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AN APPROXIMATE SOLUTION FOR ROCKET FLIGHT WITH LINEAR-TANGENT THRUST-ATTITUDE CONTROL

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ABSTRACT

This note presents an approximate solution to the equations of motion written with linear-tangent thrust attitude control. The usefulness of this solution is evaluated by comparison with accurate numerical integration. This comparison shows that the solution gives accuracies sufficient for preliminary calculations for satellite boost vehicles and has a four-to-one speed advantage over numerical integration of the same accuracy level.

INTRODUCTION

In order to perform preliminary design calculations for satellite-boosting systems, a reasonably accurate estimate of the propellant requirements is necessary. Unfortunately, obtaining such an estimate usually requires the aid of trajectory calculations. When trajectory data are obtained by numerical integration of the nonlinear equations of motion, the computation time often seems excessive for preliminary calculations. This note, therefore, presents a closed-form, approximate solution for the equations of motion which is about four times faster than stepwise numerical integration of equal accuracy.

Since the approximate solution presented (details given in the appendix) has been developed for use with satellite-boosting systems, it therefore utilizes an approximate form of the thrust-attitude schedule

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that maximizes the mass in orbit. This schedule, as has been shown by Fried (1) and other authors (e.g., (2), (3)), is

$$\tan \psi = a - bt$$

This relation, usually referred to as the "linear-tangent" schedule, compares well with results obtained by numerical integration of the exact Euler-Lagrange differential equations.

Theoretically, the linear-tangent schedule should be followed through all phases of flight. However, the large angles of attack that can occur with this schedule preclude its use at low altitudes. Consequently, the first stage of powered flight, where aerodynamic forces are concentrated, must follow some other thrust-attitude schedule, ordinarily a zero-lift schedule. It is therefore reasonable to neglect aerodynamic forces, for mathematical simplicity, during linear-tangent flight.

The vehicle burnout conditions for orbital missions are usually fixed (e.g., a circular orbit at a given altitude). For this case, it would be convenient if the closed form solution would allow an inverse-type calculation for the required trajectory parameters (e.g., a, b, and t_f) from the known end conditions. This is not possible with the solution presented, and further simplifications do not appear profitable. For example, a somewhat simpler solution given in (3), which has about 1/10 the accuracy of the present solution, also fails to allow an inverse computation. Consequently, the trajectory parameters must be solved for iteratively.

ACCURACY

Usually, selection of a burnout altitude, velocity, and attitude is necessary to specify an orbital mission. Therefore, at least that many vehicle and/or trajectory parameters must be variable to allow an iterative solution. One convenient set of variables is a, b, and r_n with given first-stage burnout conditions.

Table I shows the resulting values of ψ_0, b, and r_3 for a typical three-stage vehicle and a 150-nautical-mile-orbit mission. Also shown are the results for the same case as computed by accurate numerical integration. The errors shown in table I have been found acceptable for many preliminary design problems.

As will be seen in the appendix, certain assumptions will be required which allow the error to increase with the burning time of any stage. As an example of the errors that can be caused by these assumptions, Fig. 1 shows the error in top-stage weight ratio for three-stage satellite-boosting systems as a function of the total burning time of the upper two stages. The scatter in the points shown is due to other differences in the vehicles chosen, but the essential feature to be noted is that the error increases rapidly after $t = 400$ seconds. This corresponds to about 200 seconds per stage, which does not seem an unreasonable limitation for typical stages.

APPENDIX

Neglecting drag and using $\tan \psi = a - bt$, the equations of motion written for the coordinate system of Fig. 2 are

$$\left.\begin{aligned} \ddot{x} + \mu x/R^3 &= \frac{g_c F}{W} \Big/ \sqrt{(a - bt)^2 + 1} = f_x \\ \ddot{y} + \mu y/R^3 &= \frac{g_c F(a - bt)}{W\sqrt{(a - bt)^2 + 1}} = f_y \end{aligned}\right\} \qquad (1)$$

Assuming that the magnitude of R does not vary during one stage of powered flight, equations (1) becomes

$$\left.\begin{aligned} \ddot{x} + \mu x/R_0^3 &= f_x \\ \ddot{y} + \mu y/R_0^3 &= f_y \end{aligned}\right\} \qquad (2)$$

where R_0 is the radius at the beginning of powered flight of the stage under consideration.

