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# A Note on Convolving
# Refinable Function Vectors

## C. Conti and K. Jetter

**Abstract.** When convolving two refinable function vectors which give rise to convergent subdivision schemes, the convolved scheme is again convergent. Moreover, the conditions on the mask symbols which characterize the approximation order of the associated shift invariant spaces show that the order of the convolved space is, essentially, the sum of the order of the two spaces originating from the convolution factors.

## §1. Kronecker Convolved Function Vectors

Our previous paper [3] deals with special subdivision schemes associated with a shift invariant space of bivariate spline functions, where the "generators" of the shift invariant space are produced through convolving lower order splines of small support. The present paper gives a more detailed and more systematic analysis of this convolution process. In this way it is possible to prove that (i) the convolution of convergent subdivision schemes yields a scheme which is convergent as well, and (ii) essentially, the approximation power of a convolved shift invariant space is at least the sum of the approximation powers of the two convolution factors.

We recall that a vector $\boldsymbol{\Phi} = (\phi_1, \phi_2, \ldots, \phi_n)^T$ of (continuous, compactly supported) $d$-variate functions is called refinable, if it satisfies a refinement equation

$$\boldsymbol{\Phi} = \sum_{\alpha \in \mathbb{Z}^d} A_\alpha \; \boldsymbol{\Phi}(2 \cdot -\alpha) \; . \tag{1.1}$$

Here, the refinement matrix mask $\mathbf{A} = \left(A_\alpha\right)_{\alpha \in \mathbb{Z}^d}$ is a matrix sequence with each 'coefficient' $A_\alpha$ being a real $(n \times n)$-matrix. We allow only masks of finite support, *i.e.*, $A_\alpha = 0$ except for finitely many $\alpha \in \mathbb{Z}^d$.

Given another refinable function vector $\boldsymbol{\Psi} = (\psi_1, \psi_2, \ldots, \psi_m)^T$ of $d$-variate functions, satisfying the refinement equation

$$\boldsymbol{\Psi} = \sum_{\beta \in \mathbb{Z}^d} B_\beta \, \boldsymbol{\Psi}(2 \cdot -\beta) \tag{1.2}$$

with corresponding matrix mask $\mathbf{B} = \left(B_\beta\right)_{\beta \in \mathbb{Z}^d}$, which is a matrix sequence of $(m \times m)$-matrices $B_\beta$, we use the following Kronecker type notion of convolving the two vectors:

$$\boldsymbol{\Theta} := \boldsymbol{\Phi} * \boldsymbol{\Psi} := \begin{pmatrix} \phi_1 * \boldsymbol{\Psi} \\ \phi_2 * \boldsymbol{\Psi} \\ \vdots \\ \phi_n * \boldsymbol{\Psi} \end{pmatrix}.$$

Here, the convolution of a scalar function $\phi_i$ with the vector function $\boldsymbol{\Psi}$ is taken componentwise. This operation produces a vector function $\boldsymbol{\Theta} = (\theta_1, \theta_2, \ldots, \theta_{mn})^T$ with $m \times n$ components of type $\theta_{(i-1)m+j} := \phi_i * \psi_j$. It is not too hard to see that $\boldsymbol{\Theta}$ is refinable again,

$$\boldsymbol{\Theta} = \sum_{\gamma \in \mathbb{Z}^d} C_\gamma \, \boldsymbol{\Theta}(2 \cdot -\gamma) \,, \tag{1.3}$$

where the refinement mask $\mathbf{C} = \left(C_\gamma\right)_{\gamma \in \mathbb{Z}^d}$, which is a matrix sequence of $(nm \times nm)$-matrices, is computed as follows:

$$C_\gamma = \frac{1}{2^d} \sum_{\alpha \in \mathbb{Z}^d} A_\alpha \otimes B_{\gamma - \alpha} \,, \quad \alpha \in \mathbb{Z}^d \,. \tag{1.4}$$

Here, the symbol $\otimes$ denotes the Kronecker product of matrices.

