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APPROXIMATE BEHAVIOR MODELS FOR OPTIMUM STRUCTURAL DESIGN

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Summary

Efficient reanalysis models, which provide high quality explicit approximations for the structural behavior, are introduced. The presented algorithms are based on a series expansion which is shown to be equivalent to a simple iteration procedure. To preserve efficiency, only methods which do not involve matrix inversion have been considered. Only the decomposed stiffness matrix, known from exact analysis of the initial design, is required to obtain the approximate expressions. Two approaches of accelerated convergence are proposed to improve the quality of the approximations:

a) An approach where a scalar multiplier, used for scaling of the initial design, is chosen prior to the solution as the accelerating parameter.

b) An approach where information gathered during calculations of the series coefficients is used to improve the convergence rate.

Numerical examples illustrate the efficiency and the quality of the proposed approximations. A special attention is focused on reanalyses along a line, a problem typical to many optimal design procedures. The computational effort in this case is considerably reduced, since only a single independent variable is involved.

1. Introduction

In most optimal design procedures the behavior of the structure must be evaluated many times for successive modifications in the design variables. This operation, which involves much computational effort, is one of the main obstacles in applying optimization methods to large structural systems. Reanalysis methods, intended to analyze efficiently new designs using information obtained from previous ones, can broadly be classified as [1]:

(a) Direct methods, giving exact solutions and applicable to situations where a relatively small proportion of the structure is modified (for example, only a small number of elements are changed).

(b) Iterative methods [2,3], suitable for cases of relatively small changes in the structure. The known solution of a given design is usually used as an initial value for the iterative process. Problems of slow convergence rate or even divergence may arise for large changes in the design.

(c) Approximate methods [4-8], usually based on series expansion and require less computational effort. One problem often encountered is that the accuracy of the solution may not be sufficient. Under certain assumptions, some approximate methods are shown to be equivalent to iterative procedures.

In this study reanalysis methods for optimum structural design, based on explicit approximations of the structural behavior in terms of the independent design variables, are presented. Once the explicit model has been introduced, it can be used for multiple reanalyses of designs obtained by successive changes in the variables. The presented algorithms are based on a series of expansion which is shown to be equivalent to a simple iteration procedure.

Two conflicting factors should be considered in choosing an approximate behavior model for a specific optimal design problem:
(a) The computational effort involved, or the efficiency of the method.
(b) The accuracy of the calculations, or the quality of the approximation.
To preserve the efficiency, the presentation is limited to methods which do not involve matrix inversion. Only the available decomposed stiffness matrix, known from exact analysis of the initial design, is required to obtain the explicit expressions. However, since the proposed models are based on a single exact analysis, the accuracy of the approximations might be sufficient only for a limited region.

Two approaches of accelerated convergence are proposed, to improve the quality of the approximations:

(a) An approach where a scalar multiplier, used for scaling of the initial design, is chosen prior to the solution as the accelerating parameter. Several algorithms for selecting the value of this multiplier are proposed and their merit is demonstrated.

(b) An approach where the accelerated parameters are calculated from results obtained during the solution process. Information gathered during calculation of the series coefficients is used to improve the convergence rate.

Some numerical examples illustrate applications of the proposed procedures. A special attention is focused on reanalyses along a given line in the design space, a problem common to many optimal design procedures. The efficiency and the quality of the proposed approximations are demonstrated.

2. Problem Statement

The displacement analysis equations for a given design variables vector $\{\overset{*}{X}\}$ are

$$[\overset{*}{K}]\{\overset{*}{r}\} = \{R\} \tag{1}$$

where $[\overset{*}{K}]$ = stiffness matrix corresponding to the design $\{\overset{*}{X}\}$; $\{R\}$ = load vector whose elements are assumed to be independent of the design variables; and $\{\overset{*}{r}\}$ = nodal displacements computed at $\{\overset{*}{X}\}$. The elements of the stiffness matrix $[K]$ are some functions of the design variables $\{X\}$. Assuming a change $\{\Delta\overset{*}{X}\}$ in the design variables so that the modified design is

$$\{\overset{**}{X}\} = \{\overset{*}{X}\} + \{\Delta\overset{*}{X}\} \tag{2}$$

the corresponding stiffness matrix is given by

$$[\overset{**}{K}] = [\overset{*}{K}] + [\Delta\overset{*}{K}] \tag{3}$$

where $[\Delta\overset{*}{K}]$ = the matrix of changes in the stiffness matrix due to the change $\{\Delta\overset{*}{X}\}$.

The object in this study is to present explicit models for efficient calculation of the displacements $\{\overset{**}{r}\}$

corresponding to designs $\{\overset{**}{X}\}$, obtained by changing the value of the design variables. It is assumed that the displacements $\{\overset{*}{r}\}$ are known from analysis of the initial design. Also, $[\overset{*}{K}]$ is given in the decomposed form

$$[\overset{*}{K}] = [\overset{*}{U}]^T[\overset{*}{U}] \tag{4}$$

where $[\overset{*}{U}]$ is an upper triangular matrix.

Approximations along a given line in the design space are often required in optimal design procedures. This problem is common to many mathematical programming methods such as flasible directions or penalty function. A set of lines (or direction vectors) in the design space are determined successively by the optimization method used. In each of the given directions it is usually necessary to evaluate the constraint functions, or to repeat the analysis, many times. A line in the design space can be defined in terms of a single independent variable Y by

$$\{X\} = \{\overset{*}{X}\} + Y\{\Delta\overset{*}{X}\} \tag{5}$$

where $\{\overset{*}{X}\}$ is the given initial design, $\{\Delta\overset{*}{X}\}$ is a given direction vector in the design space, and the variable Y determines the step size. Approximations along a line require much less computations since only a single variable is involved.