Using the method of variation of parameters, equations (2) can be solved in the form

$$\left.\begin{aligned} y_f &= (y_0 + A) \cos \omega t_f + (y_0/\omega + B) \sin \omega t_f \\ x_f &= (x_0 + D) \cos \omega t_f + (x_0/\omega + E) \sin \omega t_f \end{aligned}\right\} \qquad (3)$$

where

$$\left.\begin{aligned} A &= -\int_0^{t_f} \frac{fy \sin \omega t \, dt}{\omega} \qquad & B &= \int_0^{t_f} \frac{fy \cos \omega t \, dt}{\omega} \\ D &= -\int_0^{t_f} \frac{fx \sin \omega t \, dt}{\omega} \qquad & E &= \int_0^{t_f} \frac{fx \cos \omega t \, dt}{\omega} \end{aligned}\right\} \qquad (4)$$

and

$$\omega = \sqrt{\mu/R_0^3}$$

Unfortunately there is no apparent way to integrate equations (4) without some additional assumption. Considering the magnitude of ω, it can be seen that it will never exceed $\sqrt{\mu/R_e^3} = 1.2398\times10^{-3}$. Consequently, if $t < 200$ seconds, which is a typical burning time for one stage, ωt will always be less than 12°. For such small angles, it is reasonable to assume

$$\left.\begin{array}{l} \sin \omega t \cong \omega t \\ \cos \omega t \cong 1.0 \end{array}\right\} \qquad (5)$$

Using equations (5), equations (4) can be integrated to give

$$A = V_j(\bar{t}G - \bar{t}F \sin \psi + H/b)$$

$$B = V_j(F \sin \psi - G)/\omega$$

$$D = V_j(\bar{t}F \cos \psi + G/b)$$

$$E = V_j(F \cos \psi)/\omega$$

where

$$F = \ln \left\{ r \left[\frac{1 + \cos(\psi_f - \bar{\psi})}{1 + \cos(\psi_0 - \bar{\psi})} \cdot \frac{\cos \psi_0}{\cos \bar{\psi}} \right] \right\}$$

$$G = \ln \left(\frac{\sin \psi_f + 1}{\sin \psi_0 + 1} \cdot \frac{\cos \psi_0}{\cos \bar{\psi}} \right)$$

$$H = (\cos \psi_f - \cos \psi_0)/\cos \psi_0 \cos \psi_f$$

Using the fact that the method of variation of parameters assumes

$$\dot{A} \cos \omega t + \dot{B} \sin \omega t = \dot{D} \cos \omega t + \dot{E} \sin \omega t = 0$$

the velocity components can be written as

$$\dot{y}_f = -\omega(y_0 + A) \sin \omega t_f + \omega(\dot{y}_0/\omega + B) \cos \omega t_f$$

$$\dot{x}_f = -\omega(x_0 + D) \sin \omega t_f + \omega(\dot{x}_0/\omega + E) \cos \omega t_f$$

NOMENCLATURE

a = $\tan \psi_0$

b = $d(\tan \psi)/dt$, 1/sec

F = thrust, lb

g_c = 32.174 ft/sec^2

I = specific impulse, sec

R = radius from center of Earth = $\sqrt{x^2 + y^2}$, ft

r = weight ratio = W_0/W_f

t = time measured from ignition of i^{th} stage, sec

V_j = $g_c I$, ft/sec

W = weight = $W_0 - \dot{W}$, lb

x = distance along x-axis of Fig. 2, ft

y = distance along y-axis of Fig. 2, ft

μ = gravitational constant = 1.4077×10^{16} cu ft/sec^2

ψ = angle between thrust vector and x-axis, deg

Subscripts:

e = evaluated at surface of Earth

f = conditions at burnout of i^{th} stage

n = n^{th} stage

0 = conditions at $t = 0$

Superscripts:

$(\dot{\ })$ = d/dt

$(\bar{\ })$ = conditions at $t = W_0/\dot{W}$

REFERENCES

1. Fried, B. D., "Powered Flight Trajectory of a Satellite," Jet Propulsion, vol. 27, no. 6, June 1957.

2. Okhotsimsky, D. E., and Eneyev, T. M., "Certain Variational Problems Connected with Launching of an Artificial Earth Satellite," Translation from Progress in Physical Sciences (in Russian), vol. 63, no. 1-2, Sept. 1957, by U.S. Joint Publications Research Service, Jan 25, 1958.

