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Best Approximation Algorithms: A Unified Approach

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Abstract. A generalization of the Remez algorithm is proposed. The new approach uses differential properties of the best approximation operator. The method was developed for polynomial approximation of complex-value functions. In this paper the convergence of algorithm is proved for Banach spaces.

§1. Introduction

Let us consider the best approximation operator

$$\mathbf{P} : \mathbf{B} \to \mathbf{P}_n,$$

where $\mathbf{B}$ is a Banach space (complex in general), $\mathbf{P}_n$ an n-dimensional subspace. Suppose that $\mathbf{P}_n$ is univalent and one-side differentiable in any direction [1]. This assumption is valid when:

i) $\mathbf{B} = C(Q, \mathbf{R})$ Q-compact, and $\mathbf{P}_n$ is a Chebyshev subspace (in particular, when P_n is the subspace of algebraic polynomials of degree less or equal to $n-1$ [2,9]);

ii) $\mathbf{B} = C(Q, \mathbf{C}), Q$ is finite set, and $\mathbf{P}_n$ is an n-dimensional Chebyshev subspace [3];

iii) $\mathbf{B} = L_p$, $p > 1$, and $\mathbf{P}_n$ is an n-dimensional subspace (R. Holmes and B. Kripke).

Originally the differential properties of the best approximation operator were applied to the development of algorithms in [4,5]. The goal was to generalize the Remez algorithm for complex-valued functions. In [6] the new best approximation algorithm was applied to the approximation of conformal mappings by polynomials. In [7] it was shown that for real polynomial approximation,

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such an approach generates exactly the Remez algorithm, and a stronger convergence theorem was proven using differentiation technique. In [8,1], the new approach was applied successfully to nonlinear approximation, including rational and generalized rational uniform approximations.

Here we show that the method is applicable to the best approximation from a finite-dimensional subspace in the arbitrary Banach space if the best approximation operator is one-sided differentiable.

§2. Description of the Algorithm

Suppose that the i-th step of the algorithm is performed to find the best approximation of the element $f \in \mathbf{B}$, and that the element $P_i \in \mathbf{P}_n$ is found. If $\|f - P_i\| = E(f)$, then the process is finished. Otherwise the inequality

$$\|f - P_i\| > E(f)$$

holds, and the next step should be performed.

In order to construct the next approximation P_{i+1}, we construct an auxiliary element $g_i \in \mathbf{B}$ such that the equality

$$\|f + g_i - P_i\| = \|f - P_i\| \tag{1}$$

holds. Suppose the following assumptions are true:

Assumption 1. *The mapping $G = G(f) : \mathbf{P}_n \to \mathbf{B}$, which defines the auxiliary element $g_i = G(f, P_i)$ is continuous.*

Assumption 2. *For all functionals $x \in \mathbf{B}^*$ with properties $|x(f - P_i)| = \|f - P_i\|, \|x\|_{B^*} = 1$, equality $x(g_i) = 0$ holds. Moreover, for every such extremal functional x, a weak neighbourhood $V(x) \subset \mathbf{B}^*$ exists such that*

$$Re\overline{y(f - P_i)}y(g_i) \geq 0, \quad \forall y \in V(x) \cap \{z \in \mathbf{B}^*, \|z\| = 1\}.$$

Assumption 3. *For given fixed f the mapping $D = D(f) : \mathbf{B} \to \mathbf{P}_n$, which defines the derivative $D_i = D(f, g_i)$, is continuous.*

These assumptions may be satisfied easily for real and complex uniform approximations [4,5,7]. When the derivative

$$D_i = \left.\frac{d\mathbf{P}(f + (1 - t)g_i)}{dt}\right|_{t=+0}$$

is calculated (usually as the solution of system of linear algebraic equations), the next element P_{i+1} is computed as

$$P_{i+1} = P_i + t_i D_i, \tag{2}$$

where $c\tau_i < t_i \leq \tau_i, 0 < c = const \leq 1$, and τ_i is the minimal value of t, for which

$$\hat{E}_i(\tau_i) = \min\{\hat{E}_i(t) = \|f - P_i - tD_i\|, 0 \leq t \leq 1\}.$$

