TITLE: A B-spline Tensor for Vectorial Quasi-Interpolant

A B-spline Tensor for Vectorial Quasi-Interpolant

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Abstract. The aim of this paper is to introduce new techniques and new tools for vector field approximation. We do so by building the equivalent of B-splines, which are now tensor B-splines, as shown below, and by appling to the discretization based on a regular grid of a differential operator a fundamental solution of it, as done for polynomial B-splines and polyharmonic B-splines (see [5,6]). We thus obtain quasi-interpolants in the vectorial case whose properties generalize the properties of the quasi-interpolants generated by using B-splines. All this is done in the case when the data lie on a regular infinite grid.

§1. Introduction

Fluid mechanics, meteorology and more and more other applications need approximate functions from $\mathbb{R}^3$ to $\mathbb{R}^3$. Given some discrete vectorial data, we want to get a function interpolating or approximating the data. At first glance, we may think of doing this with three independent approximations of the data (one for each component of the data). Of course this can be done, but it usually gives poor results since there is no connection between the various components of the approximation function, while the applications may require, for example, a divergence-free (or a rotational-free) function. In order to take into account this kind of connection, we want to determine the function interpolating the data and minimizing a seminorm over all vectorial functions interpolating the data. The seminorm which is minimized is based on the Helmholtz decomposition of vector fields into a rotational and a gradient part ($\rho\ \|\mathrm{div}\cdot\|^2 + \|\mathrm{rot}\cdot\|^2$). This will be presented in detail in the forthcoming thesis [2]. The weight ρ is introduced to allow the rotational part of the field to dominate the gradient part and so in the so obtained function [2].

In order to determine the interpolating vector, we need to solve a linear system which is usually large (three times the number of data) and badly conditioned. This is why in this paper we do not propose to interpolate,

Curve and Surface Fitting: Saint-Malo 1999

Albert Cohen, Christophe Rabut, and Larry L. Schumaker (eds.), pp. 201–208.

ISBN 0-8265-1357-3.

but instead approximate the data by building a quasi-interpolant based on B-splines. Note that if $(z_i)_{i=1}^n$ are vectors and if we want to get a vectorial function S such that $S(x) = \sum_i B_i(x)\, z_i$ (as we do in the scalar case with B-splines B_i), we need that the functions B_i are matrix and not scalar functions as in the case of scalar approximation in $\mathbb{R}$ or $\mathbb{R}^d$. This is why the tools we build are tensors.

Now, in order to build this "B-spline Tensor", we will use the same strategy as for polynomial and polyharmonic B-splines: we will discretize the differential operator $P_{m,\rho}(D)$ defined in [2] and apply to this discretization a fundamental solution of $P_{m,\rho}(D)$, thus obtaining a kind of approximation of the Dirac tensor $\delta\, \boldsymbol{I}_3$. All this is done on a cardinal grid (i.e. a regular infinite grid).

For our three-dimensional problem [2], we choose the 3×3 differential matrix defined by

$$P_{m,\rho}(D) = (-1)^m\, \Delta^{m-1}\left[\rho \nabla \mathrm{div}\cdot - \mathrm{rot}\,(\mathrm{rot}\cdot)\right],$$

where div is the divergence operator, ∇ is the gradient operator, rot is the rotational operator and ρ is an arbitrary positive parameter.

We remark that if $\rho = 1$, then $P_{m,1}(D) = (-1)^m\, \Delta^m\, \boldsymbol{I}_3$, where $\boldsymbol{I}_3$ is the identity matrix of $\mathbb{R}^3$, so we obtain three independent operators (one for each component), each one being as in ([5]).

In the first part of this paper we give the construction of a discretization of $P_{m,\rho}(D)$. In the second part, we define polyharmonic B-spline tensors and give the main properties. In the last part, we study the associated vector quasi-interpolants, and in particular prove their $\mathcal{P}_k$-reproduction.