In general, the elements of the stiffness matrix are some functions of Y. One common case is that the modified stiffness matrix can be expressed as

$$[K] = [\overset{*}{K}] + f(Y)[\Delta\overset{*}{K}] \tag{6}$$

In truss structures where {X} are the cross-sectional areas or in beam elements where the moments of inertia are chosen as design variables, the elements of the stiffness matrix are linear functions of Y and Eq. (6) becomes

$$[K] = [\overset{*}{K}] + Y[\Delta\overset{*}{K}] \tag{7}$$

If the elements of [K] are functions of aZ_i^b (Z_i being the naturally chosen design variables and a,b are given constants) we may use the transformation

$$X_i = aZ_i^b \tag{8}$$

and obtain the linear relationship (7). The expression of Eq. (8) is suitable, for example, for standard joists [9]. In cases where such transformations are not possible (for example, in frame elements where the stiffness matrix is a function of both moments of inertia and cross-sectional areas), still linear approximations may be used for the nonlinear terms of the stiffness matrix [10].

3. Explicit Behavior Models

The analysis equations at $\{\overset{**}{X}\}$ are

$$[\overset{**}{K}]\{\overset{**}{r}\} = \{R\} \tag{9}$$

Based on Eq. (3),

$$([\overset{*}{K}] + [\Delta\overset{*}{K}])\{\overset{**}{r}\} = \{R\} \tag{10}$$

Premultiplying by $[\overset{*}{K}]^{-1}$ and substituting

$$\{\overset{*}{r}\} = [\overset{*}{K}]^{-1}\{R\} \tag{11}$$

$$[\overset{*}{B}] \equiv [\overset{*}{K}]^{-1}[\Delta\overset{*}{K}] \tag{12}$$

yields

$$([I] + [\overset{*}{B}])\{\overset{**}{r}\} = \{\overset{*}{r}\} \tag{13}$$

Premultiplying by $([I] + [\overset{*}{B}])^{-1}$ and expanding

$$([I] + [\overset{*}{B}])^{-1} = [I] - [\overset{*}{B}] + [\overset{*}{B}]^2 - [\overset{*}{B}]^3 + \ldots \tag{14}$$

Eq. (13) becomes

$$\{\overset{**}{r}\} = ([I] - [\overset{*}{B}] + [\overset{*}{B}]^2 - [\overset{*}{B}]^3 + \ldots)\{\overset{*}{r}\} \tag{15}$$

The coefficients of this series can readily be calculated. Defining

$$\{\overset{*}{r}_1\} \equiv -\{\overset{*}{B}\}\{\overset{*}{r}\} \tag{16}$$

$$\{\overset{*}{r}_2\} \equiv -[\overset{*}{B}]\{\overset{*}{r}_1\} \tag{17}$$

etc., the series of Eq. (15) becomes

$$\{\overset{**}{r}\} = \{\overset{*}{r}\} + \{\overset{*}{r}_1\} + \{\overset{*}{r}_2\} + \ldots \tag{18}$$

For the given triangularization of Eq. (4), the calculation of the coefficient vectors $\{\overset{*}{r}_1\},\{\overset{*}{r}_2\},\ldots$ requires only forward and back substitutions. The calculation of $\{\overset{*}{r}_1\}$, for example, is carried out as follows. Substituting Eq. (12) into Eq. (16) and rearranging gives

$$[\overset{*}{K}]\{\overset{*}{r}_1\} = -[\Delta\overset{*}{K}]\{\overset{*}{r}\} \equiv \{\overset{*}{R}_1\} \tag{19}$$

We first solve for $\{\overset{*}{P}\}$ by a forward substitution

$$[\overset{*}{U}]^T\{\overset{*}{P}\} = \{\overset{*}{R}_1\} \tag{20}$$

$\{\overset{*}{r}_1\}$ is then calculated by the backward substitution

$$[\overset{*}{U}]\{\overset{*}{r}_1\} = \{\overset{*}{P}\} \tag{21}$$

The coefficient vectors $\{\overset{*}{r}_2\},\{\overset{*}{r}_3\}$ etc., can be calculated in a similar manner.

It is instructive to note that the series of Eq. (15) is equivalent to the simple iteration procedure [7,8]

$$\{\overset{**}{r}^{(k)}\} = [I] - [\overset{*}{B}]\{\overset{**}{r}^{(k-1)}\} \tag{22}$$

where k denotes the iteration cycle and

$$\{\overset{**}{r}^{(o)}\} = \{\overset{*}{r}\} \tag{23}$$

In the case of approximations along the line defined by Eq. (5), the expression of Eq. (15) will become explicit function of {r} in terms of Y. Assuming the relationship (6), we obtain

$$\{r\} = ([I] - f(Y)[\overset{*}{B}] + f^2(Y)[\overset{*}{B}]^2 - \ldots)\{\overset{*}{r}\} \tag{24}$$

If the linear dependence of Eq. (7) holds, this explicit expression becomes

$$\{r\} = ([I] - [\overset{*}{B}]Y + [\overset{*}{B}]^2Y^2 - [\overset{*}{B}]^3Y^3 + \ldots)\{\overset{*}{r}\} \tag{25}$$

or (see Eq. (18))

$$\{r\} = \{\overset{*}{r}\} + \{\overset{*}{r}_1\}Y + \{\overset{*}{r}_2\}Y^2 + \ldots \tag{26}$$

This equation can readily be used for multiple reanalyses along a line. Also, it can be shown that Eq. (26) and Taylor series expansion of the displacements are equivalent [7,8]. Other approximate methods can be used [7,8], however, these usually involve matrix inversion.