§3. Main Theorem

Theorem 1. *For given $f \in \mathbf{B} \backslash \mathbf{P_n}$, the sequence $\{P_i\}_{i=0}^{\infty}$ constructed according to the general scheme (see the previous section) converges to the element* $\mathbf{P(f)}$ *of best approximation of $f \in \mathbf{B}$.*

Proof: Let us write

$$\begin{aligned} E_i &= \|f - P_i\|, \\ E_i(t) &= \|f + (1-t)g_i - \mathbf{P}(f + (1-t)g_i)\|, \\ \tilde{E}_i(t) &= \|f + (1-t)g_i - P_i - tD_i\|, \\ \hat{E}_i(t) &= \|f - P_i - tD_i\|, \\ \alpha_i &= \hat{E}_i'(+0), \\ E_i^* &= \min\{\hat{E}_i(t), 0 \le t \le 1\} = \hat{E}_i(\tau_i), \qquad i \ge 0. \end{aligned} \tag{3}$$

The convexity of the function $E_i(t)$ implies

$$E_i'(+0) \le E_i(1) - E_i(0) = E(f) - E(f + g_i) = E(f) - \|f - P_i\| < 0.$$

Since

$$|E_i(t) - \tilde{E}_i(t)| \le \|\mathbf{P}(f + (1-t)g_i) - P_i - tD_i\| = o(t), \;\; t \to +0,$$

the equality

$$\tilde{E}_i'(+0) = E_i'(+0)$$

holds. Therefore,

$$\tilde{E}_i'(+0) \le E(f) - E_i < 0. \tag{4}$$

Now we show that there exists $\epsilon > 0$ such that

$$\hat{E}_i(t) \le \tilde{E}_i(t) + \frac{2t^2\|D_i\|^2}{\|f - P_i\|} \tag{5}$$

for all $0 \le t < \epsilon$. Suppose that the last statement is invalid. Then there is a sequence $\{t_l\}$, $\quad t_l \to +0, \quad l \to \infty$, such that

$$\|f - P_i - t_lD_i\| > \|f + (1-t_l)g_i - P_i - t_lD_i\| + \frac{2t_l^2\|D_i\|^2}{\|f - P_i\|}, \quad l \ge 1.$$

Choosing a subsequence if necessary, we may consider the weakly convergent sequence of functionals $\{x_l\} \subset \mathbf{B}^*$, $\quad \forall l \; \|x_l\| = 1$, such that

$$|x_l(f - P_i - t_lD_i)| > \|f + (1-t_l)g_i - P_i - t_lD_i)\| + \frac{2t_l^2\|D_i\|^2}{\|f - P_i\|}, \quad l \ge 1. \tag{6}$$

Let $x_0 = \lim_{l\to\infty} x_l$. Then inequality (6) implies

$$|x_0(f - P_i)| \ge \|f + g_i - P_i\| = \|(f - P_i)\|,$$

and since $\|x_0\| = 1$, we finally have

$$|x_0(f - P_i)| = \|(f - P_i)\|.$$

Hence in accordance with Assumption 2

$$x_0(g_i) = 0. \tag{7}$$

Since $\|x_l\|_{\mathbf{B}^*} = 1$, it follows from (6) that

$$|x_l(f - P_i - t_l D_i)| > |x_l(f + (1 - t_l)g_i - P_i - t_l D_i))| + \frac{2t_l^2 |x_l(D_i)|^2}{\|f - P_i\|}, \tag{6'}$$

for $l \geq 1$.

