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Division 8, National Defense Research Committee

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PROGRESS REPORT ON "THEORY OF SHOCK WAVES"

to August 31, 1942

by John von Neumann
Institute for Advanced Study
Princeton, New Jersey

Report OSRD No. 1140

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DIVISION 8*

NATIONAL DEFENSE RESEARCH COMMITTEE

OF THE

OFFICE OF SCIENTIFIC RESEARCH AND DEVELOPMENT

Progress Report on "Theory of Shock Waves"
to
August 31, 1942
by

John von Neumann
Institute for Advanced Study
Princeton, New Jersey

Report OSRD No. 1140

Date: January 29, 1943

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DIVISION 8*

NATIONAL DEFENSE RESEARCH COMMITTEE

OF THE

OFFICE OF SCIENTIFIC RESEARCH AND DEVELOPMENT

Section 8.1

Progress Report on "Theory of Shock Waves"

Service Directives OD-02 and OD-03

Endorsement (1) Dr. J. G. Kirkwood, Member of Division 8, to G. B. Kistiakowsky, Chief of Division 8. Forwarding report and noting:

> "In this report the basic mathematical problems of the theory of shock waves in compressible fluids are formulated and discussed in an illuminating fashion. Specific results obtained by von Neumann and his collaborators are discussed from the standpoint of the general theory. Details of the theory have been presented in part in previous OSRD reports by the author. Further details are promised in future reports."

(2) from G. B. Kistiakowsky, Chief Division 8, to Dr. Irvin Stewart, Executive Secretary of the National Defense Research Committee. Forwarding report and concurring with above endorsement.

This is a Final Report under Contract OEMsr-218 with the Institute for Advanced Study, Princeton, New Jersey.

*Division 8 is composed of former Sections B-1 and B-2.

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THEORY OF SHOCK WAVES

by John von Neumann,
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Princeton, New Jersey

Abstract

The basic mathematical problems of the theory of shock waves in compressible fluids are formulated and discussed. Specific results obtained are considered from the standpoint of the general theory. The material treated is the origin of explosions and the propagation of their effects. Terminal problems -- that is, problems of damage -- are not considered.

The topics included are the conservation laws and the differential equation; the role of entropy, vorticity, and the Riemann invariants; natural boundary conditions (the need for discontinuities); the conservation laws and the discontinuities; formulation of the basic problems of discontinuities; the origin of shock; the interaction of shocks (linear and oblique cases); classification of reaction shocks; and analysis of detonation. "Reaction shocks" is the term used for shock waves frequently denoted as "detonation waves."

I. INTRODUCTION

1. This report is concerned with theoretical work on various gas dynamical questions, partly of a rather general character, but are all related to the theory of explosions and the transmission of their blasts.[1/] The problems that arise in this field are numerous and of varying nature, but almost all lead up to the study of discontinuous changes of state in compressible substances, the so-called shock waves, or briefly shocks. The theoretical work done was, therefore, in the main an investigation of shocks, their origin, their interaction, and their study under various conditions.

2. Shocks are possible in any compressible substance, and under the conditions in and around an explosion all known substances must be regarded as

1/ That is, the origin of explosions and the propagation of their effects. Terminal problems, that is, problems of damage, are not considered.

compressible. Hence shocks should be investigated in gases, liquids, and solids.

Now the essential medium for the shock in a progressing explosion consists of its burnt gas products, while the most important media for the propagation of the shock (blast) after the explosion are air and water.

The propagation of blasts under water is being investigated by J. G. Kirkwood and others [see Ref. (d)] and accordingly our investigations were restricted to the first two topics, and so to shock waves in gases.[2/]

3. A shock may or may not alter the nature (that is, the equation of state) of the substance through which it passes. The latter is the case for blast waves. We shall call such shocks, which pass through a (chemically) inert substance, pure shocks. The former is the case for detonation waves, which as they pass induce the explosive chemical reaction. It is therefore customary to call this type of shock waves detonation waves. It is preferable, however, to talk of detonations only in a strictly technical sense. We shall therefore call all shocks of the first type, which induce chemical reactions, reaction shocks.

Thus the subject is subdivided into the theory of pure shocks and the theory of reaction shocks.

4. This report gives only the general outline of the problems considered and the results obtained. The details are given in several informal reports, of which two, Ref. (j), (k), have already been submitted, and several will be submitted in the future. These latter reports had to be delayed for the following reason. They are closely connected with other investigations, both experimental and theoretical, not under this contract, although connected with it. It appeared desirable -- in some cases necessary -- to wait for the completion of certain phases of that work.

2/ The propagation of an explosion in a solid or liquid explosive is prima facie a shock between that medium and a gas. But it will appear later that it is in the main behaving as a shock in a gas.

II. THE CONSERVATION LAWS AND THE DIFFERENTIAL EQUATION

5. Pure shocks, that is, discontinuous changes of the physical state where no chemical change is involved, are possible in a substance to the extent to which its compressibility is noticeable but its heat conductivity and viscosity are negligible. The properties of a compressible substance are expressed by its <u>caloric equation of state</u>, which gives its <u>specific inner energy</u> (inner energy per unit mass) $\underline{E}$ as a function of its <u>density</u> ρ, or its <u>specific volume</u> $v[=1/\rho]$, and the hydrostatic <u>pressure</u>,

$$E = F(p,v). \tag{1}$$

It is more convenient, however, to use the specific entropy (that is, entropy per unit mass) $\underline{S}$ instead of the pressure $\underline{p}$, and to express $\underline{E}$ in terms of $\underline{v}$ and $\underline{S}$,

$$E = E(S,v). \tag{2}$$

Expressions for the pressure $\underline{p}$ and the temperature $\underline{T}$ follow from Eq. (2):

$$p = -\frac{\partial E}{\partial v}; \quad \text{that is,} \quad p = p(S,v); \tag{3}$$

$$T = \frac{\partial E}{\partial S}; \quad \text{that is,} \quad T = T(S,v); \tag{4}$$

and Eq. (1) is obtained by eliminating $\underline{S}$ between Eqs. (2) and (3).

If the substance characterized by Eq. (2) is nonconductive (for heat) and nonviscous, then Eqs. (2) and (3) contain all we need to describe its behavior -- both thermic and mechanic. The differential equations by which it is governed obtain by a direct application of the <u>conservation laws</u>: of <u>mass</u>, of <u>momentum</u>, and of <u>energy</u>.

6. <u>First some formal preparations</u>. The special coordinates form a vector $X = (x,y,z)$. The state of the substance at $X = (x,y,z)$ and at the time $\underline{t}$ is given by the <u>mass velocity</u> vector $U = (u,v,w)$, and, as pointed out in the preceding section, by the specific volume $\underline{v}$ and the specific entropy $\underline{S}$.

We use vector notations.[3/] Now the total differential operator is:

(5) $$D = \frac{\partial}{\partial t} + U \cdot \nabla .$$

The statements of the conservation laws are:

Mass: $$\frac{\partial}{\partial t}\rho + \nabla \cdot (\rho U) = 0, \qquad \left(\rho = \frac{1}{v}\right)$$

Momentum: $$DU = -v\nabla p,$$

Energy: $$D[\tfrac{1}{2}(U \cdot U) + E] = -v\nabla \cdot (pU).$$

By a simple computation these give the Eulerian differential equations:

(A) $$Dv = v(\nabla \cdot U),$$

(B) $$DU = -v\nabla p,$$

and

$$DE = -pDv.$$

The last equation can be written

$$\frac{\partial E}{\partial S} DS + \left(\frac{\partial E}{\partial v} + p\right) Dv = 0;$$

that is, by Eqs. (3) and (4),

$$TDS = 0,$$

or

(C) $$DS = 0.$$

3/ For two vectors $A[=(a,b,c)]$, $L[=(\ell,m,n)]$, we have the scalar product

$$A \cdot L = a\ell + bm + cn$$

and the vector product

$$A \times L = (bn - cm, c\ell - an, am - bn).$$

Besides we have the differentiation or Nabla vector operator

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).$$

Thus

$$\text{grad } f = \nabla f, \quad \text{div } A = \nabla \cdot H, \quad \text{rot } A = \nabla \times A.$$

Equations (A) to (C), in conjunction with Eq. (3), which expresses $\underline{p}$ in terms of $\underline{S}$, $\underline{v}$, are then our equations. Note that Eqs. (A) and (C) are scalar equations, while Eq. (B) is vectorial. So we have five (differential) equations for the five dependent variables $\underline{v}$, $\underline{u}$, $\underline{v}$, $\underline{w}$, $\underline{S}$ as it should be.

III. THE ROLE OF ENTROPY

7. The differential equations (A) to (C) have a number of well-known peculiarities, which it is appropriate to mention at this point.

The path of an individual element of substance is defined by the differential equations,

$$\frac{d}{dt} X = U. \tag{6}$$

The differential equations (A) to (C) specify the total differential $\underline{D}$ of the five dependent variables $\underline{v}$, $\underline{u}$, $\underline{v}$, $\underline{w}$, $\underline{S}$, that is, the rates of change along the paths of Eqs. (6). The statement is particularly simple for Eq. (C), where this rate of change is zero. Thus Eq. (C) states that $\underline{S}$ is constant along each path (6).

