A Recursive Approach to the Construction of k-Balanced Biorthogonal Multifilters

Silvia Bacchelli, Mariantonia Cotronei, and Damiana Lazzaro

Abstract. In this paper we discuss some numerical aspects of a particular construction of balanced biorthogonal multifilters by means of the lifting scheme. This construction allows, by simply solving linear equations, to obtain multifilters which do not need prefiltering, and for which the discrete versions of polynomial preservation/annihilation properties, are respectively, satisfied by their low and high-pass branches. In particular, we conduct experiments on how a parameter which appears in our recursive definition of balancing can be chosen to suitably influence the spectral behaviour of the multifilter low-pass branch, making it more effective in image compression problems.

§1. Introduction

Multiwavelets are a new addition to the classical scalar wavelet theory, and have been extensively studied in the last six years [5,9,11,17]. The main motivation for multiwavelets is that, unlike the scalar wavelet case, they can simultaneously possess desirable properties which are found to be useful for image compression applications, such as orthogonality and symmetry, short support, linear phase, a high approximation order, a high number of vanishing moments, etc. This combination would not be possible in any real-valued scalar wavelet. In fact, all real-valued scalar wavelets, with only one scaling function and one mother wavelet, can never possess all the above properties at the same time. This flexibility of vector-valued wavelet functions is due to the fact that multiwavelets satisfy conditions in which matrix rather than scalar coefficients are involved.

However, multiwavelets lack some attributes that scalar wavelets possess, and this becomes apparent when one implements the discrete multiwavelet transform. In particular, in the scalar case, a scaling low-pass filter with an approximation order k refers to the ability of the low-pass filter to reproduce

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discrete-time polynomials up to a degree $k-1$, while the corresponding wavelet high-pass filter annihilates discrete-time polynomials up to the same degree. This property, which is very important in many applications, does not hold in the multiwavelet case. In fact, the approximation power property does not assure the preservation and annihilation of discrete-time polynomials by the low-pass and the high-pass branch of a multiwavelet-based filter bank, respectively.

Moreover, because the approximation order for multiwavelets is not accompanied by the additional properties mentioned above, in applications using multiwavelets, a preprocessing or prefiltering step is necessary to obtain an efficient signal or image compression. A detailed investigation of prefiltering methods can be found in the literature [12,21,7].

Recently, to overcome these problems, Lebrun and Vetterli, and Selesnick [15,18] introduced the concept of balanced multiwavelets. They constructed orthogonal multiwavelet bases whose multifilter coefficients satisfy the discrete version of the approximation and zero-moments properties, and, at the same time, avoid the use of prefilters in implementing the discrete multiwavelet transform. This is a great advantage because the preprocessing step is a crucial point in multiwavelet-based algorithms. In fact, this initialization can sometimes destroy the very properties a multiwavelet basis is designed to have. Nevertheless, the above authors' construction of orthogonal balanced multifilters implies the resolution of non-linear equations that are solved by the Gröbner basis method.

Following the previous authors' idea, in order to avoid the difficulties due to the above-mentioned non-linearity, in [2] we have given a simple algebraic construction of k-balanced biorthogonal multifilters making use of the well-known tool called the lifting scheme. As shown in [19], the lifting scheme provides a simple method for constructing new biorthogonal filters with requested properties, starting from an assigned set of biorthogonal analysis-synthesis filters. In [2] we have extended the lifting scheme to the multifilter case, and in so doing, we have exploited the additional degrees of freedom left in the multifilter construction after satisfying the perfect reconstruction condition in order to easily construct finite k-balanced multifilters. Our results have been stated using the algebraic framework of banded block recursive matrices, exploiting this flexible mathematical tool to translate both the k-balancing conditions and other desirable properties in terms of simple linear conditions on the multifilter coefficients.

In this paper, we discuss some numerical aspects of the procedure for the construction of biorthogonal balanced multifilters given in [2], and analyze in particular the effect of the choice of the shift constant p which appears in our definition of k-balancing on the compression capabilities of this kind of filters. In fact, shift constant p plays an important role, and it can be used as a further degree of freedom.