Now we temporarily write

$$\begin{aligned} a &= x_l(f - P_i - t_l D_i), \\ b &= (1 - t_l)x_l(g_i), \\ s &= \frac{2t_l^2 |x_l(D_i)|^2}{\|f - P_i\|} > 0. \end{aligned}$$

Using this notation, the inequality (6′) may be rewritten in the form

$$|a| > |a + b| + s,$$

which implies $|a| > |a + b|$. Consequently,

$$|a|^2 > |a + b|^2 + 2|a + b|s > |a + b|^2 + 2|a|s.$$

Thus,

$$|a|^2 + |b|^2 + 2\mathrm{Re}\bar{a}b < |a|^2 - 2|a|s,$$

and

$$2\mathrm{Re}\bar{a}b < -|b|^2 - 2|a|s. \tag{8}$$

Now we substitute the values of a, b, s in (8) and obtain

$$\begin{aligned} &2\mathrm{Re}\overline{x_l(f - P_i - t_l D_i)}x_l(g_i)(1 - t_l) \\ &\quad < -(1 - t_l)^2 |x_l(g_i)|^2 - 4|x_l(f - P_i - t_l D_i)\frac{t_l^2 |x_l(D_i)|^2}{\|f - P_i\|}. \end{aligned}$$

Since $t_l \to +0$, $x_l(f - P_i - t_l D_i) \to \|f - P_i\|$, when $l \to \infty$, there exists number l_0 such that inequalities

$$
\begin{aligned}
&2\mathrm{Re}\,\overline{x_l(f-P_i)}x_l(g_i)\\
&\quad < -(1-t_l)|x_l(g_i)|^2 + 2t_l\mathrm{Re}\overline{x_l(D_i)}x_l(g_i) - \frac{4t_l^2|x_l(f-P_i-t_lD_i)||x_l(D_i)|^2}{(1-t_l)\|f-P_i\|}\\
&\quad < -\frac{|x_l(g_i)|^2}{2} + 2t_l\mathrm{Re}\overline{x_l(D_i)}x_l(g_i) - 2t_l^2|x_l(D_i)|^2\\
&\quad = -\frac{1}{2}|x_l(g_i) + 2x_l(D_i)|^2 < 0
\end{aligned}
$$

are valid for all $l \geq l_0$. Therefore,

$$\mathrm{Re}\overline{x_l(f-P_i)}x_l(g_i) < 0, \qquad \forall l \geq l_0.$$

But taking into consideration (7), we see that this inequality contradicts Assumption 2, so (5) is proven.

Since $\tilde{E}_i(0) = \hat{E}_i(0) = E_i(0)$, (5) implies

$$\hat{E}_i'(+0)| \leq E_i'(+0) < E(f) - E_i. \tag{4'}$$

Therefore, in accordance with (2),

$$\hat{E}_i(\tau_i) = min\{\|f - P_i - tD_i\|, 0 \leq t \leq 1\} < E_i \quad \text{and} \quad \tau_i > 0.$$

So

$$E_{i+1} < E_i.$$

Hence the sequence $\{E_i\}_{i=0}^{\infty}$ converges to some value $E_* \geq E(f)$, i.e.,

$$\lim E_i = E_*. \tag{9}$$

From (2), (4′) and convexity of $E_i(t)$, it follows that

$$E_{i+1} < E_i - \frac{E_i - \hat{E}_i(\tau_i)}{\tau_i}t_i \leq E_i - c(E_i - \hat{E}_i(\tau_i)).$$

Consequently,

$$\Delta E_i = E_{i+1} - E_i \geq c(E_i - \hat{E}_i(\tau_i)).$$

Since $\Delta E_i \to 0$, also $E_i - \hat{E}_i(\tau_i) \to 0$, and

$$\lim \hat{E}_i(\tau_i) = \lim E_i = E_*. \tag{9'}$$

To complete the proof we must show that

$$E_* = E(f).$$

Suppose that this statement isn't valid, i.e.,

$$E_* > E(f). \tag{10}$$

Since the subspace $\mathbf{P}_n$ is finite-dimensional and Assumptions 1 and 3 hold, the subsequence $\{P_{i_k}\} = \{\tilde{P}_k\}$ may be chosen so that the following limits exist:

i) $\lim \tilde{P}_k = P_*$;

ii) $\lim g_{i_k} = g_*$;

iii) $\lim \alpha_{i_k} = \alpha_*$;

iv) $\lim \tau_{i_k} = \tau_*$.