Notation: Let m be a integer with $m \geq 2$. $\mathbb{P}_n$ denotes the set of polynomial with variable in $\mathbb{R}^3$ of total degree at most n and $\mathcal{P}_n = \mathbb{P}_n \times \mathbb{P}_n \times \mathbb{P}_n$. D' denotes the set of distribution of $\mathbb{R}^3$ and $\mathcal{D}' = D' \times D' \times D'$. Let $f : \mathbb{R}^3 \longmapsto \mathbb{R}$ be a scalar function of a three dimensional variable. Let

$$\mathrm{rot} f = \begin{pmatrix} \partial_2 f_3 - \partial_3 f_2 \\ \partial_3 f_1 - \partial_1 f_3 \\ \partial_1 f_2 - \partial_2 f_1 \end{pmatrix}$$

be the rotational operator, and let $\mathrm{div} f = \partial_1 f_1 + \partial_2 f_2 + \partial_3 f_3$ be the divergence operator. Let $\Delta^m = \left(\sum_{i=1}^3 \partial_i^2\right)^m$. $\mathcal{F}(g)$ is the Fourier transform of the function (or distribution) g. For all $\zeta \in \mathbb{R}^3$, $\sin(\zeta)$ denotes the vector defined by $\forall 1 \leq i \leq 3$, $\sin(\zeta)_i = \sin(\zeta_i)$. δ denotes the Dirac distribution. Let $\widetilde{v}_{m+1}$ be the function such that : $\widetilde{v}_{m+1} = \frac{\Gamma\left(\frac{1}{2}-m\right)}{2^{2m+2}\pi^{3/2}} \|\cdot\|^{2m-1}$. All tensors will be denoted with bold capital letters (i.e. $\boldsymbol{X}$,...). Let $h > 0$, and $\imath$ be such that : $\imath^2 = -1$. $(e_i)_{i=1}^3$ denotes the canonical basis of $\mathbb{R}^3$. For $k \in \mathbb{N}$ and $i = 1, 2$, or 3, $\delta_{h,i}^k$ denotes the kth divided difference of step h defined by $\left(\delta_{h,i}^1 f\right)(x) = f\left(x + \frac{h}{2} e_i\right) - f\left(x - \frac{h}{2} e_i\right)$. $\delta_{h,i}^k = \delta_{h,i}^{(k-1)} \circ \delta_{h,i}^1$. We use standard multi-index notations. If $\alpha \in \mathbb{N}^3$ and $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, then $D^\alpha = \frac{\partial^{\alpha_1}}{(\partial x_1)^{\alpha_1}} \frac{\partial^{\alpha_2}}{(\partial x_2)^{\alpha_2}} \frac{\partial^{\alpha_3}}{(\partial x_3)^{\alpha_3}}$, $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}$.

§2. Discretization of $P_{m,\rho}(D)$

The goal of this section is to define an approximation of $P_{m,\rho}(D)$ which should reproduce polynomials of largest possible degree (in the sense defined below). We now introduce the $\mathbb{P}_k$-exactness of an operator, and a $\mathbb{P}_k$-exact approximation of D^α.

Definition 1. *An operator E' is said to be a $\mathbb{P}_k$-exact (resp. $\mathcal{P}_k$-exact) approximation of an operator E iff for any function f, $E'f$ is a linear combination of translates of f and for any $p \in \mathbb{P}_k$ (resp. $p \in \mathcal{P}_k$), $E'p = Ep$.*

Remark. *$h^{-1}\partial^1_{h,i}$ is a $\mathbb{P}_2$-exact approximation of $\frac{\partial}{\partial x_i}$.*

Definition 2 . *Let $D^1_{h,i,N}$ be the operator defined by*

$$D^1_{h,i,N} = \sum_{k=0}^{N} \frac{(-1)^k}{2h} \frac{(k!)^2}{(2k+1)!} \left(\delta^1_{2h,i} \circ \delta^{2k}_{h,i}\right).$$

For any $\alpha \in \mathbb{N}^3$, we define an approximation of step h and level N of D^α *to be the operator defined by*

$$D^\alpha_{h,N} = \left(D^1_{h,1,N}\right)^{\alpha_1} \circ \left(D^1_{h,2,N}\right)^{\alpha_2} \circ \left(D^1_{h,3,N}\right)^{\alpha_3}.$$

In the same way, $P_{m,\rho}(D_{h,N})$ denotes the approximation of $P_{m,\rho}(D)$.

