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On Lacunary Multiresolution Methods of Approximation in Hilbert Spaces

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Abstract. We study lacunary multiresolution methods from the point of view of their analogy to the use of near-degenerate elements in finite and boundary element methods. The main results are characterization of the best N-term approximation of solutions of nonlinear operator equations and best N-term approximation by near-degenerate normal approximating families in Hilbert spaces.

§1. Introduction

This communication is part of a sequence of papers exploring the use of near-degenerate elements in finite- and boundary-element methods (see also [5,6] and their wavelet-based analogues, lacunary multiresolution methods. The use of near-degenerate and lacunary methods for solving operator equations is of considerable practical significance because in many important problems arising in industry, engineering and natural sciences, the use of such methods leads to a dramatic reduction of execution time and/or computer resources. The theoretical justification for the use of such methods is, however, very challenging: it has been successfully carried out only in a number of special cases, by specific techniques which vary from case to case. The purpose of this sequence of papers is to develop a general approach to overcoming the challenges of the use of lacunary multiresolution and near-degenerate finite and boundary element methods. Because of the limited space available, we shall consider only multiresolution methods for operator equations in Hilbert spaces, with an outline of the main ideas of the proofs, which in the Hilbert-space case are simpler and relatively short. A much more technically involved and detailed discussion of both near-degenerate finite elements and lacunary multiresolution methods and the important parallel between them will be given for more general types of nonlinear operators in quasi-Banach spaces in a later paper.

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§2. Approximate Solutions of Nonlinear Operator Equations

In this section we consider a general class of nonlinear operator equations, and study the numerical solutions of these equations obtained by iterative and projection methods.

Let X, Y be real Hilbert spaces. The class of nonlinear operators to be considered is the space $LH(X,Y)$ of all Lipschitz homeomorphisms F between X and Y, that is, $\exists\, F^{-1}$ on Y and $\exists\, C(F,X,Y) < \infty : \, ||F(x_1) - F(x_2)||_Y \leq C||x_1 - x_2||_X$, $\forall x_1 \forall x_2 \in X$, and analogously for F^{-1}.

Let H be a Hilbert space, such that $X \cap Y \subset H$ and $X \cap Y$ is dense on H, and let Y be the dual of X pivotal to H, i.e., the dual with respect to the duality functional defined by the scalar product of H. We shall denote this dual by $Y = X^* = X^*(H)$.

Definition 1. *Let $Y = X^*(H)$. The (generally nonlinear) operator $F : X \to X^*$ is called* Lipschitz, *if*

$$\exists C(F,X,H) < \infty : ||F(x_1) - F(x_2)||_{X^*} \leq C||x_1 - x_2||_X, \tag{1}$$

$\forall x_1 \forall x_2 \in X$, and strongly monotone, *if*

$$\exists c(F,X,H) > 0 : \langle F(x_1) - F(x_2), x_1 - x_2 \rangle_H \geq c||x_1 - x_2||_X^2, \tag{2}$$

$\forall x_1 \forall x_2 \in X$. The class $LSM = LSM(X,H)$ consists of exactly those $F : X \to X^*(H)$ which satisfy (1,2).

It can be shown that the constants C and c in (1,2) are related by $c \leq C$. It should be noted that the typical case here is $X \hookrightarrow H \hookrightarrow X^*$ or $X \hookleftarrow H \hookleftarrow X^*$, where, as usual, $A \hookrightarrow B$ or $B \hookleftarrow A$ denotes continuous embedding: $A \subset B$ and $||.||_B \leq C||.||_A$.

