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Constructive Approximation by (V,f)-Reproducing Kernels

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Abstract. In this paper we propose a constructive method to build reproducing kernels. We define the notion of (V, f) -reproducing kernel, and prove that every reproducing kernel is a (V, f) -reproducing kernel. We study the minimal approximation by these (V, f) -reproducing kernels for different choices of V and f . Examples to which our results apply include curve and surface fitting.

§1. (V, f) -Reproducing Kernels

For any set (respectively locally compact set) Ω , we denote by \mathbb{R}^Ω (respectively $C^m(\Omega)$) the space of real-valued functions (respectively m -times continuously differentiable functions) defined on Ω equipped with the topology of pointwise convergence (respectively uniform convergence on the compact subsets of Ω). Let us recall some definitions.

Definition 1.1. A real-valued function H defined on $\Omega \times \Omega$ is a reproducing kernel on $\Omega \times \Omega$ if

- 1) H is symmetric: $H(t, s) = H(s, t)$ for all $t, s \in \Omega$,
- 2) H is of positive type:

$$\sum_{k, l=1}^{k, l=N} \lambda_k \lambda_l H(t_k, t_l) \geq 0,$$

for any finite point set $\{t_k\}_{k=1}^N$ of Ω and real numbers $\{\lambda_k\}_{k=1}^N$.

Definition 1.2. A vector subspace \mathcal{H} of \mathbb{R}^Ω is said to be a hilbertian subspace of \mathbb{R}^Ω (respectively $C^m(\Omega)$) if

- 1) \mathcal{H} is a Hilbert space,
- 2) The natural injection from \mathcal{H} into \mathbb{R}^Ω (respectively $C^m(\Omega)$) is continuous.

We review some important results on reproducing kernels which are studied in [4].

Theorem 1.1.

- 1) A Hilbert space \mathcal{H} (respectively a real-valued function H defined on $\Omega \times \Omega$) is a hilbertian subspace of \mathbb{R}^Ω (respectively a reproducing kernel on $\Omega \times \Omega$) if and only if there exists one and only one reproducing kernel H on $\Omega \times \Omega$ (respectively hilbertian subspace \mathcal{H} of \mathbb{R}^Ω) such that

$$u(t) = \langle u | H(\cdot, t) \rangle_{\mathcal{H}}, \quad \forall t \in \Omega, \quad \forall u \in \mathcal{H}.$$

\mathcal{H} is called the hilbertian subspace associated with H .

- 2) For any hilbertian basis $(f_i)_{i \in I}$ of \mathcal{H} : $H(t, s) = \sum_{i \in I} f_i(t)f_i(s)$.
- 3) If H is separately m -times continuously differentiable, then \mathcal{H} is a hilbertian subspace of $\mathcal{C}^m(\Omega)$.
- 4) The vector space $\mathcal{H}_0 = \text{span}\{(H(\cdot, t))_{t \in \Omega}\}$ is dense in \mathcal{H} .

Let $(V, \langle \cdot | \cdot \rangle_V)$ be a Hilbert space, Ω be a set and f be a function from Ω into V .

Definition 1.3. For all $f : \Omega \rightarrow V$, we define a (V, f) -reproducing kernel H_f by

$$H_f(t, s) = \langle f(t) | f(s) \rangle_V, \quad \forall (t, s) \in \Omega \times \Omega. \quad (1.1)$$

We have the following result:

Theorem 1.2. H_f defined by (1.1) is a reproducing kernel on $\Omega \times \Omega$ and its associated hilbertian subspace \mathcal{H}_f of \mathbb{R}^Ω is

$$\mathcal{H}_f = \left\{ w \in \mathbb{R}^\Omega \mid \exists u \in V : w(t) = \langle u | f(t) \rangle_V, \forall t \in \Omega \right\}.$$

Proof: One can easily verify that H_f is a reproducing kernel.

