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A METHOD FOR DETERMINING INDIVIDUAL AND COMBINED WEAPONS EFFECTIVENESS MEASURES UTILIZING THE RESULTS OF A HIGH-RESOLUTION COMBAT SIMULATION MODEL

Mr. William H. Holter
General Research Corporation
McLean, Virginia

INTRODUCTION

The Gaming and Simulations Department of General Research Corporation/Operations Analysis Division has recently completed a study entitled "NATO Combat Capabilities Analysis II" (COMCAP II) under the sponsorship of ODCSOPS. One of the principal objectives of the study was to develop weapon effectiveness values (WEVs) and unit effectiveness values (UEVs) for representative U.S. and Soviet forces engaged in mid-intensity combat in Western Europe, circa 1976. The objectives of the study were attained by analyzing killer/casualty data generated by an exercise of the Division Battle Model (DBM) over some six days of simulated warfare in the European theater. This paper presents a mathematical description and justification of the methodology, which was employed in the study, for determining the effectiveness values. The paper appears as Appendix D of the COMCAP II final report.

DISCUSSION

Consider two opposing forces, Blue and Red, engaged in military combat. Suppose Blue has b distinct types of weapons and Red has r distinct types of weapons.

Let:

$n_{Bi}(t)$ = the number of Blue type i weapons remaining at time t after start of the battle (i=1,2,...,b).

$n_{Rj}(t)$ = the corresponding number of Red type j weapons (j=1,2,...,r).

V_{Bi} = the (time-independent) "value" of a Blue type i weapon.

V_{Rj} = the (time independent) "value" of a Red type j weapon.

The goal of COMCAP II is to assign numerical values to the parameters (the WEVs), V_{Bi} and V_{Rj} , such that: (1) the magnitudes of the values indicate the relative worth (in terms of combat effectiveness) of individual weapons; and (2) the resulting values of the linear combinations (the UEVs), $\sum_{i=1}^b V_{Bi} n_{Bi}(0)$ and $\sum_{j=1}^r V_{Rj} n_{Rj}(0)$, are "good" measures of the relative strengths of the opposing forces.

The methodology adopted for attaining this twofold goal is derived from the following intuitively appealing

Major Premise:

The total value of a number of weapons of a given type is directly proportional to the total value of the opposing force destroyed by those weapons per unit time.

In what follows it is first shown that the methodology arising from this premise has some interesting implications in connection with classical Lanchester theory; a justification of certain basic model assumptions is also presented; next, an iterative method for solving the resulting equations is described; and, finally, the procedure is illustrated via a numerical example.

Matrix notation is used throughout the discussion. In addition to those given above the following definitions are employed:

B = Blue's (b x r) "kill rate matrix" = $\begin{bmatrix} \alpha_{Bij} \end{bmatrix}$
 α_{Bij} = the constant rate at which a single Blue type i weapon kills Red type j weapons.

R = Red's (r x b) "kill rate matrix" = $\begin{bmatrix} \alpha_{Rji} \end{bmatrix}$
 α_{Rji} = the constant rate at which a single Red type j weapon kills Blue type i weapons.

\bar{V}_B = the column vector $[V_{Bi}]$ with b components.

\bar{V}_R = the column vector $[V_{Rj}]$ with r components.

$\bar{n}_B(t)$ = the column vector $[n_{Bi}(t)]$ with b components.

$\bar{n}_R(t)$ = the column vector $[n_{Rj}(t)]$ with r components.

The elements of the matrices, B and R, are measures of the killing power of individual firers against different types of targets. In COMCAP II, estimates of these measures are obtained by grouping DEM killer/casualty data into discrete sets of small unit engagements according to Blue posture--delay, defense, and counterattack. Specifically, for each such set of engagements,

$$\alpha_{Bij} = \frac{\sum_{m=1}^E K_{Bijm}}{\sum_{m=1}^E n_{Bim} \Delta t_m}, \text{ and}$$

$$\alpha_{Rji} = \frac{\sum_{m=1}^E K_{Rjim}}{\sum_{m=1}^E n_{Rjm} \Delta t_m},$$

requiring the additional definitions:

- E = the total number of small unit engagements in the set.
- K_{Bijm} = the total number of kills by Blue type i weapons of Red type j weapons in the mth engagement.
- n_{Bim} = the initial number of Blue type i weapons in the mth engagement.
- Δt_m = the duration of the mth engagement.
- K_{Rjim} = the total number of kills by Red type j weapons of Blue type i weapons in the mth engagement.
- n_{Rjm} = the initial number of Red type j weapons in the mth engagement.

