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A NEW LINEAR OPERATIONAL CALCULUS

Frank W. Bubb, Ph.D.

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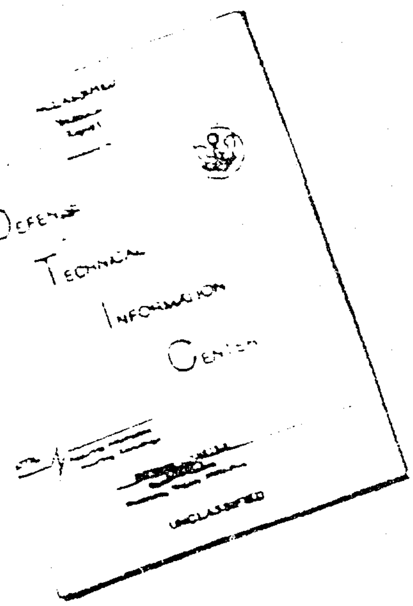

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AF Technical Report No. 6581 May 1951

A NEW LINEAR OPERATIONAL CALCULUS

Frank W. Bubb, Ph.D.

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FOREWORD

This report was prepared by Dr. Frank W. Bubb, Chief Scientist of the Office of Air Research, Hq., Wright Air Development Center. Work was completed under Expenditure Order Number 461-1. The report is one of a series to be issued on this project. Others in the series will be published as research progresses.

ABSTRACT

This paper presents a new operational calculus applicable to the approximate (and in the limit, exact) analysis and synthesis of linear physical systems (such as servomechanisms, electrical circuits, and so on). Corresponding to any time function $F(t)$, a "P-transform" $\widetilde{F}(x) = \sum_{k=-K}^{K} F(kv)x^k$ is defined, $F(kv)$ being ordinates of $F(t)$ at integral multiples of a time interval v. The relation between input $F(t)$ to <u>any</u> linear system, its output $H(t)$, and its memory function (impulsive admittance) $M(t)$, is given by the superposition integral, $H(t) = \int_{-\infty}^{\infty} F(s)M(t-s)ds$. Corresponding to this convolution integral, the relation $\widetilde{H}(x) = \widetilde{F}(x)\widetilde{M}(x)$ holds between the respective P-transforms. Conversely, this transform equation implies the convolution equation. This correspondence is identical in form to the correspondence between convolution integral and its Laplace (or Fourier) transform which is the basic theorem of the classical operational methods. The mode of constructing a table of P-transforms is indicated by working out the transforms of the elementary functions of mechanics and of circuit analysis. By way of indicating how this new operational calculus may be developed, a set of useful transform theorems are worked out for such analytical operations as differentiation, integration, and so on. Finally, to exemplify the power of this new calculus, a linear differential equation is solved, numerical results being compared with those of the analytical solution. An outstanding virtue of this new operational calculus is that it does not depart from the time domain itself. In a subsequent Air Force Technical Report No. 6586, this method is used to give a simple explanation of Wiener's theory of the smoothing and predicting filter.

PUBLICATION REVIEW

Manuscript copy of this report has been reviewed and found satisfactory for publication.

FOR THE COMMANDING GENERAL:

LESLIE B. WILLIAMS
Lt. Colonel, USAF
Chief, Office of Air Research
Research Division

CONTENTS

A NEW LINEAR OPERATIONAL CALCULUS

This paper presents an operational calculus applicable to the approximate (and in the limit, exact) analysis and synthesis of linear physical systems, such as servomechanisms, electrical circuits, electromechanisms, and the like. In contrast to the methods based upon the transform theories of Fourier and Laplace, this calculus does not depart from the time domain[1], thus allowing a more direct appeal to the intuition.

The new operational method takes origin in the notion of the generating function, used so effectively by Laplace[2] in the theory of probability. Actually, the new method is closely related to the Fourier and Laplace transform methods, but provides familiar algebraic methods of executing analyses and syntheses. Before setting forth the new methods, we recapitulate a well known bit of linear system theory which brings out the essential features of <u>any</u> operational method.

THE SUPERPOSITION INTEGRAL

Consider the conventional "black box" representation of a physical system shown in Figure 1. $F(t)$ represents the input (cause or driving function) which actuates the system. The output of the system (effect or response) is represented by $H(t)$. The system itself is characterized by its MEMORY FUNCTION $M(t)$, which is the response of the system at time t caused by a unit impulsive input applied to the system at time $t = 0$. $M(t)$ is also called the system weighting function or its impulsive admittance and is the Fourier transform of the system frequency response as well as the inverse Laplace transform of the system transfer function.

Input $F(t)$, response $H(t)$ and memory function $M(t)$ are related[3] by the superposition (convolution or faltung) integral

$$H(t) = \int_{-\infty}^{t} F(s)M(t-s)ds \tag{1}$$

This is easy to see from the following considerations, illustrated in Figure 2. The element $F(s)ds$ may be regarded as a small impulse at time s. Had this been a unit impulse, the corresponding response at time t (later by the time $t - s$) would be $M(t-s)$. The proportional response due to the non-unit impulse is then $F(s)dsM(t-s)$. The total response $H(t)$ at time t is the sum of all the responses at time t due to similar elementary impulses occurring preceding time t, namely, is the superposition integral (1).

The memory function $M(t)$ vanishes for negative t, since it is axiomatic that an effect or response cannot precede its cause (the unit impulse at $t = 0$). This means that $M(t-s) = 0$ for $s > t$; hence the upper limit t in

(1) may changed to ∞. This is also clear, since any impulse after t cannot be retroactive to produce any response at the earlier time t.

It is important to note that (1) is a very comprehensive equation. It covers the complete range of linear system analysis and synthesis covered by Laplace or Fourier transform methods. M(t) may be equivalent to the operation of differentiating, or integrating, or, in fact, may be equivalent to any ordinary integro-differential equation. Due to this fact that the superposition integral embraces all of linear system analysis and synthesis, it provides a proper basis for the development of <u>any</u> form of operational calculus.

To construct an operational calculus, one defines, corresponding to a given time function F(t), another function $\tilde{F}(x)$, called the transform of F(t). If then, corresponding to the superposition integral (1), there exists a simple relation between the transforms of F(t), M(t) and H(t), we shall have an operational method. In Laplace transform theory, the transform relation is $L[H(t)] = L[M(t)] \cdot L[F(t)]$. In Fourier transform theory the relation is identical in form, being $H(j\omega) = M(j\omega)\, F(j\omega)$. In the new method to be set forth here, the transform relation is again of this same form, $\tilde{H}(x) = \tilde{M}(x) \cdot \tilde{F}(x)$, where these respective functions are the "Polynomial-Transforms" of H(t), M(t), F(t). We must now define, of course, this new type of transform, which we abbreviate as "P-Transform".

POLYNOMIAL TRANSFORMS

Corresponding to a given time function F(t), we define now a transform $\tilde{F}(x)$, called here a P-Transform, by the equation,

$$\tilde{F}(x) = P\ F(t) = \sum_{n=-K}^{L} F(nv)x^{n} \qquad (2)$$

where the F(nv) constitutes a sequence of ordinates of F(t) equally spaced a time v apart. The respective factors x^n may, for the moment, be thought of as a means of ordering the sequence F(nv) of ordinates. The word ordinate F(nv) may, of course, include an ordered sequence of empirical values, taken for example, from a table. And the independent variable t does not, of course, have to be time.

We also define an operator P^{-1}, inverse to P as defined in (2). Thus, having given a P-transform $\tilde{F}(x)$, that is to say, its sequence of ordinates F(nv), we shall understand by the expression

$$f(t) = P^{-1}\,\tilde{F}(x) \qquad (3)$$

the interpolation of a time function f(t) having the values f(nv)=F(nv) (n = -K to L). Graphically, this amounts to drawing a curve f(t) exactly through the points $F(nv)$. We shall call $f(t)$ in (3) a DETERMINING FUNCTION or TIME FUNCTION corresponding to the P-transform $\bar{F}(x)$. The function $f(t)$ is an approximation to $F(t)$ which can be made exact. Before a determing function $f(t)$ becomes uniquely defined, we shall have to prescribe how its ordinate sequence $f(nv) = F(nv)$ is to be interpolated. For conciseness, we shall occasionally write F_n for $F(nv)$.

CARDINAL INTERPOLATION

For present purposes, interpolations will be limited to the form

$$f(t) = \sum_{n=-K}^{L} F_n L(t-nv) \tag{4}$$

where the interpolating function $L(t)$ will be taken as the CARDINAL FUNCTION[4],

$$L(t) = \frac{\sin \frac{\pi}{v} t}{\frac{\pi}{v} t} \tag{5}$$

a graph of which is shown in the upper sketch of Figure 3. This function has the properties

$$\left.\begin{aligned} L(t-nv) &= 1 \text{ for } t = nv \\ &= 0 \text{ for } t = kv \\ n,k &= \text{integers, } n \neq k \end{aligned}\right\} \tag{6}$$

$$L(t) = L(-t), \tag{7}$$

is continuous for all t and its tails attenuate hyperbolically to zero as t approaches infinity.

