complement of the straight line to this segment otherwise. In other words,  $f_N(\alpha_k)$

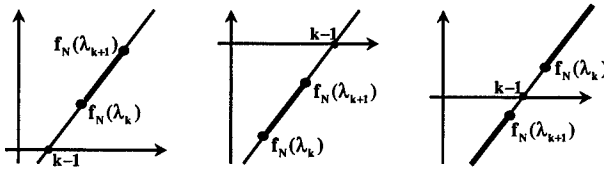


Fig. 6. Image of  $\alpha_k$  depending on the signs of  $y(\lambda_k)$  and  $y(\lambda_{k+1})$ .

is a segment if plane  $c_1 + c_2 + \dots + c_N = 0$  does not intersect edge  $\alpha_k$  of box  $B$ , and a complement of the segment otherwise (see Fig. 6).

Of course, the analogous statement holds for the image of  $\beta_k$ . In what follows, we will usually formulate only statements for  $\lambda_k$  and  $\alpha_k$ , and omit the analogous ones for  $\mu_k$  and  $\beta_k$ .

**Conclusion 6.**  $f_N(\lambda_k)$  and  $f_N(\lambda_{k+1})$  are connected by segment iff they lie in the same (upper or lower) half-plane, i.e. if  $y(\lambda_k)$  and  $y(\lambda_{k+1})$  have the same signs.

Note further, that every one of the points  $f_N(\lambda_k)$  ( $f_N(\mu_k)$ ) is a point of the concatenation of two boundary segments corresponding to  $CoeffMax_{k-1}$  and  $CoeffMax_k$  ( $CoeffMin_{k-1}$  and  $CoeffMin_k$ , respectively).

In order to make this precise and to assure that all boundary can be described in this manner, the following theorem was proved.

**Theorem 7.**  $f_N(c_1, \dots, c_N)$  belongs to the boundary of domain  $\Omega$  iff there exists  $k = 1, \dots, N$  such that  $(c_1, \dots, c_N) = \alpha_k(c_k)$  or  $(c_1, \dots, c_N) = \beta_k(c_k)$  for  $c_k \in [c_k^-, c_k^+]$ .

Again the proof (which is relatively long) is omitted the proof uses the "topological" notion of point neighborhood, boundary etc.

**Conclusion 8.** In order to describe the boundary of the domain  $\Omega$ , it suffices to move along the following contour in the space of coefficients

$$\begin{array}{ccccccccccc}
 & \nearrow & \lambda_2 & \rightarrow & \lambda_3 & \rightarrow & \dots & \rightarrow & \lambda_N & \searrow & \\
 \lambda_1 & & & & & & & & & & \mu_1 & (11) \\
 & \nwarrow & \mu_N & \leftarrow & \mu_{N-1} & \leftarrow & \dots & \leftarrow & \mu_2 & \swarrow & 
 \end{array}$$

Note that independently of the specific values of  $c_j^-$  and  $c_j^+$  ( $j = 1, \dots, N$ ), always only  $2N$  (among  $2^N$ ) definite vertices and edges of  $B$  and in definite order participate in the boundary of domain  $\Omega$ .

It remains to "fill in" the boundary of  $\Omega$  with  $\Omega$  itself. Before we formulate the main result, let us study an additional property of domain  $\Omega$  and its boundary.



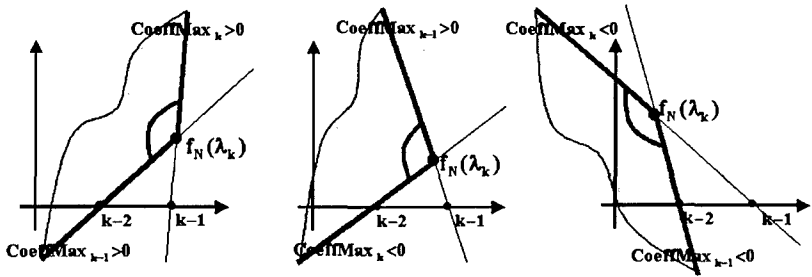


Fig. 7. Convexity of vertex  $f_N(\lambda_k)$ .

**Lemma 9.** *The domain  $\Omega$  has only convex boundary vertices.*

Indeed, it was shown above that a boundary vertex of  $\Omega$  of the form  $f_N(\lambda_k)$  is the intersection of the lines with slopes  $1/\text{CoeffMax}_{k-1}$  and  $1/\text{CoeffMax}_k$  and passing through points  $(k-2, 0)$  and  $(k-1, 0)$  respectively. Note that the following equation holds:

$$\text{CoeffMax}_{k-1} - \text{CoeffMax}_k = \text{sum}(\lambda_k) = 1/y(\lambda_k) \tag{12}$$

Assume for example that  $f_N(\lambda_k)$  lies in upper half-plane, i.e., that  $y(\lambda_k) > 0$  (the analogous consideration can be done for the lower hyperplane). Then  $\text{CoeffMax}_{k-1} > \text{CoeffMax}_k$ , and using Lemma 4 we get that in any one of three possible cases (see Fig. 7) this vertex is convex.

### §5. The Main Result - Description of the Domain $\Omega$

**Theorem 10.** *The domain  $\Omega$  has one of two possible forms depending on whether  $\text{sum}(\lambda_1) = c_1^+ + c_2^+ + \dots + c_N^+$  and  $\text{sum}(\mu_1) = c_1^- + c_2^- + \dots + c_N^-$  has the same sign or not.*

Note that this condition is equivalent to the condition that the plane  $c_1 + c_2 + \dots + c_N = 0$  intersects the box  $B$  in the space of coefficients.