Let $\tilde{H}_f : V \rightarrow \mathbb{R}^\Omega$ be defined by $(\tilde{H}_f u)(t) = \langle u | f(t) \rangle_V$. The mapping \tilde{H}_f is linear, and the inequality

$$|(\tilde{H}_f u)(t)| \leq \|u\|_V \|f(t)\|_V = \|u\|_V \|H_f(t, t)\|_V^{\frac{1}{2}},$$

for all $t \in \Omega$ and for all $u \in V$ implies that it is continuous. Let \mathcal{M} be the closure in V of the vector space $\text{span}\{(f(t))_{t \in \Omega}\}$, and $P_{\mathcal{M}}$ the orthogonal projector on \mathcal{M} . We define on $\mathcal{H}_f = \tilde{H}_f(V)$ the bilinear form

$$\langle \tilde{H}_f u | \tilde{H}_f v \rangle_{\mathcal{H}_f} = \langle P_{\mathcal{M}} u | P_{\mathcal{M}} v \rangle_V.$$

It is easy to see that this form is a scalar product on \mathcal{H}_f . Then the linear mapping $\tilde{H}_f : \mathcal{M} \rightarrow \mathcal{H}_f$ is an isometry, and consequently $(\mathcal{H}_f, \langle \cdot | \cdot \rangle_{\mathcal{H}_f})$ is a Hilbert space. For all $t \in \Omega$, the function

$$H_f(t, \cdot) : s \in \Omega \longrightarrow H_f(t, s) = \langle f(t) | f(s) \rangle_V,$$

is an element of \mathcal{H}_f , and satisfies the reproducing formula

$$(\tilde{H}_f u)(t) = \langle \tilde{H}_f u | H_f(t, \cdot) \rangle_{\mathcal{H}_f}, \quad \forall u \in V.$$

Consequently (see Theorem 1.1), \mathcal{H}_f is a hilbertian subspace of \mathbb{R}^Ω and admits H_f as reproducing kernel. \square

Theorem 1.3. Let ω be a set and c a mapping from ω to Ω . Then

$$(Hc)_f(y, z) = H_f(c(y), c(z)) = \langle f(c(y)) | f(c(z)) \rangle_V$$

is a reproducing kernel on $\omega \times \omega$.

Proof: For all $y \in \omega$, $f(c(y))$ is in V . The function $(Hc)_f$ is symmetric and is of positive type:

$$\sum_{k,l=1}^{k=N} \lambda_k \lambda_l (Hc)_f(y_k, y_l) = \langle \sum_{k=1}^{k=N} \lambda_k f(c(y_k)) | \sum_{k=1}^{k=N} \lambda_k f(c(y_k)) \rangle_V \geq 0. \square$$

Example 1.1. Let $V = L^2(a, b)$, Ω a subset of \mathbb{R} and $f(t)(x) = \exp(ckt)$ where c is a real constant. Then

$$H_f(t, s) = \begin{cases} (\exp(cb(t+s)) - \exp(ca(t+s)))/(c(t+s)), & \text{if } (t+s) \neq 0, \\ b-a, & \text{otherwise.} \end{cases}$$

Example 1.2. Let $V = L^2(\mathbb{R}^+)$, and suppose Ω is a subset of \mathbb{R}^n .

For all functions $c : \Omega \longrightarrow (0, +\infty)$, we have

(i) If $f(t)(x) = \frac{2}{\pi^{\frac{1}{4}}} \exp^{-c(t)|x|^2}$, then $H_f(t, s) = \frac{1}{\sqrt{c(t) + c(s)}}$.

(ii) If $f(t)(x) = \exp^{-c(t)x}$, then $H_f(t, s) = \frac{1}{c(t) + c(s)}$, and in particular if

$c(t) = \frac{P(t)}{Q(t)}$ (with $P(t)$ and $Q(t)$ polynomials), we obtain the rational reproducing kernel

$$H_f(t, s) = \frac{Q(t)Q(s)}{P(t)Q(s) + P(s)Q(t)}.$$

§2. (V,f)-Reproducing Kernels of Convolution Type

We consider the case where

- 1) $V = L^2(\mathbb{R}^n)$ and $\Omega = \mathbb{R}^n$.
- 2) $f(t)(x) = f(t-x)$, with f in the familar Sobolev space $H^m(\mathbb{R}^n)$.