The meaning of the inverse transform $f(t) = P^{-1}\bar{F}(x)$ may be seen clearly as follows. If we multiply $L(t-nv)$ by F_n and graph this product in proper position on the graph of $F(t)$, see Figure 3, we obtain a curve enclosing the shaded area. This curve passes exactly through this particular point (nv, F_n) and crosses the time axis at every other ordinate of the sequence F_ν. If we do the like for every ordinate F_n and add the separate interpolating curves so obtained for all values of t, we see that the sum curve (4) representing the inverse transform $f(t)$ is an approximation or interpolation, continuous for all t, of the time function $F(t)$. Note that this interpolation $f(t)$ passes exactly through all points of the ordinate sequence F_n. Note further that, by passing to the limit as $v \to 0$, $f(t)$ can be made exactly equal to $F(t)$ if $F(t)$ is continuous.

It is also of interest to note that the area under the curve $F_n L(t-nv)$ is $F_n v$. This is the correct amount of impulse to be associated with the ordinate F_n.

The cardinal function also has the following useful properties[5]. $L(t)$ is equal, up to a constant multiplier v, to its auto-convolution, namely,

$$\int_{-\infty}^{\infty} L(s)L(t-s)\,ds = v\,L(t) \tag{8}$$

From this it is easy to show that the function set $L(t-nv)$, where k,n are integers, is orthogonal, namely,

$$\int_{-\infty}^{\infty} L(s-nv)L(s-kv)\,ds = v \text{ when } n = k, \; = 0 \text{ when } n \neq k. \tag{9}$$

The properties (6) to (9) will suffice for our present purposes.

PRINCIPAL THEOREM: We are now in position to proceed with the statement and proof of our principal theorem:

PRINCIPAL THEOREM: THE CONVOLUTION OR SUPERPOSITION INTEGRAL,

$$H(t) = \int_{-\infty}^{\infty} F(s)M(t-s)\,ds \tag{1}$$

IMPLIES THE P-TRANSFORM RELATION

$$\tilde{H}(x) = v\,\tilde{F}(x)\,\tilde{M}(x) \tag{10}$$

AND CONVERSELY, IN THE SENSE OF APPROXIMATION WHICH CAN BE MADE EXACT, THE TRANSFORM RELATION IMPLIES THE SUPERPOSITION INTEGRAL.

We now prove this theorem. The time functions $F(t)$, $M(t)$ and $H(t)$ needed in (1) may be represented by the interpolations of the cardinal type (4):

$$\begin{aligned} F(s) &\cong \sum_{j=-J}^{I} F_j\,L(s-jv), \quad F_j = 0 \text{ outside } -J \leq j \leq I \\ M(t-s) &\cong \sum_{k=-K}^{L} M_k\,L(t-s-kv), \quad M_k = 0 \text{ outside } -K \leq k \leq L \\ H(t) &= \sum_{n=-N}^{M} H_n\,L(t-nv), \quad H_n = 0 \text{ outside } -N \leq n \leq M \end{aligned} \tag{11}$$

Inserting these into (1) and commuting the order of summation and integration,

$$H(t) \cong \sum_{n=-N}^{M} H_n L(t-nv) = \sum_{j=-J}^{I} F_j \sum_{k=-K}^{L} M_k \int_{-\infty}^{\infty} L(s-jv)L(t-s-kv)ds.$$

Setting $x = s-jv$ and invoking (8), we have

$$H(t) = \sum_{n=-N}^{M} H_n L(t-nv) = v \sum_{j=-J}^{I} F_j \sum_{k=-K}^{L} M_k L\left[t-(j+k)v\right].$$

Making use of (6), the last expression will yield explicit values for the H_n in terms of the F_j and M_k. Thus, let us set $t = iv$ in the last equation: on the left, all terms vanish except H_i. In the second summation on the right, all terms vanish except that for which $i = j + k$, the surviving term being M_{i-j}. Hence,

$$H_i = v \sum_{j=-J}^{I} F_j M_{i-j} \text{ which we write as}$$

$$H_n = v \sum_{j=-J}^{I} F_j M_{n-j} \tag{12}$$

This is a very useful result.

To see more clearly what (12) means, let us now form the P-transform for the sequence H_n:

$$\begin{aligned}\tilde{H}(x) &= \sum_{n=-N}^{M} H_n x^n = v \sum_{n=-N}^{M} x^n \sum_{j=-J}^{I} F_j M_{n-j} \\ &= v \sum_{j=-J}^{I} F_j x^j \sum_{n=-N}^{M} M_{n-j} x^{n-j}\end{aligned}$$

Replacing the index $n - j$ by k and changing the limits on the n summation to corresponding values of k, we have

$$\tilde{H}(x) = \sum_{n=-N}^{M} H_n x^n = v \sum_{j=-J}^{I} F_j x^j \sum_{k=-N-j}^{M-j} M_k x^k \tag{12'}$$

In the applications of this operational method, we shall frequently encounter equations of this form (12'), for which reason (12) deserves a careful study.

In equations (11), we placed certain restrictions upon the F_j, M_k and H_n, stating that these ordinates vanish outside certain ranges - it amounts to the same thing to say that outside these ranges these sets of ordinates become negligible. For the purposes of system analysis and synthesis, we can place upon the ranges concerned the further restriction that $M = I + L$, $N = J + K$. This last restriction will not be maintained in a later report dealing with noise smoothing problems, and this will require a more detailed study of the form (12').

We can now simplify (12') as follows. Equating coefficients of like powers of x on both sides of (12'), we have,

$$\frac{1}{v} H_n = F_{-J}\, x^{-J} \sum_{k=-N+J}^{M+J} M_k x^k + \cdots + F_j\, x^j \sum_{k=-N-j}^{M-j} M_k x^k + \cdots + F_I\, x^I \sum_{k=-N-I}^{M-I} M_k x^k.$$

Recalling the additional restriction that $M = I + L$, we see that the upper limits on these successive summations decrease from $M + J > L$ to $M - I = L$, and since $M_k = 0$ for all $k > L$, all these upper limits can be replaced by L. In similar manner, the successive lower limits go from $-N + J = -K$ to $-N - I < -K$, and since $M_k = 0$ for all $k < -K$, we see that all these lower limits can be replaced by $-K$. Hence, in (12') the lower and upper limits $k = -N - j$ and $k = M - j$ can be replaced respectively by $k = -K$ and $k = -L$. Accordingly, (12') becomes

$$\left.\begin{aligned} &\tilde{H}(x) = \sum_{n=-N}^{M} H_n x^n = v \sum_{j=-J}^{I} F_j x^j \sum_{k=-K}^{L} M_k x^k \\ &\text{where } M = I + L \text{ and } N = J + K \end{aligned}\right\} \qquad (13)$$

This relation may also be illustrated by multiplying out two polynomials of low degrees.

We note now that the two polynomial factors on the right side of (13) are respectively $\tilde{F}(x)$ and $\tilde{M}(x)$. Hence $\tilde{H}(x) = v\, \tilde{F}(x)\, \tilde{M}(x)$. This proves the first part of our principal theorem.

It remains to prove the converse, namely, that (1) follows from (10). Let us write (12) in the functional notation:

$$H(iv) = \sum_{jv=-Jv}^{Iv} F(jv)\, M(iv-jv)\, v$$

Now let the time interval $v = ds \to 0$. Since v is decreasing, the number of intervals must increase to cover the given ranges of variables: hence, we set $iv = t$, $jv = s$, $Jv = T_1$, and $Iv = T_2$. Passing to the limit, then,

$$H(iv) = \sum_{jv=-Jv}^{Iv} F(jv)M(iv - jv)\, v$$

gives

$$H(t) = \int_{-T_1}^{T_2} F(s)\, M(t-s)ds.$$

With proper restrictions as to convergence of this integral, already implied in (1), we can let $T_1 \to \infty$ and $T_2 \to \infty$ and so obtain (1). Hence, since (12) passes over in the limit to (1), the equation (10) which is equivalent to (12) is an approximation to (1) which can be made exact. This completes the proof of our principal theorem.

It is interesting to compare (see Appendix A) the present operational methods with the familiar methods based upon the Fourier and Laplace transform theory.

LINEAR SYSTEM THEORY

There are three typical problems in linear system theory: the analysis problem, the synthesis or design problem, and the instrument problem.

In the typical ANALYSIS PROBLEM, one has the system $M(t)$ given (the memory function being used here to characterize the system), one has given the input $F(t)$, and one has to calculate the system output $H(t)$.

In the typical SYNTHESIS PROBLEM, one has given the input $F(t)$ and the output $H(t)$, and one has to calculate the system memory function $M(t)$ - accepting $M(t)$ as the attorney or mathematical representative of the system itself, and leaving it up to the design engineer actually to make the system so it will have this memory function response to unit impulse.