Theorem 1. *(Generalization of Theorem 18.5 in [11] and strengthening of Theorem 18.5 in [15] for the case of Lipschitz operators in Hilbert spaces.) Let X and H be Hilbert spaces with the same cardinality. Then, $LSM(X,H) \subset LH(X, X^*(H))$.*

Proof: (Outline.) By duality arguments, it can be shown that the cardinality of $X^*(H)$ is equal to that of X and H. Therefore, since all spaces are Hilbertian with the same cardinality, there exist linear invertible operators $R : H \to X$ and $S : H \to X^*(H)$ which are isometric together with their inverses. Hence, the equation $F(x) = y$, $x \in X$, $y \in X^*(H)$ is equivalent to the equation $Av = w$, $v, w \in H$, where $Av = S^{-1}FR$. Now, since $F \in LSM(X,H)$, it follows from $||S^{-1}||_{X^* \to H} = ||R||_{H \to X} = 1$, that $A \in LSM(H,H)$. Therefore, by Theorem A (see below), F is bijective from X to X^*. By a condition of the theorem, F is Lipschitz; it remains to prove the same for F^{-1}. Indeed, by the strong monotonicity of F, setting $x_1 = F^{-1}(y_1)$, $x_2 = F^{-1}(y_2)$, $\forall y_1 \forall y_2 \in X^*$, we get

$$\begin{aligned} ||F^{-1}(y_1) - F^{-1}(y_2)||_X^2 &\leq \frac{1}{c}\langle y_1 - y_2, F^{-1}(y_1) - F^{-1}(y_2)\rangle_H \\ &\leq \frac{1}{c}||y_1 - y_2||_{X^*}||F^{-1}(y_1) - F^{-1}(y_2)||_X, \end{aligned}$$

which completes the proof. □

Theorem A. *(See [11], Theorem 18.5). Assume that $X = X^* = H$. Then, $LSM(H,H) \subset LH(H,H)$ and the operator $T_{\varepsilon,y}(x) = x - \varepsilon[F(x) - y]$, $x \in H$, is contractive in H for $0 < \varepsilon < \frac{2c}{C^2}$, uniformly in $y \in H$, where C and c are defined in (1,2). The best contraction factor is $1 - c^2/C^2$ and is achieved for $\varepsilon = c/C^2$.*

Following the idea of the proof of Theorem 1, Theorem A can be modified for the case when $X \neq H$. We omit the details.

In the remaining part of this section we shall consider methods for approximate solution of the equation $F(x) = y$, $x \in X$, $y \in Y$, where X, Y are Hilbert spaces.

Definition 2. *(See [12]). Let X be a Hilbert space. $G \subset X$ is called an* existence set for X, *if $\forall x \in X\ \exists g_x \in G : ||x - g_x||_X = \min_{g \in G} ||x - g||_X = E_G(x)_X$. (The best approximation g_x need not be necessarily unique.) The sequence $\{G_N\}_{N=1}^{\infty}$, $G_N \subset X$ is called* a normal approximating family in X, *if for any $N \in \mathbb{N}$, G_N is an existence set, with $G_N \subset G_{N+1}$ and $G_N - G_{N-1} \subset G_{2N}$.*

Obviously, an existence set in X is closed in X (typical example: any finite-dimensional subspace of X).

Definition 3. *Let X be a Hilbert space. The sequence $\{G_N\}_{N=1}^{\infty} : G_N \subset X$ is said to have the* strong approximation property *(SAP, for short) if $\bigcup_{N=1}^{\infty} G_N$ is dense on X in the inner-product topology of X.*

Let us consider now the Galerkin–Petrov projection methods. Let $P_N : X \to X$, $Q_N : X \to X$ be projectors with $\dim P_N(x) = \dim Q_N(X) = N$, and $P_N P_{N+1} = P_{N+1} P_N = P_N$, $Q_N Q_{N+1} = Q_{N+1} Q_N = Q_N$.

Example 1. (*Galerkin–Petrov method for monotone operators.*) For a Hilbert space X, let $Y = X^*(H)$. The equation $F(x) = y$, $x \in X$, $y \in X^*$, is replaced by $Q_N^* F(P_N x) = Q_N^* y$, where $Q_N^* : X^* \to X^*, \dim Q_N^*(X^*) = N$, is the Banach adjoint of Q_N. The $N \times N$ nonlinear system is determined by

$$\langle Q_N^* F(P_N x), Q_N h \rangle_H = \langle Q_N^* y, Q_N h \rangle_H. \tag{3}$$

By Lemma 23.1 in [15], it follows that if $F \in LMS(X,H)$, where X and H are separable, then (3) has a unique solution for N large enough.