Proposition 3. *Let $\alpha \in \mathbb{N}^3$ and $D^\alpha_{h,N}$ be defined as above. Then, for all mappings f from $\mathbb{R}^3$ to $\mathbb{R}$, there exists real constants (c_γ) such that*

$$\mathcal{F}\left(D^\alpha_{h,N} f\right) = \left(\frac{1}{h^{|\alpha|}} \sum_{|\gamma|=(2N+1)|\alpha|} c_\gamma \left(\sin(\pi h \cdot)\right)^\gamma\right) \mathcal{F}(f).$$

Proof: For every $1 \le i \le 3$,

$$\mathcal{F}\left(\delta^1_{h,i} f\right)(\zeta) = \left(\exp(\imath\pi h\zeta_i) - \exp(-\imath\pi h\zeta_i)\right) \cdot \mathcal{F}(f)(\zeta) = 2\imath \sin(h\zeta)_i \cdot \mathcal{F}(f)(\zeta)$$

By applying the Fourier's transform to the $(2N+1)|\alpha|$ centered differences of $D^\gamma_{h,N}$, we obtain the result. □

Proposition 4. *Let $\alpha \in \mathbb{N}^3$. Then*

i) $D^\alpha_{h,N}$ is a $\mathbb{P}_{|\alpha|+2N+1}$-exact approximation of D^α.

ii) $P_{m,\rho}(D_{h,N})$ is an approximation of $P_{m,\rho}(D)$ which is $\mathcal{P}_{2m+2N+1}$-exact.

Proof: We prove by induction on $|\alpha|$ that for all $f \in C^{2N+2+|\alpha|}$, there exist c_γ real constants such that

$$D^\alpha_{h,N} f = D^\alpha f + \sum_{|\gamma|=2N+2+|\alpha|} c_\gamma D^\gamma f(\zeta_\gamma).$$

In the following, for any $\gamma \in \mathbb{N}^3$, d_γ, e_γ are real constants. The proof for $|\alpha| = 1$ is due to Steffensen in ([7]). Suppose that it is true for $|\alpha| = m$. Let $|\alpha| = m+1$, $f \in C^{2N+2+m+1}$ and let us choose i such that $\alpha_i \neq 0$. Then

$$\begin{aligned}
D^\alpha_{h,N} f &= D^1_{h,i,N}\left(D^{\alpha-e_i}_{h,N} f\right) \\
&= D^1_{h,i,N}\left(D^{\alpha-e_i} f + \sum_{|\gamma|=2N+2+m} c_\gamma D^\gamma f(\zeta_\gamma)\right) \\
&= D^1_i D^{\alpha-e_i} f + \sum_{|\gamma|=2N+3} d_\gamma D^\gamma\left(D^{\alpha-e_i} f(\zeta_\gamma)\right) + \sum_{|\gamma|=2N+3+m} c_\gamma D^\gamma f(\zeta_\gamma) \\
&= D^\alpha f + \sum_{|\gamma|=2N+3+m} e_\gamma f(\zeta_\gamma).
\end{aligned}$$

Thus, we get the result if we note that $P_{m,\rho}(D)$ is a differential operator of degree $2m$. □

§3. B-spline Tensors Associated with $P_{m,\rho}(D)$

Lemma 5. *Let $\widetilde{v}_{m+1}$ be such that $(-1)^{m+1}\triangle^{m+1}\widetilde{v}_{m+1} = \delta$ in D'. Let*

$$\begin{aligned}
\widetilde{\boldsymbol{X}} &= \frac{(-1)^m}{\rho}\nabla^2\widetilde{v}_{m+1} \\
&+ (-1)^m \begin{pmatrix} \left(\partial^2_{2,2}+\partial^2_{3,3}\right)\widetilde{v}_{m+1} & -\partial^2_{1,2}\widetilde{v}_{m+1} & -\partial^2_{1,3}\widetilde{v}_{m+1} \\ -\partial^2_{1,2}\widetilde{v}_{m+1} & \left(\partial^2_{1,1}+\partial^2_{3,3}\right)\widetilde{v}_{m+1} & -\partial^2_{2,3}\widetilde{v}_{m+1} \\ -\partial^2_{1,3}\widetilde{v}_{m+1} & -\partial^2_{2,3}\widetilde{v}_{m+1} & \left(\partial^2_{1,1}+\partial^2_{2,2}\right)\widetilde{v}_{m+1} \end{pmatrix}.
\end{aligned}$$

Then $P_{m,\rho}(D)\widetilde{\boldsymbol{X}} = \delta \boldsymbol{I}_3$.