Then $H_f(t, s) = \int_{\mathbb{R}^n} f(t-x)f(s-x)dx.$

Theorem 2.1. We have the following properties:

- 1) $H_f(t, s) = h_f(t - s)$ with $h_f(\xi) = (f * \check{f})(\xi) = \mathcal{F}(|\mathcal{F}f|^2)(\xi)$, where $\check{f}(x) = f(-x)$ and $\mathcal{F}f$ is the Fourier transform of f .
- 2) $h_f \in C_0^m(\mathbb{R}^n) = \left\{ u \in C^m(\mathbb{R}^n) \mid \lim_{|t| \rightarrow \infty} (D^\alpha u)(t) = 0, \quad 0 \leq |\alpha| \leq m \right\}$.
- 3) The associated hilbertian subspace of H_f is

$$\mathcal{H}_f = f * L^2(\mathbb{R}^n) \hookrightarrow C_0^m(\mathbb{R}^n) \text{ (continuous embedding).}$$

- 4) In particular, if $|\mathcal{F}f| > 0$, then

$$\mathcal{H}_f = \left\{ w \in \mathcal{S}' \mid \mathcal{F}w \in L_{loc}^1(\mathbb{R}^n), \quad \frac{\mathcal{F}w}{\mathcal{F}f} \in L^2(\mathbb{R}^n) \right\},$$

equipped with the scalar product

$$\langle w_1 \mid w_2 \rangle_{\mathcal{H}_f} = \int_{\mathbb{R}^n} \frac{\mathcal{F}w_1(\xi) \overline{\mathcal{F}w_2(\xi)}}{|\mathcal{F}f(\xi)|^2} d\xi.$$

- 5) If f is radial, then h_f is radial: $H_f(t, s) = h_f(|t - s|)$.
- 6) For all distinct points $\{t_k\}_{k=1}^N$ in \mathbb{R}^n , the matrix $H_N = (H_f(t_k, t_l))_{1 \leq k, l \leq N}$ is invertible (strictly positive definite).

Proof:

- 1) We have

$$H_f(t, s) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-i(t-s|\xi)} |\mathcal{F}f(\xi)|^2 d\xi = \mathcal{F}(|\mathcal{F}f|^2)(t - s).$$

- 2) $f \in H^m(\mathbb{R}^n) \Rightarrow D^\alpha h_f = (D^\alpha f) * \check{f} \in C_0^0(\mathbb{R}^n)$ for $0 \leq |\alpha| \leq m$, (see [2]).
- 3) is a consequence of Theorem 1.2 and the property given in 1).
- 4) Since $\mathcal{H}_f \hookrightarrow C_0^m(\mathbb{R}^n) \hookrightarrow \mathcal{S}'$, we have the equivalences:

$$\begin{aligned} \{w \in \mathcal{H}_f\} &\Leftrightarrow \{\exists u \in L^2(\mathbb{R}^n) : \mathcal{F}w = \mathcal{F}u\mathcal{F}f\} \\ &\Leftrightarrow \{\mathcal{F}w \in L_{loc}^1(\mathbb{R}^n), \quad \frac{\mathcal{F}w}{\mathcal{F}f} \in L^2(\mathbb{R}^n)\}. \end{aligned}$$

From Theorem 1.2, we have

$$w(t) = \langle u \mid f(t - \cdot) \rangle_{L^2(\mathbb{R}^n)} = \langle P_{\mathcal{M}}u \mid f(t - \cdot) \rangle_{L^2(\mathbb{R}^n)} = (P_{\mathcal{M}}u * f)(t).$$

Then $\mathcal{F}w = \mathcal{F}P_{\mathcal{M}}u\mathcal{F}f$, and

$$\begin{aligned} \langle w_1 \mid w_2 \rangle_{\mathcal{H}_f} &= \int_{\mathbb{R}^n} P_{\mathcal{M}}u_1(x) P_{\mathcal{M}}u_2(x) dx \\ &= \int_{\mathbb{R}^n} \mathcal{F}P_{\mathcal{M}}u_1(\xi) \overline{\mathcal{F}P_{\mathcal{M}}u_2(\xi)} d\xi = \int_{\mathbb{R}^n} \frac{\mathcal{F}w_1(\xi) \overline{\mathcal{F}w_2(\xi)}}{|\mathcal{F}f(\xi)|^2} d\xi. \end{aligned}$$

5) For any orthogonal matrix A,

$$\begin{aligned}
h_f(At) &= \int_{\mathbb{R}^n} f(x)f(At-x)dx = \int_{\mathbb{R}^n} f(Ax)f(A(t-x))dx \\
&= \int_{\mathbb{R}^n} f(x)f(t-x)dx = h_f(t),
\end{aligned}$$

since $f(Ax) = f(x)$ and $|detA| = 1$.