For the definition and some properties of this Kronecker product, we refer to [4, Section 4.2]. It should be noted at least that neither the Kronecker product of matrices nor the Kronecker type convolution is commutative.

## §2. Convergence of the Convolved Subdivision Scheme

Let us also recall that a refinable function vector gives rise to a vector-valued subdivision scheme as follows: In the situation of (1.1), the subdivision operator associated with the refinable function vector $\boldsymbol{\Phi} = (\phi_1, \phi_2, \ldots, \phi_n)^T$ is defined as

$$S_A \;:\; (\ell(\mathbb{Z}^d))^n \to (\ell(\mathbb{Z}^d))^n \,,$$
$$(S_A \, \boldsymbol{\Lambda})_\alpha := \sum_{\beta \in \mathbb{Z}^d} A_{\alpha - 2\beta}^T \, \boldsymbol{\Lambda}_\beta, \quad \alpha \in \mathbb{Z}^d \,, \tag{2.1}$$

where $\ell(\mathbb{Z}^d)$ denotes the linear space of sequences indexed by $\mathbb{Z}^d$. The complete (stationary) subdivision scheme consists in the iterates of $S_A$, namely:

For a given initial vector sequence $\boldsymbol{\Lambda} \in (\ell(\mathbb{Z}^d))^n$

Put $\boldsymbol{\Lambda}^{(0)} := \boldsymbol{\Lambda}$ and $\tag{2.2}$

Compute $\boldsymbol{\Lambda}^{(k+1)} := S_A \, \boldsymbol{\Lambda}^{(k)}, \; k = 0, 1, \ldots$

Hence, the iterate $\mathbf{\Lambda}^{(k)} = (\Lambda_\alpha^{(k)})_{\alpha \in \mathbb{Z}^d}$ has components

$$\Lambda_\alpha^{(k)} := \sum_{\beta \in \mathbb{Z}^d} (A_{\alpha-2^k\beta}^{(k)})^T \Lambda_\beta^{(0)}, \quad \alpha \in \mathbb{Z}^d, \tag{2.3}$$

with the iterated matrices $\mathbf{A}^{(k)} = (A_\alpha^{(k)})_{\alpha \in \mathbb{Z}^d}$ defined by $\mathbf{A}^{(1)} := \mathbf{A}$ and

$$A_\alpha^{(k)} := \sum_{\beta \in \mathbb{Z}^d} A_\beta^{(k-1)} A_{\alpha-2\beta}, \quad \alpha \in \mathbb{Z}^d, \quad \text{for } k > 1. \tag{2.4}$$

Following [2, Section 2.4] we say that the subdivision scheme converges for $\mathbf{\Lambda} = (\lambda^1, \ldots, \lambda^n)^T \in (\ell^\infty(\mathbb{Z}^d))^n$ if there exists a continuous function $f_\Lambda : \mathbb{R}^d \to \mathbb{R}$ such that

$$\lim_{k \to \infty} \left\| f_\Lambda(\frac{\cdot}{2^k})\mathbf{e} - \mathbf{\Lambda}^{(k)} \right\|_\infty = 0 \quad \text{for} \quad \mathbf{e} = (\underbrace{1 \ 1 \ \cdots \ 1}_{n})^T. \tag{2.5}$$

Here, $\| \cdot \|_\infty$ denotes the sup-norm of the vector sequence $\Lambda = (\lambda^1, \ldots, \lambda^n)^T$ given by

$$\left\| \Lambda \right\|_\infty := \max_{i=1,\ldots,n} \|\lambda^i\|_\infty.$$

The symbol $f_\Lambda(\frac{\cdot}{2^k})$ is short for the scalar-valued sequence $\left(f_\Lambda(\frac{\alpha}{2^k})\right)_{\alpha \in \mathbb{Z}^d}$. It should be noted that, since for any convergent scheme the limit function $f_\Lambda$ is given by

$$f_\Lambda = \sum_{\alpha \in \mathbb{Z}^2} \Lambda_\alpha^T \, \Phi(\cdot - \alpha),$$

