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Quantized Frame Decompositions

M. Craizer, D. A. Fonini, Jr., and E. A. B. da Silva

Abstract. In this paper, we consider a certain type of decomposition of vectors in frames, in which the coefficients are already quantized and thus are ready for coding. This decomposition is a generalization for vectors of the usual binary expansion of real numbers, and the algorithm for obtaining it can be seen as a quantized version of the matching pursuits algorithm. We show that, in several cases, applying this algorithm is better than first finding the frame coefficients and then quantizing them.

§1. Introduction

Let $\mathcal{F} = \{e_1, e_2, \dots, e_p\}$ be a collection of unit vectors generating $\mathbb{R}^N$. This means that every $x \in \mathbb{R}^N$ can be expressed as

$$x = \sum_{i=1}^{p} a_i e_i.$$

The vectors $\{e_1, e_2, \dots, e_p\}$ may or may not be linearly independent. In the case that they are linearly dependent, the set $\mathcal{F}$ is called a frame or an overcomplete basis. In this paper, we shall call $\mathcal{F}$ a frame even if the vectors $\{e_1, e_2, \dots, e_p\}$ are linearly independent. More on frame expansions can be found in [7].

Let $q = 2p$ and, for $1 \leq i \leq p$, let $v_i = e_i$ and $v_{i+p} = -e_i$. We shall call the set $\mathcal{D} = \{v_1, \dots, v_q\}$ a codebook or a dictionary. Let α be a real number in the interval $(0,1)$. A representation of a vector $x \in \mathbb{R}^N$ in the form

$$x = \sum_{i=0}^{\infty} \alpha^i v_{k_i}$$

with $v_{k_i} \in \mathcal{D}$, will be called an $(\alpha, \mathcal{D})$-expansion. When the dictionary being used is clear by the context, we shall call this representation simply an α-expansion. Observe that the $(\alpha, \mathcal{D})$-expansion of a vector x can be seen as a decomposition of x in the frame $\mathcal{F}$.

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Define the $n^{\underline{th}}$ residual of a vector $\boldsymbol{x}$ by

$$r_n(\boldsymbol{x}) = \begin{cases} r_0(\boldsymbol{x}) = \boldsymbol{x}, & \text{if } n = 0, \\ \boldsymbol{x} - \sum_{i=0}^{n-1} \alpha^i \boldsymbol{v}_{k_i}, & \text{if } n > 0. \end{cases}$$

Given $\boldsymbol{x}$, the sequence $(k_0, k_1, \ldots)$ can be obtained recursively by the relation

$$\langle \boldsymbol{r}_n(\boldsymbol{x}), \boldsymbol{v}_{k_n} \rangle = \max_k \langle \boldsymbol{r}_n(\boldsymbol{x}), \boldsymbol{v}_k \rangle .$$

We shall call this algorithm the nearest point algorithm and it may be seen as a quantized version of the matching pursuits algorithm [4].

Denote by $\Lambda_\alpha = \Lambda_\alpha(\mathcal{D})$ the set of points of $\mathbb{R}^N$ that can be represented as an α-expansion of vectors that belong to $\mathcal{D}$, and by $\Lambda_\alpha^0 = \Lambda_\alpha^0(\mathcal{D})$ the subset of $\Lambda_\alpha(\mathcal{D})$ whose α-expansion can be obtained by the nearest point algorithm. In order for the α-expansion or the nearest point algorithm to be a suitable scheme for quantized frame decomposition, we must choose α such that Λ_α or Λ_α^0, respectively, contain an open set of $\mathbb{R}^N$. In Section 2, we shall give conditions on α that guarantee these facts.

At this point, a question arises: is it worthwhile to decompose a vector in a frame using the α-expansion, or is it better to decompose it in the usual way and then quantize the coefficients in a second step (see [6,9])? We shall answer this question by considering the rate-distortion characteristic of each scheme. We show in Section 3 that the first scheme is better, in an asymptotic sense, if and only if we can choose α satisfying

$$\frac{\log_2(2p)}{\log_2 \frac{1}{\alpha}} < p.$$

We shall also give examples where this inequality holds.

Take $\boldsymbol{x} = (x_1, \ldots, x_N) \in \mathbb{R}^N$. We can quantize $\boldsymbol{x}$ by taking the n-term binary representation of each coordinate x_i. This procedure can be considered as an n-term α-expansion using the dictionary $\mathcal{B}_N$ whose code vectors are the corners of the hypercube $[-1,1]^N$ and $\alpha = \frac{1}{2}$. So the α-expansion in an arbitrary dictionary $\mathcal{D}$ can be considered as a generalization of the usual binary expansion for vectors. The relevant question is whether there is any dictionary $\mathcal{D}$ that is better in some sense than $\mathcal{B}_N$. It is worthy of note that some special dictionaries, related to the sphere packing problem [2], have already been used for image coding, yielding better results than $\mathcal{B}_N$ ([3,8]).