6) We suppose that $f \neq 0$ in $L^2(\mathbb{R}^n)$. Since the matrix

$H_N = \left(\int_{\mathbb{R}^n} f(t_k - x)f(t_l - x)dx \right)_{1 \leq k, l \leq N}$ is a Gram matrix, it is invertible if and only if the system $\{f(t_k - \cdot)\}_{k=1}^N$ is linearly independent in $L^2(\mathbb{R}^n)$. If $\sum_{k=1}^N c_k f(t_k - x) = 0$ in $L^2(\mathbb{R}^n)$, for $c_k \in \mathbb{R}$, $1 \leq k \leq N$, then by the Fourier transform we get $\left(\sum_{k=1}^N c_k e^{-i(t_k|\xi)} \right) \mathcal{F}f(\xi) = 0$ in $L^2(\mathbb{R}^n)$.

The Lebesgue measure of the set $\mathcal{N} = \{ \xi \in \mathbb{R}^n \mid \sum_{k=1}^N c_k e^{-i(t_k|\xi)} = 0 \}$ is equal to zero. Then $\mathcal{F}f$ vanishes outside \mathcal{N} , i.e: $\mathcal{F}f \equiv 0$ in $L^2(\mathbb{R}^n)$ and by the inverse Fourier transform, $f \equiv 0$ in $L^2(\mathbb{R}^n)$, which complete the proof. \square

Example 2.1. Let $u(x) = (1 - |x|)_+$ and $\mathcal{F}u = v$. We have $v(x) = \frac{\sin^2(\frac{x}{2})}{x^2}$.

(i) Taking $f = \mathcal{F}(|u|^{\frac{1}{2}})$, $H_f(t, s) = (1 - |t - s|)_+$.

(ii) Taking $f = \mathcal{F}(|v|^{\frac{1}{2}})$, $H_f(t, s) = \frac{\sin^2(\frac{t-s}{2})}{(t-s)^2}$.

Example 2.2. (Bessel reproducing kernels) For $n \in \mathbb{N}$, $\alpha \in \mathbb{R}$ and $\alpha > n$; consider $G_\alpha \in L^2(\mathbb{R}^n)$ defined by $\mathcal{F}(G_\alpha)(x) = (1 + |x|^2)^{-\frac{\alpha}{2}}$.

(i) Taking $f = \mathcal{F}(|G_\alpha|^{\frac{1}{2}})$, $H_f(t, s) = (1 + |t - s|^2)^{-\frac{\alpha}{2}}$.

(ii) Taking $f = \mathcal{F}(|\mathcal{F}G_{n+1}|^{\frac{1}{2}})$, $H_f(t, s) = \frac{\pi^{\frac{1-n}{2}}}{2^n \Gamma(\frac{n+1}{2})} \exp(-|t - s|)$ and

$\mathcal{H}_f = H^{\frac{n+1}{2}}(\mathbb{R}^n)$ (Sobolev space).

Example 2.3. (ν -B-spline reproducing kernels) Let

1) $Y_l(x) = \frac{1}{l!} x^l_+$.

2) $\nu \in \mathcal{E}'$ (distributions with compact support) such that $\nu(p) = 0$, for all polynomial p in $\mathcal{P}_l(\mathbb{R})$.

3) $f = \nu * Y_l$.

For such functions f , we give the following theorem without proof.

Theorem 2.2. *We have:*

- 1) $f \in L^2(\mathbb{R})$.
- 2) For all u in $V^{l+1}(\mathbb{R}) = \{v \in L^2_{loc}(\mathbb{R}) / v^{(l+1)} \in L^2(\mathbb{R})\}$ (Beppo-Levi space) we have $\int_{\mathbb{R}} u^{(l+1)}(x)f(t-x)dx = (\nu * u)(t)$.
- 3) $H_f(t, s) = (-1)^l(\check{\nu} * \nu * Y_{2l+1})(t-s)$ and $\mathcal{H}_f = \nu * V^{l+1}(\mathbb{R})$.

In the particular case of divided differences, ν is defined as the m th-iterated convolution $\nu = \frac{(\delta_a - \delta_b)^{*m}}{(b-a)}$, and $\check{\nu} = \frac{(\delta_{-a} - \delta_{-b}){*m}}{(b-a)}$.

§3. Data Fitting by (V,f)-Reproducing Kernels

Let $\{t_k\}_{k=1}^N$ a set of distinct points in Ω , and define a linear operator A_N from \mathcal{H}_f into \mathbb{R}^N by $A_N(u) = (u(t_k))_{1 \leq k \leq N}$.