we can recover the components $\phi_i$, $i = 1, \ldots, n$, as follows: choose the initial sequence $\mathbf{\Lambda} = (\lambda^1, \ldots, \lambda^n)^T$ in the following way that $\lambda^i$ is a delta sequence (*i.e.*, $\lambda_\alpha^i = \delta_\alpha$ for $\alpha \in \mathbb{Z}^d$) while all other sequences $\lambda^j$, $j \neq i$, are null sequences. This special initial vector sequence and its iterates will be denoted by $\mathbf{E}_i$ and $\mathbf{E}_i^{(k)}$, $k \geq 0$, respectively. It then turns out that

$$\lim_{k \to \infty} \left\| \phi_i(\frac{\cdot}{2^k})\mathbf{e} - \mathbf{E}_i^{(k)} \right\|_\infty = 0 \quad \text{for} \quad i = 1, \ldots, n. \tag{2.6}$$

**Theorem 2.1.** *Given two convergent subdivision schemes associated with the refinable function vectors $\mathbf{\Phi}$ and $\mathbf{\Psi}$, the (Kronecker type) convolved scheme associated with $\mathbf{\Theta} = \mathbf{\Phi} * \mathbf{\Psi}$ is again convergent.*

**Proof:** Let $\mathbf{f} := (\underbrace{1 \ 1 \ \cdots \ 1}_{m})^T$, and let $\mathbf{F}_j$ denote the vector sequence (composed by $m$ sequences) with the delta sequence at position $j$ and the null sequence at all other positions. If $\mathbf{F}_j^{(k)}$, $k \geq 0$, are the iterated vectors with respect to the $\mathbf{\Psi}$-subdivision, we have

$$\lim_{k \to \infty} \left\| \psi_j(\frac{\cdot}{2^k})\mathbf{f} - \mathbf{F}_j^{(k)} \right\|_\infty = 0 \quad \text{for} \quad j = 1, \ldots, m, \tag{2.7}$$

in addition to (2.6). In order to prove the theorem, we will show that

$$\lim_{k \to \infty} \left\| ((\phi_i * \psi_j)(\frac{\cdot}{2^k}))(\mathbf{e} \otimes \mathbf{f}) - (\mathbf{E}_i * \mathbf{F}_j)^{(k)} \right\|_\infty = 0 \qquad (2.8)$$

for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Here, the convolution of two vector sequences is defined by

$$\boldsymbol{\Lambda} * \boldsymbol{\Gamma} := \begin{pmatrix} \lambda^1 \\ \lambda^2 \\ \vdots \\ \lambda^n \end{pmatrix} * \begin{pmatrix} \gamma^1 \\ \gamma^2 \\ \vdots \\ \gamma^m \end{pmatrix} := \begin{pmatrix} \lambda^1 * \boldsymbol{\Gamma} \\ \lambda^2 * \boldsymbol{\Gamma} \\ \vdots \\ \lambda^n * \boldsymbol{\Gamma} \end{pmatrix},$$

where the convolution of a scalar sequence $\lambda^i$ with the vector sequence $\boldsymbol{\Gamma}$ is taken componentwise, *i.e.*, $\lambda^i * \boldsymbol{\Gamma} = (\lambda^i * \gamma^1, \lambda^i * \gamma^2, \ldots, \lambda^i * \gamma^m)^T$. We will also use the estimate

$$\left\| \boldsymbol{\Lambda} * \boldsymbol{\Gamma} \right\|_\infty \le \min \{ \left\| \boldsymbol{\Lambda} \right\|_1 \left\| \boldsymbol{\Gamma} \right\|_\infty , \ \left\| \boldsymbol{\Lambda} \right\|_\infty \left\| \boldsymbol{\Gamma} \right\|_1 \},$$

where $\left\| \boldsymbol{\Lambda} \right\|_1 := \max_{i=1,\ldots,n} \left\| \lambda^i \right\|_1$, with the usual 1-norm of scalar sequences.