§2. Theory of Alpha-Expansions

General representation

Let $\mathcal{D} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_q\}$ be a collection of vectors that generates all $\mathbb{R}^N$, and $0 < \alpha < 1$ be a parameter. Denote by Λ_α the set of points $\boldsymbol{x} \in \mathbb{R}^N$ that can be written as

$$\boldsymbol{x} = \sum_{i=0}^{\infty} \alpha^i \boldsymbol{v}_{k_i}$$

with $v_{k_i} \in \mathcal{D}$. Let P_α be the convex hull of the vectors $\{\frac{1}{1-\alpha} v_k\}_{k=1,\dots,q}$. Observe that any $x \in \Lambda_\alpha$ is a linear combination of the vectors $\{v_k\}_{k=1,\dots,q}$ with coefficients whose sum is not larger than $\frac{1}{1-\alpha}$, and therefore $\Lambda_\alpha \subset P_\alpha$.

Define the contracting maps $f_k = f_{k,\alpha}$ by $f_k(x) = \alpha x + v_k$ for $k \in \{1,\dots,q\}$. We observe that f_k is a homotety of center $\frac{1}{1-\alpha} v_k$, which implies that $f_k(P_\alpha) \subset P_\alpha$. Therefore the set $\{f_1,\dots,f_q\}$ forms an iterated function system (IFS) [1] on P_α. It is not difficult to show that the attractor of this system is exactly Λ_α, i.e.,

$$\Lambda_\alpha = \bigcap_{n=0}^{\infty} F^n(P_\alpha),$$

where $F = F_\alpha$ is the function of sets defined by

$$F(A) = f_1(A) \cup \cdots \cup f_q(A).$$

Example 1. *Let $\mathcal{D} = \mathcal{B}_N$, the dictionary whose code vectors are the corners of the hypercube $[-1,1]^N$. For any $\alpha \geq \frac{1}{2}$, we have that $F(P_\alpha) = P_\alpha$ and therefore $\Lambda_\alpha = P_\alpha$.*

We are interested in finding the smallest value of α such that Λ_α contains an open set. In the above example, this occurs for $\alpha = \frac{1}{2}$, when in fact $\Lambda_\alpha = P_\alpha$.

Remark 1. *One can show that for any dictionary, if $\alpha \geq \frac{N}{N+1}$, then $\Lambda_\alpha = P_\alpha$, which shows that the smallest value of α such that Λ_α contains an open set is smaller than $\frac{N}{N+1}$.*

In Example 1, the smallest α such that Λ_α contains an open set satisfies also $\Lambda_\alpha = P_\alpha$. But this is not a general fact, as the following example shows.

Example 2. *Let $\mathcal{D} = \mathcal{B}_3 \cup \{(1,0,0), (-1,0,0)\}$, where $\mathcal{B}_3$ is the dictionary whose code vectors are the corners of the cube $[-1,1]^3$. If we consider $\alpha = \frac{1}{2}$, then $[-1,1]^3 \subset \Lambda_\alpha$, but Λ_α is strictly contained in P_α. This fact can be seen by observing that the centroid of the face of P_α whose vertices are $(1,0,0)$, $\frac{1}{\sqrt{3}}(1,1,1)$ and $\frac{1}{\sqrt{3}}(1,1,-1)$ are not contained in $F(P_\alpha)$, which implies that $F(P_\alpha) \neq P_\alpha$, and thus $\Lambda_\alpha \neq P_\alpha$.*

In all examples that we have considered, we observe that if Λ_α contains an open set of $\mathbb{R}^N$, then it also contains the convex hull of some of the points of the dictionary. We don't know whether this is always true, so we formulate it as a question:

Question 1. *If, for some $0 < \alpha < 1$, $\Lambda_\alpha(\mathcal{D})$ contains an open set of $\mathbb{R}^N$, then will a subdictionary $\mathcal{D}_1 \subset \mathcal{D}$ always exist such that $\Lambda_\alpha(\mathcal{D}) \supseteq P_\alpha(\mathcal{D}_1)$?*

Basic algorithm

How do we obtain the sequence of indexes $(k_0, k_1, \ldots)$ that represent a given vector $\boldsymbol{x} \in \Lambda_\alpha$? In general, the representation of a vector $\boldsymbol{x}$ is not unique. In order to define which of the sequences representing the vector x we shall look for, we consider a choice function $K : F(P_\alpha) \to \{1, \ldots, q\}$ with the following properties:

1) $f_k(V_k) \subset V_k$, for any $k \in \{1, \ldots, q\}$, where $V_k = K^{-1}(k)$.
2) If $K(\boldsymbol{x}) = k$, then $\boldsymbol{x} \in f_k(P_\alpha)$, for any $\boldsymbol{x} \in F(P_\alpha)$.