In the INSTRUMENT PROBLEM, one has given the system $M(t)$ (the measuring device), one has given its output $H(t)$ (the measurement), and one has to find out what the input $F(t)$ is or was.

All these problems are easily solved by the present operational calculus, nothing but the ordinary algebraic operations of polynomial multiplication or division being needed.

Consider the analysis problem. Having given the input $F(t)$, one picks off its graph or calculates a sequence of ordinates F_j equally spaced at integral time multiples jv, and forms the input P-transform

$$\tilde{F}(x) = \sum_{j=-J}^{I} F_j x^j .$$

In similar manner, one forms (practically by inspection) the memory P-transform

$$\tilde{M}(x) = \sum_{k=-K}^{L} M_k x^k .$$

One then multiplies these two polynomials by the familiar algorithm of ordinary algebra to obtain

$$\tilde{H}(x) = v\,\tilde{F}(x)\tilde{M}(x) = v\left(\sum_{j=-J}^{I} F_j x^j\right)\left(\sum_{k=-K}^{L} M_k x^k\right) = \sum_{n=-J-K}^{I+L} H_n x^n .$$

One then picks off the coefficients H_n, plots these at the respective points nv, fairs in a time curve $h(t)$ through these points, and accepts this curve as an adequate interpolation of the required response $H(t)$.

In the synthesis problem, one forms as above the P-transforms $\tilde{F}(x)$ and $\tilde{H}(x)$ of given input and output. One then divides, by ordinary polynomial division the first by the second to get the memory transform

$$\tilde{M}(x) = \frac{1}{v}\,\frac{\tilde{H}(x)}{\tilde{F}(x)} = \sum_{k=-N}^{M} M_k x^k ,$$

picks off and graphs the ordinate sequence M_k and obtains the interpolated time function $M(t)$.

In the instrument problem, one forms the P-transforms of the given measurement $H(t)$ and given memory $M(t)$, divides to get the polynomial

$$\tilde{F}(x) = \frac{1}{v}\,\frac{\tilde{H}(x)}{\tilde{M}(x)} ,$$

picks off this P-transform $\tilde{F}(x)$ its ordinate sequence F_n, graphs these ordinates, and obtains the interpolated measured quantity $F(t)$.

It will be noted in all three of these problems that, because of the obvious relation between P-transform and corresponding time function, one remains essentially in the time domain itself. The great intuitive value of this fact will be made apparent in a later paper (AF Technical Report No. 6586) in which the Kolmogoroff-Wiener theory of noise smoothing and predicting will be presented in terms of the new operational method. The essential ideas of noise analysis (regarded by engineers as too abstruse for practical use) take on a simple, obvious character.

The value of this new method as a practical way of getting numerical results will be illustrated later in a numerical example. This example will be delayed in order to set forth a number of useful theorems.

USEFUL THEOREMS

There are many properties of the P-transformation which are useful in applications of the present operational method. In general, these propositions correspond to parallel propositions in Laplace transform theory. Some are sufficiently obvious to be stated without proof.

Th. 1: The following operations, respectively in the time and transform domains, correspond:

$$\left.\begin{aligned} &\text{If } H(t) = F(t) + K(t), \\ &\text{then } \tilde{H}(x) = \tilde{F}(x) + \tilde{K}(x) \end{aligned}\right\} \tag{14}$$

$$\text{Th. 2:}\quad \left.\begin{aligned} &\text{If } H(t) = c\, F(t), \\ &\text{then, } \tilde{H}(x) = c\, \tilde{F}(x) \end{aligned}\right\} \tag{15}$$

$$\text{Th. 3:}\quad \left.\begin{aligned} &\text{If } H(t) = F(t-mv) \\ &\text{then } \tilde{H}(x) = x^{m}\, \tilde{F}(x) \end{aligned}\right\} \tag{16}$$

The last theorem means that multiplication of the transform $\tilde{F}(x) = PF(t)$ by x^m corresponds in the time domain to shifting the time function by amount mv into the future.

$$\text{Th. 4:}\quad \left.\begin{aligned} &\text{If } H(t) = F(-t) \\ &\text{then } \tilde{H}(x) = \textstyle\sum F(nv)\, x^{-n} \end{aligned}\right\} \tag{17}$$

This means that, corresponding to the time domain operation of reflecting the function $F(t)$ about the time origin, one merely has to change the signs of the exponents of x in the transform $\tilde{F}(x) = \sum F(nv)x^n$.

$$\text{Th. 5:}\quad \left.\begin{aligned} &\text{If } H(t) = \Delta F(t) = F(t+v) - F(t) \\ &\text{then } \tilde{H}(x) = \left(\frac{1}{x} - 1\right) \tilde{F}(x) \end{aligned}\right\} \tag{18}$$

which means that the operation of taking finite differences in the time domain corresponds to multiplication by $\left(\frac{1}{x} - 1\right)$ in the transform domain.

$$\text{Th. 6:}\quad \left.\begin{aligned} &\text{If } H(t) = t\, F(t) \\ &\text{then } \tilde{H}(x) = xv\, \frac{d\tilde{F}(x)}{dx} \end{aligned}\right\} \tag{19}$$

which means that multiplication by t in the time domain corresponds to the operation $vx\frac{d}{dx}$ in the transform domain. If one writes $\tilde{F}(x) = \sum F(nv)\, x^n$ and differentiates, the proposition readily follows.

Th. 7: A theorem for integration in the P domain goes as follows:

$$\left.\begin{aligned} &\text{If } H(t) = \frac{F(t)}{t+v} \\ &\text{then} \quad \tilde{H}(x) = \frac{1}{vx}\int_0^x \tilde{F}(x)\, dx \\ &\text{where} \quad \tilde{F}(x) = \sum_{n=0}^{\infty} F(nv)\, x^n \end{aligned}\right\} \qquad (20)$$

which follows readily by substituting the third equation into the second. The term $F(-v)x^{-1}$ would introduce a logarithmic singularity and is reserved for later study.

The amount of impulse $H(nv)$ due to the action of a driving function $F(t)$ acting over a time from 0 to nv is the definite integral

$$\int_0^{nv} F(t)dt$$

and is the area under the curve $F(t)$ from 0 to $t = nv$. Concerning the sequence of values $H_n = H(nv)$, we can state the theorem

Th. 8: If in the time domain

$$\left.\begin{aligned} &H(nv) = \int_0^{nv} F(t)\, dt \\ &\text{in the P-domain} \quad \tilde{H}(x) = \frac{v}{2}\,\frac{1+x}{1-x}\,\tilde{F}(x) - \frac{v}{2}F_0\sum_{k=1}^{\infty} x^k \\ &\text{where} \quad \tilde{F}(x) = \sum_{k=0}^{\infty} F_k x^k \\ &\text{and where } \tilde{H}(x) = \sum_{n=0}^{\infty} H_n x^n \end{aligned}\right\} \qquad (21)$$

In other words H_n is the coefficient of x^n in the second equation of (21). This theorem will be useful later in reducing ordinary linear differential equations to the polynomial operations of ordinary algebra. For this reason, it deserves a careful discussion.

The area under any portion of the curve $F(t)$ is, to a sufficient approximation, equal to the area under that part of its interpolation $\sum_{-\infty}^{\infty} F_k L(t-kv)$.

The area from 0 to $t - nv$ is then

$$H(nv) = H_n = \int_0^{nv} \sum_{k=0}^{n} F_k L(t-kv)dt$$

$$= \sum_{k=0}^{n} F_k \int_0^{nv} L(t-kv)dt$$

For conciseness, we write

$$S_k = \int_0^{nv} L(t-kv)dt \tag{22}$$

so that

$$H_n = \sum_{k=0}^{n} F_k S_k \tag{23}$$

It remains, of course, to evaluate S_k.

If we use as interpolating function the triangle shown in Figure 4, which produces the straight line approximation or polygon shown in the lower sketch, it is obvious (since the area of the triangle is v) that

$$\begin{aligned} S_k &= v \text{ for } 0 < k < n \\ &= \frac{v}{2} \text{ for } k = 0 \text{ or } n \end{aligned} \tag{24}$$

Note that of the triangular interpolating functions at $t = 0$ and $t = nv$, the half triangles outside the range 0 to nv are excluded in order to provide better approximation.

The cardinal function (5) gives approximately the same result - and this is discussed in Appendix B to avoid interrupting the continuity here.