In the case $X = X^* = H$, if N is large enough, so that $Q_N^* F(P_N H) = Q_N^* H$ holds, then, by Theorem A, (3) can be computed by quickly converging contractive iterations. For small N, the condition $Q_N^* F(P_N H) = Q_N^* H$ may fail even if F is linear and $P_N = Q_N$ (see [2], Theorem 10.1.1).

If F is twice Gateau-differentiable, then Newton's method can be used where the inverse matrix involved in each iteration is usually sparse. In general, this method needs an appropriate initial approximation x_0 to the solution of $F(x) = y$, but if F is strongly monotone and potential, that is, if there exists a real functional $f : X \to \mathbb{R}$ such that $F = \operatorname{grad} f$, then, by Theorem 5.1 in [15], f is strictly convex and the solution of (3) is equivalent to minimizing the three times Gateau-differentiable functional f. Hence, Newton's method converges to the solution of (3) for any $x_0 \in X$, the rate of convergence depending on the constant c in (2). This technique is still numerically efficient if F is only Lipschitz, and Newton's method or its various modifications be replaced by the respective variants of the more general F. Clarke's subdifferential method. In the case of potential F, the Bubnov–Galerkin method ($P_N = Q_N$) coincides with the Ritz method for minimization of f.

For projection methods (see Example 1), the strong approximation property can be written as $\lim_{N\to\infty} ||(I_X - P_N)x||_X = 0$. A typical example when $G_N = P_N X$ forms a NAF having the SAP is when P_N is obtained by multiresolution.

By Theorem 23.3 in [15], if X is separable, then $G_N = P_N X$, as defined in Example 1, has the SAP; by Lemma 23.1 in [15] and in view of $F \in LMS(X, H)$, the solution x_N of (3) exists for N large enough and $||x_N - x||_X \to 0$, where x is the solution of $F(x) = y$, $y \in X^*$.

Theorem B. *(Céa's lemma for nonlinear operators (see [13], Lemma 2.8; [11], Theorems 4.1 and 18.8.) Under the assumptions of Example 1, for $F \in LSM(X, H)$, let $x = F^{-1}(y) \in X$ be the solution of $F(x) = y \in X^*(H)$, and $x_n \in X$ be the solution of (3). Then,*

$$\exists C(F, X, H) < \infty : E_N(F^{-1}(y))_X \le ||F^{-1}(y) - x_N||_X \le C E_N(F^{-1}(y))_X.$$

This result shows that Galerkin-Petrov methods (of any type - finite element or wavelet) achieves the best approximation rates up to a constant factor.

§3. Best N-term Approximation

For the general paradigm of best N-term approximation (BNTAP) we refer to [12], section 3.5, and [8].

Definition 4. *Let X_j, Y_j, $j = 0, 1$ be Hilbert spaces, $X_1 \hookrightarrow X_0$, $Y_1 \hookrightarrow Y_0$, and let $F \in LH(X_0, Y_0) \cap LH(X_1, Y_1)$. The NAF $\{G_N\}_{N=1}^{\infty}$: $G_N \subset X_1$, is called* near-degenerate of order $(\lambda; \alpha, \beta)$, $\lambda > 0$, $\alpha \ge 0$, $\beta \ge 0$, *if it satisfies a* direct inequality *of the type*

$$\exists C < \infty : \; E_N(F^{-1}(y))_{X_0} \le C \frac{||F^{-1}(y)||_{X_1}}{N^\lambda}, \qquad \forall y \in Y_1, \tag{4}$$

where $C = C(N)$, with $C \asymp N^\alpha$; and an inverse inequality *of the type*

$$\exists D < \infty : \; ||x||_{X_1} \le D N^\lambda ||x||_{X_0}, \; x \in G_N, \tag{5}$$

where $D = D(N)$, with $D \asymp N^\beta$. The partial case $\alpha = \beta = 0$ corresponds to a non-degenerate (regular) *NAF.*