While the methods discussed so far are based on a single exact analysis at $\{\overset{*}{X}\}$, it should be recognized that better approximations could be obtained if results of two exact analyses (at $\{\overset{*}{X}\}$ and $\{\overset{**}{X}\}$) were considered. Assuming for example, quadratic and cubic interpolations, respectively, we find

$$\{r\} = \{\overset{*}{r}\} + \{\frac{\partial\overset{*}{r}}{\partial Y}\}Y + (\{\overset{**}{r}\} - \{\overset{*}{r}\} - \{\frac{\partial\overset{*}{r}}{\partial Y}\})Y^2 \tag{27}$$

$$\{r\} = \{\overset{*}{r}\} + \{\frac{\partial \overset{*}{r}}{\partial Y}\}Y + (3\{\overset{**}{r}\} - 3\{\overset{*}{r}\} - 2\{\frac{\partial \overset{*}{r}}{\partial Y}\} - \{\frac{\partial \overset{**}{r}}{\partial Y}\})Y^2 + (2\{\overset{*}{r}\} - 2\{\overset{**}{r}\} + \{\frac{\partial \overset{*}{r}}{\partial Y}\} + \{\frac{\partial \overset{**}{r}}{\partial Y}\})Y^3 \quad (28)$$

The derivatives $\{\frac{\partial r}{\partial Y}\}$ can readily be computed by several methods [1]. One possibility is to differentiate Eq. (1) with respect to Y. The result is

$$[K]\{\frac{\partial r}{\partial Y}\} = -[\frac{\partial K}{\partial Y}]\{r\} \quad (29)$$

in which both $\{r\}$, and $[K]$ in the decomposed form of Eq. (4), are known from the analysis. Thus, solution for $\{\frac{\partial r}{\partial Y}\}$ involves only calculation of the right hand side vector of Eq. (29) and forward and backward substitutions.

4. Behavior of Scaled Designs

Scaling of the initial design $\{\overset{*}{X}\}$ to obtain a modified design $\{X_\alpha\}$ is given by

$$\{X_\alpha\} = \alpha\{\overset{*}{X}\} \quad (30)$$

where α is a positive scalar multiplier. If the elements of the stiffness matrix are assumed to be linear functions of the design variables then

$$[K_\alpha] = \alpha[\overset{*}{K}] \quad (31)$$

and the displacements of the modified design are (Eq. (1))

$$\{r_\alpha\} = \frac{1}{\alpha}\{\overset{*}{r}\} \quad (32)$$

The significance of this relation is that a given design $\{\overset{*}{X}\}$ can easily be scaled by modifying α so that any desired displacement be equal to a predetermined value, an operation called scaling of the design. (The line $\alpha\{\overset{*}{X}\}$ is called a design line).

In optimal design problems it is often necessary to find the design $\{X_c\}$ along the design line $\alpha\{\overset{*}{X}\}$, with a displacement r equal to its limiting value r^u. That is (see Fig. 1)

$$\frac{1}{\alpha}\overset{**}{r} = r^u \quad (33)$$

or

$$\alpha = \frac{\overset{**}{r}}{r^u} \quad (34)$$

In cases where $\overset{**}{r}$ is calculated by an approximate behavior model (such as Eq. (15)), we may evaluate the accuracy of the displacements of the approximated design $\{\tilde{X}_c\}$ as follows. $\{\tilde{X}_c\}$ is given by

$$\{\tilde{X}_c\} = \tilde{\alpha}\{\overset{**}{X}\} \quad (35)$$

where $\tilde{\alpha}$ is determined from

$$\frac{1}{\tilde{\alpha}}\overset{\tilde{**}}{r} = r^u \quad (36)$$

or

$$\tilde{\alpha} = \frac{\overset{\tilde{**}}{r}}{r^u} \quad (37)$$

in which $\overset{\tilde{**}}{r}$ is the approximated value of $\overset{**}{r}$ at the point $\{\overset{**}{X}\}$. The approximated displacement of $\{\tilde{X}_c\}$ is $\tilde{r}(\{\tilde{X}_c\}) = r^u$ and the exact displacement at this point is (Eq. (37))

$$r(\{\tilde{X}_c\}) = \frac{1}{\tilde{\alpha}}\overset{*}{r} = \frac{\overset{**}{r}}{\overset{\tilde{**}}{r}}r^u \quad (38)$$

That is, the ratio between the approximated and the exact displacements at $\{\tilde{X}_c\}$ is

$$\frac{\tilde{r}(\{\tilde{X}_c\})}{r(\{\tilde{X}_c\})} = \frac{\overset{\tilde{**}}{r}}{\overset{**}{r}} \quad (39)$$

This result indicates that the error in the approximations at $\{\tilde{X}_c\}$ depends only on the ratio $\overset{\tilde{**}}{r}/\overset{**}{r}$ (and not on $\tilde{\alpha}$, the distance between $\{\overset{**}{X}\}$ and $\{\tilde{X}_c\}$). The combination of approximate behavior models and scaling can be used to introduce efficient optimal design procedures [10]. It will be shown in the next section how selection of the scaling multiplier α may improve the approximate behavior models.

