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Polyharmonic Splines in $\mathbb{R}^d$: Tools for Fast Evaluation

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§1. Introduction

As is now well known, hierarchical and fast multipole-like methods can greatly reduce the storage and operation counts for fitting and evaluating radial basis functions. In particular, for spline functions of the form

$$s(x) = p(x) + \sum_{i=1}^{N} \lambda_i \phi(|x - x_i|), \tag{1}$$

p a low degree polynomial, the cost of a single extra evaluation can be reduced from $O(N)$ to $O(1)$ operations, and the cost of a matrix-vector product (that is, evaluation at all centers) can be decreased from $O(N^2)$ to $O(N)$.

This paper outlines some of the mathematics required to implement methods of these types for polyharmonic splines in $\mathbb{R}^d$, d even, that is for splines s corresponding to ϕ chosen from the list

$$\phi_\ell(r) = \begin{cases} r^{2(\ell+1-d/2)}, & \ell = 0, \ldots, d/2-2, \\ r^{2(\ell+1-d/2)} \log(r), & \ell = d/2-1, \ldots \;. \end{cases} \tag{2}$$

We carry out most of our work in the general $\mathbb{R}^d$ setting and then specialize to $d = 4$. We refer the reader to our more detailed work [1] which contains all the details of this special case. We are currently working on developing all the details for the general $\mathbb{R}^d$ case.

A key technique in our development is the exploitation of the rotation group invariance of radial basis functions. This means that we exploit the fact that any kernel $k(x, y) = \phi(|x - y|)$ will be rotation invariant in the sense that

$$k(gx, gy) = k(x, y), \qquad \text{for all orthogonal } g \in O(d). \tag{3}$$

Invariance leads to many crucial simplifications and efficiencies in developing and manipulating the polyharmonic expansions which lie at the heart of the

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hierarchical and fast multipole expansions. Related development of general spherical harmonic expansions based on these techniques can be found in [6, 7,8].

We will not detail the basic framework of hierarchical and fast multipole methods within which this mathematics sits. However, we do recall that an essential component of the method is the grouping of approximations to summands like (1) into subsums, which are approximations to the influence of that part of (1) associated with centers in a single panel or cluster. The key steps to obtain them requires:

- Finding explicit Taylor/Laurent *expansions*

$$\text{For each } x, \quad \phi(|x - x_<|) = \sum_{m \geq m_0} \widetilde{p_m}(x, x_<), \quad |x_<| < |x|, \tag{4}$$

  with $\widetilde{p_m}$ homogeneous polynomials of degree m in $x_<$.
- Finding an efficient *separation* of the $x, x_<$ influence in $\widetilde{p_m}$, *i.e.*, expanding

$$\widetilde{p_m}(x, x_<) = \sum_i f_i^m(x) g_i^m(x_<), \tag{5}$$

  for some good choice of (basis) functions $\{f_i^m(x)\}$ and $\{g_i^m(x)\}$.

These expansion and separation results provide the approximations to subsums which are the far and near field expansions. Other essential components are the tools to manipulate these expansions, namely error estimates, uniqueness theorems, and translation formulae. In this paper we concentrate on the algebraic tools and give some extensive general results on (4) (Theorem 6) and (5) ((20) in Section 3) and for $\mathbb{R}^4$ we give the appropriate far and near field expansions for the ϕ_ℓ (Theorems 7 and 8), and a brief indication of the dual basis leading to (20). Analogous results for polyharmonic splines in $\mathbb{R}^2$ appear in [3]. The reader unfamiliar with the framework of the fast multipole method may wish to refer to the original paper of Greengard and Rokhlin [4], or to the introductory short course [2].

§2. Polyharmonic Functions and Homogeneous Polynomials

First we record some detailed facts relating to the Laplacian Δ, and its actions on special homogeneous functions and the logarithm of the distance. In particular, we show why our basic functions ϕ_ℓ in (2) are polyharmonic or more specifically $(\ell + 1)$-harmonic in the sense that $\Delta^{\ell+1}\phi_\ell = 0$.