Consider the approximation space

$$A_q^s(X_0) := \{f \in X_0 : ||f||_{A_q^s(X_0)} = \left(||f||_{X_0}^q + \sum_{j=0}^{\infty}[2^{js}E_{2^j}(f)_{X_0}]^q\right)^{1/q} < \infty\} \tag{6}$$

and the real interpolation space

$$(Y_0, Y_1)_{\theta,q} := \{f \in X_0 : \ ||f||_{(Y_0,Y_1)_{\theta,q}} = \left(||f||_{X_0}^q + \sum_{j=0}^{\infty}[2^{j\theta}K(2^{-j}, f; Y_0, Y_1)]^q\right)^{1/q} < \infty\}, \tag{7)}$$

where $K(t, f; Y_0, Y_1)$ is Peetre's K-functional (see [2,12]), $0 < t < \infty$, $s > 0$, $0 < \theta < 1$, $0 < q \leq \infty$ (with the usual sup-modification in (6,7) for $q = \infty$). (Recall that $X_1 \hookrightarrow X_0$, which explains the presence of the saturation term $||f||_{X_0}^q$ in (6,7).)

Theorem 2. *(Characterization of the best N-term approximation of solutions of nonlinear operator equations by near-degenerate NAF in Hilbert spaces). Assume that the conditions of Definition 4 hold. Let $0 < q \leq \infty$. Then,*

(i) if $0 \leq \alpha < \lambda$ and $s : 0 < s < \lambda - \alpha$, then, $\exists C_1 < \infty :$

$$||F^{-1}(y)||_{A_q^s(X_0)} \leq C_1[||F^{-1}(0)||_{X_1} + ||y||_{(Y_0,Y_1)_{\frac{s}{\lambda-\alpha},q}}]; \tag{8}$$

(ii) if $\beta \geq 0$ and $0 < s < \lambda + \beta$, then $\exists C_2 < \infty :$

$$||y||_{(Y_0,Y_1)_{\frac{s}{\lambda+\beta},q}} \leq C_2[||F(0)||_{Y_1} + ||F^{-1}(y)||_{A_q^s(X_0)}]. \tag{9}$$

Proof: (Outline.) By a standard technique, typical for BNTAP (see [12], Theorem 3.16 and Corollary 3.7), we prove

$$||F^{-1}(y)||_{A_q^s(X_0)} \leq c_1||F^{-1}(y)||_{(X_0,X_1)_{\frac{s}{\lambda-\alpha},q}}, \tag{10}$$

$$||F^{-1}(y)||_{(X_0,X_1)_{\frac{s}{\lambda+\beta},q}} \leq c_2||F^{-1}(y)||_{A_q^s(X_0)}. \tag{11}$$

By obtaining appropriate upper bounds for the K-functionals in the definition of $(X_0, X_1)_{\theta,q}$ and $(Y_0, Y_1)_{\theta,q}$, $0 < \theta < 1$, and using the embeddings $X_1 \hookrightarrow X_0$, $Y_1 \hookrightarrow Y_0$, it can be shown that, for Lipschitz operators F, F^{-1},

$$||F^{-1}(y)||_{(X_0,X_1)_{\frac{s}{\lambda-\alpha},q}} \leq c_3(||F^{-1}(0)||_{X_1} + ||y||_{(Y_0,Y_1)_{\frac{s}{\lambda-\alpha},q}}), \tag{12}$$

$$||F(x)||_{(Y_0,Y_1)_{\frac{s}{\lambda+\beta},q}} \leq c_4(||F(0)||_{Y_1} + ||x||_{(X_0,X_1)_{\frac{s}{\lambda+\beta},q}}) \tag{13}$$

hold. Combining (10) with (12) and (13) with (11), we arrive at (8) and (9), respectively. □

Corollary 1. *Under the conditions of Theorem 2, let* $0 < s < \lambda - \alpha$, *and assume that* $F(0) = 0_{Y_1}$, $F^{-1}(0) = 0_{X_1}$. *Then,*

$$(Y_0, Y_1)_{\frac{s}{\lambda-\alpha},q} \hookrightarrow A_q^s(X_0) \hookrightarrow (Y_0, Y_1)_{\frac{s}{\lambda+\beta},q}. \tag{14}$$

In particular, if $\alpha = \beta = 0$, *then*

$$(Y_0, Y_1)_{\frac{s}{\lambda},q} = A_q^s(X_0) \tag{15}$$

(isomorphism of the spaces, equivalence of the Hilbert norms).

