A Combined Bayes-Sampling Theory Method For Monitoring a Bernoulli Process

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We assume a population of one-shot missiles which are stored in a ready or near ready state at the physical point of their deployment. We hope that the missiles will sit in idle waiting for many years, but this allows environmental effects to degrade the missiles' capability of successful deployment. Since even a brand new missile may fail to operate properly, and there are no important physical differences between the individual missiles in the given population, we shall assume that a randomly selected missile will have a probability p_t of successful deployment, or reliability, at time t.

It is obviously important to monitor p_t, so a sample of the missiles is tested periodically. Since the testing is destructive, the population is eventually depleted by the testing. Furthermore, defects in missile design may be uncovered, so modifications may be introduced which will have a tendency to increase the reliability. For technical reasons, however, we choose to describe a test which is designed to detect a deterioration in the reliability.

No target value for the reliability is given by management, so that the testing at time t is used to determine if there has been a change in the reliability since time t-1. The following requirements are given and will be used to formalize a test of hypothesis to accomplish the goals of the testing procedure.

It is required to:

1. detect whether p_t has changed by an amount d^* since the immediately preceeding testing period, with a probability of at least π at time t=2,3,4,...

2. compensate for the sampling uncertainty in $\hat{p}_t$, the estimate of p_t, in constructing the test of hypothesis.

3. use the minimum possible sample sizes in accomplishing requirements 1 and 2, above.

Since the test data are pass-fail in nature, the binomial probability model is appropriate for describing the stochastic sample behaviour. Suppose we choose the test size to be α for the hypotheses:

$$H_0:\ p_t = p_{t-1}$$
$$H_1:\ P_t = (p_{t-1}) - d^*$$

Requirement 1, above, leads to a type II error, $\beta=1-\pi$. We are then lead to solve the following inequalities simultaneously for n_t and x_t^* as follows. Let $B(x,n;p)$ represent the cumulative binomial probability of x or fewer successes in n trials. That is,

$$B(x,n;p) = \sum_{j=0}^{x}\binom{n}{j}p^{j}(1-p)^{(n-j)}$$

Then the inequalities of interest are:

$$B(x_t^*,n_t;p_t) \leq \alpha \qquad (1)$$

$$B(x_t^*,n_t;p_t-d^*) \geq 1-\beta \qquad (2)$$

For p_{t-1} known, the null hypothesis is rejected if the current sample yields x_t^* or fewer reliable missiles. Since p_{t-1} is not known, however, (1) and (2) are solved after substituting $\hat{p}_{t-1}$ for p_{t-1} since we have no target value for it. We will account for this uncertainty by averaging the pair x_t^*, n_t with respect to the prior distribution for p_t. First, however, we shall introduce a sequential scheme to reduce the sample sizes required.

For practical reasons, the missiles are tested sequentially in time. Therefore, when a critical sample value is obtained, the sampling may be curtailed. That is, if x_t^*+1 successful tests or if $n_t-x_t^*+1$ failures are experienced before the sample is completed, then the test may be curtailed (terminated prematurely) without effecting the error distribution of the test. The curtailed sampling distribution is expressed as follows. Given p_t and x_t^*, the probability that $n_t=x$ when a curtailed sampling procedure is used is:

$$P[n_t=x|p_t] = \begin{cases} \binom{x-1}{n_t-x_t^*-1}(1-p_t)^{n_t-x_t^*}p_t^{x-(n_t-x_t^*)}, & n_t-x_t^* \leq x \leq x_t^* \\ \binom{x-1}{n_t-x_t^*-1}(1-p_t)^{n_t-x_t^*}p_t^{x-(n_t-x_t^*)} + & \\ \binom{x-1}{x-x_t^*-1}(1-p_t)^{x-x_t^*-1}p_t^{x_t^*+1}, & x_t^* < x \leq n_t \end{cases} \qquad (3)$$