Definition 6. *Let $\widetilde{\boldsymbol{X}}$ be a fundamental solution tensor of $P_{m,\rho}(D)\cdot\boldsymbol{X} = \delta\cdot\boldsymbol{I}_3$. We define the* level N and step h B-spline tensor associated with the operator *$P_{m,\rho}(D)$, to be the tensor $\boldsymbol{B}^m_{h,N,\rho}$ defined by $\boldsymbol{B}^{m,\rho}_{h,N} = h^3 P_{m,\rho}(D_{h,N})\tilde{\boldsymbol{X}}$.*

Remarks.

a) *$\boldsymbol{B}^{m,\rho}_{h,N}$ is not a symmetrical tensor.*

b) *If m is even, we can prove the existence of a differential matrix $R_{\frac{m}{2},\rho}(D)$ of degree m such that : $P_{m,\rho}(D) = R_{\frac{m}{2},\rho}(D)\,R_{\frac{m}{2},\rho}(D)$ and we can construct a symmetrical tensor $\boldsymbol{C}^m_{h,N,\rho}$ defined by*

$$\boldsymbol{C}^{m,\rho}_{h,N} = h^3 R_{\frac{m}{2},\rho}(D_{h,N})\,\tilde{\boldsymbol{X}}\,R_{\frac{m}{2},\rho}(D_{h,N}).$$

Lemma 7. *The elements of $\left(\boldsymbol{B}^{m,\rho}_{h,N}\right)_{1\leq i,j,\leq 3}$ are in the set of tempered distributions on $\mathbb{R}^3$.*

Denoting by $\mathcal{F}\left(\boldsymbol{B}_{h,N}^{m,\rho}\right)$ the Fourier transform applied to each element of the tensor $\boldsymbol{B}_{h,N}^{m,\rho}$, we obtain the following theorem:

Theorem 8. *With the above notation,*

i) $\mathcal{F}\left(\boldsymbol{B}_{h,N}^{m,\rho}\right)(0) = h^3 \cdot \boldsymbol{I}_3$,

ii) $\forall \gamma \in \mathbb{N}_*^3,\ |\gamma| \leq 2N+1\ ;\ D^\gamma \mathcal{F}\left(\boldsymbol{B}_{h,N}^{m,\rho}(0)\right) = 0$,

iii) $\forall 1 \leq j,l \leq 3,\ \left(\boldsymbol{B}_{h,N}^{m,\rho}\right)_{j,l}(t) \underset{|t|\to+\infty}{=} \mathcal{O}\left(|t|^{-2N-5}\right)$,

iv) $\forall k \in \mathbb{Z}_*^3,\ \forall \gamma \in \mathbb{N}^3,\ |\gamma| \leq 2m-1\ ;\ D^\gamma \mathcal{F}\left(\boldsymbol{B}_{h,N}^{m,\rho}\right)\left(\frac{k}{h}\right) = 0$,

v) $\boldsymbol{B}_{h,N}^{m,\rho} = \boldsymbol{B}_{1,N}^{m,\rho}\left(\frac{\cdot}{h}\right)$.

Proof: For i)–ii), we use Definition 2 and Proposition 3. By the Taylor expansion of sin, near 0 we obtain $\mathcal{F}\left(\boldsymbol{B}_{h,N}^{m,\rho}\right)(\zeta) = h^3 \cdot \boldsymbol{I}_3 + \boldsymbol{Q}(\zeta)$, where $\boldsymbol{Q}(\zeta)$ is such that for all i and j such that $1 \leq i,j \leq 3$, there exist real coefficients $c_\alpha^{i,j}$ such that $\boldsymbol{Q}_{i,j}(\zeta) = \sum_{|\alpha| \geq 2N+2m+4} c_\alpha^{i,j} \frac{\zeta^\alpha}{\|2\pi\zeta\|^{2m+2}}$. Thus, we obtain the results.

