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A TYPE OF CORRELATED DATA IN OPERATIONAL TESTING

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ABSTRACT. During a portion of a test, N gunners fired two rounds apiece. The overall proportion of hits on first rounds was very close to the overall proportion of hits on second round shots. However, an individual gunner's performance on his second shot was positively correlated with his performance on the first round.

The parameter of interest was p, the probability of hit using the firing device. The proportion of hits among the 2N shots was the natural point estimate of p. However, in calculating interval estimates for p at a given confidence level, or tests of hypothesis of the form $p \geq p_0$ at a given significance level, the situation became more subtle. Since the first round outcome did not deterministically predict the second round outcome, we clearly had more information than just the N first round shots. On the other hand, the assumption that we had 2N independent trials was not justified.

In this paper, a model is proposed for the analysis of this and similar situations. This model generalizes the "two round" case and considers data in blocks when the observations within blocks are not independent.

I. INTRODUCTION. During a portion of the test of a firing device, each gunner fired a volley consisting of two rounds. The outcome of each round was either hit (H) or miss (M) and one of the purposes of the test was to draw inferences about p, the probability of hit.

The following table depicts a typical segment of the results:

Rnd \ Gunner	1	2	3	4	5	6	7	8	9	10
1	H	H	M	M	H	M	H	H	M	H
2	H	H	H	M	H	M	H	M	M	H

Here, the overall proportion of hits on a first round is .6 and the overall proportion of hits on a second round is also .6. The probability of hit on a first round appears to be the same as the probability of hit on a second round, so the overall proportion of hits is an unbiased point estimate of p. However, the conditional probability of hit on a second round after having scored a hit on the first round of the volley is 5/6 which is greater than .6. In other words, performance on the second round is not independent of performance on the first round. Suppose n volleys were fired. We do not have 2n independent rounds. On the other hand, since the outcome on the first round did not predict the outcome on the second round deterministically, we have more information than just the n first round shots. The problem is to calculate confidence intervals and tests of hypotheses about p that reflect our true amount of knowledge realistically.

II. THE MODEL. n players are selected at random. The probability of hit for a player comes from a distribution with mean p and unknown variance σ^2. Then $P_1, \ldots, P_n$, the players' hit probabilities, are independent and identically distributed random variables with mean p.

The i'th player fires k_i shots, $k_i \geq 1, i=1,\ldots,n$. The data is $\{X_{ij}: i=1,..,n, j=1,..,k_i\}$ where $X_{ij}=1$ if the i'th player scored a hit on the j'th trial and 0 otherwise. If $i \neq j$ then X_{ir} and X_{js} are independent. X_{ir} and X_{is} are correlated but are conditionally independent Bernoulli variables with parameter p_i given $\{P_i = p_i\}$.

III. THE TEST STATISTIC. Set $G_i = \sum_{j=1}^{k_i} X_{ij}$, $i=1,..,n$ and let $T = \sum_{i=1}^{n}(G_i/k_i)/n$. Then, using the law of conditional expectation, $E(G_i)=EE(G_i|P_i) = E(k_iP_i) = k_ip$ so that T is an unbiased estimate of p.

$E(G_i^2) = EE(\sum_{j=1}^{k_i} X_{ij}^2 + \sum_{j \neq r} X_{ij}X_{ir} | P_i) =$

$E(k_iP_i+k_i(k_i-1)P_i^2) = k_ip+k_i(k_i-1)(p^2+\sigma^2)$ so that

$$Var(G_i) = k_i(p-p^2) +\sigma^2 \quad k_i(p-p^2)+\sigma^2(k_i^2-k_i). \tag{1}$$

If we set $A= \Sigma\, 1/k_i$ then

$$Var(T)=(A(p-p^2)+\sigma^2(n-A))/n^2 \tag{2}$$

To utilize T as a test statistic, it is necessary to estimate Var (T).

The following lemma is easy to verify: If $Y_1, \ldots Y_n$ are independent with a common mean and Var $(Y_i)=\sigma_i^2$, $i=1,..,n$ then $E\sum_{i=1}^{n}(Y_i-\bar{Y})^2= (n-1)/n \sum_{i=1}^{n} \sigma_i^2$ Applying the lemma with $Y_i=G_i/k_i$ and using (1),

$$E \sum_{i=1}^{n} (G_i/k_i-T)^2=((n-1)/n)(A(p-p^2)+\sigma^2(n-A)). \tag{3}$$

Letting $D = \Sigma (G_i/k_i - T)^2$, it follows from (2) and (3) that $D/(n(n-1))$ is an unbiased estimate of Var T. The statistic that is proposed is, then, T/E where $E = \sqrt{D/n(n-1)}$. If $P[U < x] = 1 - \alpha/2$ for U standard normal then $T - Ex \leq p \leq T + Ex$ is an approximate $1-\alpha$ confidence interval for p. Another application would be to test the hypothesis H_0: $p \geq .9$ vs. H_1: $p < .9$ using the rejection criterion $(T-.9)/E \leq -x$ to achieve a significance level of approximately $\alpha/2$.

IV. A REFINEMENT. If $C_1, \ldots, C_n$ are any real numbers such that $\sum_{i=1}^{n} C_i k_i = 1$ then $T^* \equiv \Sigma C_i G_i$ is an unbiased estimate of p. The choice of $C_i = 1/(nk_i)$ was made to facilitate estimating the variance of T^*.

This corresponds to weighting each player equally. Another possibility would be $C_i = 1/N$, $N = \Sigma k_i$, ie. weighing each shot equally. Using Lagrange multipliers to minimize ΣC_i^2 Var G_i subject to the condition $\Sigma C_i k_i = 1$ yields the result $C_i = K/(p - p^2 + \sigma^2(k_i - 1))$ where K is a constant of proportionality.

V. A SIMULATION. Since normal approxomation was used, a simulation was run to test the accuracy of this method. A situation was considered in which four players were selected. Their probabilities of success were distrubuted uniformly on [.5,1] so that the overall probability of success was .75. Each player fired 5 shots. 95% confidence intervals were constructed using both the proposed statistic and using (4) $T \pm 1.96\sqrt{T(1-T)/N}$ i.e. neglecting the heterogeneity of the players. The program calculated the proportion of times the confidence interval contained .75, the true value of p.

For three runs, the results were .97, .96 and .97 for the proposed interval and .81, .77 and .78 using (4).

APPENDIX - SIMULATION PROGRAM

```
5   X=0:Y=0
10  DIM P(4), X(4,5), G(4)
15  CNT=0
20  FOR I=1 to 4

30  P(I)=.5*RND(1)+.5
40  FOR J=1 to 5

50  X(I,J)=0

60  H=RND(1)

70  IF H <=P(I) THEN X (I, J) =1

80  NEXT J: NEXT I
85  T=0
90  FOR I= 1 to 4
100 G(I)=0
110 FOR J=1 to 5
120 G(I)=G(I)+X(I,J) : NEXT J
130 T=T+G(I) : NEXT I

140 T=T/20
150 D=0
160 FOR I=1 to 4 : D=D+(G(I)/4-T)^2

170 NEXT I
180 E=SQR (D/12)

200 IF ABS (T-.75)<=1.96*E THEN X=X+1

210 IF ABS (T-.75)<=1.96*SQR (T*(1-T)/20) THEN Y=Y+1

220 CNT=CNT+1

230 IF CNT <500 THEN 20

240 PRINT "XBAR="; X/500; "YBAR=";Y/500

250 END
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