Note that this special case corresponds to sublinear operators.

Remark. If the dependence of C in (4) and/or D in (5) on N is weaker than polynomial, e.g., logarithmic, then the left-hand and/or right-hand embedding in (14) can be sharpened by setting $\alpha = 0$ and/or $\beta = 0$ and modifying the index q. We omit the details.

Multiresolution Galerkin-Petrov methods for monotone operators (Example 1) are included as partial cases in Theorem 2 and Corollary 1. For monotone operators, we have $Y_0 = X_0^*(H_0)$, $Y_1 = X_1^*(H_1)$, where H_0, H_1 are Hilbert spaces with $H_0 \hookleftarrow H_1$ which are *sufficiently far away from each other so that* $X_0 \hookleftarrow X_1$ *and* $X_0^*(H_0) \hookleftarrow X_1^*(H_1)$ *hold simultaneously.* Here $X_1 \cap Y_1$ is assumed to be dense in H_0 and H_1. The projectors P_N and Q_N in Example 1 are assumed generated by multiresolution, which ensures that $G_N - G_{N-1} \subset G_{2N}$.

In the rest of this section and in the next section we shall discuss how to reduce the rates α and β in Theorem 2 and Corollary 1 to zero in the presence of near-degeneracy. To this end, we shall study the analogue of the phenomenon of near-degeneracy with multiresolution methods based on biorthogonal wavelets.

One equivalent norm in the inhomogeneous potential spaces H^s (cf., e.g., [14,4] for $p = q = 2$) is given by

$$||f||_{H^s} \asymp \{ \sum_{k\in\mathbb{Z}^n} |\alpha_{0k}|^2 + \sum_{j=0}^{\infty} 2^{2js} \sum_{k\in\mathbb{Z}^n} \sum_{l=1}^{2^n-1} |\beta_{jk}^{[l]}|^2 \}^{1/2}, \tag{16}$$

with $0 < s < r$, where in [8] r is the Lipschitz regularity of the compactly supported scaling functions $\varphi \in H^r$, $\tilde{\varphi} \in H^r$ and wavelets $\psi^{[l]} \in H^r$, $\tilde{\psi}^{[l]} \in H^r$ of the biorthonormal wavelet bases, with respect to which $f \in H^s$ can be expanded as follows:

$$f(x) = \sum_{k\in\mathbb{Z}^n} \alpha_{0k}\varphi_{0k}(x) + \sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}^n} \sum_{l=1}^{2^n-1} \beta_{jk}^{[l]}\psi_{jk}^{[l]}(x), \quad \text{a. e. } x, \tag{17}$$

where $\alpha_{0k} = \langle f, \tilde{\varphi}_{0k}\rangle_{L_2}$, $\beta_{jk}^{[l]} = \langle f, \tilde{\psi}_{jk}^{[l]}\rangle_{L_2}$. Each hypercube in the Calderon-Zygmund decomposition of $\mathbb{R}^n$ and Stein's construction of Whitney-type extension operators corresponds to $2^n - 1$ basis functions $\tilde{\psi}_{jk}^{[l]}$, $\psi_{jk}^{[l]}$, $l = 1, \dots, 2^n - 1$, in each of the two biorthonormal bases. The convergence in (17) is in the norm topology of H^s, but also Lebesgue a.e. on the domain Ω of the functions. Ω may be $\mathbb{R}^n$, hyperrectangle, correspond to the periodic case, or even general Lipschitz-graph domain. We refer to the currently most advanced work on this topic [3], as well as to the extensive account [4] (for the case of *homogeneous* potential spaces, see [7], in the special case $p = q = 2$).