Lemma 1. *Let $|\cdot|$ be the 2-norm on $\mathbb{R}^d$, d even.*

i) *If $f : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ is a non–trivial harmonic function that is homogeneous of integral degree m, then*

$$\Delta(|\cdot|^{2\ell} f) = 2\ell\,(d + 2\ell + 2m - 2)\,|\cdot|^{2(\ell-1)} f. \tag{6}$$

Hence $|\cdot|^{2\ell} f$ *is polyharmonic of exact order*

$$\begin{cases} \ell+1, & \text{for } \ell \geq 0, m > -\frac{d}{2} \text{ or } m < 1-\ell-\frac{d}{2}, \\ \ell+m+\frac{d}{2}, & \text{for } \ell < 0, m \geq 1-\ell-\frac{d}{2}. \end{cases}$$

In particular $|\cdot|^{2\ell}$ *is* $\begin{cases} (\ell+1)\text{-harmonic} & \text{for } \ell \geq 0, \\ (\ell+d/2)\text{-harmonic} & \text{for } -d/2 < \ell < 0. \end{cases}$

ii) $|\cdot|^{2\ell}\log|\cdot|$ *is* $(\ell+d/2)$*-harmonic for* $\ell \geq 0$. *More generally,*

$$\Delta|\cdot|^{2\ell}\log|\cdot| = |\cdot|^{2(\ell-1)}(2\ell(2\ell+d-2)\log|\cdot| + (4\ell+d-2)). \tag{7}$$

Proof: For the first part of (i), just apply the product rule for the Laplacian,

$$\Delta(fg) = (\Delta f)g + 2(\nabla f)\cdot(\nabla g) + f(\Delta g),$$

and the Euler relation for a function f that is homogeneous of degree m,

$$x \cdot \big(\nabla f\big)(x) = mf(x).$$

Then observe how many applications of Δ are required to reduce one of the multipliers to 0. Specializing to the case $f = 1$ yields the last result of (i).

The first part of (ii) follows from (7) in combination with (i) and its proof. (7) follows from (i), the product rule for the Laplacian, and the computation of $\nabla \log|\cdot|$ and $\Delta \log|\cdot|$. □

From the detailed eigenvalue-like information on the Laplacian map in (6), we can get a decomposition theorem for Π_n, the homogeneous polynomials of degree n, in terms of the spherical harmonics of degree n:

$$\mathbb{H}_n = \{p \in \Pi_n : \ \Delta p = 0\} = \ker(\Delta) \cap \Pi_n.$$

This is useful in understanding the structure of the homogeneous polynomial terms in any Taylor/Laurent type series expansions for the $\phi(|x-x_i|)$. In view of Lemma 1 the decomposition splits Π_n into its harmonic, biharmonic, triharmonic, etc., parts.

Lemma 2. $\Pi_n = \bigoplus_{\ell=0}^{\lfloor n/2 \rfloor} |\cdot|^{2\ell}\mathbb{H}_{n-2\ell}$. *In particular,* $\mathbb{H}_n \cap |\cdot|^2\Pi_{n-2} = \{0\}$.

Proof: Note that for $n = 0, 1$, $\Pi_n = \mathbb{H}_n$, so the base for an inductive proof is true. Assume the decomposition for $n-2$, some $n \geq 2$, so for each $p \in \Pi_n$,

$$\Delta p = \sum_{\ell=0}^{\lfloor (n-2)/2 \rfloor} |\cdot|^{2\ell} h_{n-2-2\ell}, \text{ some } h_i \in \mathbb{H}_{n-2-2\ell}.$$

Then by (6), if

$$h_n = p - \sum_{\ell=0}^{\lfloor (n-2)/2 \rfloor} \frac{1}{2}(\ell+1)^{-1}(d+2n-2(\ell+2))^{-1}|\cdot|^{2(\ell+1)}h_{n-2(\ell+1)},$$

then $\Delta h_n = 0$. So $h_n \in \mathbb{H}_n$ and the decomposition of Π_n is proved by induction. □