5. Improved Approximations by Scaling

The modified design $\{\overset{**}{X}\}$ can be expressed as (see Fig. 2)

$$\{\overset{**}{X}\} = \{\overset{*}{X}\} + \{\Delta\overset{*}{X}\} = \{X_\alpha\} + \{\Delta X_\alpha\} \quad (40)$$

and the corresponding stiffness matrices are (Eq. (31))

$$[\overset{**}{K}] = [\overset{*}{K}] + [\Delta\overset{*}{K}] = [K_\alpha] + [\Delta K_\alpha] = \alpha[\overset{*}{K}] + [\Delta K_\alpha] \quad (41)$$

Substituting Eq. (41) into Eq. (9), premultiplying by $[\overset{*}{K}]^{-1}$ and rearranging yields

$$([I] + \frac{1}{\alpha}[\overset{*}{K}]^{-1}[\Delta K_\alpha])\alpha\{\overset{**}{r}\} = \{\overset{*}{r}\} \quad (42)$$

Substituting $[\Delta K_\alpha]$ from Eq. (41) into Eq. (42) gives

$$([I] + \frac{1-\alpha}{\alpha}[I] + \frac{1}{\alpha}[\overset{*}{K}]^{-1}[\Delta\overset{*}{K}])\alpha\{\overset{**}{r}\} = \{\overset{*}{r}\} \quad (43)$$

Defining

$$[B_\alpha] \equiv \frac{1-\alpha}{\alpha}[I] + \frac{1}{\alpha}[\overset{*}{K}]^{-1}[\Delta\overset{*}{K}] = \frac{1-\alpha}{\alpha}[I] + \frac{1}{\alpha}[\overset{*}{B}] \quad (44)$$

substituting into Eq. (43) and expanding $([I]+[B_\alpha])^{-1}$, we obtain the following series for $\{\overset{**}{r}\}$ in terms of α (see Eq. (15))

$$\{\overset{**}{r}\} = \frac{1}{\alpha}([I]-[B_\alpha]+[B_\alpha]^2-[B_\alpha]^3+\ldots)\{\overset{*}{r}\} \quad (45)$$

For $\alpha = 1$ we find $[B_\alpha] = [\overset{*}{B}]$ and the series of Eqs. (45) and (15) become equivalent. Different direction vectors $\{\Delta X_\alpha\}$ may be selected in the plane of $\{\overset{*}{X}\}$ and $\{\overset{**}{X}\}$ for various α values. While it is usually difficult to predict which α will provide improved convergence, some possibilities are summarized in Table 1. In cases a,b,c, the value of α is chosen so that the resulting direction $\{\Delta X_\alpha\}$ is perpendicular to $\{\overset{*}{X}\}$, $\{\overset{**}{X}\}$, and to the bisector of angle θ, respectively. The criterion in cases d,e is chosen such that the elements on the principal diagonal of $[\Delta K_\alpha]$ or the second term in the series of Eq. (45) be equal zero. In both cases the multipliers α_i are chosen separately for each displacement. Results obtained for different α values will be compared in the numerical examples of section 7.

The scaling multiplier α affects both the direction vector $\{\Delta X_\alpha\}$ and the step size $|\Delta X_\alpha|$, where (see Eqs. (30) and (40))

$$\{\Delta X_\alpha\} = \{\overset{**}{X}\} - \alpha\{\overset{*}{X}\} \quad (46)$$

The smallest step size is determined by (case Table 1)

$$\alpha\{\overset{*}{X}\}^T\{\Delta X_\alpha\} = 0 \quad (47)$$

or, after rearranging

$$\alpha = \frac{\{\overset{*}{X}\}^T\{\overset{**}{X}\}}{\{\overset{*}{X}\}^T\{\overset{*}{X}\}} \tag{48}$$

and the corresponding step size is

$$|\Delta X_\alpha| = (|\overset{**}{X}|^2 - \alpha^2|\overset{*}{X}|^2)^{\frac{1}{2}} \tag{49}$$

Another parameter which represents the step size for a given direction is the angle θ, where

$$\cos\theta = \frac{\{\overset{*}{X}\}^T\{\overset{**}{X}\}}{|\overset{*}{X}|\ |\overset{**}{X}|} \tag{50}$$

Evidently, for any given direction $\{\Delta X_\alpha\}$ a better convergence will be obtained for smaller θ values.

It is instructive to note that a scalar multiplier β can be chosen instead of α such that (Fig.2)

$$\{X_\beta\} = \beta\{\overset{**}{X}\} \tag{51}$$

Defining

$$[B_\beta] \equiv (\beta-1)[I] + \beta[\overset{*}{B}] \tag{52}$$

we may obtain the series

$$\{\overset{**}{r}\} = \beta([I] - [B_\beta] + [B_\beta]^2 - [B_\beta]^3 + \ldots)\{\overset{*}{r}\} \tag{53}$$

Comparing Eqs. (52),(53) with Eqs. (44),(45), it can be observed that identical results would be obtained by both series if $\alpha = 1/\beta$. In this case the directions $\{\Delta X_\alpha\}$ and $\{\Delta X_\beta\}$ are parallel, and the convergence rate will be identical for any given θ.

6. Convergence Considerations

Problems of slow convergence or divergence may be encountered in applying the series of Eq. (15). The series converges if, and only if [11],

$$\lim_{k\to\infty} [\overset{*}{B}^k] = [0] \tag{54}$$

A sufficient criterion for the convergence of the series is that

$$\|\overset{*}{B}\| \leq 1 \tag{55}$$

where $\|\overset{*}{B}\|$ is the norm of $[\overset{*}{B}]$. It can be shown that

$$\rho([\overset{*}{B}]) \leq \|\overset{*}{B}\| \tag{56}$$

in which $\rho([\overset{*}{B}])$ is the spectral radius of matrix $[\overset{*}{B}]$, defined as the largest eigenvalue $|\lambda_1|$

$$\rho([\overset{*}{B}]) = |\lambda_1| = \max_i |\lambda_i| \tag{57}$$

From Eqs. (55) and (56) we have

$$\rho([\overset{*}{B}]) \leq 1 \tag{58}$$

It should be noted that the existence of a norm such that $\|\overset{*}{B}\| > 1$ does not preclude the convergence of the series.