We prove iii). By using the above expression of $\boldsymbol{Q}$, $D^\gamma \mathcal{F}\left(\boldsymbol{B}_{h,N}^{m,\rho}\right)_{j,l}$ are integrable near 0 if and only if $2N+2-|\gamma| > -3$. Using Proposition 3 and Lemma 5, there exist real coefficients $d_\alpha^{i,j,k}$ and for all i and j such that $1 \leq i,j \leq 3$

$$\mathcal{F}\left(\boldsymbol{B}_{h,N}^{m,\rho}\right)_{i,j}(\zeta) = \sum_{k=1}^{3} \frac{1}{h^{2m}} \sum_{|\alpha|=2m(2N+1)} d_\alpha^{i,j,k} \frac{(\sin(\pi h\zeta))^\alpha}{\|2\pi\zeta\|^{2m+2}} \zeta_k \zeta_j.$$

As a consequence, the elements of $\mathcal{F}\left(\boldsymbol{B}_{h,N}^{m,\rho}\right)$ are bounded at infinity by rational fractions of degree $-2m$. $D^\gamma \mathcal{F}\left(\boldsymbol{B}_{h,N}^{m,\rho}\right)_{j,l}$ are integrable near infinity if and only if $-2m-|\gamma| < -3$. Then, $D^\gamma \mathcal{F}\left(\boldsymbol{B}_{h,N}^{m,\rho}\right)_{j,l}$ are integrable in $\mathbb{R}^3$ iff $2N+2-|\gamma| > -3$ and $-2m-|\gamma| < -3$. Thus, $D^\gamma \mathcal{F}\left(\boldsymbol{B}_{h,N}^{m,\rho}\right)_{j,l} \underset{|t|\to+\infty}{=} o\left(|t|^{-2N-4}\right)$, for all $1 \leq j,l \leq 3$. Using [4], the last expression may be strengthened to

$$D^\gamma \mathcal{F}\left(\boldsymbol{B}_{h,N}^{m,\rho}\right)_{j,l} \underset{|t|\to+\infty}{=} \mathcal{O}\left(|t|^{-2N-5}\right), \qquad 1 \leq j,l \leq 3.$$

We now establish iv). For all j in $\mathbb{Z}^3$, γ in $\mathbb{N}^3$ and positive h, we have $D^\gamma \mathcal{F}\left(\boldsymbol{B}_{h,N}^{m,\rho}\right)\left(\frac{j}{h}\right) = 0$ iff $D^\gamma \mathcal{F}\left(P_{m,\rho}(D_{h,N})\right)(0) = 0$. Now by using Proposition 3, we have, for all tensors $\boldsymbol{Y}$ and all γ in $\mathbb{N}^3$ such that $|\gamma| \leq 2N+2m+1$:

$$D^\gamma\Big(\mathcal{F}\big(P_{m,\rho}(D_{h,N})\,\boldsymbol{Y}\big)\Big)(0) = D^\gamma\Big(\mathcal{F}\big(P_{m,\rho}(D)\big)(0) \cdot \mathcal{F}(\boldsymbol{Y})(0)\Big).$$

Furthermore, $\mathcal{F}(P_{m,\rho}(D))$ is a polynomial matrix of degree $2m$ so $\forall|\gamma| \leq 2m-1$, $D^\gamma \mathcal{F}(P_{m,\rho}(D))(0) = 0$. This gives iv).

Finally, we prove v). According to the Fourier transform's properties, v) is equivalent to

$$\mathcal{F}\left(\boldsymbol{B}_{h,N}^{m,\rho}\right)(\zeta) = \mathcal{F}\left(\boldsymbol{B}_{1,N}^{m,\rho}\left(\frac{\cdot}{h}\right)\right)(\zeta) = h^3 \cdot \mathcal{F}\left(\boldsymbol{B}_{1,N}^{m,\rho}\right)(h\cdot\zeta), \qquad \forall \zeta \in \mathbb{R}^3.$$

Now, from the definition of $\boldsymbol{B}_{h,N}^{m,\rho}$, we derive for any ζ in $\mathbb{R}^3$,

$$\begin{aligned}\mathcal{F}\left(\boldsymbol{B}_{1,N}^{m,\rho}\right)(h\cdot\zeta) &= \mathcal{F}\left(P_{m,\rho}(D_{1,N})\tilde{\boldsymbol{X}}\right)(h\cdot\zeta)\\ &= \mathcal{F}\Big(P_{m,\rho}(D_{1,N})\Big)(h\cdot\zeta)\cdot\mathcal{F}(P_{m,\rho})(h\cdot\zeta)^{-1}.\end{aligned}$$