Expanding (23) and using (24),

$$H_n = S_0F_0 + S_1F_1 + S_2F_2 + \cdots + S_{n-1}F_{n-1} + S_nF_n$$

$$= \frac{v}{2}F_0 + vF_1 + vF_2 + \cdots + vF_{n-1} + \frac{v}{2}F_n$$

which is also obvious from a calculation of the trapezoidal areas of the interpolation as shown in Figure 4. This result can be written as

$$H_n = \frac{v}{2}\left[(2F_0 + 2F_1 + 2F_2 + \cdots + 2F_{n-1} + F_n) - F_0\right] \tag{25}$$

The quantity in the parentheses ( ) is the coefficient of x^n in the product of the two polynomials $(\sum_{k=0}^{\infty} F_k x^k)(1 + 2x + 2x^2 + 2x^3 + \cdots \text{ad.inf.})$, which can be readily shown by a bit of pencil work. Furthermore, a division by $1-x$ will show that $(1 + 2x + 2x^2 + 2x^3 + \cdots \text{ad.inf.}) = \frac{1+x}{1-x}$. The remaining term F_0 in the bracket of (25) is the coefficient of x^n in the polynomial $F_0 \sum_{k=1}^{\infty} x^k = F_0 \frac{x}{1-x}$ where the term for $k = 0$ is omitted to avoid cancelling the x^n term of the previous part. Hence, it appears that H_n is the coefficient of x^n in the P-transform.

$$\left.\begin{aligned} \tilde{H}(x) &= \frac{v}{2}\ \frac{1+x}{1-x} \sum_{k=0}^{\infty} F_k x^k - \frac{F_0 v}{2} \sum_{k=1}^{\infty} x^k \\ &= \frac{v}{2}\ \frac{1+x}{1-x}\ \tilde{F}(x) - \frac{v}{2}\ F_0 \sum_{k=1}^{\infty} x^k \\ &= \frac{v}{2}\ \frac{1+x}{1-x}\ \tilde{F}(x) - \frac{F_0 v x}{2(1-x)} \end{aligned}\right\} \qquad (21')$$

from which Theorem 8 follows.

A better approximation than (25) to the area or impulse H_n is given[6] by Simpson's one-third rule

$$H_n = \frac{v}{3}\left[F_0 + 4F_1 + 2F_2 + 4F_3 + 2F_4 + \cdots + 4F_{n-1} + F_n\right] \qquad (26)$$

where n is <u>even</u>. By direct execution of the operations below, it can be shown that

$$\tilde{H}(x) = \frac{v}{3}\ \frac{1+4x+x^2}{1-x^2}\ \tilde{F}(x) - \frac{v}{3}\ F_0 \sum_{\substack{k=1 \\ k\ \text{even}}}^{\infty} x^k. \qquad (27)$$

Hence, (27) may be used in place of the second equation of (21) - one should recall then that n is to be taken only as an <u>even</u> number.

A large class of interpolating functions from which additional approximations may be derived are given by Schoenberg[4].

If one writes the first equation of (21) in the form $H(t) = \int_0^t F(t)dt$, it follows that $F(t) = \dot{H}(t) = \frac{dH(t)}{dt}$. In this case, the constant lower limit plays no part. According, we may drop the term in F_0 from the second equation of (21). Solving this second equation for $\tilde{H}(x) = \tilde{F}(x)$, we have the theorem, Th. 9: Corresponding to the time domain operation of differentiation,

$$\left.\begin{array}{ll} \dot{H}(t) = \dfrac{dH(t)}{dt} & \\ \text{we have in the P-domain} & \tilde{\dot{H}}(x) = \dfrac{2}{v}\,\dfrac{1-x}{1+x}\,\tilde{H}(x) \\ \text{where} & \tilde{\dot{H}}(x) = \displaystyle\sum_{k=0}^{\infty} \dot{H}(kv)x^{k} \end{array}\right\} \qquad (28)$$

A still better P-domain operation corresponding to differentiation in the time domain is obtained by a similar inversion of (27), thus

$$\tilde{\dot{H}}(x) = \frac{3}{v}\,\frac{1-x^{2}}{1+4x+x^{2}}\,\tilde{H}(x) \qquad (29)$$

and this may be used in place of the second equation in (28).

These theorems might be continued, more or less paralleling those of Laplace transform theory. Enough have been given, however, to sketch out the procedure for developing the new operational calculus.

An example (coming under Th. 8) will now be worked out to illustrate the arithmetic convenience of the new methods.

NUMERICAL EXAMPLE

As an example under Th. 8, we calculate the ordinates $H(nv)$ of the integral $\int_0^{t=nv} \sin t\, dt$. Taking the time interval as $v = \frac{\pi}{10} = 18°$, the P-transform of sin t, taken from a trigonometric table, is

$$\tilde{F}(x) = 0 + .31x + .59x^{2} + .81x^{3} + .95x^{4} + x^{5} + .95x^{6} + \text{etc.}$$

From (21), the transform operator is $\frac{v}{2}\,\frac{1+x}{1-x} = .157\,\frac{1+x}{1-x}$. Hence, noting that $F(0) = \sin 0 = 0$, we have

$$\tilde{H}(x) = 0.157\,\frac{1+x}{1-x}\,(0 + .31x + .59x^{2} + .81x^{3} + .95x^{4} + 1x^{5} + .95x^{6} + \cdots)$$

With a bit of pencil work on the side (one may multiply first by 1 + x and then divide by 1 - x or vise versa), the reader will readily verify the results calculated here, the second gives values calculated from the analytical solution

$$\int_0^{t=nv} \sin t\, dt = 1-\cos nv.$$

H_0	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8	H_9	H_{10}	H_{11}	$H_{12}\ldots$
0	.05	.19	.41	.69	.98	1.30	1.57	1.79	1.94	1.98	1.94	1.79
0	.05	.19	.41	.69	1.00	1.31	1.59	1.81	1.95	2.00	1.95	1.81

These results are sufficiently accurate to illustrate the simplicity and precision of the new methods. As an additional bit of practice, the reader may use, instead of the integrating operator above, the better one given in (27), namely,

$$\frac{v}{3}\ \frac{1+4x+x^2}{1-x} = .1047\ \frac{1+4x+x^2}{1-x^2}.$$

A TABLE OF P-TRANSFORMS

For reasons brought out so clearly by Gardner and Barnes[7] in the case of Laplace transform theory, we find here that a table of P-transforms are useful. A few examples only can be worked out here, and these will be limited to the elementary functions usually encountered in linear system theory. The calculations amount, of course, to direct applications of the definition

$$\tilde{F}(x) = P\left[F(t)\right] = \sum_{n=-N}^{M} F(nv)\ x^n \qquad (2)$$

of the P-transform.

The unit step function $S(t)$ is defined by the equation

$$\begin{aligned} S(t) &= 1 \text{ for } t \geq 0 \\ &= 0 \text{ for } t < 0 \end{aligned} \qquad (30)$$

The corresponding P-transform $\tilde{S}(x)$ is $\tilde{S}(x) = 1 + x + x^2 + \cdots + x^n + \text{etc}$

$= \sum_{n=0}^{\infty} x^n$. But $1 + x + x^2 + \cdots \text{ad.inf.} = \frac{1}{1-x}$, hence

$$\tilde{S}(x) = \frac{1}{1-x} \qquad (31)$$

in a closed and simple form.

The truncated exponential function is defined by

$$E(t) = \left.\begin{aligned} &e^{-\alpha t} \quad \text{for } t \geq 0 \\ &= 0 \qquad \text{for } t < 0 \end{aligned}\right\} \tag{32}$$

The ordinate sequence is $E(nv) = e^{-\alpha nv} = (e^{-\alpha v})^n$; hence, $\tilde{E}(x) = \sum_{n=0}^{\infty} e^{-\alpha nv} x^n$ $= \sum_{n=0}^{\infty} (e^{-\alpha v} x)^n$. Accordingly, we can write

$$\left.\begin{aligned} &\tilde{E}(x) = \sum_{n=0}^{\infty} (ax)^n = \frac{1}{1-ax} \\ &\text{where } a = e^{-\alpha v} \end{aligned}\right\} \tag{33}$$

Note that by setting $\alpha = 0$, and hence $a = 1$, we recover the previous result for the unit step function.

In similar fashion, by calculating $P\left[e^{-\alpha t} e^{j\omega t}\right]$ and separating real and imaginary parts, we find for the truncated damped sinusoids the polynomial transforms,

$$P\left[e^{-\alpha t} \sin \omega t\right] = \frac{(a \sin \omega v)\, x}{1-(2a \cos \omega v)\, x + a^2 x^2} \tag{34}$$

where $a = e^{-\alpha v}$,

$$P\left[e^{-\alpha t} \cos \omega t\right] = \frac{1-(a \cos \omega v)\, x}{1-(2a \cos \omega v)\, x + a^2 x^2} \tag{35}$$

Application of Theorem 6 (Equations 19) will introduce a factor t into any one of the above functions, corresponding to which one applies to the P-transform the operation $vx \frac{d}{dx}$. As an example of this, consider the ramp function

$$R(t) = t\, \mathcal{S}(t) \tag{36}$$

Its transform is $\tilde{R}(x) = vx \frac{d}{dx} \tilde{S}(x) = vx \frac{d}{dx} \left(\frac{1}{1-x}\right)$. Hence

$$\tilde{R}(x) = \frac{vx}{(1-x)^2} = v(0+x+2x^2+\cdots+nx^n + \text{etc}) \tag{37}$$

The respective coefficients 0, v, 2v,...,nv, etc. in the last expression are obviously the sequence of ordinates of this ramp function.