If $\underline{S}$ happens to be constant on some three-dimensional surface[4/] which all paths (6) intersect -- for example, at all points with a certain $t = t_o$ -- then the above statement implies that it is an absolute constant. In this case, therefore, Eq. (C) may be replaced by

$$S = S_o \qquad (S_o \text{ a constant}). \tag{C^1}$$

Note that the condition which is required for the validity of Eq. (C^1) -- constancy of $\underline{S}$ on a suitable three-dimensional surface -- is in the nature of a boundary condition. That is, it may be satisfied in consequence of a suitable boundary condition, and on the other hand a boundary condition may perfectly well conflict with Eq. (C^1), and thereby remove the implication of Eq. (C^1) by Eq. (C).

These observations are of importance, because they show that Eq. (C^1) is not an integral of the differential equations (A), (B), and (C), although

---

4/ In the four-dimensional space-time of $\underline{x}$, $\underline{y}$, $\underline{z}$, $\underline{t}$.

it looks like one. An integral is an equation that follows from the differential equations under all conditions, while Eq. (C^1) obtains only when suitable boundary conditions are assigned. We call such an equation a <u>pseudo</u> <u>integral</u>.

8. The pseudo integral Eq. (C^1), to the extent to which it is valid, allows us to express p as a function of v by means of Eq. (3),

$$(7) \qquad p = \phi(v) \qquad [\phi(v) = p(S_0,v)].$$

Equation (7) has the appearance of an equation of state, but it can be regarded as such only in a very limited sense. Indeed (i) the validity of Eq. (7) is dependent upon the very restricted validity of the pseudo integral (C^1); (ii) even when valid, Eq. (7) contains the constant S_0 which is not determined by the nature of the substance [whereas Eqs. (1) to (4) are], but arbitrarily assigned by the boundary conditions.[5/]

In certain cases, however, Eq. (7) becomes an equation of state in the true sense. This occurs, when $p(S,v)$ does not depend on S. According to Eq. (3) this is equivalent to assuming that Eq. (2) has the form

$$(2^1) \qquad E = E(S,v) \equiv A(S) + B(v).$$

Then Eqs. (3) and (4) become

$$(3^1) \qquad p = -\frac{\partial E}{\partial v} = -\frac{\partial}{\partial v} B(v),$$

$$(4^1) \qquad T = \frac{\partial E}{\partial S} = \frac{\partial}{\partial S} A(S);$$

that is, pressure and specific volume on the one hand and temperature and specific entropy on the other form two pairs, such that the members of each pair determine each other directly without any interference from the other pair. The energy is simply additive with respect to the contributions of these two pairs, that is, there is no interaction energy between them.

J. G. Kirkwood and H. Bethe have shown [Ref. (d), I, pp. 17 to 19] that this assumption is reasonably verified under the conditions of

---

5/ We are, of course, describing the peculiar relationship of the <u>adiabatic</u> <u>law</u> -- expressed by Eq. (7) -- to the equation of state.

underwater blasts. Thus the validity or invalidity of Eq. (2^1) corresponds to a certain extent to the division between liquids and gases.[6/]

Although our interest is, as stated before, with shocks in gases, it will prove useful to keep the possibility of Eqs. (2^1) to (4^1) in mind.

9. To conclude this subject, for the time being, we observe this. When Eqs. (2^1) to (4^1) hold, then Eqs. (A) and (B) form a closed system, not involving $\underline{S}$ at all. When $\underline{v}$, $\underline{u}$, $\underline{\mathcal{v}}$, $\underline{w}$ are obtained from Eqs. (A) and (B), then Eq. (C) yields, as a secondary operation, $\underline{S}$. In other words:

> When Eqs. (2^1) to (4^1) hold, then the conservation laws of mass and momentum [that is, Eqs. (A) and (B)] suffice to determine everything except the specific entropy $\underline{S}$. The conservation law of energy [that is, Eq. (C)] then determines $\underline{S}$: it states, as in the general case, that $\underline{S}$ is constant along each path (6).

IV. VORTICITY AND THE RIEMANN INVARIANTS

10. Equations (A) to (C) possess further well-known pseudo integrals. Their validity, however, is even more conditional than that one of Eq. (C^1). Specifically, they depend on the validity of the $\underline{S}$ pseudo integral -- that is, on the possibility of inferring Eq. (C^1) from (C); or rather, on the existence of a fixed relation

$$p = \phi(S), \tag{7}$$

which, as we saw, holds in the general case only when Eq. (C^1) does, but in the special case, Eqs. (2^1) to (4^1), also without Eq. (C^1).

6/ For an ideal gas

$$RT = pv, \quad E = \frac{R}{\gamma - 1} T$$

and

$$S = \frac{R}{\gamma - 1} \ln (p, v^{\gamma}),$$

where

$$\frac{R}{\gamma - 1} = c_v.$$

Consequently Eq. (2) becomes

$$E = E(S,v) \equiv \frac{1}{\gamma - 1} v^{-(\gamma-1)} e^{\frac{\gamma-1}{R} S}.$$

This is the opposite extreme from Eq. (2^1).

Thus we assume now the validity of an Eq. (7) for all x, y, z, t. This entails the consequences pointed out in Par. 8 for the special case given by Eqs. (2¹) to (4¹): we need only consider Eqs. (A), (B), and v, u, $\mathcal{v}$, w -- Eq. (C) and S have no influence on the results in that sphere.

11. A simple computation, based on Eqs. (A) and (B) alone, without using Eq. (7), gives

$$D[v(\nabla \times U)] = -v(\nabla v \times \nabla p). \tag{8}$$

Now Eq. (7) gives

$$\nabla p = \frac{\partial \phi}{\partial v} \nabla v$$

so that the vectors ∇p and ∇v are parallel, and consequently $\nabla v \times \nabla p = 0$. Then Eq. (8) becomes

$$D[v(\nabla \times U)] = 0. \tag{9}$$

This brings about the same situation for $v(\nabla \times U)$ as was observed for S in Par. 7: $v(\nabla \times U)$ is constant along each path (6), and if it happens to be constant on a suitable three-dimensional surface -- for example, for a certain $t = t_0$ -- then it is an absolute constant, that is, then Eq. (9) becomes

$$v(\nabla \times U) = V_0 \qquad (V_0 \text{ a constant vector}). \tag{10}$$

Thus Eq. (10) is also a pseudo integral; but it depends not only on the usual boundary-condition properties, but also on the validity of Eq. (7) [see Par. 10].

The quantity $v(\nabla \times U)$ occurring in Eq. (10) is the specific vorticity vector (vorticity per unit mass; $\nabla \times U$ is the vorticity per unit volume).

12. Being a vector equation, Eq. (10) really comprises three pseudo integrals. However, if the physical problem under consideration has really two, or even one, dimension instead of three -- that is, if everything depends only on the coordinates x, y, or even only on the coordinate x -- then this number is reduced. Indeed, in the two-dimensional case only

one component of $v(\nabla \times U)$ is not identically zero -- the z-component -- and in the one-dimensional case none. So we see that if the physical problem under consideration has three, two, one dimensions, then Eq. (10) stands for three, one, zero pseudo integrals, respectively.

In the last mentioned case, the one-dimensional case where $v(\nabla \times U)$ fails completely, there exist however two other pseudo integrals. They are dependent on Eq. (7) [see Par. 10] just like $v(\nabla \times U)$, but their paths are different from (6). They have no analogues for three and two dimensions.

These integrals obtain as follows. Using Eq. (7), define[7/]

$$c = c(v) = \sqrt{-\frac{d\phi}{dv}}\, v, \tag{11}$$

$$\omega = \omega(v) = \int \sqrt{-\frac{d\phi}{dv}}\, dv, \tag{12}$$

where c is the velocity of sound (relative to the substance), while the interpretation of ω is not so simple. Now assume that everything depends on x alone. Then a simple computation, based on Eqs. (A), (B), and (7), gives

$$\left[\frac{\partial}{\partial t} + (u \mp c)\frac{\partial}{\partial x}\right](u \pm \omega) = 0. \tag{13}$$

The form of Eq. (13) suggests the introduction of the characteristics defined by

$$\frac{d}{dt}x = u \mp c \tag{14}$$

in place of the paths (6). Now we have the same situation for $u \pm \omega$ and Eq. (14) as was observed for $v(\nabla \times U)$ and (6) in Par. 11: $u \pm \omega$ is constant along each characteristic (14), and if it happens to be constant on a suitable three-dimensional surface -- for example, for a certain $t = t_0$ -- then it is an absolute constant. That is, then Eq. (13) becomes

$$u + \omega = a_0 \quad \text{or} \quad u - \omega = b_0 \qquad (a_0,\ b_0 \text{ constants}). \tag{15}$$

7/ $-\frac{d\phi}{dv} > 0$, since ϕ, that is, p, decreases when v increases.