Starting from Lazy multifilters, k-balanced multifilters of order 2 and 4 are constructed, and their effectiveness in image compression is tested on the Lena image. Using numerical experiments, we observe that the p parameter influences the shape of the Fourier transform of the scalar filters associated with the low-pass matrix coefficients, and we determine the value of p in such a way that the spectral behaviour of the newly constructed low-pass filters is as close as possible to the optimal shape. With this selection of p, we obtain the best compression results.

We remark that the aim of this paper is essentially to show the flexibility of our tool in building multifilters which do not need prefiltering and which are easily found by solving simple linear equations.

§2. Balanced Biorthogonal Multifilters

Let $\{\mathcal{H} = \sum_i \boldsymbol{H}_i t^i, \mathcal{W} = \sum_i \boldsymbol{W}_i t^i\}$ and $\{\widetilde{\mathcal{H}} = \sum_i \widetilde{\boldsymbol{H}}_i t^i, \widetilde{\mathcal{W}} = \sum_i \widetilde{\boldsymbol{W}}_i t^i\}$ be two pairs of block Laurent polynomials associated, respectively, with the analysis and the synthesis phase of a FIR multifilter bank, where $\{\boldsymbol{H}_i\}$, $\{\boldsymbol{W}_i\}$, $\{\widetilde{\boldsymbol{H}}_i\}$, $\{\widetilde{\boldsymbol{W}}_i\}$ are finite sequences of $r \times r$ matrices. In the following section we will refer to $\{\mathcal{H}, \mathcal{W}\}$, $\{\widetilde{\mathcal{H}}, \widetilde{\mathcal{W}}\}$ as analysis multifilters and synthesis multifilters, respectively, where we can think of them either as the sequences of matrix coefficients or as their associated block Laurent polynomials.

Let $R(t^2, \mathcal{H}), R(t^2, \mathcal{W}), R(t^2, \widetilde{\mathcal{H}}), R(t^2, \widetilde{\mathcal{W}})$ be the block banded Hurwitz matrices whose generating functions are $\mathcal{H}, \mathcal{W}, \widetilde{\mathcal{H}}, \widetilde{\mathcal{W}}$, respectively. With these matrices, we can give an algebraic description of the action of the analysis-synthesis system on a block Laurent polynomial σ given as input, in the following way:

Analysis:

$$[\sigma^{(0)}] = R(t^2, \mathcal{H})[\sigma]$$
$$[\sigma^{(1)}] = R(t^2, \mathcal{W})[\sigma]$$

Synthesis:

$$[\hat{\sigma}] = R(t^2, \widetilde{\mathcal{H}})^T [\sigma^{(0)}] + R(t^2, \widetilde{\mathcal{W}})^T [\sigma^{(1)}],$$

where $\boldsymbol{\sigma}^{(0)}$ and $\boldsymbol{\sigma}^{(1)}$ represent the output of the analysis phase, while $\hat{\boldsymbol{\sigma}}$ represents the output of the synthesis phase and therefore of the whole FIR system.

Given any pair of multifilters $\{\mathcal{H}, \mathcal{W}\}$, define the 2-decimated matrix

$$\Delta_{(\mathcal{H},\mathcal{W})} = \begin{bmatrix} \mathcal{H}_0 & \mathcal{H}_1 \\ \mathcal{W}_0 & \mathcal{W}_1 \end{bmatrix},$$

whose elements are the 2-decimated block series related to $\mathcal{H}, \mathcal{W}$, that is

$$\mathcal{H}_k = \sum_i \boldsymbol{H}_{2i+k} t^i, \ \mathcal{W}_k = \sum_i \boldsymbol{W}_{2i+k} t^i, \quad k = 0, 1.$$