Multiple application of Theorems 6, 1 and 2 to the above functions and their transforms will yield the transforms of any polynomial in t times the original function - such as $(2t^3+5t^2-3)\,e^{-5t}$. This corresponds, of course, to a multiple pole (degenerate eigen functions) on the complex plane of the Laplace transform.

The unit impulse $\delta(t)$ at time $t = 0$ is defined by the conditions

$$\begin{aligned} &\delta(t) = 0 \text{ for } t \neq 0 \\ &\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \end{aligned} \tag{38}$$

From (4) we can write $\delta(t) = \sum_{k=-\infty}^{\infty} \delta(kv)L(t-kv)$ which (since $\delta(kv) = 0$ for $k \neq 0$) reduces to $\delta(t) = \delta(0)\,L(t)$. Using this result in the second equation of (38) gives $\int_{-\infty}^{\infty} \delta(t)\, dt = \delta(0) \int_{-\infty}^{\infty} L(t)dt = v\,\delta(0) = 1$. Hence $\delta(0) = \frac{1}{v}$. Insertion of this into the transform $\widetilde{\delta}(x) = \sum_{-\infty}^{\infty} \delta(kv)\, x^k$ gives

$$\widetilde{\delta}(x) = \frac{1}{v} \tag{39}$$

The time origin of any of the above functions may be changed from 0 to mv by use of our "shifting" Theorem 3. The corresponding operation in the P-domain is multiplication by x^m. For example, the transform of the delta function located "at" mv, namely, $\delta(t-mv)$ is

$$P\left[\delta(t-mv)\right] = \frac{x^m}{v} \qquad (39\ )$$

These results may be collected in a table, one column listing the time function, the other the corresponding P-transform. Such a table can be made quite extensive and would serve the same general purposes as a table of Laplace transforms or a table of Fourier transforms. The few items listed above suffice for our present purpose of showing how such a table can be constructed.

We now apply these methods to the solution of ordinary linear differential equations.

LINEAR DIFFERENTIAL EQUATIONS

The few theorems set down above enable us to solve ordinary linear differential equations. The general method of attack can be clearly indicated by a couple of simple examples.

Consider the first order equation with its initial condition,

$$\frac{dH}{dt} + BH = F(t)$$

$$\text{with } H(0) = H_o \tag{40}$$

Integrating from 0 to t,

$$\int_o^t \dot{H}(t)dt + B\int_o^t H(t)dt = \int_o^t F(t)dt$$

$$= H(t) - H(0) + B\int_o^t H(t)dt$$

Inserting the initial condition $H(0) = H_o$ and write this as $H_oS(t)$ where $S(t)$ is the unit step function, we have

$$H(t) + B\int_o^t H(t)dt = \int_o^t F(t)dt + H_oS(t) \tag{41}$$

which is now in proper condition to apply the P-transformation.

Theorems 1 and 2 show that P[ ] is a linear operation (applies distributively and commutes with a constant); hence, we have

$$P\left[H(t)\right] + BP\left[\int_o^t H(t)dt\right] = P\left[\int_o^t F(t)dt\right] + H_oP\left[S(t)\right] \tag{42}$$

The first term on the left is simply $\tilde{H}(x)$. The second term on the left, by Theorem 8, is $B\left[\frac{v}{2}\frac{1+x}{1-x}\tilde{H}(x) - \frac{v}{2}\frac{H_ox}{1-x}\right]$, the last term of which is a minor correction (vanishing in the limit as $v \to 0$) and will be omitted. The first term on the right is similarly $\frac{v}{2}\frac{1+x}{1-x}\tilde{F}(x)$. The last term on the right, see (31), is $H\tilde{S}(x) = \frac{H_o}{1-x}$. Putting these values in place of the respective terms in (42), and solving for $\tilde{H}(x)$, we have

$$\tilde{H}(x) = \frac{\frac{v}{2}\frac{1+x}{1-x}}{1 + \frac{Bv}{2}\frac{1+x}{1-x}} \cdot \tilde{F}(x) + \frac{\frac{H_o}{1-x}}{1 + \frac{Bv}{2}\frac{1+x}{1-x}} \tag{43}$$

The coefficient of $\tilde{F}(x)$ corresponds to the transfer function of Laplace transform theory and to the frequency response of Fourier transform theory. The second term in (43) is due to the initial condition. In the so-called normal solution (initially quiescent system with dependent variable and all its derivatives zero) the second term in (43) vanishes - which is usually the case treated in Laplace transform theory.

It is interesting to note the correspondence here with usual operational methods. Thus, writing (40) in operational notation, pH + BH = F, and solving in the form

$$H = \frac{\frac{1}{p}}{1 + \frac{B}{p}} \; F,$$

we see that our "transfer" operator in (43) could have been obtained simply by regarding $\frac{1}{p}$ as our integrating operator $\frac{v}{2}\,\frac{1+x}{1-x}$ --- or $p = \frac{2}{v}\,\frac{1-x}{1+x}$ as our differentiating operator.

But equation (43) can be simplified still more. Clearing out the 1-x and simplifying, we have

$$\tilde{H}(x) = \frac{v(1+x)\tilde{F}(x) + 2H_o}{(Bv+2)+(Bv-2)x} \;. \qquad (43\;)$$

which is an easily manipulated form.

At this stage in our process of finding H(t), one could throw $\tilde{H}(x)$ into a sum of elementary types to be found in a table of P-transforms, and write out the inverse transformations to obtain H(t). The procedure here would follow the familiar pattern of Laplace transform theory. But this procedure is not necessary.

An outstanding virtue of this P-transform method lies in its easy handling of numerical problems. For example, in the case of the above differential equation, every quantity except x on the right side of (43) would be given. The denominator in this expression can be (by dividing by Bv+2) put into the form 1+Ax. The numerator reduces, after collecting coefficients of like powers of x, to a simple polynomial. After division by 1+Ax, we get then a simple polynomial $\tilde{H}(x) = H_o + H_1 x + H_2 x^2 + \cdots + H_n x^n + \cdots$etc., where all the H_n are mere numbers. This sequence, $H_o, H_1, H_2, \ldots, H_n, \ldots$etc., of coefficients give the successive ordinates of H(nv) = H(t), and may be plotted to give a graph closely approximating our desired H(t). Thus, no formal work whatever is required to pass from $\tilde{H}(x)$ to H(t) - a mere inspection of $\tilde{H}(x)$ gives H(t), and vice versa.

The above procedure for solving a linear differential equation subject to its initial conditions may be formalized into the following steps:

(1) Apply to the differential equation the definite integration $\int_0^t (\quad)dt$, as many times as the order of the equation. This introduces the initial conditions into our solution and puts the equation into proper form for applying our P-transformation.

(2) Apply to the result of Step 1 the P-transformation, make use of Theorems 1 to 8 to reduce every term containing H(t), its derivatives or integrals into terms containing $\widetilde{H}(x)$.

(3) Apply to the result of Step 2 the indicated operations of ordinary algebra including polynomial multiplication and division, thus reducing $\widetilde{H}(x)$ to the simple polynomial form $H(x) = H_0 + H_1 x + H_2 x^2 + \text{etc.}$

(4) Pick off the coefficients (or time ordinates) H_0, H_1, H_2, H_3, etc., plot at the successive points 0, v, 2v, 3v, etc. on a time axis, fair a curve through these points to obtain an approximation to the required solution H(t).

Consider now the second order equation with normal boundary conditions

$$\frac{d^2H}{dt^2} + A\frac{dH}{dt} + BH = F(t) \qquad \text{with } H_0 = \dot{H}_0 = 0 \tag{44}$$

The system is assumed to be initially quiescent (dependent variable and all its derivatives being zero at t=0) to conform to the usual case treated in Laplace transform theory. All this assumption does is to omit a number of additional terms which can be easily handled but which clutter up the equations.

Applying step 1, we integrate the equation from 0 to t, with the result

$$\dot{H}(t) + AH(t) + B\int_0^t H(t)dt = \int_0^t F(t)dt$$

where we omit a term $(\dot{H}_o + AH_o)\delta(t)$ from the right side due to initial values, because $\dot{H}_o = H_o = 0$. Integrating a second time,

$$H(t) + A\int_0^t H(t)dt + B\int_0^t dr\int_0^r H(s)ds = \int_0^t dr\int_0^r F(s)ds \qquad (44')$$

where a term $H_0\delta(t)$ is again omitted because H_o vanishes.