Some additional consequences of (6) come when we study what happens for negative ℓ. Here we have noted that $|\cdot|^{2-2m-d}f$ is harmonic whenever f is m-homogeneous and harmonic. But bringing the factor of $(|\cdot|^{-2})^m$ inside the m-homogeneous function f shows $|\cdot|^{2-d}f(\cdot/|\cdot|^2)$ is harmonic for any homogeneous harmonic function f. In fact this construction is independent of the homogeneity order of f. In general the Kelvin transform, defined by $Kf(x) = |x|^{2-d}f(x/|x|^2)$, which arises from inversion in the sphere followed by multiplication by $|x|^{2-d}$, maps harmonic functions to harmonic functions. On Π_n $Kf = |\cdot|^{2-d-2n}f$. Associated with the Kelvin transform are the spaces of negative degree $2-d-m$

$$\bigoplus_{\ell=0}^{\infty} |\cdot|^{2\ell} K\mathbb{H}_{m+2\ell} = \bigoplus_{\ell=0}^{\infty} |\cdot|^{2-d-2m-2\ell}\mathbb{H}_{m+2\ell}, \tag{8}$$

which are useful for analysing Laurent (far field) series. An application of Lemma 1 shows that Equation (8) displays the space under consideration split into its harmonic, biharmonic, triharmonic, etc., parts.

§3. Rotation Invariance and Simplified Taylor Expansions

The decompositions of polynomial spaces in the previous section already simplify the Taylor/Laurent type expansions (4) we need to determine. To make further progress, we want to exploit the *rotation invariance* of $\phi(|x-x_i|)$. When we come to combine subsums in (1), we will want to fix x and concentrate on rotations (orthogonal matrices) which fix x. When we are given a pole $p \in \mathbb{R}^d$, we let $G_p = \{g : g \in O(d),\ gp = p\}$ denote the rotations about the ray through p. So the function $f_\phi^x(x_<) = \phi(|x - x_<|)$ satisfies $f_\phi^x(gx_<) = f_\phi^x(x_<)$, for all $g \in G_x$. We refer to any function f which is unchanged by rotations in G_p as a p-zonal function. In particular we have the p-zonal harmonics

$$\mathbb{H}_n^p = \{h \in \mathbb{H}_n :\ h(gy) = h(y) \text{ for all } g \in G_p\}, \tag{9}$$

and the p-zonal homogeneous polynomials Π_n^p. Now the Taylor/Laurent expansion of f_ϕ^x as in (4), will have $\widetilde{p_m}(x, gx_<) = \widetilde{p_m}(gx, gx_<) = \widetilde{p_m}(x, x_<)$, for $g \in G_x$ since the homogeneous terms must remain unchanged under rotations (see Theorem 6). Thus these terms will be x-zonal polynomials as a function of $x_<$. What is the general stucture of Π_n^x and $\mathbb{H}_n^x$?

Theorem 3. *Fix a pole $x \in \mathbb{R}^d\backslash\{0\}$. Let $\chi_0^x(\cdot) = 1$, $\chi_1^x(\cdot) = 2\langle x, \cdot\rangle$. Then there exist a unique set of constants a_m, $m > 1$, such that the inductively defined sequence of homogeneous polynomials,*

$$\chi_{m+1}^x = \chi_1^x\chi_m^x - a_{m+1}|x|^2|\cdot|^2\chi_{m-1}^x, \qquad m \geq 1, \tag{10}$$

consists of harmonic functions. Moreover,

i) *χ_m^x is an m-homogeneous x-zonal harmonic function, which is rotation invariant in the sense that $\chi_m^{gx}(g\cdot) = \chi_m^x(\cdot)$ for all $g \in O(d)$.*

ii) *The constants* a_{m+1} *are independent of* x. *Hence*

$$\chi_m^x(x_<) = \chi_m^{x_<}(x), \qquad 0 \neq x, x_< \in \mathbb{R}^d, \tag{11}$$

and the $\chi_m^x(x_<)$ *are also homogeneous and harmonic as functions of* x.