Definition 1. *We say that the pairs* $\{\mathcal{H}, \mathcal{W}\}$, $\{\widetilde{\mathcal{H}}, \widetilde{\mathcal{W}}\}$ *are* biorthogonal multifilters *or duals to each other or, equivalently, that they satisfy the* Perfect Reconstruction (PR) property *if*

$$\Delta^{T*}_{(\widetilde{\mathcal{H}},\widetilde{\mathcal{W}})} \times \Delta_{(\mathcal{H},\mathcal{W})} = \Delta_{(\mathcal{H},\mathcal{W})} \times \Delta^{T*}_{(\widetilde{\mathcal{H}},\widetilde{\mathcal{W}})} = I.$$

In this case, if furthermore $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ admit a convergent subdivision scheme, then it is possible to define corresponding multiscaling functions and multiwavelets from the well-known matrix two-scale relations:

$$\mathbf{\Phi}(x) = \sqrt{2}\, R(t^2, \mathcal{H})\, \mathbf{\Phi}(2x), \quad \widetilde{\mathbf{\Phi}}(x) = \sqrt{2}\, R(t^2, \widetilde{\mathcal{H}})\, \widetilde{\mathbf{\Phi}}(2x),$$

$$\mathbf{\Psi}(x) = \sqrt{2}\, R(t^2, \mathcal{W})\, \mathbf{\Phi}(2x), \quad \widetilde{\mathbf{\Psi}}(x) = \sqrt{2}\, R(t^2, \widetilde{\mathcal{W}})\, \widetilde{\mathbf{\Phi}}(2x),$$

where $\mathbf{\Phi}(x), \mathbf{\Psi}(x), \widetilde{\mathbf{\Phi}}(x), \widetilde{\mathbf{\Psi}}(x)$ represent the vector containing the translates of the r-vectors $\boldsymbol{\phi} = [\phi_0, \ldots, \phi_{r-1}]^T$, $\boldsymbol{\psi} = [\psi_0, \ldots, \psi_{r-1}]^T$, $\tilde{\boldsymbol{\phi}} = [\tilde{\phi}_0, \ldots, \tilde{\phi}_{r-1}]^T$, $\tilde{\boldsymbol{\psi}} = [\tilde{\psi}_0, \ldots, \tilde{\psi}_{r-1}]^T$.

We now extend the concept of balancing order (introduced in [15]) to biorthogonal multifilters. We require that the multifilters associated with the analysis system must satisfy the discrete versions of both the polynomial preserving and zero moment properties.

Definition 2. *A pair of multifilters $\{\mathcal{H}, \mathcal{W}\}$ related to the analysis phase of a FIR system is said to be* balanced of order k *(or k-balanced), if there exists at least one real number p such that the following relations hold:*

$$\left.\begin{aligned} R(t^2, \mathcal{H}) \times [\pi_n] &= \sqrt{2}\, 2^n [(\pi + p)_n], \\ R(t^2, \mathcal{W}) \times [\pi_n] &= 0, \end{aligned}\right\} \quad n = 0, \ldots, k-1, \tag{1}$$

where $[\pi_n]$ and $[(\pi + p)_n]$ are bi-infinite column vectors which can also be seen as r-block vectors associated with the formal block series

$$\pi^n = \sum_i \begin{bmatrix} (ri)^n \\ (ri+1)^n \\ \vdots \\ (ri+r-1)^n \end{bmatrix} t^i, \quad (\pi + p)^n = \sum_i \begin{bmatrix} (ri+p)^n \\ (ri+1+p)^n \\ \vdots \\ (ri+r-1+p)^n \end{bmatrix} t^i.$$

In [2] an equivalent condition to (1) has been given which turns out to be more useful in practice:

Theorem 3. *A pair of FIR multifilters $\{\mathcal{H}, \mathcal{W}\}$ is balanced of order k if and only if*

$$\sum_{j=l_1}^{l_2} H_j \begin{bmatrix} (rj)^n \\ (rj+1)^n \\ \vdots \\ (rj+r-1)^n \end{bmatrix} = \sqrt{2}\, 2^n \begin{bmatrix} p^n \\ (p+1)^n \\ \vdots \\ (p+r-1)^n \end{bmatrix}, \tag{2}$$

$$\sum_{j=m_1}^{m_2} W_j \begin{bmatrix} (rj)^n \\ (rj+1)^n \\ \vdots \\ (rj+r-1)^n \end{bmatrix} = 0, \tag{3}$$

for $n = 0, \ldots, k-1$.