**Case 1.** If  $\text{sum}(\lambda_1)$  and  $\text{sum}(\mu_1)$  have the same sign, then  $\Omega$  is a convex bounded polygon inside the contour (11), (see Fig. 8 and 9 - left parts. In the figures we will write  $\bar{\lambda}_k$  instead of  $f_N(\lambda_k)$  in order to make the figures clearer and more compact).

Indeed, let us assume that  $\text{sum}(\mu_1) > 0$ . Then  $y(\lambda_k) > 0$  and  $y(\mu_k) > 0$  for every  $k = 1, \dots, N$  (all vertices of  $\Omega$  lie in the upper half-plane). According to Conclusion 6, the boundary of  $\Omega$  in this case consists of segments which form a convex (Lemma 9) bounded polygon.

**Case 2.** If  $\text{sum}(\mu_1) < 0$  and  $\text{sum}(\lambda_1) > 0$ , then  $\text{sum}()$  changes its sign exactly once when upper or lower chain of the contour (11) is traversed, say at the segment  $\alpha_k = [\lambda_k, \lambda_{k+1}]$  at the upper chain and segment  $\beta_p = [\mu_p, \mu_{p+1}]$  at the lower chain. Then the domain  $\Omega$  is a union of two convex unbounded

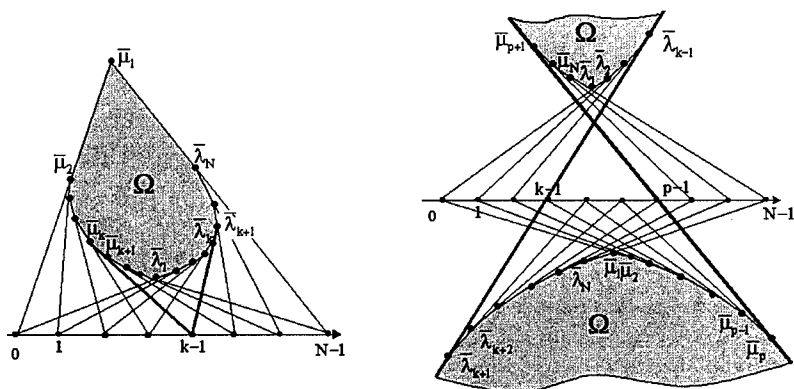


Fig. 8. The domain  $\Omega$  in the general case.

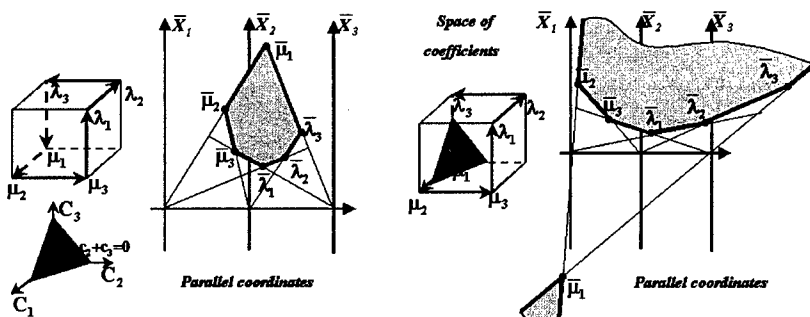


Fig. 9. An example of the domain  $\Omega$  in the 3D case.

polygons. The first is inside the part of the contour that belongs to the upper half-plane, and the second is inside the part of the contour that belongs to the lower half-plane.

Upper and lower parts are bounded by infinite rays with the same slopes (corresponding to  $CoeffMax_k$  and  $CoeffMin_p$ ), i.e. it is a *convex bounded polygon in the projective plane* (see Fig. 8 and 9 - right parts).

### §6. Example of an Affine Subspace of Lower Dimension

We now show how the general result can be applied to the construction of  $p$ -flats of lower dimensions, for the "approximate" line in 3D case. For example, if we would like to describe the range of the indexed point  $\bar{\pi}_{13} = \left( \frac{2c_3^{(13)}}{c_1^{(13)} + c_3^{(13)}}, \frac{1}{c_1^{(13)} + c_3^{(13)}} \right)$  which enters in the representation of the line, then we can reduce it to the consideration of  $f_3$  by putting  $c_2 \in [0, 0]$ . We get "void" connections instead of edges  $\alpha_2$  and  $\beta_2$ , and finally get a convex quadrilateral instead of convex hexagon (see Fig. 10).

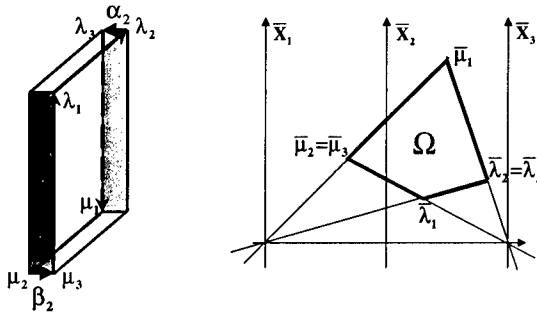


Fig. 10. Range of  $\bar{\pi}_{12}$  - the first indexed point corresponding to line in 3D case.

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