Also: a "heterogeneous force" is defined as a force comprising weapons with differing characteristics - tanks, TOWs, rifles, etc.; a "homogeneous force" is defined as a force comprising identical weapons; dots are used to denote time derivatives; and superscript T denotes matrix transposition. Other definitions are provided as needed.

Connection Between the Methodology and Lanchester Theory

Using the notation just defined, Lanchester's square law for the attrition of heterogeneous forces engaged in combat may be stated mathematically as

$$\dot{\bar{n}}_R(t) = -B^T \bar{n}_B(t) \quad (1)$$

$$\dot{\bar{n}}_B(t) = -R^T \bar{n}_R(t); \quad (2)$$

i.e., the rate at which targets of a given type are attrited is equal to a weighted sum of the numbers of firers of a given type on the opposing side, the weights being the rates at which the individual firers kill the targets. Denote the total strength of the Blue force at time t by $U_B(t)$, a weighted sum of the number of Blue weapons,

$$U_B(t) = \bar{V}_B^T \bar{n}_B(t) \quad (3)$$

and the corresponding strength of the Red force by $U_R(t)$, a similar sum,

$$U_R(t) = \bar{V}_R^T \bar{n}_R(t) \quad (4)$$

where \bar{V}_B and \bar{V}_R are the yet-to-be-determined vectors of the Blue and Red WEVs. (Note that, if \bar{V}_B and \bar{V}_R are selected "properly," $U_B(0)$ and $U_R(0)$ are the Blue and Red UEVs.) Further, as a direct consequence of the major premise stated earlier, the relationships between the Red and Blue WEVs may be written

$$\beta_B \bar{V}_B = B \bar{V}_R \quad (5)$$

and

$$\beta_R \bar{V}_R = R \bar{V}_B \quad (6)$$

where β_B and β_R are positive constants also to be determined.

Using equations (1), (3), and (4), equation (5) transforms successively to

$$\bar{V}_R^T B^T = \beta_B \bar{V}_B^T$$

$$- \bar{V}_R^T (B^T \bar{n}_B(t)) = - \beta_B (\bar{V}_B^T \bar{n}_B(t))$$

$$\bar{V}_R^T \dot{\bar{n}}_R(t) = - \beta_B U_B(t)$$

and

$$\dot{U}_R(t) = - \beta_B U_B(t). \quad (7)$$

Similarly, using equations (2), (3), and (4), equation (6) transforms to

$$\dot{U}_B(t) = - \beta_R U_R(t). \quad (8)$$

Equations (7) and (8) have the form of Lanchester's square law for the attrition of homogeneous forces, where β_B is the rate at which an "average" Blue weapon kills "average" Red weapons, and β_R is the rate at which an "average" Red weapon kills "average" Blue weapons. Thus, equations (3) - (6) (assuming that equations (5) and (6) can be solved to yield unique values of β_B , β_R , \bar{V}_B and \bar{V}_R) imply that one can go from a heterogeneous Lanchester model represented by equations (1) and (2) to an equivalent homogeneous Lanchester model represented by equations (7) and (8). This interesting (and important) fact was first noted by Dare and James¹ and subsequently elaborated upon by Thrall² and Anderson.^{3*}

¹Dare, D.P., and James, B.A.P., "The Derivation of Some Parameters for a Corps/Division Model from a Battle Group Model," Defense Operation Analysis Establishment Memorandum 7120, Ministry of Defense, West Byfleet, United Kingdom, July 1971 (CONFIDENTIAL).

²Thrall, Robert M., and Associates, Final Report to US Army Strategy and Tactics Analysis Group (RMT-200-R4-33), May 1972.