Thus Eq. (15) does indeed furnish two more pseudo integrals, which again depend not only on the usual boundary-condition properties, but also on the validity of Eq. (7) [see Par. 10].

The quantities $u \pm \omega$ occurring in Eq. (13) are the Riemann invariants.

13. Summary. There exist several pseudo integrals, S, $v(\nabla \times U)$, $u \pm \omega$ -- the specific entropy, the specific vorticity, and (in one dimension only) the Riemann invariants. In three, two, one dimensions these are four, two, three pseudo integrals.

The importance of these pseudo integrals in solving the differential equations (A) to (C) is well known:

(i) When S is constant, we have a relation (7), with many useful applications, one of which is the emergence of the other pseudo integrals.

(ii) When $v(\nabla \times U)$ is constant, the possibility with the widest applications is that it is zero. Then $\nabla \times U = 0$, and this means that there exists a velocity potential, that is, a scalar function $\phi = \phi(x,y,z,t)$ with $U = \nabla \phi$.

(iii) When either $u \pm \omega$ is constant, then an explicit relation between u and v obtains, considerably facilitating the determination of the solution. When both $u \pm \omega$ are constant, then u and v are immediately known.

These techniques are familiar in the literature, so we need not go into detail.

We wish, however, to point out this: while S has a certain precedence over the other pseudo integrals [see (i) above or Par. 10], all these pseudo integrals operate in the main in the same way. This will become even more conspicuous when we begin to study the influence of discontinuities. All the foregoing pseudo integrals will be affected in the same, characteristic way.

It is important to keep this in mind, because S, $v\nabla \times U$, $u \pm \omega$, are quantities of very different physical nature, and hardly ever classified or visualized together. They belong nevertheless together, and this insight helps considerably in understanding the role of discontinuities.

V. NATURAL BOUNDARY CONDITIONS. THE NEED FOR DISCONTINUITIES

14. Every physical problem that is governed by differential equations possesses what may be called its natural boundary conditions, that is, conditions under which one can expect by ordinary physical intuition, by commonsense, that one and only one solution must exist.

In such a case the mathematical verification of this intuitive assertion ought to be possible. In fact, one of the most effective criteria for the appraisal of the value and finality of a mathematical formulation of a physical problem is just this: whether it provides one and only one solution for natural boundary conditions.

In the gas dynamical problem governed by the differential equations (A) to (C), examples of such natural boundary conditions are easy to find. A "box" of a prescribed shape C_t, changing with time t, provides one. We may prescribe the state of the substance in C_o for $t = 0$, and that it follow the changing shape C_t for all $t > 0$. Specifically:

(i) For $t = 0$ and $X = (x,y,z)$ in the interior of C_o, the quantities v, U, S have given values.

(ii) For $t > 0$ and $X = (x,y,z)$ on the boundary of C_t, the component of U normal to C_t at X is equal to the normal velocity of C_t at X.[8]

If the present mathematical setup of the theory is to be regarded as really satisfactory, then it should secure one and only one solution of Eqs. (A) to (C) with conditions (i) and (ii) for any family of C_t.

The problem in this general form is of extreme difficulty. However, if the v, U, S in condition (i) are assigned constant values, then it simplifies greatly: obviously all pseudo integrals S, $v(\nabla \times U)$,[9] $u \pm \omega$,[10] become available.

8/ Since we assume the substance to be nonviscous, we must allow for gliding along the boundary of C_t.

9/ For three or two dimensions. The constancy of U implies, of course, that $v(\nabla \times U) = 0$.

10/ For one dimension.

15. The discussion of an arbitrary family C_t has been carried out in the literature for the one-dimensional case, with the following result.

When the motion of the boundary of C_t is generally receding (that is, expanding the substance in its interior), then there exists a unique solution. An exception must be made for the case when recession of C_t is too fast (considerably supersonic), but this is satisfactorily explained by the physical consideration that in such a case the substance will not follow all changes of the boundary of C_t, but form a <u>free surface</u> in the interior.

When the motion of the boundary of C_t is anywhere advancing (that is, compressing the substance in its interior), then there exists no solution. The motion of C_t may be perfectly regular, even analytical; the difficulty persists nevertheless. In fact, if the velocities of C_t are always continuous, then there exists a unique solution for a certain time: it is only a finite time after the advancing (compressive) motion of C_t has begun, and at a finite distance in the interior of C_t, that the solution breaks down.

This breakdown of the continuous behavior of the substance, governed by the differential equations (A) to (C), is well attested by experiments: in a compressible substance every compressive influence produces states that exhibit all symptoms of discontinuity -- to the extent to which conductivity and viscosity can be disregarded. In this way the <u>pure shocks</u> come into existence.

Thus the theory based on Eqs. (A) to (C) is incomplete. Account must be taken of the possibilities of free surfaces and of discontinuities. The free surfaces, however, affect only the boundary conditions, but not the differential equations (A) to (C). They, therefore, do not interest us any further. The discontinuities, on the other hand, upset the mechanism of Eqs. (A) to (C), and for this reason it is necessary to give them our attention.

VI. THE CONSERVATION LAWS AND THE DISCONTINUITIES CLASSIFICATION

16. The simplest possible discontinuity consists of a surface $\mathscr{S}$ in space, such that v, U, S are continuous on both sides of $\mathscr{S}$, but (possibly) discontinuous when crossing $\mathscr{S}$.

Consider a point $X = (x,y,z)$ on $\mathscr{S}$ (all this at a definite time t), and the element of $\mathscr{S}$ around X. Denote the two sides of $\mathscr{S}$ by 1 and 2, and the corresponding values of v, U, S, p, E (at x, y, z, t) by v_1, U_1, S_1, p_1, E_1, and v_2, U_2, S_2, p_2, E_2. Denote the normal of $\mathscr{S}$, that is, a vector of unit length, orthogonal to $\mathscr{S}$ (at x, y, z, t), with the orientation $1 \longrightarrow 2$, by n. The surface $\mathscr{S}$ may be moving; denote its normal velocity (at x, y, z, t, in the direction n) by s.

We must now state the laws that replace the differential equations (A) to (C) at this discontinuity. These are based on the same physical principles from which Eqs. (A) to (C) obtained in Par. 6: the conservation laws of mass, momentum, and energy.

It is convenient to introduce the mass flow μ: the mass which crosses $\mathscr{S}$ in the direction of $1 \longrightarrow 2$ (that is, n) per unit surface per unit time.

The statements of the conservation laws are:

Mass: $(U_1 \cdot n) - s = \mu v_1, \quad (U_2 \cdot n) - s = \mu v_2;$

Momentum: $\mu(U_1 - U_2) = -(p_1 - p_2)n;$

Energy: $\mu[\frac{1}{2}(U_1 \cdot U_1) + E_1 - \frac{1}{2}(U_2 \cdot U_2) - E_2] = -[p_1(U_1 \cdot n) - p_2(U_2 \cdot n)].$

By simple computations these yield the following equations.

When $p_1 \neq p_2$, the Rankine-Hugoniot equations:

(A_s) $$\mu = \pm\sqrt{\frac{p_1 - p_2}{v_2 - v_1}},$$

(B_s) $$U_1 - U_2 = \pm\sqrt{(p_1 - p_2)(v_2 - v_1)}\,n,$$

The signs in the two formulae must $\left\{\begin{matrix}\text{disagree}\\ \text{agree}\end{matrix}\right.$ when $p_1 \gtrless p_2$

(C_s) $$E_1 - E_2 = \tfrac{1}{2}(p_1 + p_2)(v_2 - v_1).$$

When $p_1 = p_2$, the contact discontinuity equations:

(A_c) $$\mu = 0,$$

(B_c) $$(U_1 \cdot n) = (U_2 \cdot n),$$

(C_c) $$p_1 = p_2.$$

There is no need to discuss these equations in detail: Eqs. (A_s) to (C_s) have received sufficient attention in the literature, and Eqs. (A_c) to (C_c) are fairly trivial. We restrict ourselves to the following observations:

(i) $\underline{s}$ can now be expressed with the help of the original conservation law of mass;

(ii) the discontinuity of $\underline{U}$ [that is, $U_1 - U_2$] is normal to $\underline{\mathcal{f}}$ in the first case [use Eq. (B_s)], and tangential to it in the second case [use Eq. (B_c)];

(iii) the two cases are also characterized by $\mu \neq 0$ or $\mu = 0$, that is, by the presence or absence of a mass flow across the discontinuity surface $\underline{\mathcal{f}}$.

17. The circumstance that we wish to emphasize is this: although Eq. (A_s) to (C_s) and (A_c) to (C_c) are based on the same physical principles as Eqs. (A) to (C) -- the conservation laws of mass, momentum, and energy (see Par. 6 and Par. 16) behave nevertheless in an entirely different manner with respect to the pseudo integrals $\underline{S}$, $v(\nabla \times U)$, $u \pm \omega$.