Using Proposition 3, we have

$$\mathcal{F}(P_{m,\rho}(D_{1,N}))(h\cdot\zeta) = h^{2m}\mathcal{F}(P_{m,\rho}(D_{h,N}))(\zeta),$$

$$\mathcal{F}(P_{m,\rho}(D))^{-1}(h\cdot\zeta) = h^{-2m}\mathcal{F}(P_{m,\rho}(D))^{-1}(\zeta),$$

and thus $\forall \zeta \in \mathbb{R}^3$,

$$\begin{aligned}\mathcal{F}\left(\boldsymbol{B}_{1,N}^{m,\rho}\right)(h\cdot\zeta) &= \mathcal{F}(P_{m,\rho}(D_{h,N}))(\zeta)\cdot\mathcal{F}(P_{m,\rho}(D))^{-1}(\zeta)\\ &= h^{-3}\cdot\mathcal{F}\left(B_{h,N}^{m,\rho}\right)(\zeta). \quad \square\end{aligned}$$

Remarks.

a) *These polyharmonic B-spline tensors may be considered as a regularisation of the Dirac distribution tensor $\delta\,\boldsymbol{I}_3$.*

b) We obtain the same properties with $\boldsymbol{C}_{h,N}^{m,\rho}$.

§4. Associated Vector Quasi-Interpolant

Given vectorial data $(z_j)_{j\in\mathbb{Z}^3}$, in this section we define a vector field S approximating the data $(jh, z_j)_{j\in\mathbb{Z}^3}$ (i.e. such that $S(jh) \simeq z_j$ for all j in $\mathbb{Z}^3$), by using the above defined tensor B-splines. This vector generalizes the polyharmonic B-spline quasi-interpolant (see [5,6]).

Definition 9. *Let $\boldsymbol{B}_{h,N}^{m,\rho}$ be the level N and step h B-spline tensor associated with $P_{m,\rho}(D)$. For all $j \in \mathbb{Z}^3$, let $z_j \in \mathbb{R}^3$, and let $z = (z_j)_{j\in\mathbb{Z}^3}$. Then the* vector quasi-interpolant of step h and level N associated with the operator $P_{m,\rho}(D)$ and the $(jh, z_j)_{j\in\mathbb{Z}^3}$ data, *is the vector function defined by*

$$S_{h,N;z}^{m,\rho} = \sum_{j\in\mathbb{Z}^3} \boldsymbol{B}_{h,N}^{m,\rho}(\cdot - jh)\ z_j.$$

Theorem 10. *Let $l = \inf\{2N+1, 2m-1\}$, and suppose there exists $p \in \mathcal{P}_l$ such that for all j in $\mathbb{Z}^3$, $z_j = p(jh)$. Then, $S^{m,\rho}_{h,N;z} = p$. We say that the vector quasi-interpolant of step h and level N reproduces $\mathcal{P}_l$. As a particular case, $S^{m,\rho}_{h,m-1}$ reproduces $\mathcal{P}_{2m-1}$.*

Proof: The proof is based mainly on Poisson's egality and Theorem 8. It follows along the same lines as the proof in [3] in the scalar case. □

Remarks.

a) *$2m-1$ is the maximal order of reproduction.*

b) *We obtain the same properties if we define the vector quasi-interpolant using by the symmetrical tensor $C^{m,\rho}_{h,N}$.*

c) *A similar problem is studied in $\mathbb{R}^2$ in ([1]), where using another discretization of $P_{2,\rho}(D)$, the authors obtain a vector quasi-interpolant which is $\left(\mathbb{P}_1\left(\mathbb{R}^2\right)\right)^2$-reproducing.*

Theorem 11. *Let f be a vector function of $C\left(\mathbb{R}^3\right)^k$-class and all partial derivatives of order k being bounded over $\mathbb{R}^3$. Let $S^{m,\rho}_{h,N;f}$ be the above defined vector quasi-interpolant associated to the $\Big(jh, f(jh)\Big)_{j\in\mathbb{Z}^3}$ data. Then*

$$\sup_{t\in\mathbf{R}^3} \|S^{m,\rho}_{h,N;f}(t) - f(t)\| \underset{h\to 0}{=} \begin{cases} \mathcal{O}\left(h^k\right) & \text{if } k \le 2N+1, \\ \mathcal{O}\left(h^{2N+2}|\ln(h)|\right) & \text{if } k \ge 2N+2. \end{cases}$$

Proof: The proof follows that of Theorems 4.11, 5.1 and 5.6 in [3]. □

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