³Anderson, Lowell B., "A Method for Determining Linear Weighting Values for Individual Weapons Systems," Institute for Defense Analyses, Improved Methodologies for General Purpose Forces Planning (New Methods Study) Working Paper WP-4, December 1971.

*The author is indebted to Dr. Anderson for bringing to his (the author's) attention the earlier works of Thrall, and Dare and James.

Equations (5) and (6) may be combined to yield

$$(\beta_B \beta_R I_r - B R) \bar{V}_R = \bar{O}_r \quad (9)$$

and

$$(\beta_B \beta_R I_b - B R) \bar{V}_B = \bar{O}_b \quad (10)$$

where I_r and I_b are, respectively, r^2 and b^2 identity matrices; \bar{O}_r and \bar{O}_b are correspondingly-dimensioned null column vectors. As further noted by Dare and James,¹ Spudich,⁴ Thrall,² and Anderson,³ these equations, in most cases, determine the product $\beta_B \beta_R$ uniquely and the components of V_B and V_R to within an arbitrary scaling factor for each of the vectors.*

In general, two additional scaling relationships must be specified in order to permit a unique determination of values of β_B , β_R , V_B and V_R . Among the relationships that have been assumed in other studies, where, it must be emphasized, the goals were not necessarily the same as those of COMCAP II, are those of

$$\text{Spudich}^4: \beta_B = \bar{V}_R^T \bar{n}_R(0), \beta_R = \bar{V}_B^T \bar{n}_B(0) \quad (11)$$

Dare and James¹:

$$\sum_{i=1}^b V_{Bi} = \sum_{j=1}^r V_{Rj} = 1 \quad (12)$$

and

Thrall²:

$$\beta_B = \sum_{j=1}^r V_{Rj}, \beta_R = \sum_{i=1}^b V_{Bi} \quad (13)$$

⁴Spudich, John, "The Relative Kill Productivity Exchange Ratio Technique," Booz-Allen Applied Research, Inc., Combined Arms Research Office, no date given. (A similar discussion is presented in TAB E, Appendix II to Annex L of the TATAWS III Study, Headquarters, US Army Combat Developments Command, Tank, Antitank and Assault Weapons Requirements Study (U), Phase III, December 1968 (SECRET-NOFORN).

* Provided the matrices BR and RB are "irreducible," there is one and only one value of the product $\beta_B \beta_R$ that leads to nonnegative values of the components of V_B and V_R - namely, the maximum eigenvalue of BR and RB (it is the same for both). The matrices are "reducible" (not irreducible) if at least two opposing weapons types are not interacting directly with the other participants in the battle. In the COMCAP II DBM exercise the problem of reducibility did not arise. For a thorough discussion of matrix reducibility and its implications in weapon effectiveness analyses see Thrall.²

In COMCAP II the relationships are taken to be

$$a_B = a_R (=1/c), V_{B1} = 1 \quad (14)$$

where the M60A3 tank is assigned the role of the Blue type 1 weapon.

There are two principal arguments for employing equations (14), rather than (11), (12), or (13), in COMCAP II. The arguments are presented below.

Recall that, at the outset, it was stated that the first part of the goal of COMCAP II is to assign numerical values to the parameters (the WEVs) V_{Bi} and V_{Rj} such that the magnitudes of the values indicate the relative worth of individual weapons. Equations (14) result in WEVs for both Blue and Red that are all measured relative to the worth of the same weapon - the M60A3 tank. Thus, if by using equations (14) it turns out that $V_{R2} = V_{B2} = V_{B3} = 2$: one can infer that a Red type 2 weapon, a Blue type 2 weapon, and a Blue type 3 weapon are equally effective, and each is worth two M60A3s. On the other hand, if by using equations (11), (12), or (13) it turns out that $V_{R2} = V_{B2} = V_{B3} = 4$: one can only infer that a Blue type 2 weapon and a Blue type 3 weapon are equally effective; nothing can be inferred about their effectiveness as compared to a Red type 2 weapon. The point being made here is this: equations (14) lead to a set of relative values, the relativity extending not only to the weapons within a Blue force or a Red force but across forces as well; equations (11), (12), or (13) also lead to relative values, but the relativity extends only to the weapons within a force, not across forces.