Consider first $\underline{S}$ and Eqs. (A_s) to (C_s). Combining Eq. (C_s) with Eq. (2), Eq. (3) gives

$$\frac{E(S_1,v_1) - E(S_2,v_2)}{v_1 - v_2} = \frac{1}{2}\left[\frac{\partial E}{\partial v}(S_1,v_1) + \frac{\partial E}{\partial v}(S_2,v_2)\right]. \tag{16}$$

Now this equation shows, that $v_1 \rightarrow v_2$ implies $S_1 \rightarrow S_2$, that is, that if the $\underline{v}$-discontinuity is small, then the $\underline{S}$-discontinuity is also small. Indeed, it can be shown that $S_1 - S_2$ is third order in $v_1 - v_2$. [See, for example, Ref. (a), p. 8.] But in general $S_1 \neq S_2$ when $v_1 \neq v_2$. Bethe has shown [Ref. (a), pp. 10 to 12], that if the substance has an equation of state (2) fulfilling a few plausible requirements, then Eq. (16) implies

$$S_1 \gtrless S_2 \quad \text{for} \quad v_1 \lessgtr v_2, \text{ respectively.} \tag{17}$$

It is easy to verify these assertions for an ideal gas, using the formulae given in footnote 6.

So we see that while $\underline{S}$ remains constant along the paths (6) of the substance as long as we have the continuous regime given by Eqs. (A) to (C), this fails to be the case in the discontinuous regime in which Eqs. (A_s) to (C_s) hold. Also, if $\underline{S}$ is constant on one side of $\underline{\mathcal{S}}$, even this will not in general be true on the other side, unless $\underline{\mathcal{S}}$ is plane and moving with the same velocity everywhere.

Thus $\underline{S}$ ceases to be a pseudo integral as soon as a discontinuity $\underline{\mathcal{S}}$ satisfying Eqs. (A_s) to (C_s) is crossed -- but this disturbance is a third-order effect if the discontinuity at $\underline{\mathcal{S}}$ is small.

Considering their dependence on the pseudo-integral character of $\underline{S}$, the quantities $v(\nabla \times U)$, $u \pm \omega$, cannot be pseudo integrals either. The disturbance is again a third-order effect if the discontinuity at $\underline{\mathcal{S}}$ is small.

The failure of $v(\nabla \times U)$ to be a pseudo integral in this situation has, among others, this consequence. Even if conditions are constant on one side of $\underline{\mathcal{S}}$, and hence $\nabla \times U$ vanishes (see footnote 8), $\nabla \times U$ will be nonvanishing on the other side of $\underline{\mathcal{S}}$ unless $\underline{\mathcal{S}}$ is plane, cylindrical, or spherical. That is, a discontinuity surface of unsymmetric nature produces vorticity. [See Ref. (c), pp. 362 to 369.]

18. Before we go any further, let us give some more attention to the fact that $\underline{S}$ changes at the crossing of a discontinuity surface. In the older literature of the subject this caused considerable confusion. [See, for example, Ref. (c), pp. 189 to 207, including Ref. to Sébert and Hugoniot.]

The situation is this: Eq. (C) states that the specific entropy of an individual element of substance never changes in the course of its continuous motion, that is, that this motion remains always <u>thermodynamically reversible</u>. Now Eqs. (A) to (C) expressed only the conservation laws of matter, momentum, and energy. Hence the computation which gave Eq. (C) its present form, really proved this: for a compressible, nonconductive, nonviscous substance the conservation of matter, momentum, and energy implies also that of entropy -- that is, thermodynamic reversibility -- as long as the motion is continuous.

The result given in inequality (17) then proves that this implication no longer holds good when this motion (or rather its $\underline{v}$, $\underline{U}$, $\underline{p}$, $\underline{E}$) becomes discontinuous. This is very odd. The implication of one conservation law by another one is usually an algebraical fact which should not be affected by such differences. But it is nevertheless so.

Consequently the <u>entropy theorem</u>, which took care of itself in the continuous case, must be given special consideration in the discontinuous case. The entropy must not decrease during the motion of an individual element of substance. That is, for $\mu \gtrless 0$ we must forbid $S_1 \gtrless S_2$, respectively -- that is, by inequality (17) we must forbid $v_1 \lessgtr v_2$. This means that never $\mu(v_1 - v_2) < 0$. Now a simple consideration based on Eqs. (A_s), (B_s), and inequality (17) yields this.

The entropy theorem requires that the sign + be always used in Eq. (B_s). That is, the sign $\pm$ must be used in Eq. (A_s) for $p_1 \lessgtr p_2$, that is, for $v_1 \gtrless v_2$.

If this condition is fulfilled, we call $\underline{\mathcal{Y}}$ a <u>positive shock</u>; if it is not, a <u>negative shock</u>. Hence positive shocks alone are permissible.

As mentioned above, this change of $\underline{S}$ in a shock was questioned in the older literature. Doubts were expressed as to whether the conservation of energy, that is, Eq. (C_s), should not be sacrificed rather than the conservation of entropy. The latter amounts to Eq. (7), that is, to

$$(18) \qquad p_1 = \phi(v_1), \quad p_2 = \phi(v_2)$$

and Eq. (C_s) and Eq. (18) are generally conflicting.[11/] The question arose as to which of these two adiabatic laws of footnote 11 should be considered valid.

11/ Thus for an ideal gas (see footnote 6) putting

$$\frac{p_2}{p_1} = \xi, \quad \frac{v_2}{v_1} = \eta,$$

Eq. (18) is the well-known <u>ordinary adiabatic law</u>,

$$\xi = (\eta)^{-\gamma},$$

while Eq. (C_s) is the <u>Rankine-Hugoniot adiabatic law</u>

$$\xi = \frac{(\gamma + 1) - (\gamma - 1)\eta}{(\gamma + 1)\eta - (\gamma - 1)}.$$

There can be no doubt that it is Eq. (C_s): the energy must be conserved, and entropy must only not decrease. The irreversibility of Eqs. (A_s) to (C_s) is odd but not at all absurd. All continuity arguments used in the literature are invalid. The (irreversible) discontinuities $\underline{\mathcal{S}}$ are not limiting forms of (reversible) continuous motions, since no compressive motion can remain continuous.12/

There is, however, one addendum to this. If Eq. (7) -- that is, Eq. (18) -- holds, because the equation of state has the special form Eqs. (2^1) to (4^1) discussed in Par. 8, then its validity is absolute. Now in this case we saw in Par. 9 that the motion of the substance is governed by Eqs. (A) and (B) alone, while Eq. (C) stands apart. It determines only the behavior of $\underline{S}$. Similarly, Eqs. (A_s), (B_s), and (7) -- that is, Eq. (18) -- may then be used to determine the motion of the substance, and Eq. (C_s) stands apart, dealing with $\underline{S}$ only. That is, the motion is determined in each case as if there were no conservation of energy, and by using Eq. (7) -- that is, Eq. (18). But the energy is, of course, conserved -- by conserving the entropy according to Eq. (C) in the continuous case, and by changing it appropriately according to Eq. (C_s) in the discontinuous one.

19. Consider next Eqs. (A_c) to (C_c). In this case no substance crosses the discontinuity [see (iii) in Par. 16]; hence there arise no questions in connection with the pseudo integrals $\underline{S}$, $v(\nabla \times U)$. In the one-dimensional case, the pseudo integrals $u \pm \omega$ may have to be treated differently on the two sides of $\underline{\mathcal{S}}$, but this does not lead to any serious difficulties either.

The following point, however, is worth emphasizing. There exists here a fundamental difference between the one-dimensional case, and the three- and two-dimensional ones.

In the first case only $\underline{v}$ can be discontinuous at $\underline{\mathcal{S}}$, since here Eq. (B_c) implies $U_1 = U_2$. Since $\underline{p}$ is continuous by Eq. (C_c), this involves by Eq. (3) a discontinuity in $\underline{S}$ -- that is, different adiabatic laws [Eq. (7)]

12/ The discontinuities $\underline{\mathcal{S}}$ are limiting forms of continuous motions, if the substance is endowed with a small conductivity, or viscosity, and this allowed to tend to zero. Such considerations corroborate the increase of entropy in $\underline{\mathcal{S}}$, although this aspect of the subject has not been studied quite exhaustively. [See, for example, Ref. (g), pp. 587 to 607.]

on both sides of $\mathcal{S}$. This implies that when there is an absolute reason for the validity of Eq. (7) -- that is, when the equation of state has the special form Eqs. (2^1) to (4^1) discussed in Par. 8 -- then this kind of discontinuity cannot occur. But this is true in one dimension only!