It can be shown that such a function always exists. This choice function K determines a function $g : F(P_\alpha) \to P_\alpha$ given by

$$g(\boldsymbol{x}) = \frac{\boldsymbol{x} - \boldsymbol{v}_{K(\boldsymbol{x})}}{\alpha}.$$

It is not difficult to show that

$$\Lambda_\alpha = \cap_{n=1}^{\infty} g^{-n}(P_\alpha).$$

This implies that if $\boldsymbol{x} \in \Lambda_\alpha$, then $g^i(\boldsymbol{x}) \in \Lambda_\alpha$, for any $i \geq 0$.

By the last paragraph, given $\boldsymbol{x} \in \Lambda_\alpha$, we can choose the sequence $(k_0, k_1, \ldots)$ by the relation $k_i = K(g^i(\boldsymbol{x}))$. We shall call this the *basic algorithm.* This algorithm always works, but it is computationally expensive. So we propose another algorithm, computationally feasible, called the nearest point algorithm.

Nearest point algorithm

The nearest point algorithm is used for obtaining a sequence $(k_0, k_1, \ldots)$ representing a vector $\boldsymbol{x} \in \Lambda_\alpha$. It can be seen as a quantized version of the matching pursuits algorithm.

It is determined by the choice function K_0 defined by the property that $K_0(\boldsymbol{x})$ is the code vector in $\mathcal{D}$ nearest to $\boldsymbol{x}$. We denote by $V_{0,k}$ the set $K_0^{-1}(k)$ and by g_0 the function

$$g_0(\boldsymbol{x}) = \frac{\boldsymbol{x} - \boldsymbol{v}_{K_0(\boldsymbol{x})}}{\alpha}.$$

This choice function certainly satisfies Property 1 above, but Property 2 can fail. It is not difficult to see that Property 2 holds if and only if $V_{0,k} \subset f_k(P_\alpha)$, for every $k \in \{1, \ldots, q\}$.

Let $\Lambda_\alpha^0(\mathcal{D}) = \cap_{n=1}^{\infty} g_0^{-n}(P_\alpha)$. We have that $\Lambda_\alpha^0(\mathcal{D}) \subset \Lambda_\alpha(\mathcal{D})$, but they are not necessarily equal. One can verify that $\Lambda_\alpha^0(\mathcal{D}) = \Lambda_\alpha(\mathcal{D})$ if and only if the choice function K_0 satisfies Property 2 above.

Example 3. *Let $\mathcal{D} = \{(1,0), (0,1), (-1,0)\}$. For any $0 < \alpha < 1$, the segment $(0,\delta), 0 < \delta < 1$ is not contained in $f_2(P_\alpha)$. Therefore $g_0(0,\delta)$ is not in P_α, which implies that this segment is not in $\Lambda_\alpha^0(\mathcal{D})$. On the other hand, if $\alpha \geq \frac{2}{3}$, $\Lambda_\alpha(\mathcal{D}) = P_\alpha$.*

We have also observed in examples that if $\Lambda_\alpha^0(\mathcal{D})$ contains an open set of $\mathbb{R}^N$, then it must contain the convex hull of some code vectors. This prompts the following question:

Question 2. *If, for some $0 < \alpha < 1$, $\Lambda^0_\alpha(\mathcal{D})$ contains an open set of $\mathbb{R}^N$, then will there be a subdictionary $\mathcal{D}_1 \subset \mathcal{D}$ such that $\Lambda^0_\alpha(\mathcal{D}) \supseteq P_\alpha(\mathcal{D}_1)$?*

§3. Comparison between Alpha-Expansions and the Decompose-Quantize Procedure

In this section we shall compare the α-expansion in a frame with the 2-step procedure of first decomposing in the frame and then quantizing the coefficients so obtained in a second step. We shall do this by comparing the rate-distortion functions of each scheme.

Let $\mathcal{F} = \{e_1, e_2, \ldots, e_p\}$ be a frame in $\mathbb{R}^N$, and take $x \in \mathbb{R}^N$ with $\|x\| \leq M$. We shall assume that the coefficients $(a_1, a_2, \ldots, a_p)$ of the decomposition x in the frame $\mathcal{F}$ satisfy $|a_i| \leq C_1 M$, for some constant C_1 that depends only on $\mathcal{F}$. We shall consider here the quantization of these coefficients by binary expansions. If each coefficient is represented by n bits, the total number of bits used is $R = np$, and the maximum square error per coefficient is given by $\left[C_1 M \left(\frac{1}{2}\right)^{n-1}\right]^2$. If we multiply this by p, we obtain the total maximum square distortion D. Therefore, we can write the rate-distortion relation