Definition 5. *Let $j_1 \in \mathbb{N}$. A* non-degenerate wavelet-based projector *(NWP) is denoted by P_{j_1} and defined by*

$$P_{j_1} f(x) = \sum_k \alpha_{0k} \varphi_{0k}(x) + \sum_{j=0}^{j_1 - 1} \sum_k \sum_{l=1}^{2^n - 1} \beta_{jk}^{[l]} \psi_{jk}^{[l]}(x), \quad x \in \Omega, \tag{18}$$

cf. (17). A near-degenerate wavelet-based projector *(NDWP) is denoted by $\tilde{P}_{j_1}$ and defined by*

$$\tilde{P}_{j_1} f(x) = \sum_k \alpha_{0k} \varphi_{0k}(x) + \sum_k \sum_{j=0}^{J(j_1,k)-1} \sum_{l=1}^{2^n - 1} \beta_{jk}^{[l]} \psi_{jk}^{[l]}(x), \quad x \in \Omega, \tag{19}$$

$$\forall k \; J(j_1, k) > J(j_1 - 1, k), \; J(j_1, k) \geq j_1; \; \exists k_{j_1} : J(j_1, k_{j_1}) = j_1. \tag{20}$$

In other words, for a NWP $J(j_1, k) = j_1 = \text{const}$, uniformly in k. Thus, NDWP's are a specific partial case of lacunary wavelet-based projectors (see the concluding remarks in [4], subsection 6.2), lacunarity being with respect to the NWP corresponding to $J_1 := \max_k J(j_1, k)$.

Example 2. One example when near-degenerate FEM or lacunary wavelet-based projectors of NDWP type are needed is in the error analysis of numerical solutions in the immediate neighbourhood of the boundary $\partial\Omega$ (see, e.g., [11], Fig. 3.14, 3.15, 6.14, 6.15, 8.12). Then it is desirable to ensure that the local approximation rates near and on $\partial\Omega$ *do not deteriorate* compared to the local approximation rates in the interior of Ω. Indeed, assume that $\partial\Omega$ is regular enough (Lipschitz or smoother). Then, by the trace theorem (see, e.g., [1,9]), if $f \in H^s(\Omega)$, $\Omega \subset \mathbb{R}^n$, then the restriction of f on $\partial\Omega$ is less regular, namely, $f_{|\partial\Omega} \in H^{s-1/2}(\partial\Omega)$ holds. Then, the local approximation rate achieved via NWP, given in (18), is $O(2^{-j_1 s})$ in the interior of Ω and only $O(2^{-j_1(s-1/2)})$ near $\partial\Omega$. To achieve the desired uniform distribution of the error in the interior and near the boundary when f is smooth enough ($s > 1/2$), NDWP given in (19,20) should be employed, with $J(j_1, k) \approx j_1$ for k corresponding to the interior of Ω, and with $J(j_1, k) \asymp C_1 j_1 + C_2$ otherwise, where

$$C_2 \geq 0, \qquad C_1 = 1 + \frac{1}{2(s - 1/2)} > 1. \tag{21}$$

In the context of Theorem 2 and Corollary 1, if $X_j = H^{s_j}$, $j = 0, 1$, with $s_0 \leq s_1$ so that $X_1 \hookrightarrow X_0$ is fulfilled, then it can be verified that for NWP both the direct inequality $||f - P_{j_1}f||_{X_0} \leq C_1 2^{-j_1\lambda}||f||_{X_1}$ and the inverse inequality $||P_{j_1}f||_{X_1} \leq C_2 2^{j_1\lambda}||P_{j_1}f||_{X_0}$ hold, with $\lambda = s_1 - s_0$ and with C_1, C_2 independent of j_1. Hence, in Definition 4 $\alpha = \beta = 0$ is attained. On the contrary, for NDWP satisfying (21) the constants C_1 and C_2 depend on j_1 and $\alpha > 0$, $\beta > 0$ holds.