3. Ross, Stanley, "Composite Trajectory Yielding Maximum Coasting Apogee Velocity," ARS Journal, vol. 29, no. 11, Nov. 1959.

FIGURE LEGENDS

Fig. 1. - Variation of third-stage weight-ratio error with total, upper-stage burning time for satellite-boosting vehicles. Orbit altitude, 150 nautical miles.

Fig. 2. - Earth-centered, nonrotating (inertial) coordinate system.

TABLE I. - ERROR IN FLIGHT PARAMETERS FOR A THREE-STAGE SATELLITE-BOOSTING VEHICLE. ORBIT ALTITUDE, 150 NAUTICAL MILES

	Numerical integration	Approximate solution	
		Error	% Error
r_3	2.3261208	-0.0459212	1.97
ψ_0	48.714°	+0.560°	1.30
b	0.00323681	+0.00004885	1.51

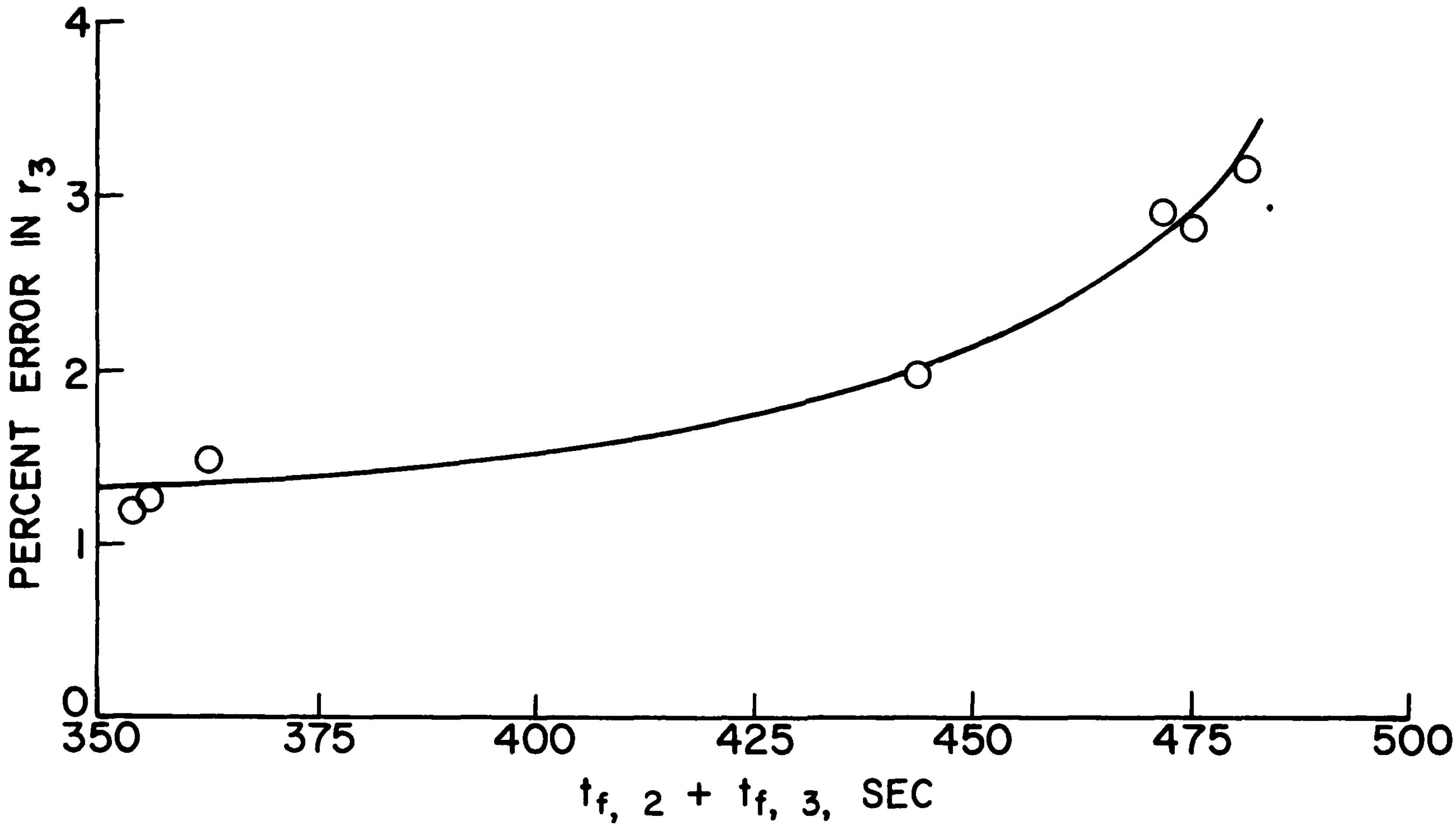
PERCENT ERROR IN r_3
0
1
2
3
4
350
375
400
425
450
475
500
$t_{f,\,2} + t_{f,\,3}$, SEC