§3. Construction with the Lifting Scheme

The lifting scheme (introduced by Sweldens [19]) is a flexible tool for the construction of biorthogonal bases. In [2] an extension to the multifilter setting has been given. In short, given a set $\{\mathcal{H}, \widetilde{\mathcal{H}}, \mathcal{W}, \widetilde{\mathcal{W}}\}$ of biorthogonal multifilters, then the new multifilters

$$\begin{cases} \widetilde{\mathcal{H}}^{new} = \widetilde{\mathcal{H}} + (S \circ t^2)\widetilde{\mathcal{W}} \\ \mathcal{W}^{new} = \mathcal{W} - (S^{*T} \circ t^2)\mathcal{H}, \end{cases} \tag{4}$$

where S is any block Laurent polynomial, gives rise to a new set $\{\mathcal{H}, \widetilde{\mathcal{H}}^{new}$, $\mathcal{W}^{new}, \widetilde{\mathcal{W}}\}$ of biorthogonal multifilters.

Analogously, by simply changing the roles of the previous multifilters,

$$\begin{cases} \mathcal{H}^{new} = \mathcal{H} + (\widetilde{S} \circ t^2)\mathcal{W} \\ \widetilde{\mathcal{W}}^{new} = \widetilde{\mathcal{W}} - (\widetilde{S}^{*T} \circ t^2)\widetilde{\mathcal{H}} \end{cases} \tag{5}$$

gives rise to a new set $\{\mathcal{H}^{new}, \widetilde{\mathcal{H}}, \mathcal{W}, \widetilde{\mathcal{W}}^{new}\}$ of biorthogonal multifilters. We call (4) and (5) respectively the lifting scheme and the dual lifting scheme.

In [2] some useful conditions are given which allow the new multifilters to inherit symmetry/antisymmetry properties from the starting multifilters.

We can take advantage of the previous scheme to construct new balanced biorthogonal multifilters. In fact, unlike the orthogonal case where the balancing and the orthogonal conditions give rise to non-linear equations, which in [15,18], for example, are solved with a Gröbner basis approach, our balancing conditions (2) and (3) applied to the lifted (or dual lifted) multifilters give rise to linear conditions. The main steps of our approach are:

1) Construct the new low-pass multifilter coefficients, using the dual lifting scheme;
2) Apply the balancing condition (2), and solve the linear equations to find the coefficients of the unknown dual lifting matrix polynomial;
3) Construct the new high-pass multifilter coefficients, with the lifting scheme;
4) Apply the balancing condition (3), and solve the linear equations to find the coefficients of the unknown lifting polynomial;
5) Construct the corresponding dual low and high-pass multifilters.

It is important to note that in applying the balancing condition (2), a value must be assigned to the shift parameter p. In our experiments, it turns out that p influences on the effectiveness of the multifilters in their applications.

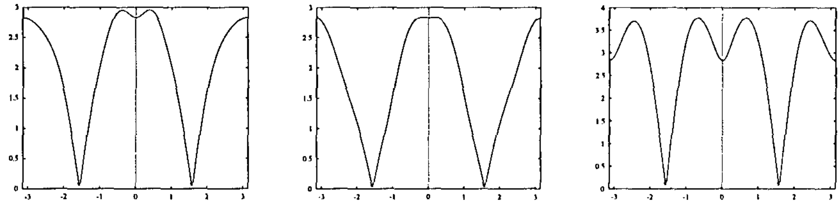

Fig. 1. $|\hat{h}^0| + |\hat{h}^1|$ associated to the Lazy 2-balanced low-pass multifilter with a varying p: from left to right, $p = -1, 0.25, 1$.

§4. Examples

In the following example, we start from Lazy multifilters to obtain balanced biorthogonal multifilters of order 2 and 4, which, furthermore, are of the type symmetric/antisymmetric (see [2]).