iii) *If* $x \neq 0$, $\{|\cdot|^{2\ell}\chi_{m-2\ell}^x(\cdot),\ \ell = 0, \ldots, \lfloor m/2 \rfloor\}$, *form a basis for* Π_m^x. *In particular,* χ_j^x *is the unique (up to a scalar multiple) element of* $\mathbb{H}_j^x$.

iv) *For* m, n *nonnegative integers, the kernels*

$$\begin{aligned} k_{m,\ell,n}(x, x_<) = &|x|^{2\ell+m-\kappa}|x_<|^{2\ell+n-\kappa}\chi_{\kappa-2\ell}^x(x_<), \quad \kappa = \min(m,n), \\ &m \equiv n \bmod 2, \quad \ell = 0, \ldots, \lfloor \kappa/2 \rfloor, \end{aligned} \tag{12}$$

form a basis for the space of all rotation invariant polynomial kernels, $p_{m,n}(x, x_<)$ *which are homogeneous of degree* m *in* x, *and degree* n *in* $x_<$.

Proof: The proof of the existence of a_m and (i), (ii), and (iii) is by induction. For $m = 0$ (i) and (iii) are trivially true. Let $0 \neq h \in \mathbb{H}_1^x$. Then h has the form $h(\cdot) = c\langle p, \cdot\rangle$. Since h is x-zonal $c\langle p, \cdot\rangle = c\langle p, g\cdot\rangle = c\langle g^{-1}p, \cdot\rangle$, for all $g \in G_p^x$. This implies p has the same direction as x. Hence h is a multiple of χ_1^x and (i) and (iii) follow for $m = 1$.

Now induction shows that (10) defines m-homogeneous x-zonal polynomials which are m-homogeneous in x, for any choice of a_{m+1}. Also they are rotation invariant. To complete the inductive step for (i) with a fixed x we need only show that there is a unique a_{m+1} that makes $\chi_{m+1}^x(\cdot)$ harmonic.

From the homogeneity in x, we may assume $|x| = 1$. Since $\chi_1^x\chi_m^x$ is a homogenous polynomial of degree $m+1$, Lemma 2 asserts that there exist unique homogenous harmonic polynomials $q_{m+1-2\ell}$ such that $\chi_1^x\chi_m^x = \sum_{\ell=0}^{\lfloor (m+1)/2 \rfloor} |\cdot|^{2\ell} q_{m+1-2\ell}$. Since $\nabla\chi_1^x = 2x^T$, the product rule for the Laplacian and the inductive assumption that χ_m^x is harmonic show that

$$\Delta(\chi_1^x\chi_m^x) = 4x^T \cdot \nabla\chi_m^x = 4\,\partial_x\chi_m^x, \quad |x| = 1, \tag{13}$$

where ∂_x denotes the directional derivative in the (fixed) direction x. Since $\Delta\partial_x\chi_m^x = \partial_x\Delta\chi_m^x = 0$, it follows that $\chi_1^x\chi_m^x$ is bi-harmonic and

$$\Delta(\chi_1^x\chi_m^x) = \Delta(|\cdot|^2 q_{m-1}) = 2(d + 2(m-1))q_{m-1}.$$

Since Δ maps x-zonal functions to x-zonal functions it follows that $q_{m-1} \in \mathbb{H}_{m-1}^x$ and therefore by part (iii) of the inductive hypothesis q_{m-1} is a multiple of χ_{m-1}^x. Thus the existence and uniqueness of a_{m+1} making χ_{m+1}^x harmonic is proved.

We now turn to the inductive step in the proof of (ii). Using the rotation invariance part of the inductive hypothesis, $\chi_{m+1}^x(g^{-1}\cdot)$ is

$$\begin{aligned} \chi_1^x(g^{-1}\cdot)\chi_m^x(g^{-1}\cdot) - a_{m+1}|x|^2|g^{-1}\cdot|^2\chi_{m-1}^x(g^{-1}\cdot) = \\ \chi_1^{gx}\chi_m^{gx} - a_{m+1}|gx|^2|\cdot|^2\chi_{m-1}^{gx}. \end{aligned} \tag{14}$$

Since rotations and Δ commute, the left-hand side of the above is harmonic. Thus the right-hand side of (14) equals χ_{m+1}^{gx} and a_{m+1} is independent of g. Using homogeneity it is also independent of $|x|$. Hence a_{m+1} is independent of x. The symmetry in $x, x_<$ of (10) then implies (11), and hence the homogeneity and harmonicity of $\chi_{m+1}^{x}(x_<)$ as a function of x.