Some procedures have been proposed to predict the eigenvalue λ_1. One possibility, based on the use of Rayleigh quotient, is [12]

$$\lambda_1 \simeq \frac{\{\overset{*}{r}_k\}^T[\overset{*}{B}]\{\overset{*}{r}_k\}}{\{\overset{*}{r}_k\}^T\{\overset{*}{r}_k\}} = \frac{\{\overset{*}{r}_k\}^T\{\overset{*}{r}_{k+1}\}}{\{\overset{*}{r}_k\}^T\{\overset{*}{r}_k\}} \tag{59}$$

where $\{\overset{*}{r}_k\}$ are the vectors of the series (see Eq. (18)). A better estimation would be obtained for large k values.

Dynamic acceleration

In cases of slow convergence rate dynamic acceleration methods, which make use of previous terms in the series, can be employed. If the iterative process has a slow convergence rate, successive errors will generally exhibit an exponential decay in the later stages of the iteration (see Fig. 3). Aitken's δ^2 process [13] is one approach to predict the asymptotic limit to which the predictions for each displacement r_j is tending. Assume the extrapolation expression

$$\overset{**}{r}_j \simeq a + be^{-kc} \tag{60}$$

where a, b, c are constants and k is the iteration number (or number of terms in the series). The final solution ($k\to\infty$) is determined from the three successive estimates

$$\begin{aligned} r_j^{(k)} &= a + be^{-kc} \\ r_j^{(k+1)} &= a + be^{-(k+1)c} \\ r_j^{(k+2)} &= a + be^{-(k+2)c} \end{aligned} \tag{61}$$

The result is

$$\overset{**}{r}_j \simeq a = \frac{r_j^{(k)}\, r_j^{(k+2)} - (r_j^{(k+1)})^2}{r_j^{(k)} - 2r_j^{(k+1)} + r_j^{(k+2)}} \tag{62}$$

or, alternatively

$$\overset{**}{r}_j \simeq a = r_j^{(k+2)} + s_j(r_j^{(k+2)} - r_j^{(k+1)}) \tag{63}$$

where

$$s_j = \frac{r_j^{(k+1)} - r_j^{(k+2)}}{r_j^{(k)} - 2r_j^{(k+1)} + r_j^{(k+2)}} \tag{64}$$

If Aitken's acceleration is applied at the wrong time the denominator of Eq. (62) could be zero or very small. In such circumstances the method will either fail to yield a prediction or else give a predicted value which is grossly in error. The "wrong time" may be interpreted as either too soon, before an exponential decay is established, or too late when rounding errors affect the predictions.

Jennings[12] proposed a modified version of Aitken's acceleration, used by several authors [14,15]. While the prediction of Eq. (62) is calculated for each displacement r_j separately, a common acceleration parameter is introduced for all variables in the modified method. The result is

$$\{\overset{**}{r}\} = \{r^{(k+2)}\} - \frac{\lambda_1}{1+\lambda_1}(\{r^{(k+2)}\} - \{r^{(k+1)}\}) \tag{65}$$

in which

$$\frac{\lambda_1}{1+\lambda_1} = \frac{(\{r^{(k)}\} - \{r^{(k+1)}\})^T(\{r^{(k+1)}\} - \{r^{(k+2)}\})}{(\{r^{(k)}\} - \{r^{(k+1)}\})^T(\{r^{(k)}\} - 2\{r^{(k+1)}\} + \{r^{(k+2)}\})} \tag{66}$$

or

$$\lambda_1 = -\frac{(\{r^{(k+2)}\} - \{r^{(k+1)}\})^T(\{r^{(k+1)}\} - \{r^{(k)}\})}{(\{r^{(k+1)}\} - \{r^{(k)}\})^2} \tag{67}$$

The convergence rate is governed by the magnitude of λ_1. This method is particularly effective when only one eigenvalue of matrix $[\overset{*}{B}]$ has modulus close to unity. It is possible to apply the acceleration after two or more iterations and to repeat the procedure frequently.

7. Numerical Examples

Ten-bar truss. The truss shown in Fig. 4 is subjected to a single loading condition (all dimensions are in kips and inches) and the initial design is $\{X\} = \{6.0\}$. The following three cases of changes in the design were solved:

Case 1: $\{\Delta\overset{*}{X}\}^T = \{4.0,4.0,4.0,4.0,6.0,6.0,1.2,1.2,1.2,1.2\}$

Case 2: $\{\Delta\overset{*}{X}\}^T = \{-3.0,-3.0,-3.0,-3.0,9.0,9.0,9.0,9.0,9.0,9.0\}$

Case 3: $\{\Delta\overset{*}{X}\}^T = \{18.0,18.0,18.0,18.0,18.0,18.0,36.0,36.0,36.0,36.0\}$

The angles θ for the three cases (Eq. (50)) are 11.3^o, 30^o, and 11.5^o, respectively.
Results obtained for cases 1,2, by Aitken's δ^2 method (Eq. (62)) and the modified acceleration method (Eq. (65)), assuming $k = 2,3,4$, are given in Table 2. While the approximations for case 1 are excellent, some errors can be observed in case 2. The iteration history for the latter case, with Aitken's δ^2 method applied after iterations 4 and 6, is shown in Figs. 5 and 6. It can be seen that no convergence of the vertical displacements could be achieved without Aitken's method. Applying scaling by the five methods of Table 1 combined with Aitken's process ($k = 2,3,4$) for case 2 may improve the convergence, as shown in Table 3. The best results have been obtained by methods b and d. The effect of α on the spectral radius (Eq. (59)) is illustrated in Fig. 7. The divergence for $\alpha=1$ is explained by the relatively large $|\lambda_1|$ value ($|\lambda_1|=1.5$). Assuming $\alpha=1.96$, the value of $|\lambda_1|$ is reduced to 0.75.