We restrict to the case $r = 2$, and define Lazy multifilters as follows:

$$2\mathcal{H} = \widetilde{\mathcal{H}} = \sqrt{2}\, I, \quad \mathcal{W} = 2\widetilde{\mathcal{W}} = \sqrt{2}\, I\, t.$$

In order to show the influence of p on the performance of the new multifilters, we introduce the notation

$$h^m_{2k+n} = [\boldsymbol{H}_k]_{m,n}, \quad k \in \mathbb{Z},\ m, n = 0, 1,$$

which give the two low-pass scalar filters $h^0 = \{h^0_k\}_{k\in\mathbb{Z}}, h^1 = \{h^1_k\}_{k\in\mathbb{Z}}$ obtained by reorganizing the set of 2×2 low-pass matrix multifilter $\{\boldsymbol{H}_k\}_{k\in\mathbb{Z}}$, as a multichannel scalar filter bank.

As shown in the following figures, the shift constant p influences the shape of the Fourier transforms of $h^{0,new}, h^{1,new}$, making them more or less suitable for application problems.

In Figure 1, we show 3 graphs of the sum $|\hat{h}^{0,new}| + |\hat{h}^{1,new}|$, with p varying in $\{-1, 1/4, 1\}$. It can be seen that the choice $p = 1/4$ gives visually a better low-pass behaviour. In this case the new coefficients (except for a factor $\sqrt{2}$) are

$$\mathcal{H}^{new} = \begin{bmatrix} \frac{1}{4} & \frac{-1}{8} \\ \frac{-3}{8} & \frac{1}{4} \end{bmatrix} t^{-1} + \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} \frac{1}{4} & \frac{1}{8} \\ \frac{3}{8} & \frac{1}{4} \end{bmatrix} t,$$

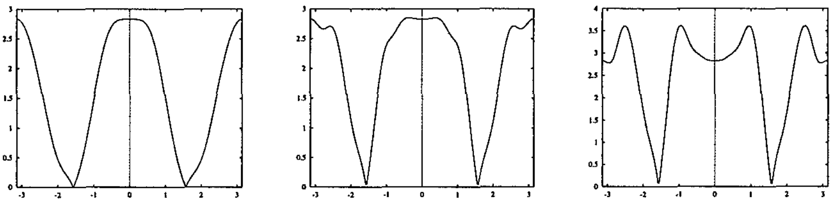

Fig. 2. $|\hat{h}^0| + |\hat{h}^1|$ associated to the Lazy 4-balanced low-pass multifilter with a varying p: from left to right, $p = 0, 0.4, 0.8$.

$$\mathcal{W}^{new} = \begin{bmatrix} -\frac{5}{64} & \frac{1}{32} \\ \frac{3}{32} & -\frac{5}{64} \end{bmatrix} t^{-1} + \begin{bmatrix} -\frac{1}{4} & -\frac{1}{16} \\ -\frac{3}{16} & -\frac{1}{4} \end{bmatrix} + \begin{bmatrix} \frac{21}{32} & 0 \\ 0 & \frac{21}{32} \end{bmatrix} t +$$

$$\begin{bmatrix} -\frac{1}{4} & \frac{1}{16} \\ \frac{3}{16} & -\frac{1}{4} \end{bmatrix} t^2 + \begin{bmatrix} -\frac{5}{64} & -\frac{1}{32} \\ -\frac{3}{32} & -\frac{5}{64} \end{bmatrix} t^3,$$

$$\widetilde{\mathcal{H}}^{new} = \begin{bmatrix} -\frac{5}{64} & -\frac{3}{32} \\ -\frac{1}{32} & -\frac{5}{64} \end{bmatrix} t^{-2} + \begin{bmatrix} \frac{1}{4} & -\frac{3}{16} \\ -\frac{1}{16} & \frac{1}{4} \end{bmatrix} t^{-1} + \begin{bmatrix} \frac{21}{32} & 0 \\ 0 & \frac{21}{32} \end{bmatrix} +$$

$$\begin{bmatrix} \frac{1}{4} & \frac{3}{16} \\ \frac{1}{16} & \frac{1}{4} \end{bmatrix} t + \begin{bmatrix} -\frac{5}{64} & \frac{3}{32} \\ \frac{1}{32} & -\frac{5}{64} \end{bmatrix} t^2,$$

$$\widetilde{\mathcal{W}}^{new} = \begin{bmatrix} -\frac{1}{4} & -\frac{3}{8} \\ -\frac{1}{8} & -\frac{1}{4} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} t + \begin{bmatrix} -\frac{1}{4} & \frac{3}{8} \\ \frac{1}{8} & -\frac{1}{4} \end{bmatrix} t^2.$$