$$R = \frac{p}{2}\log_2\left(\frac{4C_1^2 M^2}{D}\right).$$

Let us consider now the α-expansion procedure. If we approximate $x \in \Lambda_\alpha$, $\|x\| \leq M$, by its n-term α-expansion $(v_{i_0}, v_{i_1}, \ldots, v_{i_{n-1}})$, the maximum square distortion is given by $D = [C_2 M \alpha^n]^2$, where C_2 is a constant that depends only on $\mathcal{F}$. The number of bits necessary to code this sequence is $R = n\log_2(2p)$, and thus we have the rate-distortion relation

$$R = \frac{\log_2(2p)}{2\log_2\left(\frac{1}{\alpha}\right)}\log_2\left(\frac{C_2^2 M^2}{D}\right).$$

We conclude that asymptotically, the α-expansion is better than the decompose-quantize procedure if we can choose α such that Λ_α contains an open set and

$$\frac{\log_2(2p)}{\log_2\left(\frac{1}{\alpha}\right)} \leq p \tag{1}$$

Example 4. *Let $F = \left\{\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right), (1,0), \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)\right\}$ and take $\alpha = \frac{1}{2}$. Then $\Lambda_\alpha = P_\alpha$ (which in this case is a hexagon) and*

$$\frac{\log_2(2p)}{\log_2\frac{1}{\alpha}} = \log_2 6 < 3,$$

which implies that in this case the α-expansion is better than the decompose-quantize procedure.

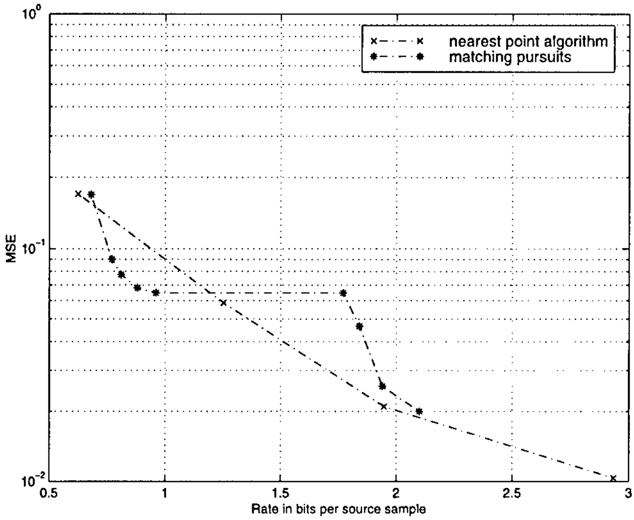

Fig. 1. The 8-dimensional hypercube as codebook.

Remark 2. *If for a given frame $\mathcal{F}$, Λ_α contains an open set, then this remains true for any other frame $\mathcal{F}_1$ obtained from $\mathcal{F}$ by adding some more vectors. Hence, even if relation (1) does not hold for $\mathcal{F}$, it will hold for highly redundant frames $\mathcal{F}_1$ containing $\mathcal{F}$.*

Remark 3. *By Remark 1, $\alpha \geq \frac{N}{N+1}$ implies that $\Lambda_\alpha = P_\alpha$. Therefore, if we consider frames satisfying*

$$\frac{\log_2(2p)}{p} \leq \log_2\left(\frac{N+1}{N}\right),$$

the α-expansion scheme will be better than the decompose-quantize procedure.

Experimental results

In order to directly compare our method (the nearest point algorithm) with some established results, we look at some examples presented in [6] (3.4.2, pp. 41–45). To this end, we used a similar source and the same codebook.

A zero-mean gaussian AR source with correlation coefficient $\rho = 0.9$ was used to provide the data points. Vectors were formed by blocks of N samples. Rate was measured as the first order entropy of the index stream produced by the algorithm – similarly to [6].

In Fig. 1, we used the vertices of the 8-dimensional hypercube as the codebook. In this case, α was set to 0.501 and $\| v_k \| = 1.4142$ (that is $\frac{\sqrt{8}}{2}$).

As can be seen from this data, our method does give some performance benefits on low bit-rates.

Although our algorithm can be very expensive in terms of computational effort, so are other greedy algorithms like matching pursuits. But there are some well structured codebooks which lend themselves to fast calculation of the steps involved – like the one used in this example.

§4. Conclusions

In this paper we have further developed the theory of α-expansions and applied it in the context of quantized frame expansions.

We have shown that α-expansions perform asymptotically better, in a rate-distortion sense, than the decompose-quantize method. In addition, preliminary experimental results indicate that this method also compares favorably to the decompose-quantize method in practical cases. This was verified by direct comparison between our method and quantized matching pursuits from [6].

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