We now turn to the inductive step in the proof of (iii). Since $\Pi_m^x = \Pi_m^{x/|x|}$ and $\chi_\kappa^{ax} = a^\kappa \chi_\kappa^x$ we may assume $|x| = 1$. It suffices to show that $\dim \Pi_{m+1}^x \leq \lfloor m/2 \rfloor + 1$ since, by Lemma 1, $\{|\cdot|^{2\ell}\chi_{m+1-2\ell}^x,\ \ell = 0, \ldots, \lfloor m/2 \rfloor\}$ is an independent set in Π_{m+1}^x. Since we can rotate e_1 to x by some orthogonal map, which will isomorphically map Π_{m+1}^x to $\Pi_{m+1}^{e_1}$, we prove our dimensionality statement for $x = e_1$. To analyze the value $f(y)$ of any e_1-zonal function f, we choose orthogonal $g_\pm \in G_{e_1}$ which transform y into the coordinate plane spanned by the first two basis vectors. The two possible values for the transformed y are

$$g_\pm y = \langle y, e_1 \rangle e_1 \pm \sqrt{|y|^2 - \langle y, e_1 \rangle^2}\, e_2.$$

Then $f(y) = f(g_\pm y) = f(y_1, \pm\sqrt{|y|^2 - y_1^2}, 0, \ldots, 0)$. In particular, f must be even in its second variable. If $f \in \Pi_{m+1}^{e_1}$,

$$f(y) = \sum_{\ell=0}^{\lfloor (m+1)/2 \rfloor} c_\ell y_1^{m+1-2\ell} \left(\sqrt{|y|^2 - y_1^2} \right)^{2\ell}, \qquad \text{for some } c_\ell.$$

Hence the functions $y_1^{m+1-2\ell}(|y|^2 - y_1^2)^\ell$ span $\Pi_{m+1}^{e_1}$.

For (iv) we just note that $p_{m,n}(x, \cdot) = |x|^m p_{m,n}(x/|x|, \cdot) \in \Pi_n^{x/|x|}$ by the homogeneity assumptions. Thus the basis facts from (iii) imply there are functions $b_{n,\ell}(x/|x|)$ with

$$p_{m,n}(x, \cdot) = \sum_{\ell=0}^{\lfloor n/2 \rfloor} b_{n,\ell}(x/|x|)|x|^m |\cdot|^{2\ell} \chi_{n-2\ell}^{x/|x|}.$$

The rotation invariance of $p_{m,n}$ and of the terms $|x|^m |\cdot|^{n-2\ell}\chi_{2\ell}^{x/|x|}$ implies $b_{n,\ell}(gx/|gx|) = b_{n,\ell}(x/|x|)$ for all rotations g. Rotating $x/|x|$ to e_1 shows $b_{n,\ell}(x/|x|) = b_{n,\ell}(e_1)$, *i.e.*, the $b_{n,\ell}$ are constants. Moreover, the homogeneity in x of χ_j^x shows

$$p_{m,n}(x, \cdot) = \sum_{\ell=0}^{\lfloor n/2 \rfloor} b_{n,\ell} |x|^{2\ell-(n-m)} |\cdot|^{2\ell} \chi_{n-2\ell}^x. \tag{15}$$

Since the left hand side is a polynomial of degree m in x, and the $|\cdot|^{2\ell}\chi_{n-2\ell}^x$ are independent, each $|x|^{2\ell-(n-m)}\chi_{n-2\ell}^x$ associated with a nonzero coefficient must be a polynomial of degree m in x. Hence, $n - m = 2j$ must be even. Also, applying the second part of Lemma 2, $2(\ell - j) = 2\ell - (n - m) \geq 0$. If $m \geq n$ the proof of (iv) is done. If $m < n$, then reindexing the sum in terms of $(\ell - j)$ yields (12). $\square$

The following result is known [5], but is included for the sake of completeness.