This completes the first argument for employing equations (14) in COMCAP II.

The second argument--a rather lengthy one--is based on a consideration of equations (7) and (8): Lanchester's square law for homogeneous forces. The solutions to these equations for $U_B(t)$ and $U_R(t)$ as functions of time are well known (see Morse and Kimball⁵ for example). They are

$$\frac{U_B(t)}{U_B(0)} = \cosh(\sqrt{a_B a_R} t) - \frac{1}{\sqrt{G}} \sinh(\sqrt{a_B a_R} t) \quad (15)$$

for the Blue force, and

$$\frac{U_R(t)}{U_R(0)} = \cosh(\sqrt{a_B a_R} t) - \sqrt{G} \sinh(\sqrt{a_B a_R} t) \quad (16)$$

for the Red force, where G is defined as

$$G = \frac{a_B U_B^2(0)}{a_R U_R^2(0)}. \quad (17)$$

⁵Morse, Philip M., and Kimball, George E., "Methods of Operations Research," The M.I.T. Press, Cambridge, Massachusetts, August 1963.

Dividing each side of equation (15) by the corresponding side of (16) and taking derivatives with respect to time it is readily shown that the designation of the "superior" force is determined by the value of G. For example, $G > 1$ implies that

$$d \left(\frac{U_B(t)}{U_R(t)} \right) > 0$$

and Blue is the superior force since the ratio of the strength of the Blue force to the strength of the Red force increases monotonically with time. On the other hand, $G < 1$ implies that

$$d \left(\frac{U_R(t)}{U_B(t)} \right) < 0,$$

and, by the same reasoning, Red is the superior force. (If $G = 1$, $U_B(t) / U_R(t) = U_B(0) / U_R(0)$ for all t and neither side has the advantage.)

Now, from equations (3) and (5) it may be deduced that

$$U_B^2(0) = (\bar{v}_R^T B^T \bar{n}_B(0)) (\bar{v}_B^T \bar{n}_B(0))$$

and, from equations (4) and (6) that

$$U_R^2(0) = (\bar{v}_B^T R^T \bar{n}_R(0)) (\bar{v}_R^T \bar{n}_R(0)).$$

It follows, then, that

$$G = \left(\frac{\bar{v}_R^T B^T \bar{n}_B(0)}{\bar{v}_R^T \bar{n}_R(0)} \right) \bigg/ \left(\frac{\bar{v}_B^T R^T \bar{n}_R(0)}{\bar{v}_B^T \bar{n}_B(0)} \right). \quad (18)$$

The implications of this equation are quite interesting. It is evident from the equation that the value of G is independent of the method by which the vectors \bar{v}_B and \bar{v}_R are scaled. Therefore, the relationships (11) - (14) (or any other scaling relationships for that matter) all lead to the same value of G; they also all lead to the same value of the right-hand side of equation (15), and the same value of the right-hand side of equation (16).

Again recall that, at the outset, it was stated that the second part of the goal of COMCAP II is to determine the vectors V_B and V_R such that the linear combinations $\bar{v}_B^T \bar{n}_B(0)$ and $\bar{v}_R^T \bar{n}_R(0)$ (i.e., the UEVs $U_B(0)$ and $U_R(0)$) are "good" measures of the relative strengths of the Blue and Red force. If equations (14) are assumed, it follows from equation (17) that

$$\frac{U_B(0)}{U_R(0)} = \sqrt{G} . \quad (19)$$

Under this assumption, then, $U_B(0) > U_R(0)$ implies that Blue is superior; $U_B(0) < U_R(0)$ implies that Red is superior; and $U_B(0) = U_R(0)$ implies that the forces are equal. The assumption of equations (14), therefore, leads to values of $U_B(0)$ and $U_R(0)$ that are, in fact, "good" measures and a meaningful "force ratio," F , may be defined as

$$F = \frac{U_B(0)}{U_R(0)} . \quad (20)$$

Blue is superior, inferior, or equal to Red accordingly as $F > 1$, $F < 1$, or $F = 1$.