Now, the iterated matrices $\mathbf{C}^{(k)}$ for the $\boldsymbol{\Theta}$-subdivision can be expressed by the iterated matrices $\mathbf{A}^{(k)}$ and $\mathbf{B}^{(k)}$ for the $\boldsymbol{\Phi}$- and $\boldsymbol{\Psi}$-schemes as follows:

$$C_\alpha^{(k)} = \frac{1}{2^{d \cdot k}} \sum_{\delta \in \mathbb{Z}^d} A_\delta^{(k)} \otimes B_{\alpha-\delta}^{(k)} , \quad \alpha \in \mathbb{Z}^d. \qquad (2.9)$$

Thus, taking $\mathbf{E}_i * \mathbf{F}_j$ as a starting vector for the $\boldsymbol{\Theta}$-subdivision, the iterated vectors are given by

$$(\mathbf{E}_i * \mathbf{F}_j)^{(k)} = \frac{1}{2^{d \cdot k}} \mathbf{E}_i^{(k)} * \mathbf{F}_j^{(k)} , \quad k \ge 0, \qquad (2.10)$$

whence

$$\left( (\phi_i * \psi_j)(\frac{\cdot}{2^k}) \right)(\mathbf{e} \otimes \mathbf{f}) - (\mathbf{E}_i * \mathbf{F}_j)^{(k)}$$

$$= \left( (\phi_i * \psi_j)(\frac{\cdot}{2^k}) \right)(\mathbf{e} \otimes \mathbf{f}) - \frac{1}{2^{d \cdot k}} \left( \phi_i(\frac{\cdot}{2^k})\mathbf{e} \right) * \left( \psi_j(\frac{\cdot}{2^k})\mathbf{f} \right)$$

$$+ \frac{1}{2^{d \cdot k}} \mathbf{E}_i^{(k)} * \left( \psi_j(\frac{\cdot}{2^k})\mathbf{f} - \mathbf{F}_j^{(k)} \right)$$

$$- \frac{1}{2^{d \cdot k}} \left( \mathbf{E}_i^{(k)} - \phi_i(\frac{\cdot}{2^k})\mathbf{e} \right) * \left( \psi_j(\frac{\cdot}{2^k})\mathbf{f} \right).$$

We estimate the three vector sequences in the preceding three lines as follows: the first term is a vector sequence where each component of the vector consists of the sequence $(e_\alpha^{(k)})_{\alpha \in \mathbb{Z}^d}$ with

$$e_\alpha^{(k)} := \int_{\mathbb{R}^d} \left( \phi_i(x) \right) \left( \psi_j(\frac{\alpha}{2^k} - x) \right) dx - \frac{1}{2^{d \cdot k}} \sum_{\delta \in \mathbb{Z}^d} \left( \phi_i(\frac{\delta}{2^k}) \right) \left( \psi_j(\frac{\alpha - \delta}{2^k}) \right).$$

This is the error of a tensor product rectangular rule applied to the convolution integral, and due to the continuity and the compact support of $\phi_i$ and $\psi_j$ we get

$$\lim_{k \to \infty} \sup_{\alpha \in \mathbb{Z}^d} |e_\alpha^{(k)}| = 0.$$

For the third term,

$$\frac{1}{2^{d \cdot k}} \left\| \left( \mathbf{E}_i^{(k)} - \phi_i(\frac{\cdot}{2^k})\mathbf{e} \right) * \left( \psi_j(\frac{\cdot}{2^k})\mathbf{f} \right) \right\|_\infty \leq \left\| \mathbf{E}_i^{(k)} - \phi_i(\frac{\cdot}{2^k})\mathbf{e} \right\|_\infty \frac{1}{2^{d \cdot k}} \left\| \psi_j(\frac{\cdot}{2^k})\mathbf{f} \right\|_1,$$

and the bound tends to zero as $k \to \infty$, since

$$\frac{1}{2^{d \cdot k}} \left\| \psi_j(\frac{\cdot}{2^k})\mathbf{f} \right\|_1 = \frac{1}{2^{d \cdot k}} \sum_{\alpha \in \mathbb{Z}^d} \left| \psi_j(\frac{\alpha}{2^k}) \right| \to \int_{\mathbb{R}^d} |\psi_j(x)| \, dx.$$