As in step 2 above, we now apply the P-transformation. The only new item here is the double integral. This is handled by a double application of Theorem 8. Thus, writing out the result in full, we have

$$P\left[\int_0^t dr\int_0^r F(s)ds\right] = \frac{v}{2}\frac{1+x}{1-x}\left[\frac{v}{2}\frac{1+x}{1-x}\tilde{F}(x) - \frac{v}{2}\frac{F_o x}{1-x}\right] - \frac{vx}{2(1-x)}\int_0^o F(t)dt$$

The last term vanishes (unless we have an impulse at t=0, a case we omit here for brevity). The last term inside the bracket is only a small correction (vanishing as $v\to 0$) which may be carried along by those who wish, but which will be omitted here for brevity. Thus, we shall have

$$P\left[\int_o^t dr\int_o^r F(s)ds\right] = \left(\frac{v}{2}\frac{1+x}{1-x}\right)^2\tilde{F}(x) \qquad (45)$$

A still better transform for a double integral is obtained by using the transform operator of (27). Let us compromise here and use both, the operator of (27) once and that of (21) once. This gives

$$\begin{aligned} P\left[\int_o^t dr\int_o^r F(s)ds\right] &= \frac{v}{3}\frac{1+4x+x^2}{1-x^2}\cdot\frac{v}{2}\frac{1+x}{1-x}\tilde{F}(x) \\ &= \frac{v^2}{6}\frac{1+4x+x^2}{(1-x)^2}\tilde{F}(x) \end{aligned} \qquad (45\ )$$

where, as above, we have omitted one term which vanishes and another which, for sufficiently small v, is negligible.

Putting the result (45) in for double integrals, as well as the more familiar transforms for the other terms, equation (44) transforms into,

$$\tilde{H}(x) + A\left(\frac{v}{2}\frac{1+x}{1-x}\right)\tilde{H}(x) + B\left(\frac{v}{2}\frac{1+x}{1-x}\right)^2\tilde{H}(x) = \left(\frac{v}{2}\frac{1+x}{1-x}\right)^2\tilde{F}(x)\ .$$

Solving this for $\tilde{H}(x)$ gives

$$\tilde{H}(x) = \frac{\left(\frac{v}{2}\frac{1+x}{1-x}\right)^2}{1 + A\left(\frac{v}{2}\frac{1+x}{1-x}\right) + B\left(\frac{v}{2}\frac{1+x}{1-x}\right)^2}\ \tilde{F}(x) \tag{46}$$

The parallel here between the polynomial coefficient of $F(x)$ and the corresponding transfer function of Laplace theory will be immediately recognized.

In place of the operator $\frac{v}{2}\frac{1+x}{1-x}$ used in (46), one may use the operator $\frac{v}{3}\ \frac{1+4x+x^2}{1-x^2}$ of equation (27) - or one may use $\frac{v}{2}\ \frac{1+x}{1-x}$ as the first order operator and the operator $\frac{v^2}{6}\ \frac{1+4x+x^2}{(1-x)^2}$ of (45) for the second order.

By clearing the denominators $2(1-x)$ from (46), we immediately simplify this equation to

$$\tilde{H}(x) = \frac{v^2\ (1+2x+x^2)\ \tilde{F}(x)}{(4+2Av+Bv^2) - (8-2Bv^2)x + (4-2Av+Bv^2)\ x^2} \tag{46'}$$

Further reduction of $\tilde{H}(x)$ to the simple polynomial form $H(x) = H_0 + H_1 x + H_2 x^2 +$ etc. by polynomial division would appear to be profitable if we have numerical values to put into (46'). The time function $H(t)$ is immediately specified, as before, by its ordinate sequence H_0, H_1, H_2, etc. picked off by inspection of $\tilde{H}(x)$.

The P-transform solution of the two simple equations worked out above was given in considerable detail. These examples should make the pattern clear so that one should be able to write down by inspection the transform equation for the normal solution of a linear equation of any order. As a final example, we write out the result by inspection for the third order equation

$$\frac{d^3H}{dt^3} + A\ \frac{d^2H}{dt^2} + B\frac{dH}{dt} + CH = F(t)$$

or in operational form

$$(p^3 + Ap^2 + Bp + c)\ H = F\ .$$

From this, $\quad \tilde{H} = \dfrac{\tilde{F}}{p^3 + Ap^2 + Bp + c}$

in which we can now replace p by our differentiator $\frac{2}{v}\ \frac{1-x}{1+x}$ or one of its substitutes. One may also throw the equation into the form

$$\tilde{H} = \frac{(\frac{1}{p})^3 F}{1 + A(\frac{1}{p}) + B(\frac{1}{p})^2 + c(\frac{1}{p})^3}$$

and replace $\frac{1}{p}$ by the integrator $\frac{\tau}{2}\ \frac{1+x}{1-x}$ or one of its substitutes.

In case one is interested in a problem where the initial values of H and all its derivatives do not vanish, the more careful procedure used in solving our first equation above should be followed in order to get these initial conditions introduced properly into the problem. Additional care must be taken if one wishes to carry along the small corrections which were dropped.

In order to emphasize again the practical arithmetic advantages of our P-transform method, a numerical example is worked out below.

EXAMPLE: The following example illustrates the arithmetic convenience of the polynomial-transform method - as well as the other advantages inherent in all operational methods.

Consider the simple servomechanism[8] shown in Figure 5. The torque equation is $T = J\ddot{\theta}_o + F\dot{\theta}_o$, the controller equation is $T = KE$, and the error equation is $E = \theta_i - \theta_o$. Eliminating θ_o and T, we obtain for the error the differential equation

$$J\ddot{E} + F\dot{E} + KE = J\ddot{\theta}_i + F\dot{\theta}_i$$

Transforming parameters in the usual way by writing $\omega_n = \sqrt{K/J}$ = circular frequency of free undamped vibration, $F_C = 2\sqrt{KJ}$ = critical damping coefficient, and $C = F/F_C$ = damping ratio (which for conciseness we take here as 1), the differential equation becomes

$$\ddot{E} + 2\omega_n\dot{E} + \omega_n^2 E = \ddot{\theta}_i + 2\omega_n\dot{\theta}_i$$

The operational form of this equation is

$$E = \frac{p^2 + 2\omega_n p}{p^2 + 2\omega_n p + \omega_n^2}\ \theta_i$$

In order to make the problem numerical, let us take $\omega_n = 20$, and the driving function as the step velocity $\theta_i = \omega_i tS(t) = tS(t)$. Then

$$E = \frac{p^2 + 40p}{p^2 + 40p + 400}\ t \qquad (47)$$

The analytical solution of this equation (normal solution with $E(0) = \dot{E}(0) = 0$) is

$$E = \frac{1}{10}\left[1 - e^{-20t}(1 + 10t)\right] \tag{48}$$

It is this equation (47) which we solve here by the P-transform method. We shall then compare our numerical results with those of the analytical solution (48).

We take the time interval as $v = 0.05$. Our differentiating P-operator P-operator becomes $\frac{2}{v}\,\frac{1-x}{1+x} = 40\,\frac{1-x}{1+x}$. The ordinate sequence of the driving function is $t = 0.05\,(0,1,2,3,\ldots etc.)$; hence, the transform of the driving function is $\tilde{\Theta}_i(x) = 0.05\left[0+x+2x^2+3x^3+4x^4+\ldots etc.\right] = P[t]$. Inserting these into (47) gives

$$\tilde{E}(x) = \frac{1600\left(\frac{1-x}{1+x}\right)^2 + 1600\,\frac{1-x}{1+x}}{1600\left(\frac{1-x}{1+x}\right)^2 + 1600\,\frac{1-x}{1+x} + 400}\;0.05\,(x+2x^2+3x^3+\ldots etc.)$$

$$= \frac{1}{5}\,\frac{(1-x)^2 + (1+x)(1-x)}{4(1-x)^2 + 4(1+x)(1-x) + (1+x)^2}\;(x+2x^2+3x^3+\ldots etc.)$$

$$= 0.4\,\frac{(1-x)(x+2x^2+3x^3+\ldots etc.)}{9 - 6x + x^2}$$

$$= 0.4\,\frac{x + x^2 + x^3 + \ldots etc.}{9 - 6x + x^2}$$

$$= .044x + .074x^2 + .089x^3 + .096x^4 + .098x^5 + .099x^6 + \ldots + 1.00x^{\infty}.$$

The coefficients here are the respective ordinates of the required time function $E(t)$ at .05, .10, .15, .20,...etc. The following table, which compares these results with those calculated from the analytical solution (48), shows the practical degree of accuracy of our polynomial transform method.

TIME	.00	.05	.10	.15	.20	.25	.30	.....	∞
ANALYSIS	.000	.041	.073	.087	.094	.097	.099	....	.100
P-TRANSFORM	.000	.044	.074	.089	.096	.098	.099	....	.100

Figure 6 shows a graph of the driving function $\theta_i(t) = t$, and shows how easy it is to write down by inspection the driving transform $\Theta_i(x) = .05x + .10x^2 + .15x^3 + \text{etc.} = .05\,(x + 2x^2 + 3x^3 + \ldots \text{etc.}$ This figure also shows a graph of the analytical solution together with points showing the ordinates calculated above by the P-transform method.

The above dissertation on linear differential equations ought to be greatly elaborated. Our short treatment serves, however, to outline how the new calculus may be applied to linear differential equations, and outlines how a more complete presentation can be worked out.