The effect of scaling on the convergence is demonstrated in case 3 (Table 4). Applying the modified acceleration method ($k=2,3,4$), no convergence could be obtained for $\alpha=1$. Assuming $\alpha=4.94$ (method c, Table 1), the convergence is fast; a solution very close to the exact one is obtained for three terms in the series. Similar results could be reached with the criterions of methods a,b in Table 1.

Forty-seven-bar truss. The truss shown in Fig. 8 was solved for the following data (all dimensions are in kips and inches):
Initial design $\{X\} = \{0.5\}$.

$\Delta\overset{*}{X}_i = 0.5$ $\quad (i=1,\ldots,8)$

$\Delta\overset{*}{X}_i = 0$ $\quad (i=9,10,27\text{-}30,37\text{-}40,45\text{-}47)$

$\Delta\overset{*}{X}_i = 0.2$ $\quad (i=11\text{-}20,41\text{-}44)$

$\Delta\overset{*}{X}_i = 0.4$ $\quad (i=21\text{-}26)$

$\Delta\overset{*}{X}_i = -0.15$ $\quad (i=31\text{-}36)$

Notation of the displacements is as follows: $r_1, r_3, \ldots$ are the horizontal displacements and $r_2, r_4, \ldots$ are the vertical ones. Results obtained by Aitken's method and the modified acceleration method (in both cases $k=2,3,4$) are given in Table 5. It can be noted that despite the different order of magnitude of the various displacements, relatively small errors have been obtained. To illustrate the effect of the step size Y, the displacements r_{39}, r_{41}, r_{43} have been calculated along the line $\{X\} = \{X\}+Y\{\Delta\overset{*}{X}\}$ for $Y = 0.25, 0.50, 0.75, 1.0$. Results obtained after four iterations ($k=4$) are shown in Fig. 9. The effect of Aitken's δ^2 method on the iteration history for $Y=1.0$ is illustrated in Fig. 10 and the evaluation of $|\lambda_1|$ is demonstrated in Fig. 11. The final value $|\lambda_1| = 0.99$ explains the slow convergence rate of the series.

8. Concluding Remarks

Approximate behavior models for efficient reanalysis have been presented. The algorithms are based on a series expansion which is equivalent to a simple iteration procedure. A single exact analysis is sufficient to introduce the series coefficients and matrix inversion is not required throughout. The proposed approximations might be sufficient only for a limited region, in the neighborhood of the initial design.
Two approaches have been proposed to improve the quality of the approximations:

a) An approach where a scalar multiplier, used for scaling of the initial design, is chosen prior to the solution as the accelerating parameter. Several algorithms for selecting the value of this multiplier are proposed and their potential for improving the series convergence is demonstrated. It is shown how the scaling multiplier affects both the direction vector in the design space and the step size.

b) An approach where the accelerated parameters are determined from results obtained during the solution process. Information gathered during calculations of the series coefficients is used to introduce extrapolation expressions. The two methods of Aitken's δ^2 process and a modified method of Aitken's acceleration are presented. It is shown how these methods provide high quality results in cases of poor convergence rate or divergence of the series.

In the typical problem of reanalyses along a given line in the design space, the methods discussed in this study involve much less computational effort. Multiple reanalyses along a line can efficiently be introduced in terms of a single independent variable.

Acknowledgement

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Table 1: Various Possibilities for the Selection of α

Case	Criterion for Determining α	Condition	α
a	$\{\Delta X_\alpha\}$ perpendicular to $\alpha\{\overset{*}{X}\}$	$\alpha\{\overset{*}{X}\}^T\{\Delta X_\alpha\} = 0$	$\frac{\{\overset{*}{X}\}^T\{\overset{**}{X}\}}{\{\overset{*}{X}\}^T\{\overset{*}{X}\}}$
b	$\{\Delta X_\alpha\}$ perpendicular to $\{\overset{**}{X}\}$	$\{\overset{*}{X}\}^T\{\Delta X_\alpha\} = 0$	$\frac{\{\overset{**}{X}\}^T\{\overset{**}{X}\}}{\{\overset{*}{X}\}^T\{\overset{**}{X}\}}$
c	$\{\Delta X_\alpha\}$ perpendicular to the bisector of θ	$\lvert\alpha\overset{*}{X}\rvert = \lvert\overset{**}{X}\rvert$	$(\frac{\{\overset{**}{X}\}^T\{\overset{**}{X}\}}{\{\overset{*}{X}\}^T\{\overset{*}{X}\}})^{\frac{1}{2}}$
d	$\Delta K_{ii_\alpha} = 0$	$\Delta K_{ii_\alpha} = \overset{**}{K}_{ii} - \alpha_i \overset{*}{K}_{ii} = 0$	$\alpha_i = \frac{\overset{**}{K}_{ii}}{\overset{*}{K}_{ii}}$
e	second term in the series = 0	$(1-\alpha_i)\overset{*}{r}_i + \{\overset{*}{B}_i\}^T\{\overset{*}{r}\} = 0$	$\alpha_i = 1 + \frac{\{\overset{*}{B}_i\}^T\{\overset{*}{r}\}}{\overset{*}{r}_i}$

Table 2: Results (×100), Ten-Bar Truss.