In the case of NDWP, is it possible to somehow reduce α and β to zero, thereby achieving isomorphism in (14)? It turns out that *the answer is positive*, and below we shall propose a general method how to achieve this.

Our approach will be consider *more general spaces* X_0, X_1 *than* H^s, so that, for the new X_0 and X_1, $\alpha = \beta = 0$ holds. Consider the Hilbert space $H^{s,w}$ with norm

$$||f||_{H^{s,w}} \asymp (\sum_k |\alpha_{0k}|^2 + \sum_{j=0}^{\infty}\sum_k 2^{2w(j,k)s} \sum_{l=1}^{2^n-1} |\beta_{jk}^{[l]}|^2)^{1/2}.$$

The spaces from this scale still admit atomic decomposition via the same Riesz bases of biorthonormal wavelets as H^s. The weight $w(j,k)$ is positive, monotonously increasing function in j for each fixed k, and depends on the choice of $J(j_1,k)$ in (20). The definition of $w(j,k)$ is

$$w(J(j_1,k),k) = j_1, \tag{22}$$

$$w(j,k) = j_1 - 1,\ \ j = J(j_1-1,k),\ J(j_1-1,k)+1,\ \ldots,\ J(j_1,k)-1, \tag{23}$$

$\forall j_1 \in \mathbb{N}\ \forall k \in \mathbb{Z}^n$.

Now, take $X_j = H^{s_j,w}$, $j = 0,1$, with $s_0 < s_1$. It can be seen that $X_1 \hookrightarrow X_0$ holds, and we can consider this pair of spaces in the context of Theorem 2 and Corollary 1.

Corollary 2. *Under the conditions of Corollary 1, assume that* $X_j = H^{s_j,w}$, $j = 0,1$, *where* $w = w(j,k)$ *is the left inverse (see (22,23)) of* $J(j,k)$ *as defined in (20). Assume also that* $N = 2^{j_1}$ *and* $G_N = \tilde{P}_{j_1}X_0$, *where the NDWP* $\tilde{P}_{j_1}$ *is defined in (19), with the same* $J(j,k)$ *in (20). Let* $s : 0 < s < \lambda = s_1 - s_0$. *Then (15) holds.*

Proof: (Outline.) It can be verified that the bounds

$$||f - \tilde{P}_{j_1}f||_{H^{s_0,w}} \leq (\sum_{j=j_1}^{\infty} \sum_{k:w(j,k)\geq j_1} 2^{2w(j,k)s_0} \sum_l |\beta_{jk}^{[l]}|^2)^{1/2}, \tag{24}$$

$$||\tilde{P}_{j_1}f||_{H^{s_1,w}} \leq (\sum_k |\alpha_{0k}|^2 + \sum_{j=0}^{J_1} \sum_{k:w(j,k)\leq j_1} 2^{2w(j,k)s_1} \sum_l |\beta_{jk}^{[l]}|^2)^{1/2}, \tag{25}$$

hold. (Recall that $j_1 = \min_k J(j_1,k)$, $J_1 = \max_k J(j_1,k)$.) After some computations, (24) and (25) imply

$$||f - \tilde{P}_{j_1}f||_{X_0} \leq C_1 2^{-j_1\lambda}||f||_{X_1},\ \ \forall f \in X_1, \tag{26}$$

$$\|\tilde{P}_{j_1} f\|_{X_1} \le C_2 2^{j_1 \lambda} \|\tilde{P}_{j_1} f\|_{X_0}, \quad \forall f \in X_0, \tag{27}$$

with $\lambda = s_1 - s_0$, and the constants C_1 and C_2 in (26,27) *do not depend on* j_1, i.e., for this choice of the spaces X_0, X_1 in Definition 4 $\alpha = \beta = 0$ holds. The result now follows from Corollary 1. □

Thus, we have solved the problem of characterizing the best approximation spaces induced by NDWP defined in (19) and (20). In this approach we remained entirely within the classical BNTAP. There is also another approach which goes beyond the general BNTAP, by abandoning the use of the real interpolation functor. This approach leads to atomic decomposition of Wiener amalgam spaces and will be considered elsewhere.

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