Finally, for the second term

$$\frac{1}{2^{d \cdot k}} \left\| \mathbf{E}_i^{(k)} * \left( \psi_j(\frac{\cdot}{2^k})\mathbf{f} - \mathbf{F}_j^{(k)} \right) \right\|_\infty \leq c_k \left\| \psi_j(\frac{\cdot}{2^k})\mathbf{f} - \mathbf{F}_j^{(k)} \right\|_\infty \to 0$$

as $k \to \infty$. This follows from

$$c_k := \frac{1}{2^{d \cdot k}} \left\| \mathbf{E}_i^{(k)} \right\|_1 \leq \frac{1}{2^{d \cdot k}} \left\| \phi_i(\frac{\cdot}{2^k})\mathbf{e} \right\|_1 + \frac{1}{2^{d \cdot k}} \left\| \phi_i(\frac{\cdot}{2^k})\mathbf{e} - \mathbf{E}_i^{(k)} \right\|_1,$$

since the first term on the right-hand side converges to $\int_{\mathbb{R}^d} |\phi_i(x)| \, dx$, and the second term is bounded due to (2.6), the uniform continuity of the compactly supported function $\phi_i$, and the fact that for some compact set $K \subset \mathbb{R}^d$ (in fact we can take the support of $\phi_i$):

$$(\mathbf{E}_i^{(k)})_\alpha = \mathbf{0} \quad \text{for } \frac{\alpha}{2^k} \notin K.$$

This latter property is a consequence of the compact support of the matrix mask $\mathbf{A}$, since we start the iteration with a 'delta'-sequence $\mathbf{E}_i$. In conclusion, (2.8) holds true. $\square$

## §3. Approximation Order

Approximation orders of shift invariant spaces have been studied quite intensively in the past few years. Concerning definitions and notation we refer to the recent survey [5]. A characterization of approximation power in terms of the mask symbol is also given there. In case of the refinement equation (1.1), this mask symbol is defined by

$$\mathbf{H}_\Phi(\xi) := \frac{1}{2^d} \sum_{\alpha \in \mathbb{Z}^d} A_\alpha \, e^{-2\pi i \alpha \cdot \xi}. \tag{3.1}$$

The result is as follows: For given $k \in \mathbb{N}$ we say that $\mathbf{H}_\Phi$ *satisfies condition* $(Z_k)$ if there exists a row vector $\mathbf{v} = (\tau_1, \ldots, \tau_n)$ of trigonometric polynomials such that

$$\mathbf{v}(0) \, \hat{\Phi}(0) \neq 0, \quad \mathbf{v}(0) \, \mathbf{H}_\Phi(0) = \mathbf{v}(0), \qquad (3.2.a)$$

and

$$D^\mu \Big( \mathbf{v}(\cdot) \mathbf{H}_\Phi\big(\tfrac{\cdot}{2}\big) \Big) \Big|_\beta = 0 \quad \text{for } |\mu| < k \text{ and } \beta \in E_0', \qquad (3.2.b)$$

where $E_0'$ is short for the set of corners of the cube $[0,1]^d$ with the origin removed. [5, Theorem 3.2.8.(i)] then asserts that *condition* $(Z_k)$ *implies that* $S_\Phi$ *has* $L_2$- *approximation order* $k$, *for* $f \in W_2^k(\mathbb{R}^d)$.