CONCLUSION

The polynomial - transform method outlined in this paper is worthy of a complete development. It has the virtue of great simplicity. The analysis follows the natural mode of thought, since it remains always in the time domain (the P-transform itself being so obviously connected with the time domain). The very real difficulties in Laplace and Fourier transform theory in calculating direct and inverse transforms are avoided completely here, the connection between time function and its P-transform being so obvious that if one has either function one can write down the other by inspection. The arithmetic convenience of the new methods is obvious, since actual calculations involve only a high school knowledge of algebra. And finally, whereas in the older methods the synthesis and instrument types of problem are commonly regarded as much more difficult than the analysis problem, a noteworthy feature of the polynomial-transform method is the trivial distinction between these types.

A paper will appear later applying the new methods to non-linear problems. Useful work on non-linear problems has already been done by Madwed[1].

A paper will appear shortly applying the P-transform methods to "noise" problems (random disturbances in electrical circuits, servomechanisms, instruments, and so on). The Wiener - Kolmogoroff theory of smoothing and predicting filters will be reduced to a simple, immediately obvious set of algebraic operations.

APPENDIX A

Consider the P-transform

$$\hat{F}(x) = \sum_{n=-N}^{N} F(nv)\ x^n$$

Let $nv = t$, $v = dt \to 0$, $Nv = \infty$, and set

$$x = e^{i\omega v},$$

then, passing to the limit,

$$v\hat{F}(x) = v\sum_{-N}^{N} F(nv)\ (e^{i\omega v})^n = \sum_{-Nv}^{Nv} F(nv)\, e^{i\omega(nv)} \cdot v$$

$$\longrightarrow \int_{-\infty}^{\infty} F(t)\, e^{i\omega t}\, dt = \hat{F}(\omega)$$

where $\hat{F}(\omega)$ is the Fourier transform of $F(t)$. Thus, we see a close relation between our P-transform and the Fourier transform. Had we taken $x = e^{-sv}$ and kept the summation unilateral from 0 to ∞, the limiting form would have been the Laplace transform.

Consider now the inverse Fourier transform,

$$F(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{F}(\omega)\, e^{-i\omega t}\, d\omega$$

This is, as shown above, the limiting value of

$$f(t) = \frac{1}{2\pi} \int_{-\frac{\pi}{v}}^{\frac{\pi}{v}} \left(v\sum_{-N}^{N} F(nv)\ x^n\right) e^{-i\omega t}\, d\omega$$

as $v \to 0$. Replacing x by $e^{i\omega v}$, and interchanging order of summation and integration, we have

$$f(t) = \frac{v}{2\pi} \sum_{-N}^{N} F(nv) \int_{-\frac{\pi}{v}}^{\frac{\pi}{v}} e^{i\omega(nv-t)}\, d\omega$$

$$= \frac{v}{2\pi} \sum_{-N}^{N} F(nv) \frac{e^{i\frac{\pi}{v}(nv-t)} - e^{-i\frac{\pi}{v}(nv-t)}}{i(nv-t)}$$

$$= \sum_{n=-N}^{N} F(nv) \frac{\sin\frac{\pi}{v}(t-nv)}{\frac{\pi}{v}(t-nv)}$$

which is a determining or time function corresponding to the P-transform $\bar{F}(x)$. Thus, the inverse Fourier transform leads to the inverse P-transform. Conversely, one can start with the inverse P-transform and, upon passing to the limit, obtain the inverse Fourier transform.

APPENDIX B

In the proof of Theorem 8, the cardinal function itself may be used instead of the triangular interpolation function. Thus, in calculating the coefficients S_k of equation (22), we set

$$S_k = \int_0^{nv} \frac{\sin \frac{\pi}{v}(t-kv)}{\frac{\pi}{v}(t-kv)} \, dt$$

This can be written as

$$S_k = \frac{v}{\pi} \int_{t=0}^{nv} \frac{\sin \frac{\pi}{v}(t-kv)}{\frac{\pi}{v}(t-kv)} \; D\left[\frac{\pi}{v}(t-kv)\right]$$

Changing the variable of integration to $u = \frac{\pi}{v}(t-kv)$ and replacing the limits by the corresponding values of u, we have

$$S_k = \frac{v}{\pi} \int_{-k\pi}^{(n-k)\pi} \frac{\sin u}{u} \, du = \frac{v}{\pi}\left[\int_0^{(n-k)\pi} \frac{\sin u}{u} \, du + \int_0^{k\pi} \frac{\sin u}{u} \, du\right]$$

$$= \frac{v}{\pi}\left\{ \mathrm{Si}\left[(n-k)\pi\right] + \mathrm{Si}\,(k\pi)\right\}$$

The sine-integral function ($\mathrm{Si}\, x = \int_0^x \frac{\sin u}{u} \, du$) increases[9] from 0 at $x = 0$ and oscillates with decreasing amplitude about the value $\frac{\pi}{2}$, approaching $\frac{\pi}{2}$ as x increases, the greatest overshoot being about 9% with the amplitude of overshoot decreasing approximately hyperbolically.

With this information about the sine-integral function, consider then the result,

$$S_k = \frac{v}{\pi}\left\{ \mathrm{Si}\left[(n-k)\pi\right] + \mathrm{Si}(k\pi)\right\}.$$

For $k = 0$, the second term vanishes and $S_0 = \frac{v}{\pi}\,\mathrm{Si}(n\pi)$ which approximates $\frac{v}{\pi} \cdot \frac{\pi}{2} = \frac{v}{2}$ even for moderate values of n. Similarly, for $k = n$, the first term vanishes and $S_n = \frac{v}{\pi} Si(n\pi)$ which can again be taken as $\frac{v}{2}$. For $k = \frac{n}{2}$, $S_{\frac{n}{2}} = \frac{2v}{\pi} Si(\frac{n}{2}\pi)$, so that $S_{\frac{n}{2}} = v$ approximately. In fact, due to the rapid increase of $Si(x)$ from 0 to $\frac{\pi}{2}$ as x increases from 0, all intermediate values of S_k can be taken as approximately v. Accordingly, we have approximately,

$$S_k = v \text{ for } 0 < k < n$$
$$= \frac{v}{2} \text{ for } k = 0 \text{ or } n$$

confirming our result (24).

REFERENCES

(1) The initial paper presenting the principal ideas of the operational methods developed here was presented in a highly condensed form by Dr. R. G. Piety in a paper, "A Linear Operational Calculus of Empirical Functions", before the March 1951 meeting of the IRE in New York City.

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A much fuller treatment from this point of view is presented by Madwed, "Number Series Method of Solving Linear and Non-Linear Differential Equations", Report 6445-T-26, Instrumentation Laboratory, MIT.

(2) Marquis de Laplace, "Theorie Analytique des Probabilities", Courcier, Paris, 1812. Generating polynomials formed the basis of this great work of Laplace, although they have been neglected since his time.

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An excellent discussion of generating functions also appears in, "An Introduction to Probability Theory and Its Applications", W. Feller, Wiley, 1950.

(3) James, Nichols and Phillips, "Theory of Servomechanisms", Vol. 24, Rad. Lab. Series, McGraw-Hill.

(4) "Contributions to the Problem of Approximation of Equidistant Data by Analytic Functions", by I. J. Schoenberg, Quar. App. Math., April and June 1946.

Considerable use is made of the cardinal function by Wheeler, "The Interpretation of Amplitude and Phase Distortion in Terms of Paired Echoes", Proc. IRE, June 1939.

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(5) Shown by direct integration in, "On an Integral Equation", G. H. Hardy, Proc. Lon. Math. Soc. 27, 7 (1909).

See also, "On the Cardinal Function of Interpolation Theory", by W. L. Ferrar, Proc. Roy. Soc. Edin., May 1925, June 1926, April 1927.

(6) Karman and Biot, "Mathematical Methods in Engineering", McGraw-Hill.

(7) Gardner and Barnes, "Transients in Linear Systems", John Wiley and Sons.

(8) See "Servomechanism Fundamentals", by Lauer. Lesnick, Matson, p. 76, McGraw-Hill.

(9) See the book, "Frequency Analysis, Modulation and Noise", by Goldman, McGraw-Hill.