In the second case $\underline{v}$ may again be discontinuous at $\mathcal{S}$, but Eq. (B_c) allows also any discontinuity of the component $\underline{U}$ tangential to $\mathcal{S}$, that is, we may have gliding of the two sides along $\mathcal{S}$ (see footnote 8). Now it is well known that this type of discontinuity is the equivalent of a vorticity sheet.[13/]

It follows that we must expect such a discontinuity to originate where there is reason to expect the creation of a concentrated form (sheet) of vorticity. Now it appeared at the end of Par. 17 that a discontinuity surface $\mathcal{S}$ of the type satisfying Eqs. (A_s) to (C_s), when of unsymmetric nature produces vorticity. There $\mathcal{S}$ was continuously curved and accelerated, and the vorticity created was continuously disturbed. Hence if the curvature or the acceleration of $\mathcal{S}$ is concentrated on an infinitesimal stretch -- that is, if $\mathcal{S}$ has an edge or corner, or if it has to undergo a discontinuous change in velocity -- then a vorticity sheet may be expected. Thus a discontinuity satisfying Eqs. (A_c) to (C_c) may be expected to originate where a discontinuity of the type satisfying Eqs. (A_s) to (C_s) exhibits any one of the above traits.

In one dimension a similar argument could be made, by using $\underline{S}$ instead of $v(\nabla \times U)$ -- and this alternative is effective in three or two dimensions also. However, as we observed further above, the special form given by Eqs. (2^1) to (4^1) of the equation of state excludes discontinuities of the type satisfying Eqs. (A_c) to (C_c) in one dimension, but not in three or in two dimensions.

VII. FORMULATION OF THE BASIC PROBLEMS OF DISCONTINUITIES

20. By comparison of these facts with the difficulties pointed out in Par. 15, it appears reasonable to try the theory in a new form, which

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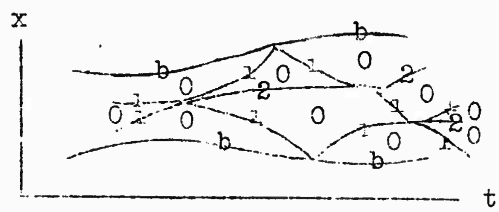
13/ Like $\mathcal{S}$, it is two-dimensional in the three-dimensional case, and one-dimensional in the two-dimensional one.

allows for discontinuities of the two types, those satisfying Eqs. (A_s) to (C_s) and those satisfying Eqs. (A_c) to (C_c), besides the areas in which the differential equations (A) to (C) are fulfilled.[14/]

In other words the four-dimensional x,y,z,t-space-time must be divided by three-dimensional surfaces $\underline{\mathcal{S}}$, $\underline{\mathcal{S}'}$, $\underline{\mathcal{S}''}$, ..., into distinct domains $\underline{A}$, $\underline{A'}$, $\underline{A''}$, ... In each one of these domains there is continuity, the differential equations (A) to (C) being valid. The separating interfaces $\underline{\mathcal{S}}$, $\underline{\mathcal{S}'}$, $\underline{\mathcal{S}''}$, ... represent discontinuities, either of the <u>first kind</u>, that is, satisfying Eqs. (A_s) to (C_s), or of the <u>second kind</u>, that is, satisfying Eqs. (A_c) to (C_c).

From the remarks of Par. 15 we conclude further that the interfaces of the first kind may begin in the interior of the $\underline{A}$, $\underline{A'}$, $\underline{A''}$, ... domains, with free (two-dimensional) edges. From the remarks of Par. 19 interfaces of the second kind should begin only at (two-dimensional) edges formed by two already existing interfaces of the first kind.

In the two-dimensional case space-time is three-dimensional, all the above dimensions are reduced by one, and so the words domain, surface, edge assume their usual geometric meaning -- making things easier to visualize. In the one-dimensional case space-time is two-dimensional; all the above dimensions are reduced by two, and we can even give a schematic drawing of the conditions to be expected (Fig. 1).



b, boundaries of C_t
0, areas A, A', A", ...
1, interfaces $\mathcal{S}$, $\mathcal{S}'$, $\mathcal{S}''$, ... of the first kind
2, interfaces $\mathcal{S}$, $\mathcal{S}'$, $\mathcal{S}''$, ... of the second kind

Fig. 1.

In applying Eqs. (A_s) to (C_s) to the interfaces of the first kind it is also necessary to remember the conclusion of Par. 18, according to which only positive shocks are allowed.

14/ The free surfaces, mentioned at the beginning of Par. 15 are really special cases of Eqs. (A_c) to (C_c) with $p_1 = p_2 = 0$ and with zero density ($1/v = \rho = 0$) on the empty side.

21. The considerations of Par. 20 are of a highly heuristic nature; the conclusions reached are only surmises. The mathematical corroboration would consist of showing that the present formulation of our problem has always one and only one solution when natural boundary conditions are prescribed. This would necessitate giving a definition of what a natural boundary condition is that is in harmony with physical intuition and sufficiently general to include all plausible situations. As a preliminary check, however, the special setup of the "box" C_t as discussed in Pars. 14 and 15 should be analyzed.

The simplest possible case of this setup has been solved in the literature: one dimension, constant values and rest at $t = 0$ (see the end of Par. 14), C_t semi-infinite, its one boundary point at rest at $x = 0$ for $0 < t \leqq t_0$ and then set into motion with a discontinuous change of velocity for $t > t_0$; $x = u_0(t - t_0)$.

For $u_0 < 0$ this is an expansive motion; for $u_0 > 0$ it is a compressive one. In the first case there exists one, and only one, solution with no discontinuities. In the second case no such solution exists, but there exists one, and only one, with a discontinuity of the first kind beginning at the boundary point $x = 0$, $t = t_0$. This is a positive shock. A similar discontinuous solution would exist in the first case only if negative shocks, too, were allowed.

So we see that it is necessary to allow positive shock discontinuities in order to have at least one solution in each natural problem. It is necessary to forbid negative shock discontinuities in order to have no more than one solution. This takes care of the discontinuities of the first kind. The discontinuities of the second kind are presumably necessary in order to be able to continue the solutions beyond the edges formed by discontinuity surfaces of the first kind — that is, their intersections (see Par. 20 and Fig. 1).

Thus the setup arrived at for partly thermodynamic reasons is also plausible from a purely mechanical point of view.

22. For a more general motion

$$x = f(t) \tag{19}$$

of the boundary point of C_t, and for the case when C_t is finite and has two boundary points, only very fragmentary results exist. A good deal can be predicted qualitatively -- but the properly mathematical theory is extremely incomplete.

Assuming, as one should, that the boundary velocities in Eq. (19) are continuous, that is, that $\frac{df}{dt}$ is continuous, the discontinuities must be expected to begin in the interior, and not on the boundary. (See Pars. 15 and 20 and Fig. 1.)

Before any exhaustive mathematical theory can be attempted, it is necessary to acquire an insight into the nature of the various elementary constituents which combine to give the complex picture presented, for example, on Fig. 1. The matters to be considered are therefore these:

(i¹) How does a discontinuity surface begin in the interior?

(ii¹) How do two discontinuity surfaces intersect; that is, what phenomena originate at such an intersection edge?

As we saw in Par. 20, it seems probable that the primary discontinuities, originating according to (i¹), are of the first kind -- those of the second kind should come from (ii¹). [For vortex sheets this was proved in Ref. (c), pp. 355 to 361.] Combining this with the observations made subsequently, (i¹) can be modified as follows:

(i¹¹) How does a discontinuity surface of the first kind -- a positive shock -- begin in the interior, if $\frac{df}{dt}$ is continuous and compressive.15/

Since there is no flow of matter across a discontinuity of the second kind [see (ii) in Par. 16], two such discontinuities cannot intersect. So we must have at least one discontinuity of the first kind in (ii¹). When this intersects one of the second kind, there arises a problem which we need not consider in the framework of this first orientation. In some cases it is quite easy to solve, and in the others it is essentially equivalent to a special case of the next case. The last case, intersection of

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15/ Compressivity means that the acceleration of the boundary is directed toward the substance -- that is, for a lower boundary point in x, df/dt increases; for an upper boundary point in x, df/dt decreases.

two discontinuities of the first kind, is the really interesting one. Combining these observations with the conclusions of (ii) in Par. 20, we come to replace (ii[1]) by this statement:

> (ii[11]) How do two discontinuity surfaces of the first kind intersect, that is, what phenomena originate at such an intersection edge? In particular: how do the discontinuities of the second kind begin there?

VIII. THE ORIGIN OF A SHOCK

23. The mathematical approach to (i[11]) is very difficult because the shock $\mathcal{S}$ will be accelerated, and the problem is of determining $\mathcal{S}$ together with the solution of Eq. (A) -- a quite unusual type of mixed differential equation unknown boundary problem. It is possible, however, to determine the point X_0 where the shock $\mathcal{S}$ begins, and the conditions in the neighborhood of that point. They are singular, and the description of this singularity is the problem.

The existing literature on this question is unsatisfactory, partly because the apparent conflict between the conservation laws for energy and entropy were usually not treated properly. [See the last part of Par. 18, and Ref. (c), pp. 207 to 217.]

If df/dt is continuous, but d^2f/dt^2 is allowed to be discontinuous, we obtain a situation which is typified by $f(t) = 0$ for $0 < t \leqq t_0$ and $f(t) = a_0(t - t_0)^2$ for $t > t_0$. This is the case that was usually considered in the literature. The solution in the above sense was completely determined by J. Calkin in connection with the contract under which this report was written. A detailed report on this subject will be submitted shortly.