**Lemma 3.1.** *If the mask symbol* $\mathbf{H}_\Phi$ *satisfies condition* $(Z_k)$ *(with the row vector* $\mathbf{v}$*) and the mask symbol* $\mathbf{H}_\Psi$ *satisfies condition* $(Z_\ell)$ *(with the row vector* $\mathbf{w}$*) then the mask symbol* $\mathbf{H}_\Theta$ *of the convolved function vector* $\Theta := \Phi * \Psi$ *satisfies condition* $Z_{k+\ell}$ *(with the row vector* $\mathbf{z} := \mathbf{v} \otimes \mathbf{w}$*).*

**Proof:** We apply the identity

$$\mathbf{H}_\Theta = \mathbf{H}_\Phi \otimes \mathbf{H}_\Psi \qquad (3.3)$$

several times. Condition $(3.2.a)$ holds for the convolved function vector, since

$$(\mathbf{v} \otimes \mathbf{w})(0) \, (\widehat{\Phi * \Psi})(0) = \mathbf{v}(0)\hat{\Phi}(0)\mathbf{w}(0)\hat{\Psi}(0) \neq 0,$$

and by $(3.3)$,

$$\mathbf{z}(0)\mathbf{H}_\Theta(0) = (\mathbf{v}(0)\mathbf{H}_\Phi(0)) \otimes (\mathbf{w}(0)\mathbf{H}_\Psi(0)) = \mathbf{v}(0) \otimes \mathbf{w}(0) = \mathbf{z}(0).$$

In order to verify condition $(3.2.b)$,

$$D^\mu \Big( \mathbf{z}(\cdot)\mathbf{H}_\Theta\big(\tfrac{\cdot}{2}\big) \Big) \Big|_\beta = 0 \quad \text{for } |\mu| < k + \ell \text{ and } \beta \in E_0',$$

we make use of the Leibniz-type formula

$$D^\mu \Big( (\mathbf{v} \otimes \mathbf{w})(\cdot)(\mathbf{H}_\Phi \otimes \mathbf{H}_\Psi)\big(\tfrac{\cdot}{2}\big) \Big)$$
$$= \sum_{0 \leq |\gamma| \leq |\mu|} \binom{\mu}{\gamma} D^\gamma \Big( \mathbf{v}(\cdot)\mathbf{H}_\Phi\big(\tfrac{\cdot}{2}\big) \Big) \otimes D^{\mu-\gamma} \Big( \mathbf{w}(\cdot)\mathbf{H}_\Psi\big(\tfrac{\cdot}{2}\big) \Big)$$

at every point $\beta \in E_0'$. □

In order to derive an approximation order result from this lemma, we refer to the precise statement of [5, Theorem 3.2.8]. A sample result is as follows: *If the shift invariant spaces associated with* $\Phi$ *and* $\Psi$ *have approximation orders* $k$ *and* $\ell$, *respectively, and if the Gramians* $\mathbf{G}_\Phi$ *and* $\mathbf{G}_\Psi$ *satisfy the regularity condition given in [5, Theorem 3.2.8.(ii)], then the 'convolved' shift invariant space has (at least) approximation order* $k + \ell$. However, in general the 'convolved' Gramian $\mathbf{G}_\Theta$ does not satisfy this regularity condition.

### §4. An Example: Bivariate $C^1$ Cubics on a 3-directional Mesh

In [3], we have given an example of piecewise $C^1$-cubics on the four-directional mesh. Following [1], we consider piecewise $C^1$-cubics on the three-directional mesh generated by the lines $x = k$, $y = l$, with $k, l \in \mathbb{Z}$ when adding the diagonals $x - y = m$, $m \in \mathbb{Z}$. A basis of this space is given by the two functions $\theta_1 = B_{111} * \chi_{T_1}$ and $\theta_2 = B_{111} * \chi_{T_2}$, with $B_{111}$ the linear three-directional box-spline (or "Courant" element), and $\chi_{T_1}, \chi_{T_2}$ the characteristic functions of the two triangles $T_1, T_2$ obtained by cutting the unit square $[0, 1]^2$ by the 'north-east' diagonal. Thus, $\Theta = \Phi * \Psi$ with $\Phi := (B_{111})$ and $\Psi := (\chi_{T_1}, \chi_{T_2})^T$.