From Assumption 3 it follows that also

$$\lim D_{i_k} = D_*$$

exists, and

$$\lim \|f - P_{i_k} - tD_{i_k}\| = \|f - P_* - tD_*\| \tag{11}$$

uniformly for $t \in [0,1]$. Equalities (9) and (9') imply that at least one of following statements

$$\alpha_* = \lim \alpha_{k_i} = 0; \tag{12}$$

or

$$\tau_* = \lim \tau_{k_i} = 0 \tag{13}$$

is valid.

Using the assumption (10) and the scheme of the algorithm, we construct the auxiliary element $\tilde{g} \neq 0$ for the approximation P_*. Due to Assumption 1, we have

$$g_* = \tilde{g}.$$

For the following two convex functions

$$E_*(t) = \|f + (1-t)g_* - \mathbf{P}(f + (1-t)g_*)\|,$$

$$\hat{E}_*(t) = \|f - P_* - tD_*\|,$$

where

$$D_* = \frac{d\mathbf{P}(f + (1-t)g_*)}{dt}\,|_{t=+0},$$

analogously to $(4')$, we obtain inequalities

$$E'_*(+0) = \hat{E}'_*(+0) < E(f) - E_* < 0 \tag{14}$$

and $\tilde{\tau} > 0$, where τ is a minimal value of t, for which

$$\hat{E}_*(\tilde{\tau}) = \min\{\hat{E}_*(t),\ 0 \le t \le 1\}.$$

From (11), it follows that $\tau_* = \tilde{\tau}$ and therefore (13) is impossible. So (12) is true. But (14) implies that there is an integer k_0 such, that for every $k, k > k_0$, the inequality

$$\alpha_{i_k} < \frac{E(f) - E_*}{2} < 0$$

is valid, and therefore

$$\alpha_* < 0.$$

This inequality contradicts (12). Hence assumption (10) is invalid. The theorem is proven. □

§3. Applications

As was mentioned above, the proposed algorithm may be considered as a wide generalization of the classical Remez algorithm. Applied to polynomial approximations of complex-valued functions, the method generates an algorithm which possesses in general a linear convergence as numerical experiments show (see also [6]). For finite sets, the convergence of the algorithm is quadratical.

When applied to real polynomial approximations, the method generates exactly the Remez algorithm [7]. But even in this case an approach which uses differential properties of the best approximation operator allows better estimations of convergence.

Theorem 2. ([7]). *Let* $\mathbf{P_n}$ *be a* n*-dimensional Chebyshev subspace in* $C[a,b]$, *and let* $\Delta_h^2(u,x)$ *be the second difference of the function* u *at the point* x *with step* h. *If the function* $f \in C[a,b] \backslash \mathbf{P_n}$ *has the best approximation* $\mathbf{P}(f) \in \mathbf{P_n}$ *such that the difference* $f - \mathbf{P}(f)$ *possesses exactly* $n+1$ *extremal points* $x_0, x_1, \ldots, x_n$ *and the inequality*

$$|\Delta_h^2(f - \mathbf{P}(f), x_j)| \geq \gamma h^2, \ \gamma = const, \quad j = 0, 1, \ldots, n$$

holds in points x_j, *then the Remez algorithm for* f *converges quadratically.*

In [7] a modified Remez algorithm for twice continuously differentiable functions is proposed. A procedure for extremal points calculation, using differential properties, is developed to reduce the complexity of the most complicated part of the Remez algorithm.

This method may be applied to the best polynomial L_p- approximation.

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