Alternatively, if equations (11) are assumed, it follows from equations (5), (6) and (18) that

$$F = \frac{U_B(0)}{U_R(0)} = G . \quad (21)$$

Hence, the assumption of equations (11) also leads to a meaningful force ratio, F . However, by comparing equations (21) and (17) it is evident that the assumption of equations (11) is tantamount to assuming that

$$A_B U_B(0) = A_R U_R(0),$$

or, equivalently, that the total Red strength destroyed per unit time is equal to the total Blue strength destroyed per unit time - an assumption that lacks credibility.

Finally, if either equations (12) or (13) are assumed, the resulting ratio, F , is not meaningful. The value of F under either of these assumptions gives no indication whatsoever as to which side is the superior force; under either of these assumptions the calculated value of F , for example, can be considerably less than unity while the corresponding value of G is considerably greater than unity.

This completes the second argument.

In light of the preceding arguments, assumptions (14) are clearly superior to the three alternatives considered, insofar as their applicability to the COMCAP II study is concerned. That is not to say, however, that the alternatives would not be useful in other studies where the goals are different from those of COMCAP II.

The justification of assumptions (14) having been established, equations (15) and (16) may be written

$$\frac{U_B(t)}{U_B(0)} = \cosh(t/c) - \frac{1}{F} \sinh(t/c) \quad (22)$$

and

$$\frac{U_R(t)}{U_R(0)} = \cosh(t/c) - F \sinh(t/c) \quad (23)$$

where

$$c = 1/\beta_B = 1/\beta_R$$

$$V_{Bl} = 1$$

$$F = \frac{U_B(0)}{U_R(0)} = \frac{\bar{V}_B^T \bar{n}_B(0)}{\bar{V}_R^T \bar{u}_R(0)}$$

and the values of c , \bar{V}_B , and \bar{V}_R are obtained by solving equations (5) and (6). Eliminating the explicit use of the parameter, t , from equations (22) and (23) leads to the "state equation" relating Blue's strength to Red's corresponding strength at any instant after the start of battle,

$$\sqrt{\frac{1 - \left[\frac{U_R(t)}{U_R(0)} \right]^2}{1 - \left[\frac{U_B(t)}{U_B(0)} \right]^2}} = F. \quad (24)$$

For a given value of F , if one specifies a fraction of the initial strength remaining on one side - say a "break threshold" - the corresponding fraction remaining on the opposing side may be determined from this equation.

Equations (22) - (24) should prove useful in calculating the attrition of forces in highly aggregated war games.

In sum, then, through the use of calculated weighting factors (the WEVs), the COMCAP II methodology converts two opposing heterogeneous forces into two opposing homogeneous forces, both comprising identical weapons. The only difference in the opposing homogeneous forces lies in their respective initial numbers of weapons (the UEVs). The force with the larger UEV is the superior force. The conversion to homogeneous forces permits one to use classical Lanchester theory to compute the relative attrition of forces in highly aggregated war games.

Solution of the WEV Equations

We turn next to the solutions of equations (5), (6), and (14)

$$\beta_B \bar{V}_B = B \bar{V}_R$$

$$\beta_R \bar{V}_R = R \bar{V}_B$$

$$\beta_B = \beta_R = 1/c$$

$$V_{B1} = 1$$

for the components \bar{V}_B and \bar{V}_R (the WEVs) and the constant, c , (the reciprocal of the average kill rate).

These equations may be combined to yield

$$\bar{V}_B = c^2 BR \bar{V}_B, \quad (25)$$

a relationship involving only c and the Blue WEVs. Let $\lambda = c^2$. A rapidly converging algorithm leading to unique values of λ and the components of \bar{V}_B and \bar{V}_R is given by the following sequence of operations, where the superscript (j) denotes values at the end of the j th iteration.

Step 1. Set $j = 1$.

Step 2. Set all the components of $\bar{V}_B^{(j)}$ equal to unity.