Theorem 4. *Define a rotation invariant inner product (pairing) for functions on the unit ball $B \subset \mathbb{R}^d$ by*

$$[f,h] = c_V \int_{\{|y|\leq 1\}} f(y)h(y)dy, \quad c_V^{-1} = \ vol\{|y| \leq 1\}. \tag{16}$$

i) *If f, h are homogeneous of degrees m,n, respectively with $fh \in L^1(B)$, then $[f,h] = 0$ if and only if $\int_{\{|y|=1\}} f(y)h(y)dA = 0$, i.e., the integral of fh over the sphere S^{d-1} is zero.*

ii) *If $m \neq n$, then $|\cdot|^{2i}\mathbb{H}_n$ and $|\cdot|^{2j}\mathbb{H}_m$ are orthogonal with respect to this inner product(pairing), provided $m+n+2(i+j) > -d$.*

Proof: For (i) just introduce polar coordinates $(r,y) \in [0,1] \times S^{d-1}$. Then, by homogeneity and scaling properties of the area of $\{x : \ |x| = r\}$,

$$[f,h] = c_V \int_{r=0}^{1} r^{m+n+d-1} \int_{S^{d-1}} f(y)h(y)dA\,dr, \qquad m+n+d > 0,$$

and the result follows. By (i) it suffices to prove (ii) when $i = j = 0$, since integrals of a product $|\cdot|^{2(i+j)}fh$ on S^{d-1} do not depend on i,j. Let $f \in \mathbb{H}_m$ and $h \in \mathbb{H}_n$. Then by Green's Theorem and the Euler relation for homogeneous functions

$$\begin{aligned} 0 &= \int_{\{|y|\leq 1\}} \Big(f(y)\Delta h(y) - h(y)\Delta f(y)\Big)dy \\ &= \int_{\{|y|=1\}} \Big(f(y)\nabla h(y) - h(y)\nabla f(y)\Big) \cdot \mathbf{n}dA = \int_{\{|y|=1\}} (n-m)f(y)h(y)dA. \end{aligned}$$

Thus $\int_{\{|y|=1\}} f(y)h(y)dA = 0$, and the analogous relation holds for integration over the ball by part (i). □

An application of the above gives

Lemma 5. *The constants a_m, $m \geq 2$, in the 3-term recurrence (10) defining the x-zonal harmonics χ_m^x of Theorem 3 are positive.*

Proof: By Theorem 4 and (10),

$$\begin{aligned} 0 = [\chi_{m+1}^x, \chi_{m-1}^x] &= [\chi_1^x\chi_m^x, \chi_{m-1}^x] - a_{m+1}[|\cdot|^2\chi_{m-1}^x, \chi_{m-1}^x] \\ &= [\chi_m^x, \chi_1^x\chi_{m-1}^x] - a_{m+1}[|\cdot|^2\chi_{m-1}^x, \chi_{m-1}^x] \\ &= [\chi_m^x, \chi_m^x + a_m|\cdot|^2\chi_{m-2}^x] - a_{m+1}[|\cdot|^2\chi_{m-1}^x, \chi_{m-1}^x] \\ &= [\chi_m^x, \chi_m^x] - a_{m+1}[|\cdot|^2\chi_{m-1}^x, \chi_{m-1}^x]. \end{aligned}$$

Hence, $a_{m+1} = [\chi_m^x, \chi_m^x]/[|\cdot|^2\chi_{m-1}^x, \chi_{m-1}^x] > 0$. □