Figure 1

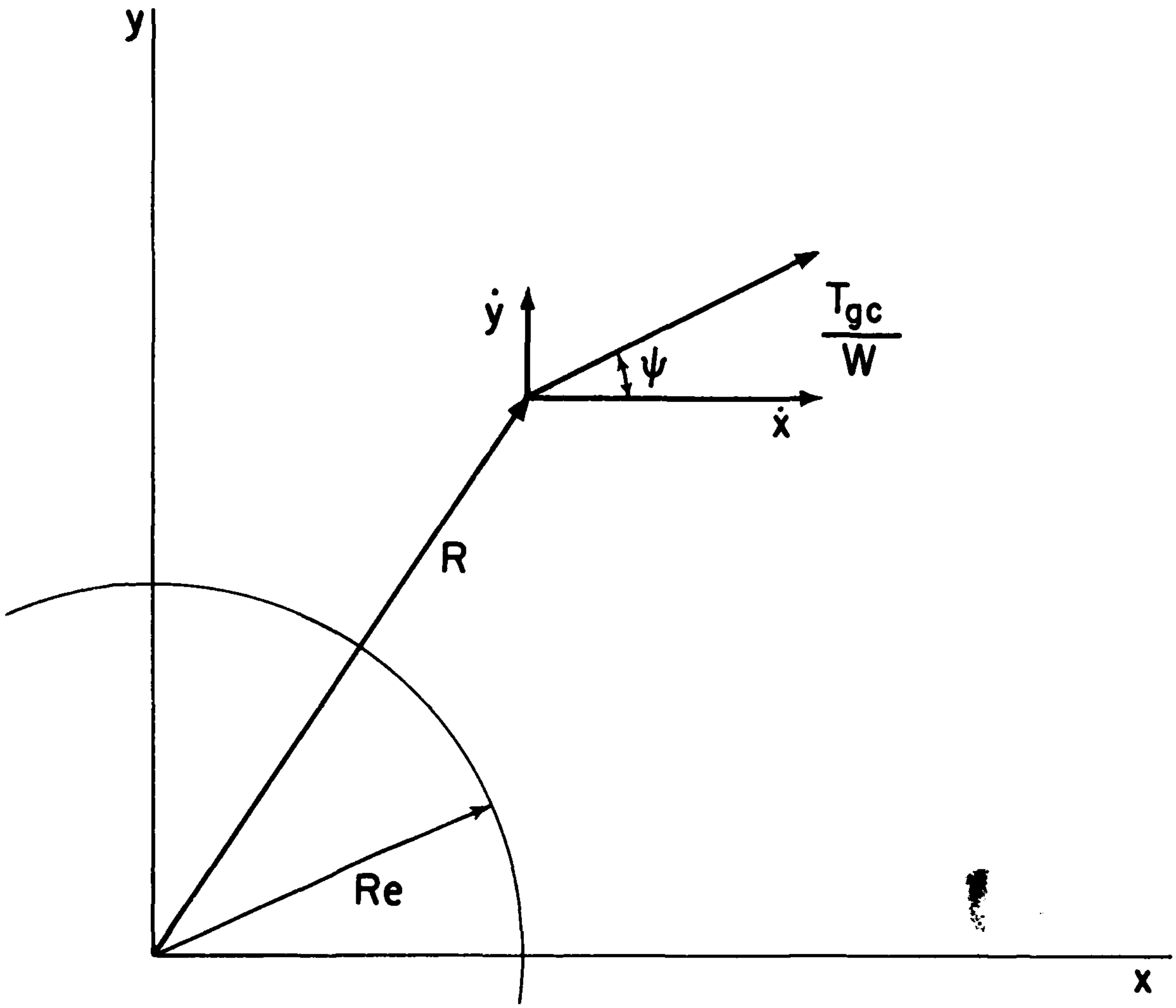

Figure 2