Step 3. Calculate successively:

$$\bar{W}^{(j)} = BR \bar{V}_B^{(j)},$$

$$\lambda^{(j)} = \frac{1}{W_1^{(j)}} \quad (W_1 \text{ is the first component of } \bar{W}),$$

$$\bar{V}_B^{(j+1)} = \lambda^{(j)} \bar{W}^{(j)}.$$

Step 4. Repeat Step 3, incrementing j by 1 at each iteration, until $\lambda^{(j+1)} = \lambda^{(j)}$ to within a specified degree of accuracy. The process converges to a unique value of λ and the vector \bar{V}_B with $V_{B1} = 1$.*

Step 5. Calculate:

$$c = \sqrt{\lambda}$$

$$\bar{V}_R = cR \bar{V}_B.$$

The iterative procedure is a variation of Hildebrand's⁶ method for determining the maximum eigenvalue of a matrix. Proof of the convergence of the method is given on Pages (68-87) of the reference.

*As previously discussed, it is assumed that the matrix BR is irreducible. See Thrall.²

⁶Hildebrand, F. B., "Methods of Applied Mathematics," Prentice-Hall, Inc., Englewood Cliffs, New Jersey, June 1963.

Sample Problem

The ultimate worth of any mathematical model is determined by the degree to which it can be used to solve real-world problems. To illustrate how the foregoing discussion might be used in a practical sense the following sample problem is posed and solved.

Consider a single battle involving two distinct types of weapons on either side with $n_{B1}(0) = n_{R1}(0) = 100$, $n_{B2}(0) = n_{R2}(0) = 25$; i.e.,

$$\bar{n}_B(0) = \bar{n}_R(0) = \begin{bmatrix} 100 \\ 25 \end{bmatrix}.$$

From previously accumulated battle data, Blue's kill rate matrix, B, has been estimated to be

$$B = \begin{bmatrix} \alpha_{B1j} \end{bmatrix} = \begin{bmatrix} .1 & .05 \\ .05 & .1 \end{bmatrix}$$

and Red's kill rate matrix, R, by

$$R = \begin{bmatrix} \alpha_{Rj1} \end{bmatrix} = \begin{bmatrix} .05 & .05 \\ .1 & .05 \end{bmatrix}$$

where the rates are measured in kills per weapon per hour. Using the COMCAP II methodology and the related Lanchester equations answer the following questions:

Question 1. What are the relative values (the WEVs) of the individual weapons in the battle, assuming $V_{B1} = 1$?

Question 2. What are the relative strengths of the forces at the beginning of the battle (the UEVs), and the force ratio, F?

Question 3. The break threshold of both sides is set at 30 percent loss of strength. What is the percent loss of strength of the "winner" when the "loser" breaks?

Question 4. How long does the battle last before the loser breaks?

Solution

Question 1

First perform the matrix multiplication:

$$BR = \begin{bmatrix} .1 & .05 \\ .05 & .1 \end{bmatrix} \begin{bmatrix} .05 & .05 \\ .1 & .05 \end{bmatrix} = \begin{bmatrix} .01 & .0075 \\ .0125 & .0075 \end{bmatrix}$$

Set $\bar{V}_B(1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Now perform the iterations:

$$\bar{W}(1) = BR \bar{V}_B(1) = \begin{bmatrix} .01 & .0075 \\ .0125 & .0075 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} .0175 \\ .0200 \end{bmatrix}$$

$$\lambda(1) = \frac{1}{W_1(1)} = \frac{1}{.0175} = 57.143$$

$$\bar{V}_B(2) = \lambda(1) \bar{W}(1) = 57.143 \begin{bmatrix} .0175 \\ .0200 \end{bmatrix} = \begin{bmatrix} 1 \\ 1.1429 \end{bmatrix}$$

$$\bar{W}(2) = BR \bar{V}_B(2) = \begin{bmatrix} .01 & .0075 \\ .0125 & .0075 \end{bmatrix} \begin{bmatrix} 1 \\ 1.1429 \end{bmatrix} = \begin{bmatrix} .01857 \\ .02107 \end{bmatrix}$$

$$\lambda(2) = \frac{1}{W_1(2)} = \frac{1}{.01857} = 53.8502$$

$$\bar{V}_B(3) = \lambda(2) \bar{W}(2) = 53.8502 \begin{bmatrix} .01857 \\ .02107 \end{bmatrix} = \begin{bmatrix} 1 \\ 1.1346 \end{bmatrix}$$