Definition 3.1. For all $z_N \in \mathbb{R}^N$ and $\epsilon \in [0, 1]$ we define a spline to be any solution of the following minimal approximation problem:

$$(P_\epsilon(z_N)) : \quad \inf_{u \in \mathcal{C}_\epsilon} \left((1 - \epsilon)\langle u | u \rangle_{\mathcal{H}_f} + \epsilon \|A_N u - z_N\|_{\mathbb{R}^N}^2 \right),$$

where

$$\mathcal{C}_\epsilon = \begin{cases} A_N^{-1}\{z_N\}, & \text{if } \epsilon = 0 \text{ (Interpolation),} \\ \mathcal{H}_f, & \text{if } \epsilon \in]0, 1[\text{ (Smoothing).} \end{cases}$$

The following theorem gives the spline in the case $\epsilon \neq 0$.

Theorem 3.1. For all $(\epsilon, z_N) \in]0, 1[\times \mathbb{R}^N$, the problem $P_\epsilon(z_N)$ (Smoothing) admits a unique solution

$$\sigma^\epsilon(t) = \sum_{k=1}^{k=N} \lambda_k^\epsilon H_f(t, t_k),$$

where the coefficients ${}^t \Lambda^\epsilon = (\lambda_1^\epsilon, \dots, \lambda_N^\epsilon) \in \mathbb{R}^N$ are the solution of the system

$$(H_N + \frac{\epsilon}{1-\epsilon} I_N) \Lambda_N^\epsilon = z_N,$$

with $H_N = (H_f(t_k, t_l))_{1 \leq k, l \leq N}$ and I_N is the identity matrix.

Proof: 1) From the continuous embedding: $\mathcal{H}_f \hookrightarrow \mathbb{R}^\Omega$ (see Theorem 1.2), we deduce that A_N is continuous. 2) $A_N(\mathcal{H}_f)$ is closed as a vector subspace of \mathbb{R}^N . Then from the general spline theory (see [1,3]) we get the theorem. \square

Theorem 3.2. *The following two properties are equivalent:*

- 1) For all $z_N \in \mathbb{R}^N$, the problem $P_0(z_N)$ admits a unique solution.
- 2) The system $\{f(t_k)\}_{k=1}^N$ is linearly independent in V .

Proof: For all $z_N \in \mathbb{R}^N$, the problem $P_0(z_N)$ admits a unique solution if and only if the matrix $H_N = (H_f(t_k, t_l))_{1 \leq k, l \leq N}$ is invertible (see [1,3]). Since the matrix $H_N = (\langle f(t_k) | f(t_l) \rangle_V)_{1 \leq k, l \leq N}$ is a Gram matrix, it is invertible if and only if the system $\{f(t_k)\}_{k=1}^N$ is linearly independent in V . \square

Furthermore, for the particular case $V = L^2(\mathbb{R}^n)$ and $f \in H^m(\mathbb{R}^n)$, Theorem 3.2 and the property (6) of Theorem 2.1 imply the following theorem.

Theorem 3.3. *For all $f \in H^m(\mathbb{R}^n)$ and $z_N \in \mathbb{R}^N$, the problem $P_0(z_N)$ (Interpolation) admits a unique solution*

$$\sigma^0(t) = \sum_{k=1}^{k=N} \lambda_k^0 H_f(t, t_k),$$

where the coefficients ${}^t\Lambda^0 = (\lambda_1^0, \dots, \lambda_N^0) \in \mathbb{R}^N$ are the solution of the system

$$H_N \Lambda_N^0 = z_N,$$

with $H_N = (H_f(t_k, t_l))_{1 \leq k, l \leq N}$ and I_N is the identity matrix.

§4. Data Fitting Preserving Polynomials

Let $\mathcal{P}_d(\mathbb{R}^n)$ the vector space of polynomials of degree at most d . We suppose:

(H1) For all $p \in \mathcal{P}_d(\mathbb{R}^n)$ the subset $\{t_k\}_{k=1}^N$ of \mathbb{R}^n is such that

$$\{p(t_k) = 0, \quad 1 \leq k \leq N\} \iff p \equiv 0.$$

(H2) $\mathcal{H}_f \cap \mathcal{P}_d(\mathbb{R}^n) = \{0\}$.