In order to obtain $P[n_t=x]$, we compute the average with respect to the prior probability for p_t, given by $g(p_t|H)$. In the absence of information to the contrary, the conjugate prior in the binomial case, is not only convenient, but also natural. This prior is the Beta distribution given by:

$$g(p|a,b,H) = B^{-1}(a,b)p^{a-1}(1-p)^{b-1}, \quad a\geq 1, \; b\geq 1,$$

where,

$$B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b) \; ,$$

$\Gamma(x)$ is the gamma function [1, p. 255], and H refers to the experimental hypothesis relevant to our situation. The averaging process yields:

$$P[n_t=x] = \begin{cases} \binom{x-1}{n_t-x_t^*-1} B^{-1}(a,b)\; B(x-n_t+x_t^*+a,\, n_t-x_t^*+b) & \text{for } n_t-x_t^*\leq x\leq x_t^* \\[2ex] \binom{x-1}{n_t-x_t^*-1} B^{-1}(a,b)\; B(x-n_t+x_t^*+a,\, n_t-x_t^*+b) \; + & \\[2ex] \binom{x-1}{x-x_t^*-1} B^{-1}(a,b)\; B(x_t^*+1+a,\, x-x_t^*-1+b) & \text{for } x_t^*<x\leq n_t \end{cases} \tag{4}$$

The expected sample sizes, $E[n_t]$, can be obtained by computing:

$$E[n_t] = \sum_{x=0}^{n} xP[n_t=x]$$

A full Bayesian treatment of the problem is developed as follows. Equations (1) and (2) are averaged with respect to the prior as shown below.

$$\int_0^1 B(x_t^*,n_t;p_t)g(p_t|H)dp_t \leq \alpha \tag{5}$$

$$\int_0^1 B(x_t^*,n_t;p_t-d)g(p_t|H)dp_t \geq 1-\beta \tag{6}$$

Integrals (5) and (6) may be re-expressed in closed form which allows them to be solved iteratively for $\hat{x}_t$ and $\hat{n}_t$. These values are then used in equations (3) for computing the expected sample sizes. We point out that (5) is related to the predictive distribution which is used for model checking or informal hypothesis testing in the Bayesian context [2, p.385].

Generally, prior distributions on unknown parameters involve parameters of their own which, in turn, depend on the experimental conditions or hypotheses. In our example the parameters are 'a' and 'b'. The experimental hypotheses and specific parametric values for our situation are obtained and applied by using the following line of reasoning. Before the initial test, little or no a-priori information is available about p_t so a flat prior distribution is assumed. The uniform prior corresponds to the parameter values $a=b=1$, and essentially assigns equal weights to all values of p_t in the interval $(0,1)$. After the first test sample has been obtained, say x_1 and n_1, the posterior distribution is a Beta distribution with parameters $a+x_1$ and $b+n_1-x_1$. The mode of the posterior may be used as an estimate for p_1. This is given by $p_1=(a+x_1-1)/(a+b+n_1-2)$, and as noted previously, is the value against which the second sample is tested. The complete testing strategy is outlined below.

1. Before testing begins, the prior distribution is defined. This should be based on engineering knowledge and experience and developmental history. Since it is not usually possible to obtain that information from engineers, it is imperative to provide a reasonable alternative. For this we suggest using an initial sample, corresponding to time t=0. The implied prior for the initial sample is the uniform distribution of the Beta family.

2. The monitoring procedure begins with the first test sample and proceeds as follows. At time t(=1,2,3,...) the prior distribution, $g_t(.)$, is the posterior distribution from the test at time t-1, or $h_{t-1}(.)$. The mode of the prior is the value for p_{t-1} in the null hypothesis against which the sample at time t is tested.

3. The sample size and critical value for the test is obtained from equations (1) and (2). If the sample results on an acceptance of the null hypothesis, then the sample values are used to update the prior, resulting in the posterior distribution. A new modal value for p is obtained which will be used in the test at time t+1, and a new sample size and critical value are obtained.

4. If the sample results in a rejection of the null hypothesis at time t, then the current prior is discarded, and the current sample is used to determine the prior for the following test of hypothesis.

The authors wish to acknowledgs helpful discussions with Prof. George Box, Prof. Michael Woodroofe, and Dr. Daniel Willard.

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