$$\bar{W}(3) = BR \bar{V}_B(3) = \begin{bmatrix} .01 & .0075 \\ .0125 & .0075 \end{bmatrix} \begin{bmatrix} 1 \\ 1.1346 \end{bmatrix} = \begin{bmatrix} .01851 \\ .02101 \end{bmatrix}$$

$$\lambda(3) = \frac{1}{W_1(3)} = \frac{1}{.01851} = 54.0250$$

$$\bar{V}_B(4) = \lambda(3) \bar{W}(3) = 54.0250 \begin{bmatrix} .01851 \\ .02101 \end{bmatrix} = \begin{bmatrix} 1 \\ 1.1351 \end{bmatrix}$$

$$\bar{W}(4) = BR \bar{V}_B(4) = \begin{bmatrix} .01 & .0075 \\ .0125 & .0075 \end{bmatrix} \begin{bmatrix} 1 \\ 1.1351 \end{bmatrix} = \begin{bmatrix} .01851 \\ .02101 \end{bmatrix}$$

$$\lambda(4) = \frac{1}{W_1(4)} = \frac{1}{.01851} = 54.0250 = \lambda(3)$$

Therefore,

$$\lambda = 54.0250$$

$$\bar{V}_B = \begin{bmatrix} 1 \\ 1.1351 \end{bmatrix}$$

Calculate:

$$c = \sqrt{\lambda} = \sqrt{54.0250} = 7.3502$$

$$\bar{V}_R = cR \bar{V}_B = 7.3502 \begin{bmatrix} .05 & .05 \\ .1 & .05 \end{bmatrix} \begin{bmatrix} 1 \\ 1.1351 \end{bmatrix} = \begin{bmatrix} .7850 \\ 1.1525 \end{bmatrix}.$$

The resulting WEVs are, therefore,

$$V_{B1} = 1$$

$$V_{B2} = 1.1351$$

$$V_{R1} = .7850$$

$$V_{R2} = 1.1525.$$

Question 2

$$\text{Blue's UEV} = U_B(0) = \bar{V}_B^T \bar{n}_B(0) = \boxed{1 \quad 1.1351} \begin{bmatrix} 100 \\ 25 \end{bmatrix} = 128.4$$

$$\text{Red's UEV} = U_R(0) = \bar{V}_R^T \bar{n}_R(0) = \boxed{.7850 \quad 1.1525} \begin{bmatrix} 100 \\ 25 \end{bmatrix} = 107.3$$

Blue is the superior force - it has the equivalent of 128.4 Blue type 1 weapons while Red has only 107.3. The initial force ratio is

$$F = \frac{U_B(0)}{U_R(0)} = \frac{128.4}{107.3} = 1.20.$$

Question 3

Blue is the superior force. Since both sides have set their break thresholds at 30 percent loss of strength, Red is the loser. When Red breaks, Blue's corresponding percent loss is determined by first solving equation (24)

$$\sqrt{\frac{1 - [1 - .3]^2}{1 - [U_B(t)/128.4]^2}} = 1.20$$

for $U_B(t)/128.4$. This leads to

$$\frac{U_B(t)}{128.4} = .8036.$$

So, Blue has suffered $1 - .8036 = 19.6\%$ reduction in strength. Blue's strength, when Red breaks, is $(.8036)(128.4) = 103.2$, and Red's strength is $(.7)(107.3) = 75.1$.

Notice that if Blue had set its break threshold at something less than 19.6 percent loss in strength it would be the loser (assuming Red had set its threshold at 30 percent losses) even though it has the superior force.

Question 4

Equation (23) may be rearranged to yield the battle time, t , as a function of the fraction of the initial Red strength remaining. Performing the necessary algebraic manipulations, we arrive at

$$t = c \log_e \left[\sqrt{\frac{U_R^2(t) + F^2 - 1}{U_R^2(0)} - \frac{U_R(t)}{U_R(0)}} \right].$$

Now, in the present example, $c = 7.35$, $F = 1.20$, and $U_R(t)/U_R(0) = .7$. Substituting these values into the equation leads to

$t \approx 2$ hours of battle until Red breaks.

REFERENCES

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