Now part (iii) of Theorem 3 leads quite directly to the structure of near and far field expansions of general *rotation invariant* kernels $\psi(x, x_<)$. The heuristic that a far field expansion of $\psi(x, x_<)$ with respect to x can be found from a Taylor expansion with respect to $x_<$, has been known to us for some while. Theorem 6 below gives a proof that the underlying idea is correct inimportant special cases. In fact, we have the following result for such ψ which are jointly homogeneous ($\psi(ax, ax_<) = a^{2n}\psi(x, x_<)$) for some even integral power and are analytic about $x_< = 0$, such as

$$\psi_n(x, x_<) = |x - x_<|^{2n}(\log(|x - x_<|^2) - \log(|x|^2)). \tag{17}$$

Theorem 6. *Let $\psi(x, x_<)$, $x, x_< \in \mathbb{R}^d$ be rotation invariant, jointly homogeneous of degree $2n$ and analytic in $x, x_<$, for $|x_<| < |x|$. Then there exist constants $c^n_{m,\ell}$ such that the Taylor expansion of ψ about $x_< = 0$ has the form*

$$\psi(x, x_<) = \sum_{m=0}^{\infty} \sum_{\ell=0}^{\lfloor m/2 \rfloor} c^n_{m,\ell} |x|^{2(n+\ell-m)} |x_<|^{2\ell} \chi^x_{m-2\ell}(x_<) \tag{18}$$

$$= \sum_{m=0}^{\infty} \sum_{\ell=0}^{\lfloor m/2 \rfloor} c^n_{m,\ell} |x|^{2(n+\ell-m)} |x_<|^{2\ell} \chi^{x_<}_{m-2\ell}(x). \tag{19}$$

When ψ is $(k+1)$-harmonic in $x_<$, the upper limit on ℓ in (18) or (19) is $\min\{k, \lfloor m/2 \rfloor\}$. If $\psi(x, 0) = 0$ then the lower limit on m in (18) or (19) is 1.

Proof: The terms $\widetilde{p_m}(x, x_<)$ in (4), the Taylor series of $\psi(x, x_<)$ with respect to $x_<$, are degree m homogeneous polynomials in $x_<$. When any Taylor series is grouped by homogeneity with respect to $x_<$, each group is uniquely determined. Since only the term $\widetilde{p_m}(gx, gx_<)$ in the series for $\psi(gx, gx_<)$ has homogeneity m in $x_<$, the rotation invariance implies that $\widetilde{p_m}$ is also rotation invariant. Similarly the joint homogeneity of ψ yields $\widetilde{p_m}(ax, ax_<) = a^{2n}\widetilde{p_m}(x, x_<)$. Since for any $x, x_<$ there is a rotation g (or reflection if $d = 2$) which interchanges the rays through x and $x_<$, *i.e.*, $g(x/|x|) = (x_</|x_<|)$ and $g(x_</|x_<|) = (x/|x|)$, it follows that

$$\begin{aligned} |x|^{2(m-n)}\widetilde{p_m}(x, x_<) &= |x|^{2(m-n)}\widetilde{p_m}(|x|x_</|x_<|, |x_<|x/|x|) \\ &= \frac{|x|^{2m}}{|x_<|^{2n}}\widetilde{p_m}\left(x_<, \left(\frac{|x_<|}{|x|}\right)^2 x\right) = |x_<|^{2(m-n)}\widetilde{p_m}(x_<, x). \end{aligned}$$

Since the final right side in this string of equalities is an m-homogeneous polynomial in x, we see that the terms in (4) have the form $|x|^{2(n-m)}p_m(x, x_<)$ with p_m a rotation invariant m-homogeneous polynomial in each of $x, x_<$. Hence (18) follows by Theorem 3.(iii). □

The *separation* properties in (5) can now be achieved by further use of rotation invariance. Each of the subspaces $|\cdot|^{2\ell}\mathbb{H}_j$, $j + 2\ell = n$, which occur in the decomposition of Π_n is rotation invariant. Hence it has a (unique) rotation invariant reproducing kernel

$$k(x, y) = |x|^{2\ell}|y|^{2\ell} \sum_{i=0}^{\dim \mathbf{H}_j} f_i(x)\tilde{f}_i(y), \tag{20}$$