If d^2f/dt^2 is also continuous, and df/dt increasing, then the shock originates under entirely different conditions. This was established by J. Calkin. A report on the details of this case -- which are rather unexpected -- will follow.

The first setup -- d^2f/dt^2 discontinuous -- can never be anything but an approximation. It would be a useful one if its result approximated that of the second setup -- d^2f/dt^2 continuous. Since it does not, but the second case has a qualitatively different solution, we conclude that the first setup must be rejected. That is, the solution of the second setup gives the desired answer to (i^{11}).16/

24. A variant of (i^{11}) which deserves consideration is the following. Consider an arrangement, whereby the "box" [that is, its $f(t)$] is compressed for $t > t_0$, as discussed in Par. 23, but only during a finite time interval $t_0 < t < t_1$, and brought to rest again for $t \geqq t_1$. It is known that this initiates a positive shock in the interior, as described before, but that the shock will lose intensity subsequently owing to the expansive motion necessitated by bringing $f(t)$ to rest. This phenomenon is mathematically most difficult.

Now let the interval $t_0 < t < t_1$ be very short, but the motion of $f(t)$ during this period very violent. One may try to arrange the data so that this motion injects into the substance an energy $e_0 (> 0, < \infty)$ and then make $t - t_0 \rightarrow 0$, while the value of e_0 is held fixed. This amounts to injecting a fixed amount of energy $e_0 (> 0, < \infty)$ into the substance during an infinitesimally brief period.

The problem is of a certain practical interest since it is equivalent to describing the decay of a very violent, instantaneously originated, blast wave in air.

It was solved -- in three and in two dimensions, as well as in one -- in a report submitted by the author previously in connection with this contract [Ref. (j)].

The procedure used there has since found applications in some other problems of similar nature. [See, for example, Ref. (i), and the author's report on "boosting," mentioned at the end of Par. 38.]

16/ In fact even the first setup may lead to a solution which belongs to the second type if $f(t)$ for $t > t_0$ is of the form $a_0(t - t_0)^2 + b_0(t - t_0)^3 + c_0(t - t_0)^4 + \ldots$ with any one of $b_0, c_0 \ldots$ sufficiently great in comparison to a_0.

IX. THE INTERACTION OF SHOCKS: LINEAR CASE

25. Let us now consider (ii^{11}) in Par. 22, that is, the intersection of two discontinuity surfaces of the first kind. This may also be described as the collision of two positive shocks.

The physical picture is that of two shocks moving into a domain of continuity and getting into contact with each other. In order to have as elementary a setup as possible, one may imagine that $\underline{v}$, $\underline{U}$, $\underline{S}$ are constant in the domain ahead of both shocks, and also (although with other values) in the domains behind the two shocks. The shocks are then plane, and have constant velocities.

In the one-dimensional case the two shocks move in opposite or in parallel directions. In the latter case it can be shown that the shock which is behind the other one (in their common direction of motion) must be faster than the forward one and finally catch up with it.[17/] Thus the two shocks must collide in each case, and they are not in contact before that collision. This is the linear collision of two shocks.

In the three- or two-dimensional case the conditions are the same if the two shock fronts (discontinuity surfaces) are parallel planes. We choose their common normal as the $\underline{x}$-axis and everything is obviously independent of $\underline{y}$,$\underline{z}$. We still have linear shocks.

Assume now that they are not parallel. Choose the plane containing their two normals as the $\underline{x}$,$\underline{y}$-plane. Then everything is still independent of $\underline{z}$ and so the problem is two dimensional. In this case the two shock fronts intersect at all times. That is, the two shocks have been in contact -- collision -- all along. This is the oblique collision of two shocks.

Summary. (ii^{11}) is the problem of the collision of two positive shocks. This problem is either linear, one dimensional, in which case

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17/ In this case the three domains are not as indicated above but are a domain ahead of the first shock, a domain between the two shocks, a domain behind the second shock. The second one disappears as the two shocks catch up.

the collision occurs at a definite instant; or it is oblique, two dimensional, in which case the collision is going on continuously at all times.

26. We add a remark concerning the possibility of a discontinuity surface running into a boundary, that is, the case of reflection.

Let us again assume that the discontinuity surface and the boundary are planes. Then the influence of the boundary is equivalent to what would be the influence of a mirror image of the original discontinuity surface, reflected by the wall. That is, reflection is equivalent to the collision of two symmetric discontinuity surfaces. Hence our discussions of Par. 25 apply again, and reflections, too, can be subdivided into linear reflections (one-dimensional) and oblique reflections (two-dimensional).

27. Let us return to the collisions, and consider first the linear type.

We are in one dimension; therefore we must expect at the point of collision, among other things, the beginning of a discontinuity of the second kind -- except when the equations of state have the special form of Eqs. (2^1) to (4^1). It is also easy to see that this discontinuity of the second kind must disappear for reasons of symmetry if the two colliding discontinuity surfaces are symmetric.

The problem has been solved fully when the substance is an ideal gas with $\gamma \leqq 5/3$. The only further phenomena originating at the collision are these: two positive shocks if the two colliding shocks are in opposite directions; one positive shock if they are in the same direction. Apart from these, and from the discontinuity of the second kind mentioned above, the substance has no discontinuities and obeys the differential Eqs. (A) to (C).

These results form the content of a report to be submitted by the author.

We restate this result. Two positive shocks in a linear head-on collision produce two positive shocks; if one catches up with the other, then they produce one positive shock.

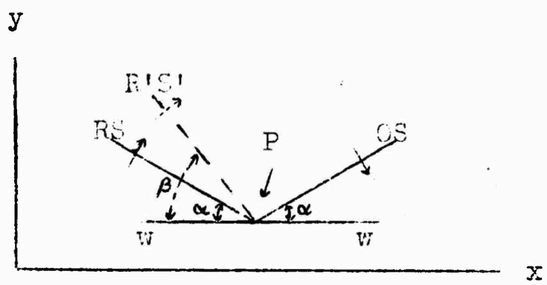
It would be interesting to determine for which equations of state this result is generally true. This would involve investigations along the lines of Bethe's work [Ref. (a), mentioned in Par. 17]. For ideal gases the condition is, as mentioned above, $\gamma \leq 5/3$. It seems remarkable that this inequality, which is justifiable by molecular-kinetic considerations, emerges here in a purely macroscopic context.

We add that some of the older literature on this subject is in error because of a failure to recognize the role of the discontinuities of the second kind.

X. THE INTERACTION OF SHOCKS: OBLIQUE CASE

28. We now pass to the collisions of the oblique type. For the sake of simplicity, we discuss the symmetric case, that is, that one of oblique reflection (see Par. 26). In this case the discontinuity of the second kind does not arise, and there are some minor technical simplifications. But the characteristic difficulties of the problem, which we are going to consider now, are essentially the same as in the general case.

Consider first the oblique reflection of a very weak shock, that is, a sound wave. In this case the original shock and the wall produce a second, reflected shock which forms the same angle with the wall as the original one (Fig. 2). [This is x,y-space not x,y,t-space time. We have pointed out before that this problem is essentially two dimensional.]



w---w, wall
OS, original shock
RS, reflected shock (sonic case)

Fig. 2.

If the original shock is not sonic, there will be complications -- but it is easy to predict their nature. The gas behind the original shock is easily seen to move in the direction of that shock -- hence it has a component to the right in Fig. 2 -- and to have a higher sound velocity than the gas ahead of the original shock. Hence the reflected shock must be expected to be faster, even in relation to the original one, than it would be in the sonic case. That is, it will be pushed forward to a position like R'S' on Fig. 2, that is, its angle β with the wall will exceed the angle of the original (or the sonic-reflected) shock with the wall.

It is natural to make this exact, by applying Eqs. (A_s) to (C_s) to these two shocks. The quantities v, U, S are constant ahead of the original shock.[18/] It is also natural to try to make v, U, S constant in the domain between the two shocks, and similarly in the domain behind the second shock.[17/]

If this is done the same number of equations and variables obtain, but the equations are of a high algebraic order. It is found that these equations can be solved, unless the angle α is too near to $\pi/2$.[19/] However, there exist then two solutions of the type R'S'.[20/] While it is usually possible to tell which of these two is the physically real one, this duplicity is nevertheless somewhat disquieting.

But when α is near to $\pi/2$ -- that is, for a nearly glancing incidence -- the situation becomes even stranger: there exists no solution.[21/]

Attempts to find a solution by other, more complicated arrangements of plane shocks have invariably failed. The reality of the phenomenon is, however, beyond question. The existence of an "abnormal" type of reflection for strong shocks and nearly glancing incidence has been established experimentally by E. Mach [Ref. (f)].