Case	Method	$\overset{**}{r}_1$	$-\overset{**}{r}_2$	$-\overset{**}{r}_3$	$-\overset{**}{r}_4$	$\overset{**}{r}_5$	$-\overset{**}{r}_6$	$-\overset{**}{r}_7$	$-\overset{**}{r}_8$
1	Eq. (62)†	0.086	0.440	0.094	0.453	0.071	0.207	0.073	0.219
	Eq. (65)†	0.086	0.439	0.094	0.453	0.071	0.207	0.073	0.220
	Exact	0.086	0.440	0.094	0.453	0.071	0.207	0.073	0.219
2	Eq. (62)†	0.290	0.813	0.310	0.824	0.237	0.284	0.243	0.294
	Eq. (65)†	0.275	0.818	0.295	0.827	0.225	0.278	0.231	0.287
	Exact	0.290	0.876	0.310	0.887	0.237	0.303	0.243	0.313

† k = 2,3,4.

Table 3: Results (×100), Ten-Bar Truss Case 2 Eq. (62)† for Various α Values.

Case††	α	$\overset{**}{r}_1$	$-\overset{**}{r}_2$	$-\overset{**}{r}_3$	$-\overset{**}{r}_4$	$\overset{**}{r}_5$	$-\overset{**}{r}_6$	$-\overset{**}{r}_7$	$-\overset{**}{r}_8$
a	1.70	0.291	3.041	0.309	19.000	0.237	0.194	0.243	0.214
b	2.26	0.292	0.878	0.309	0.889	0.237	0.304	0.243	0.314
c	1.96	0.291	0.915	0.309	0.929	0.237	0.335	0.243	0.350
d	separate	0.290	0.877	0.310	0.887	0.237	0.303	0.243	0.313
e	separate	0.290	0.756	0.310	0.757	0.237	0.222	0.243	0.217
Exact	-	0.290	0.876	0.310	0.887	0.237	0.303	0.243	0.313

† k = 2,3,4

†† See Table 1.

Table 4: Effect of Scaling, Ten-Bar Truss, Case 3 (×100)

j	α = 1		α = 4.90		Exact Solution
	$r_j^{**(4)}$	Eq. (65)†	$r_j^{**(4)}$	Eq. (65)†	
1	9.928	-0.762	0.035	0.035	0.035
2	-155.200	-0.150	-0.137	-0.137	-0.137
3	-8.372	0.859	0.040	0.040	0.040
4	-157.600	-0.019	-0.143	-0.143	-0.143
5	7.561	-0.632	0.029	0.029	0.029
6	-94.430	-0.870	-0.056	-0.056	-0.056
7	-7.079	0.665	-0.031	-0.031	-0.031
8	-97.040	-0.756	-0.061	-0.061	-0.061

† k= 2,3,4

Table 5: Results 47-Bar Truss (×100)

j	r_j^*	r_j^{**}			j	r_j^*	r_j^{**}		
		Eq. (62)†	Eq. (65)†	Exact			Eq. (62)†	Eq. (65)†	Exact
5	1.34	1.31	1.31	.31	25	3.63	4.07	4.04	4.04
6	0.32	0.24	0.24	0.24	26	-4.51	-3.82	-3.84	-3.84
7	1.56	1.49	1.49	1.49	27	3.17	3.61	3.57	3.57
8	-1.21	-1.19	-1.19	-1.19	28	-0.02	-0.78	-0.78	-0.78
9	3.87	3.50	3.50	3.50	29	0.99	2.20	2.16	2.16
10	0.14	0.08	0.08	0.08	30	-12.94	-9.84	-9.83	-9.85
11	4.04	3.73	3.72	3.73	31	0.19	1.62	1.59	1.58
12	-1.96	-1.92	-1.92	-1.92	32	5.94	3.19	3.19	3.20
13	6.33	5.79	5.79	5.79	33	-4.78	-1.64	-1.63	-1.67
14	-0.56	-0.85	-0.85	-0.86	34	-27.64	-19.78	-19.64	-19.78
15	6.74	6.39	6.38	6.39	35	-3.89	-1.15	-1.16	-1.18
16	-2.25	-2.28	-2.28	-2.28	36	12.04	-9.84	-9.83	-9.85
17	7.76	7.48	7.47	7.48	37	-3.01	-0.65	-0.68	-0.69
18	-2.26	-2.71	-2.71	-2.71	38	-4.95	-4.14	-4.15	-4.15
19	7.73	7.43	7.42	7.43	39	-2.56	-0.20	-0.24	-0.24
20	-1.48	-1.50	-1.50	-1.50	40	0.42	-0.46	-0.46	-0.46
21	6.42	6.00	5.98	5.98	41	-2.56	-0.20	-0.24	-0.24
22	-3.29	-3.21	-3.22	-3.22	42	5.94	3.19	3.19	3.20
23	6.58	6.08	6.06	6.06	43	-2.56	-0.20	-0.24	-0.24
24	-0.90	-1.21	-1.21	-1.21	44	11.45	6.84	6.84	6.85

† k = 2,3,4.