Now, $\Phi = (B_{111})$ satisfies a scalar refinement equation (1.1) with refinement mask

$$
\mathbf{A} = (A_\alpha)_{\alpha \in \mathbb{Z}^2} =
\begin{pmatrix}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots \\
\cdots & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & \cdots \\
\cdots & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{pmatrix},
$$

and $\Psi = (\chi_{T_1}, \chi_{T_1})^T$ satisfies the refinement equation (1.2) with matrix mask

$$
\mathbf{B} = (B_\alpha)_{\alpha \in \mathbb{Z}^2} =
\begin{pmatrix}
 & \vdots & \vdots & \vdots & & \vdots & \vdots & \\
\cdots & 0 & 0 & 0 & & 0 & 0 & \cdots \\
\cdots & 0 & 0 & \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0 & \cdots \\
\cdots & 0 & 0 & \begin{pmatrix} \mathbf{1} & 0 \\ \mathbf{0} & 1 \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} & 0 & \cdots \\
\cdots & 0 & 0 & 0 & & 0 & 0 & \cdots \\
 & \vdots & \vdots & \vdots & & \vdots & \vdots &
\end{pmatrix}.
$$

Here, the indexing of the 'coefficients' is such that the **boldface** entry is at position $\alpha = (0, 0)$.

It follows that $\Theta = \Phi * \Psi$ satisfies the refinement equation (1.3), with the matrix mask $\mathbf{C}$ as displayed on the following page, and Theorem 2.1 provides the convergence property for the associated subdivision scheme.

Concerning approximation order, Lemma 3.1 can be applied by putting $k = 2$, $\ell = 1$, and $\mathbf{z} := (1) \otimes (1\ 1)$. As a consequence, the approximation order for bivariate $C^1$-cubics on the the three-directional mesh is at least 3, and this is the precise approximation order (as was proved in [1]).

$$
C = \begin{pmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots \; 0 & 0 & 0 & 0 & 0 & 0 \; \cdots \\
\cdots \; 0 & 0 & \begin{pmatrix}0 & 0\\ \frac{1}{8} & \frac{1}{8}\end{pmatrix} & \begin{pmatrix}\frac{1}{8} & 0\\ \frac{1}{8} & \frac{1}{4}\end{pmatrix} & \begin{pmatrix}\frac{1}{8} & 0\\ 0 & \frac{1}{8}\end{pmatrix} & 0 \; \cdots \\
\cdots \; 0 & \begin{pmatrix}0 & 0\\ \frac{1}{8} & \frac{1}{8}\end{pmatrix} & \begin{pmatrix}\frac{1}{4} & 0\\ \frac{1}{4} & \frac{1}{2}\end{pmatrix} & \begin{pmatrix}\frac{1}{2} & \frac{1}{8}\\ \frac{1}{8} & \frac{1}{2}\end{pmatrix} & \begin{pmatrix}\frac{1}{4} & \frac{1}{8}\\ 0 & \frac{1}{8}\end{pmatrix} & 0 \; \cdots \\
\cdots \; 0 & \begin{pmatrix}\frac{1}{8} & 0\\ \frac{1}{8} & \frac{1}{4}\end{pmatrix} & \begin{pmatrix}\frac{1}{2} & \frac{1}{8}\\ \frac{1}{8} & \frac{1}{2}\end{pmatrix} & \begin{pmatrix}\frac{1}{2} & \frac{1}{4}\\ 0 & \frac{1}{4}\end{pmatrix} & \begin{pmatrix}\frac{1}{8} & \frac{1}{8}\\ 0 & 0\end{pmatrix} & 0 \; \cdots \\
\cdots \; 0 & \begin{pmatrix}\frac{1}{8} & 0\\ 0 & \frac{1}{8}\end{pmatrix} & \begin{pmatrix}\frac{1}{4} & \frac{1}{8}\\ 0 & \frac{1}{8}\end{pmatrix} & \begin{pmatrix}\frac{1}{8} & \frac{1}{8}\\ 0 & 0\end{pmatrix} & 0 & 0 \; \cdots \\
\cdots \; 0 & 0 & 0 & 0 & 0 & 0 \; \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{pmatrix} .
$$

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