We remark that in the case $V = L^2(\mathbb{R}^n)$ and $f \in H^m(\mathbb{R}^n)$, the hypothesis (H2) is satisfied because $\mathcal{H}_f \subset C_0^m(\mathbb{R}^n)$ (see Theorem 2.1(2)), and

$$C_0^m(\mathbb{R}^n) \cap \mathcal{P}_d(\mathbb{R}^n) = \{0\}.$$

Let \mathcal{H}_f^d be the Hilbert direct sum: $\mathcal{H}_f^d = \mathcal{H}_f \oplus \mathcal{P}_d(\mathbb{R}^n)$. We denote by Π_f the orthogonal projector from \mathcal{H}_f^d onto \mathcal{H}_f , and we define on \mathcal{H}_f^d the linear mapping $A_N(u) = (u(t_k))_{1 \leq k \leq N} \in \mathbb{R}^N$. For all $(\epsilon, z_N) \in [0, 1] \times \mathbb{R}^N$, we consider the following minimal approximation problem in \mathcal{H}_f^d :

$$(P_\epsilon(z_N)) : \quad \inf_{u \in \mathcal{C}_\epsilon} \left((1 - \epsilon) \langle \Pi_f(u) | \Pi_f(u) \rangle_{\mathcal{H}_f} + \epsilon \|A_N u - z_N\|_{\mathbb{R}^N}^2 \right),$$

where

$$C_\epsilon = \begin{cases} A_N^{-1}\{z_N\}, & \text{if } \epsilon = 0 \text{ (Interpolation),} \\ \mathcal{H}_f^d, & \text{if } \epsilon \in]0, 1[\text{ (Smoothing),} \\ \mathcal{P}_d(\mathbb{R}^n), & \text{if } \epsilon = 1 \text{ (Least squares).} \end{cases}$$

The hypothesis (H1) implies that the problem $P_1(z_N)$ admits a unique solution. In the case $\epsilon \in]0, 1[$ (Smoothing) we have the following theorem:

Theorem 4.1. For all $(\epsilon, z_N) \in]0, 1[\times \mathbb{R}^N$, the problem $P_\epsilon(z_N)$ admits a unique solution σ^ϵ . In the case $\epsilon \in]0, 1[$, the solution σ^ϵ is given by

$$\sigma^\epsilon(t) = \sum_{k=1}^{k=N} \lambda_k^\epsilon H_f(t, t_k) + \sum_{i=1}^{i=n_d} b_i^\epsilon p_i(t),$$

where the coefficients ${}^t\Lambda^\epsilon = (\lambda_1^\epsilon, \dots, \lambda_N^\epsilon) \in \mathbb{R}^N$ and ${}^tB^\epsilon = (b_1^\epsilon, \dots, b_{n_d}^\epsilon) \in \mathbb{R}^{n_d}$ are the solution of the system

$$\begin{pmatrix} H_N + \frac{\epsilon-1}{\epsilon} I_N & E \\ {}^tE^\epsilon & 0 \end{pmatrix} \begin{pmatrix} \Lambda^\epsilon \\ B^\epsilon \end{pmatrix} = \begin{pmatrix} z_N \\ 0 \end{pmatrix},$$

with

1) $H_N = (H_f(t_k, t_l))_{1 \leq k, l \leq N}$ and I_N is the identity matrix,

2) $E = (E_{k,i})_{\substack{1 \leq k \leq N \\ 1 \leq i \leq n_d}}$ with $E_{k,i} = p_i(t_k)$ and $(p_i)_{1 \leq i \leq n_d}$ is a basis of $\mathcal{P}_d(\mathbb{R}^n)$.

In particular, if there exists $p \in \mathcal{P}_d(\mathbb{R}^n)$ such that $\{p(t_k) = z_{N,k}, 1 \leq k \leq N\}$, then $\sigma^\epsilon = p$ (preserving polynomials property).

Proof: Theorem 4.1 is a consequence of general spline theory (see [1,3]):

1) A_N is continuous since \mathcal{H}_f^d is a hilbertian subspace of \mathbb{R}^Ω .

2) Π_f is continuous and $\Pi_f(\mathcal{H}_f^d) = \mathcal{H}_f$ is closed since Π_f is an orthogonal projector.

3) $\ker A_N \cap \ker \Pi_f = \{0\}$: derives from the hypothesis (H1) and the fact that $\ker \Pi_f = \mathcal{P}_d(\mathbb{R}^n)$.

4) $\ker A_N + \ker \Pi_f$ is closed since $\ker \Pi_f$ is a finite dimensional vector space.

□

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