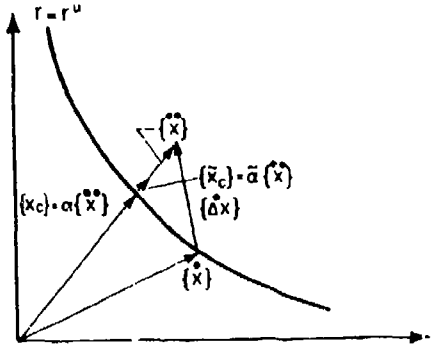


FIG. 1

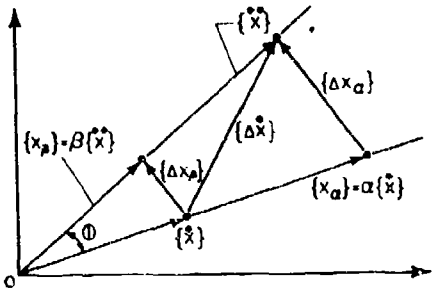
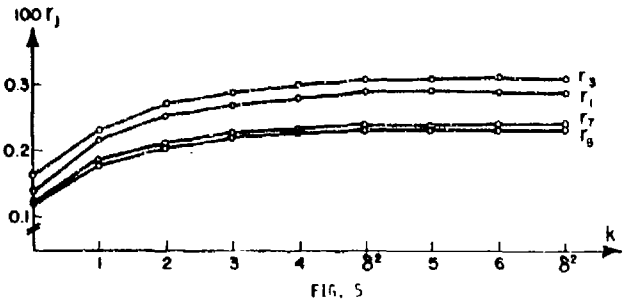


FIG. 2



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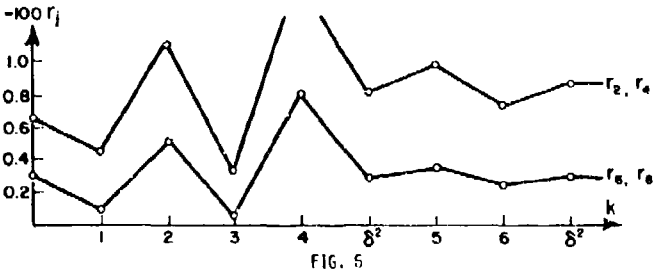
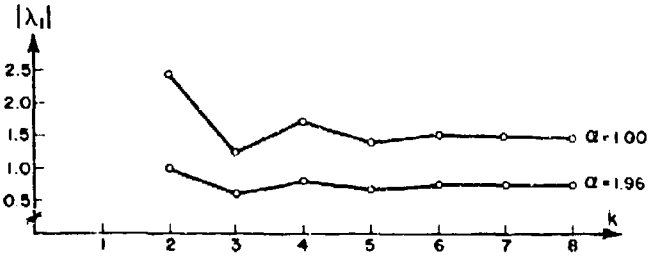


FIG. 6



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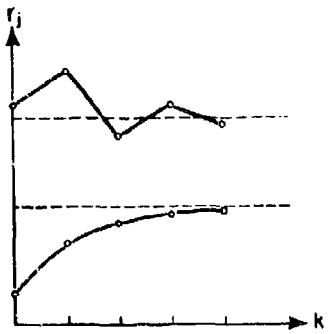

FIG. 3

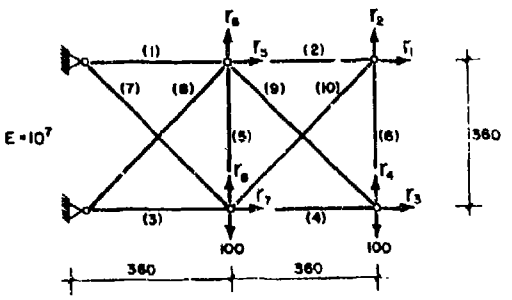

FIG. 4

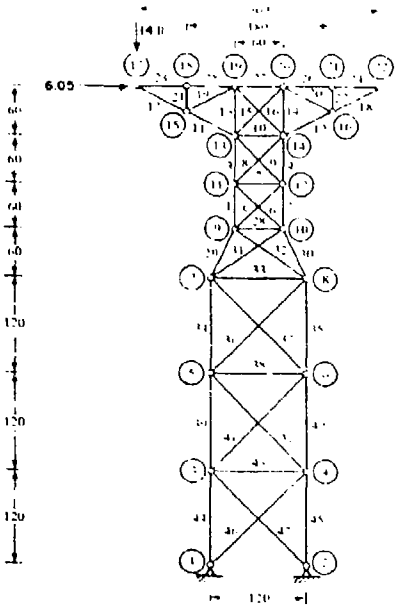

FIG. 8

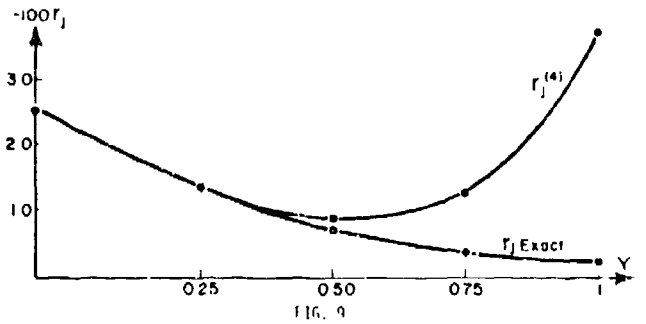

FIG. 9

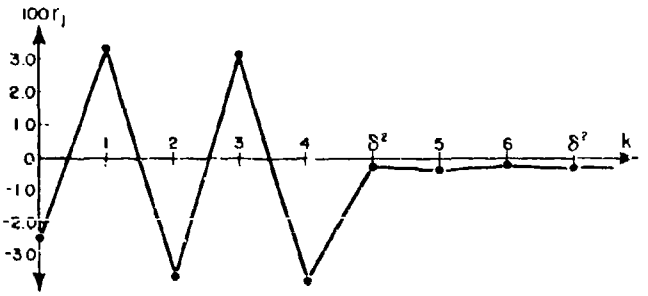

FIG. 10

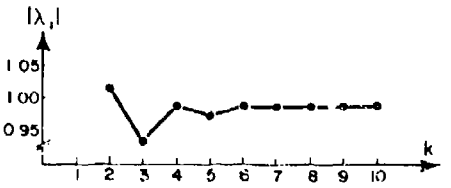

FIG. 11