A second example (Fig. 2) shows the behaviour of the scalar low-pass filters associated to the Lazy 4-balanced multifilters, with different choices for the parameter p. In this case, the choice $p = 0.4$ provides the best behaviour of the low-pass filters.

We have experimented with the above multifilters in an image compression example (on the Lena image), by making use of a multiwavelet-based embedded coding [6]. Results obtained with the best choices of 2 and 4-balanced Lazy multifilters are compared, at same compression ratio 1:16, with those produced by Chui-Lian (CL) [5] and Geronimo-Hardin-Massopust (GHM) [9] multifilters. CL and GHM multifilter both have approximation order 2, but need prefiltering. For comparison purposes, this prefiltering step has been

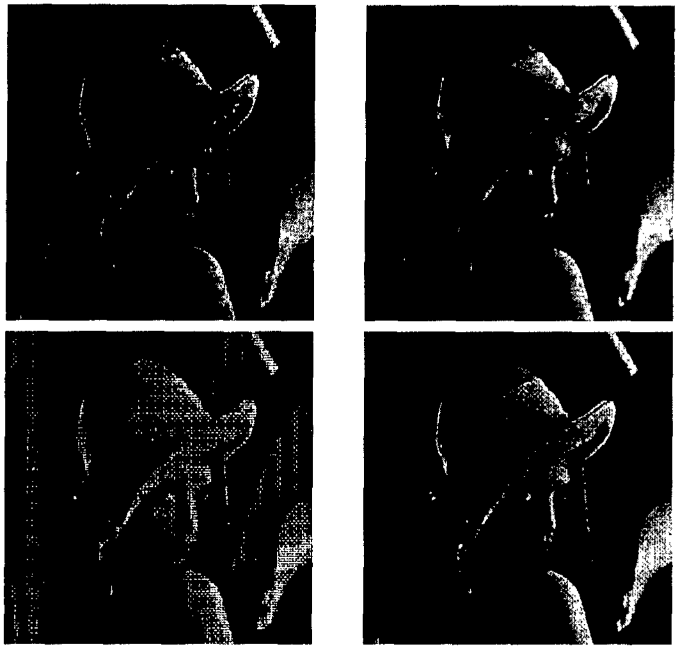

Fig. 3. Reconstructions of Lena compressed with different bases. From the top left corner: 2-balanced Lazy multifilter with $p = 0.25$; 4-balanced Lazy multifilter with $p = 0.4$; CL without prefiltering; GHM without prefiltering.

C.R.	2-bal. Lazy	4-bal. Lazy	CL without pref.	GHM without pref.
16	26.61	28.12	11.79	21.72

Tab. 1. PSNR values (in dB) with different multifilters.

omitted, in order to show how a prefiltering is absolutely necessary when dealing with non-balanced multifilters.

These results are shown in Table 1. Figure 3 shows the reconstruction of Lena compressed with the above-mentioned bases. It can be seen from the table and the figure that 2-balanced Lazy multifilters behave better than CL and GHM. Better results are of course achieved by the 4-balanced multifilters.

In the above experiments, we have not taken into account the orthonormal balanced multiwavelets of Lebrun-Vetterli [16] (which definitely give the best results, due to their good spectral properties), since our aim was not to construct the best possible filters, but to show the flexibility of our tool in building multifilters which do not need prefiltering and which are easily found by solving simple linear equations. One can obtain more effective filters with this procedure by extending the length of the lifting polynomials, and by using one of the many well-known good strategies for filter construction.

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