where $\{|\cdot|^{2\ell} f_i\}$ and $\{|\cdot|^{2\ell}\tilde{f}_i\}$ are any bases for this subspace which are bi–orthogonally dual with respect to some rotation invariant inner product, *e.g.* the inner product (16) (see [8].) But by (12) in Theorem 3.(iii), since $k(x,y)$ is (exactly) $(\ell+1)$-harmonic as a function of y, and is homogeneous of degree $2\ell+j$ in both x, y,

$$k(x,y) = c_{j,\ell}|x|^{2\ell}|y|^{2\ell}\chi_j^x(y), \tag{21}$$

for some $c_{j,\ell} > 0$. Equating (20) and (21) provides separation of the influence of x, $x_<$ in the expression for $\chi_j^x(x_<)$. A consequence is separation of x, $x_<$ in the far and near field expansions given by Theorem 6, thus allowing the combination of the expansions for several centers $x_< = x_i, i = 1, \ldots, N$, into one expansion about 0.

§4. Expansions in $\mathbb{R}^4$

In this section we use the results of Section 3 in the $\mathbb{R}^4$ case to outline the explicit *expansion* formulae for the ϕ_ℓ in (2) for $d = 4$. We start with the far field expansion of the potential function $|x - x_<|^{-2}$.

Theorem 7. *For $x, x_< \in \mathbb{R}^4$ with $|x_<| < |x|$,*

$$|x - x_<|^{-2} = \sum_{m=0}^{\infty} |x|^{-2(m+1)} c_m \chi_m^{x_<}(x), \quad c_m = 1. \tag{22}$$

Proof: Since $|x - x_<|^{-2}$ is harmonic in $\mathbb{R}^4$, an expansion of this form holds for some constants c_m by Theorem 6. Using $\chi_m^{x_<}(x) = \chi_m^x(x_<)$, multiplication by $|x - x_<|^2 = |x|^2 + |x_<|^2 - \chi_1^x(x_<)$ yields

$$1 = \sum_{m=0}^{\infty} (c_m - c_{m-1})|x|^{-2m}\chi_m^x(x_<)$$
$$+ (c_{m-2} - c_{m-1}a_m)|x|^{-2(m-1)}|x_<|^2\chi_{m-2}^x(x_<),$$

when the recurrence (10) is used and the geometrically convergent expansion is rearranged to group terms of common homogeneity in $x_<$. Then equating coefficients using (iii) of Theorem 3 shows $c_0 = 1$, $c_m = c_{m-1}$ and $c_{m-2} = c_{m-1}a_m$. These must be consistent so $a_m = c_m = 1$ for all m. □

We now outline the expansion of $\psi_n(x, x_<)$ from (17). This gives us the bulk of the far field expansion for ϕ_{n+2}.

Theorem 8.

$$\psi_n(x, x_<) = \sum_{m=1}^{\infty} \sum_{\ell=0}^{\min\{n+1, \lfloor m/2 \rfloor\}} c_{m,\ell}^n |x|^{2(\ell+n-m)}|x_<|^{2\ell}\chi_{m-2\ell}^{x_<}(x), \tag{23}$$

where the non–zero coefficients $c_{m,\ell}^n$ are given by the formulae $c_{m,\ell}^0 = -\frac{(-1)^\ell}{m}$, and the recurrence $c_{m,\ell}^{n+1} = c_{m,\ell}^n - c_{m-1,\ell}^n - c_{m-1,\ell-1}^n + c_{m-2,\ell-1}^n$.

Proof: The form of all the expansions follow from (18), since $\psi_n(x, 0) = 0$. The explicit determination of the $c_{m,\ell}^0$, the $n = 0$ case, is done in Lemma 4.4

of [1]. The recurrence for the $c^n_{m,\ell}$ follows as in Theorem 7 from (10) with $a_m = 1$ upon multiplication of the ψ_n case by $|x - x_<|^2$. The details are in Lemma 4.6 of [1]. □

The explicit construction of bases for $\mathbb{H}_j$ (and dual bases) which are needed for the separation results can also be significantly simplified by use of the rotation invariance perspective. A detailed development in $\mathbb{R}^4$ is in our previously cited work.

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