FIGURE 1

Black-box Representation of a Physical System

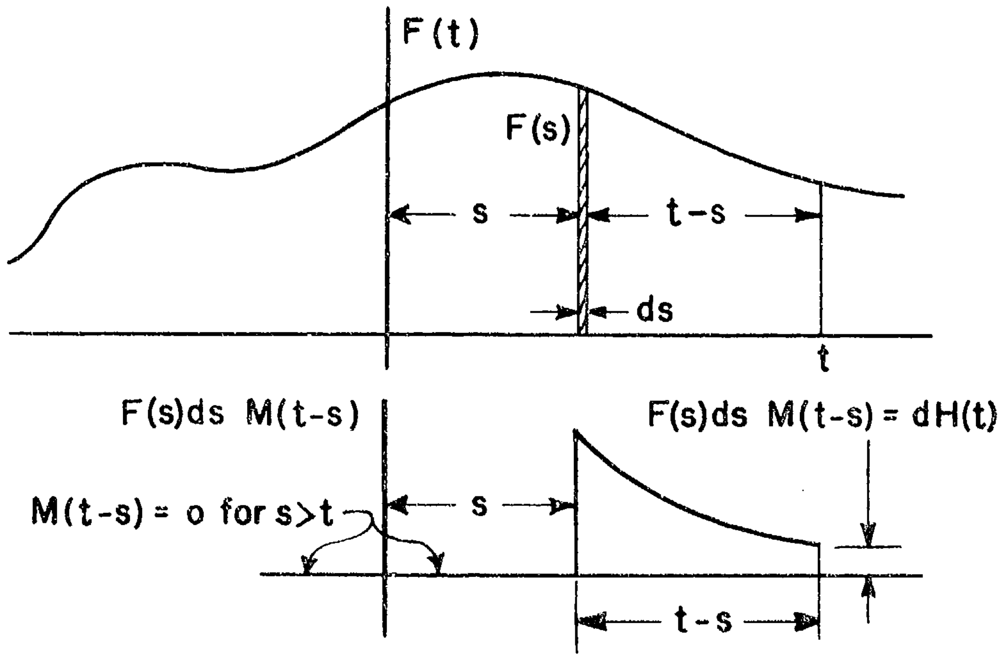
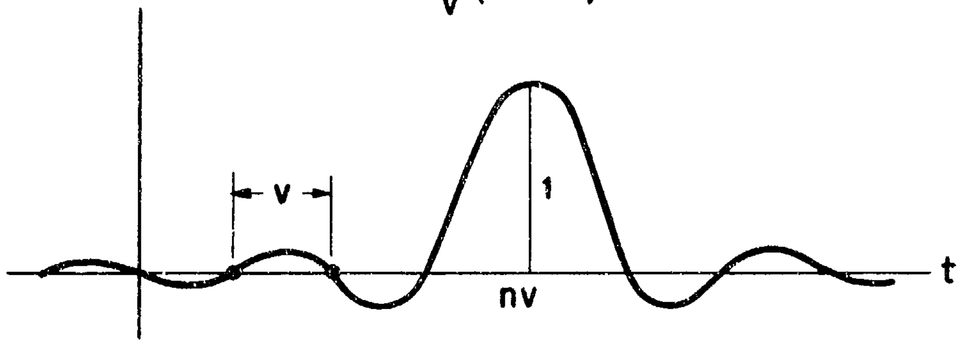


FIGURE 2

$$L(t-nv) = \frac{\sin \frac{\pi}{v}(t-nv)}{\frac{\pi}{v}(t-nv)}$$



$$f(t) = \sum_{k=o}^{m} F_k L(t-kv)$$

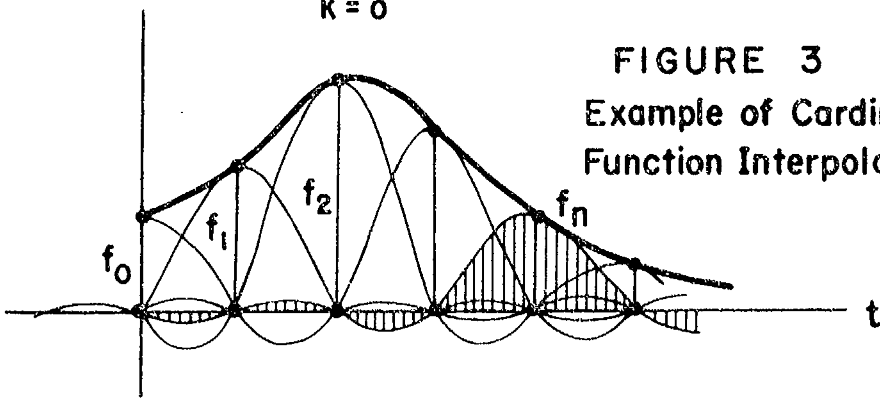


FIGURE 3
Example of Cardinal Function Interpolation

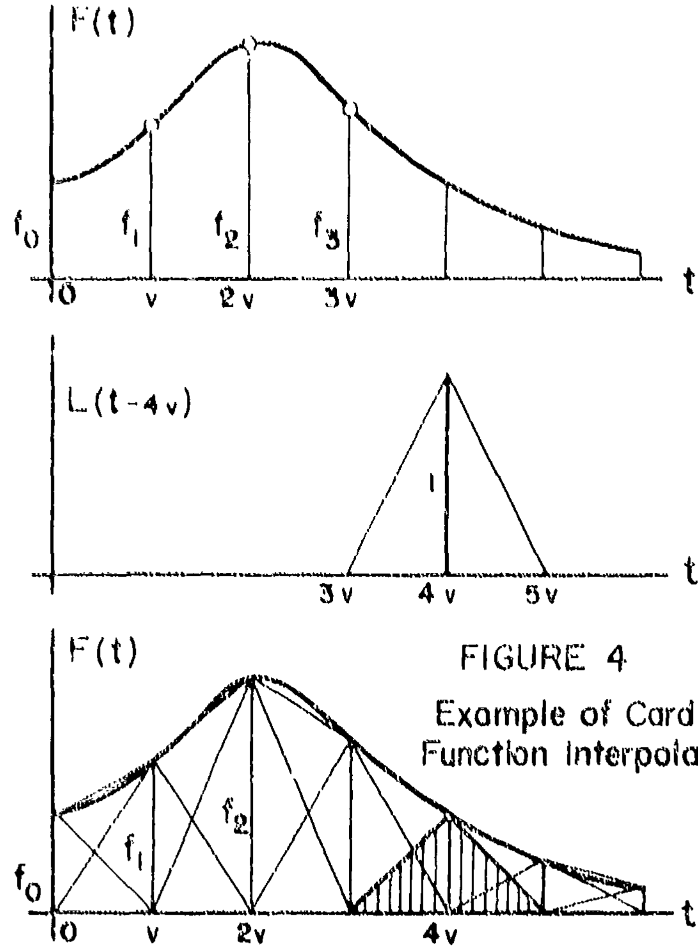


FIGURE 4
Example of Cardinal Function Interpolation

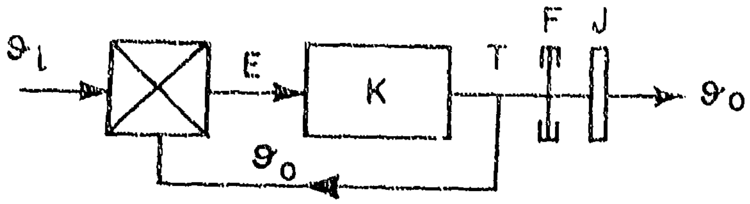


FIGURE 5

Schematic of a Simple Servomechanism

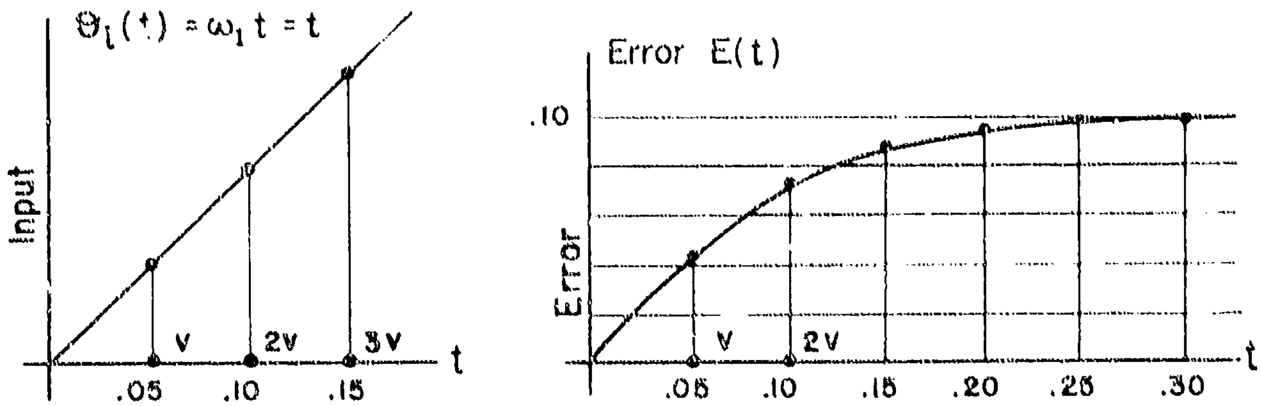


FIGURE 6

Servomechanism Input and Error Spectra

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Dear Ms. Akers,

This concerns Technical Report ~~AD~~ ADB183516, A New Linear Operational Calculus by Frank W. Bub, Ph.D., May 1951. This technical report, previously Unclassified/Limited Distribution, is now releasable to the public. The attached AFMC Form 559 verifies that it was reviewed by release authorities at Air Force Research Lab Air Vehicles Directorate (AFRL/VA).

Please call me at (937) 904-8189 if you have any questions.

Sincerely

Lynn Kane

Lynn Kane
Freedom of Information Act Analyst
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Base Information Management Division

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