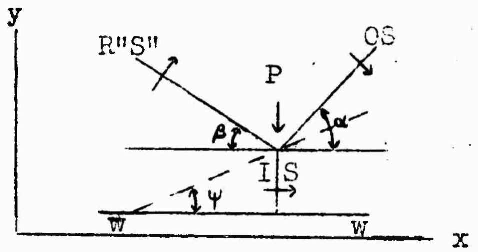
18/ Of course, $U = 0$ ahead of the original shock. There is no reason to restrict U in the domain between the two shocks. Behind the reflected shock, U must be parallel to the wall.

19/ The weaker the original shock, the nearer α may come to $\pi/2$. Of course $\alpha = \pi/2$ itself must be excluded. In this case no reflection occurs at all.

20/ If the shock is very weak, then one solution has its β near to α, the other near to $\pi/2$. The first one yields a weak shock, the second one a strong shock. The physically realized case is therefore the first one -- except possibly for some very special situations.

21/ This phenomenon was observed before on a similar problem by Epstein [Ref. (b)]. In the case of oblique reflection it was first mentioned by E. Teller (oral communication to the author).

29. The experimental evidence, [Refs. (f), (m)] is sufficient to establish qualitatively the number and the nature of the shocks that intervene in this "abnormal" reflection (Fig. 3). Thus a new shock, the intermediate shock IS enters into the picture. The original shock and the reflected shock meet at a point P, which is no longer at the wall, as in the "normal" reflection of Fig. 2, but moving in the interior. The experiments show, furthermore, that P is moving into the interior, away from the wall, along the dashed line of Fig. 3.



w---w, wall
OS, original shock
R"S", reflected shock
IS, intermediate shock

Fig. 3.

If the mathematical analysis is now applied, the following facts appear.

(i) The reflected shock near P and the intermediate shock must be curved.

(ii) For weak shocks, at least, $\beta < \alpha$ and not $\beta > \alpha$ as in the "normal" reflection of Fig. 2.

Therefore we must expect a rather complicated motion of the substance behind the reflected and the intermediate shocks, which has vorticity -- that is, neither S nor $v(\nabla \times U)$ constant. Besides, a discontinuity of the second type -- a vorticity sheet -- should issue from P into the same domain.

All this leads to very difficult mathematical problems, even for ideal gases. Assuming a strong shock, and a very nearly glancing incidence -- that is, $\alpha \sim \pi/2$ -- approximate solutions can be determined: "zero" order quite easily, "first" order with considerable difficulty. They corroborate in detail the qualitative statements made above.[22]

22/ Since this phenomenon is not stationary it is necessary to discuss it from the beginning -- where the original shock first hits the wall. Owing to the obliqueness of the reflection this necessitates (see Par. 25) some changes in the geometry of the picture.

These investigations, for ideal gases, are contained in a report which will be submitted shortly by the author.

A direct comparison of the results of the mathematical analysis with the experiments has only been possible so far to a limited extent. Consider very nearly glancing incidence -- that is, $\alpha \sim \pi/2$. Denote the velocity of the original shock by <u>s</u>, the mass velocity of the gas behind it by <u>u</u>, and the velocity of sound there by <u>c</u>. Then

$$\text{tg}\,\psi \longrightarrow \frac{\sqrt{c^2 - (s-u)^2}}{s} \quad \text{for } \alpha \longrightarrow 0. \tag{20}$$

This formula appears to be in reasonable agreement with the experiments [Ref. (m)].

30. The experiments as well as the mathematical analysis show that the intermediate shock <u>IS</u> is very flat as long as the velocity of the original shock does not exceed about 3 times sound velocity. They also show that even for shocks which are less than 10 percent above sound velocity, $\underline{\alpha}$ can deviate as much as $\pi/3$ from $\pi/2$ before the intermediate shock <u>IS</u> disappears and the reflection becomes normal. In this respect recent experiments of Charters and Thomas, Ballistic Research Laboratory, Aberdeen Proving Ground, are particularly convincing.

As the intensity of the (original) shock increases, the intermediate shock seems to become more and more convex. There are reasons to believe that this convexity may progress to the extent of giving the intermediate shock the character of a protuberance when the original shock has 10 to 20 times sound velocity, as it may in explosions. This phenomenon, if real, may be connected with some important blast effects. It was studied further in several memoranda of the author to the Navy Bureau of Ordnance [Ref. (ℓ)].

XI. REACTION SHOCKS. CLASSIFICATION

31. A reaction shock involves a chemical change; that is, the equation of state, Eq. (2) -- and with it Eqs. (3) and (4) -- is expressed by different functions on both sides of the discontinuity. That is, the

conservation laws of mass, momentum, and energy may be formulated in the same way as in Par. 16, but they will contain two different functions:

(21) $$\bar{E}_1 = E_1(S_1,v_1), \quad \bar{E}_2 = E_2(S_2,v_2),$$

The difference between the two functions in Eq. (21) expresses the chemical change.

The results of Par. 16 still apply, if this proviso is made. Thus we have again two types of discontinuities: those described by Eqs. (A_s) to (C_s), and those described by Eqs. (A_c) to (C_c). The conclusions (i), (ii) of Par. 16 are also still valid.

It follows that no flow of matter occurs across the discontinuities of the second kind [Eqs. (A_c) to (C_c)]. Hence there is really no chemical reaction in this case. Two chemically different substances are contiguously existing, separated by the discontinuity surface. Actually this is the normal form for the coexistence of two phases, since the difference in the equations of state -- hence in Eq. (3) -- prevents continuity of all of $\underline{v}$, $\underline{S}$, $\underline{p}$.

For the discontinuities of the first kind [Eqs. (A_s) to (C_s)], on the other hand, there is a flow of matter across the discontinuity. A chemical reaction is therefore the substratum of this picture, and the picture is only legitimate to the extent to which this reaction can be treated as instantaneous.

Summary. A discontinuity (reaction shock) of the first kind describes a chemical reaction, to the extent to which it can be treated as instantaneous, which must be induced by one of the discontinuities ($\underline{p}$, $\underline{T}$, $\underline{U}$?) accompanying the shock.

A discontinuity of the second kind involves no reaction at all; it describes the normal form of co-existence of two different phases.

32. It follows from the above that the really interesting objects for further study are the reaction shocks which are discontinuities of the first kind, governed by Eqs. (A_s) to (C_s). We shall therefore restrict ourselves to these.

Inspecting Eqs. (A_s) to (C_s) once more, it appears that p_1, v_1 and p_2, v_2 are linked by Eq. (C_s) only. Of course, the equations of state, Eq. (2), expressed by Eq. (21), are then better replaced by Eq. (1), expressed by

$$E_1 = F_1(p_1, v_1), \quad E_2 = F_2(p_2, v_2). \tag{22}$$

If Eq. (C_s) is fulfilled, then Eqs. (A_s) and (B_s) can be used to determine the other quantities which are of interest.

Assuming that the state of the substance into which the reaction shock is penetrating -- say that on the side 2 -- is known, we have this situation: p_2, v_2 are known; p_1, v_1 are linked by Eq. (C_s).

This connection of p_1, v_1 can be depicted by a curve in the $\underline{p}, \underline{v}$-plane, the Rankine-Hugoniot curve. It should be remembered that this curve depends on the choice of p_2, v_2.

Obviously $p_1 = p_2$, $v_1 = v_2$ fulfills Eq. (C_s) only when the two functions $F_1(p,v)$ and $F_2(p,v)$ are identical -- that is, when we have a pure shock. In other words: the point p_2, v_2 lies on the Rankine-Hugoniot curve only when there is no reaction -- for a pure shock.

For an exothermic reaction, that is, $F_1(p_1, v_1) > F_2(p_1, v_1)$ [not $F_2(p_2, v_2)$!], it is easy to verify that p_2, v_2 lies below the Rankine-Hugoniot curve. The conditions are shown in Figs. 4 and 5.

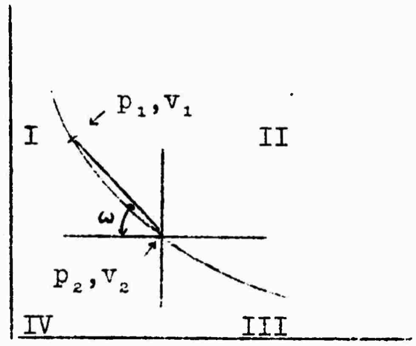


Fig. 4. Pure shock.

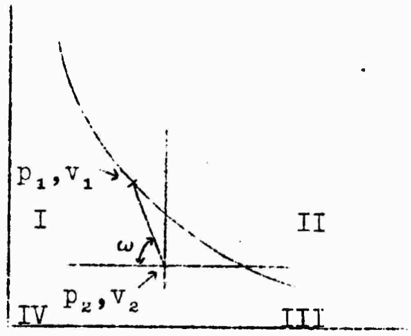


Fig. 5. Reaction shock.

33. By Eq. (B_s)

(23) $$\mu = \sqrt{\frac{p_1 - p_2}{v_2 - v_1}} = \sqrt{tg\,\omega};$$

hence $tg\,\omega \geqq 0$. Consequently ω must lie in the quadrants I or III. For a pure shock this is automatically true (Fig. 4), but for a reaction shock it excludes a certain part of the curve, which lies in quadrant II (Fig. 5).

Besides this, for a pure shock the lower part of the curve -- in quadrant III -- is clearly a negative shock. We saw that these must be forbidden since they would cause a decrease of entropy (see Par. 18). Hence only the curve in quadrant I has reality.

In the case of a reaction shock the situation is different. First, the thermodynamics of the chemical reaction which is involved here would have to be gone into in considerable detail before anything could be excluded on thermodynamic grounds. Second, there is definite evidence as to the reality of at least part of the curve in quadrant III in this case. Third, we saw in Par. 20 that there is no known application of the theory where pure shocks in quadrant III (that is, negative shocks) are needed to produce a solution, while we shall see that they are definitely necessary in the main problem involving reaction shocks (see Par. 36).

34. The parts I and III of the Rankine-Hugoniot curve of a reaction shock are distinguished by simple criteria. In the former the reaction increases p and $\rho = 1/v$ and it is easy to show that the shock velocity s exceeds the sound velocity c_2 of p_2, v_2. In the latter all this is reversed.

For an explosive reaction part I is undoubtedly describing states of detonation, while it is customary, and probably justified, to identify the states described by part III with those of burning or deflagration.[23] At any rate we are going to use these expressions in the sense indicated.

23/ The variable and somewhat erratic behavior of actual deflagration makes the latter identification less certain than the former one.

We can say, therefore, that in a detonation the pressure and density of the reacted substance are higher than those of the unreacted one, and the detonation is faster than any sonic signal that might precede it -- that is, no such signal can precede it. In a deflagration all this is reversed.

One also concludes easily from the above [or from Eq. (B_s)] that in a detonation the burnt gases follow in the same direction as the detonation wave, while in a deflagration their direction is opposite. That is, a detonation absorbs its own flame, while a deflagration emits one.

The first statement may sound paradoxical, but all moving-film photographs of these phenomena corroborate it. A detonation produces a narrow luminous strip, a deflagration a wide, expanding flame. Of course, when a flame is emitted from a detonation -- which is the superficially visible phenomenon -- the detonation is over and the subsequent expansion of the burnt gases has set in.

XII. ANALYSIS OF DETONATIONS

35. Returning to Figs. 4 and 5, we have to comment upon the fact that they each represent a one-dimensional manifold of possible values p_1, v_1 -- that is, of shocks. This is natural for the pure shock, Fig. 4, which must be supported by a compression behind it, and whose intensity will therefore depend upon the intensity of that support. For the reaction shock, Fig. 5, it is again plausible that support, or the opposite, will modify the shock. However, there should be a point on the curve of Fig. 5 representing a reaction shock unsupported and unhindered -- that is, in equilibrium.

The problem of finding the equilibrium point on the curve of Fig. 5 is one of some difficulty. It has been given a good deal of attention in the literature, and it is rather generally agreed that the hypothesis of Chapman and Jouguet is correct. The equilibrium point is that one where the line $p_2, v_2 \longrightarrow p_1, v_1$ is tangent to the curve. There is no doubt that this question cannot be settled without investigating the mechanical situation in the burnt gas farther behind the detonation front; and also the details of the chemical reaction, which was so far described as occurring instantaneously within that front, but which must actually occupy a zone of finite extension in space and time.

The physical assumptions on which the Chapman-Jouguet hypothesis rests, its domain of validity, and its proof were analyzed in a report submitted previously by the author in connection with this contract [see Ref. (k)].

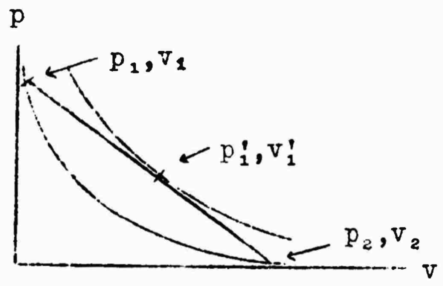
36. In this connection we wish to point out one fact that has not received so far the attention it appears to deserve.

Consider the explosive reaction and take its finite duration, that is, its noninstantaneous character, into account.

The reaction must nevertheless be initiated by an abrupt change of some significant quantity [$\underline{p}$, $\underline{T}$, $\underline{U}$? (see Par. 31)]. However, at the moment when this discontinuity passes over an element of a substance, the chemical reaction there is just beginning. That is, for the purpose of this discontinuity, the substance might as well be chemically inert -- the discontinuity at the first moment is a pure shock. If we define the shock in a broader way, so that it includes the entire reaction zone, then it is, of course, a reaction shock.

In the equilibrium form of detonation both phenomena -- the first, initiating shock and the entire reaction zone -- must have the same velocity.

So we must superpose Figs. 4 and 5, and use two points p_1,v_1 and p_1',v_1' -- corresponding to mere excitation and to complete reaction. Since both have the same velocity, and the same mass flow, both give the same angle $\underline{\omega}$ by Eq. (23); that is, p_1,v_1 and p_1',v_1' lie on the same line from p_2,v_2. The situation is shown in Fig. 6. This figure shows that the intact substance at p_2,v_2 is transformed by



the pure shock into the excited one at p_1,v_1 and that after the reaction is over, its state is p_1',v_1'. The ultimate detonation pressure p_1' is lower than the excitation pressure p_1 -- which is rather strange. However, the details are even more peculiar.

We can also take a view of the shock which excludes from it the excitation process, but includes the entire reaction zone proper. The conservation laws of matter, momentum, and energy -- that is, Eqs. (A_s) to (C_s) -- must remain true. That is, p_1',v_1' must also lie on the Rankine-Hugoniot curve of p_1,v_1. Now this curve is not shown in Fig. 6 (those on that figure belong to p_2,v_2) -- but this much is clear: $p_1' < p_1$; that is, this reaction shock decreases the pressure. Let us therefore replace p_1,v_1 and p_2,v_2 in Fig. 5 by our p_1',v_1' and p_1,v_1 and recall the discussion of Par. 34. Then we must conclude that this reaction shock has to be classified as a deflagration.

So we see that the process, which as a whole is a detonation, can be dissolved into several parts if the finite duration of the reaction is taken into account. It is then seen to consist of two parts:

(i) A pure shock, which initiates the reaction, but still takes place entirely in the inert substance.

(ii) The chemical reaction which follows, and which is best described as a deflagration.

37. This view, that the entire -- undissolved -- process is a detonation which when analyzed dissolves into a pure shock and a deflagration, may seem paradoxical. However, a comparison with the qualitative characterizations of Par. 34 shows that it is quite reasonable.

Thus a detonation increases pressure and density; a deflagration decreases it. Indeed, the whole reaction does increase them both, but the (pure shock) excitation sets in with a higher increase than the ultimate one, and so the reaction proper decreases them.

A detonation is preceded by no sonic signal of its coming; a deflagration is. Indeed, the whole reaction is preceded by no such signal, but its second phase (the reaction proper) is -- by the first phase, the (pure shock) excitation.

38. The mathematical theory must take account of the details of excitation and of the reaction proper -- while we have treated these in a very global way. Besides, this picture should be applied to the nonequilibrium forms of detonation as well. (See Par. 35.)

In this connection it is important to distinguish between two possibilities. The equilibrium detonation may produce sufficient $\underline{p}$, $\underline{U}$, $\underline{t}$ to initiate the detonation, or it may not. Let us call the first type of detonation an active, and the second type a passive one.

An active type detonation can presumably be initiated at sub-equilibrium rates, that will "pick up" to equilibrium. A passive type detonation is simply unable to exist in equilibrium. It must be initiated above it, "boosted," and it will then gradually decay toward equilibrium. And since it cannot exist in equilibrium it will "peter out" before this happens, that is, after a definite finite time dependent upon the strength of the "booster."

These qualitative indications can be substantiated mathematically. This will be done in two subsequent reports by the author, dealing with active and with passive type detonations, respectively. It is hoped that they will contribute to the understanding of the nonequilibrium forms of detonation -- the "picking up" of the active type, and the "boosting" and "petering out" of the passive one.

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For a detailed bibliography of the subject see Ref. (e), p. 50.

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ABSTRACT

Mathematical problems of theory of shock waves in compressible fluids is outlined. Subject is sub-divided into theory of pure shocks and theory of detonation waves and reaction shocks. Study of linear case and oblique case in interaction of shocks and analysis of detonation is given. Only general outline of problems is shown; details appear in supplementary reports.

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The basic mathematical problems of the theory of shock waves in compressible fluids are formulated and discussed in an illuminating fashion. The topics discussed in connection with this study are the conservation laws and the differential equation, the role of entropy, vorticity and the Riemann invariants, natural boundary conditions (the need for discontinuities), the conservation laws and the discontinuities, formulation of the basic problems of discontinuities, the origin of a shock, the interaction of shocks (linear case), the interaction of shocks (oblique case), reaction shocks, and analysis of detonations.

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